Generalized Substitutions and Stepped Surfaces

Pierre Arnoux[1], Valérie Berthé[2], and Damien Jamet[2]

[1] Institut de Mathématique de Luminy, CNRS UMR 6206, 163 avenue de Luminy, 13288 Marseille Cedex 9 - FRANCE.
arnoux@iml.univ-mrs.fr
[2] LIRMM-UMR 5506, Université Montpellier II, 161 rue Ada, 34392 Montpellier Cedex 5 - FRANCE.
{berthe,jamet}@lirmm.fr

Abstract A substitution is a non-erasing morphism of the free monoid. The notion of multidimensional substitution of non-constant length acting on multidimensional words introduced in [AI01,ABS04] is proved to be well-defined on the set of two-dimensional words related to discrete approximations of irrational planes. Such a multidimensional substitution can be associated to any usual Pisot unimodular substitution. The aim of this paper is to try to extend the domain of definition of such multidimensional substitutions. In particular, we study an example of a multidimensional substitution which acts on a stepped surface in the sense of [Jam04,JP04].

1 Introduction

Sturmian words are known to be codings of digitizations of an irrational straight line [KR04,LOTH02]. One could expect from a generalization of Sturmian words that they correspond to a digitization of a hyperplane with irrational normal vector. It is thus natural to consider the following digitization scheme corresponding to the notion of arithmetic planes introduced in [Rev91]: this notion consists in approximating a plane in $\mathbb{R}^3$ by selecting points with integral coordinates above and within a bounded distance of the plane; more precisely, given $\mathbf{v} \in \mathbb{R}^3$, $\mu, \omega \in \mathbb{R}$, the lower (resp. upper) discrete hyperplane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is the set of points $\mathbf{x} \in \mathbb{Z}^d$ satisfying $0 \leq \langle \mathbf{x}, \mathbf{v} \rangle + \mu < \omega$ (resp. $0 < \langle \mathbf{x}, \mathbf{v} \rangle + \mu \leq \omega$). Moreover, if $\omega = \sum |v_i| = |\mathbf{v}|_1$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said to be standard.

In this latter case, one approximates a plane with normal vector $\boldsymbol{v} \in \mathbb{R}^3$ by square faces oriented along the three coordinates planes; for each of the three kinds of faces, one defines a distinguished vertex; the standard discrete plane $\mathfrak{P}(\mathbf{v}, \mu, |\mathbf{v}|_1)$ is then equal to the set of distinguished vertices; after projection on the plane $x + y + z = 0$, along $(1, 1, 1)$, one obtains a tiling of the plane with three kinds of diamonds, namely the projections of the three possible faces. One can code this projection over $\mathbb{Z}^2$ by associating to each diamond the name of the projected face. These words are in fact three-letter two-dimensional Sturmian words (see e.g. [BV00]).

A generalization of the notion of stepped plane, the so-called discrete surfaces, is introduced in [Jam04]. Roughly speaking, a discrete surface is a union of pointed faces such that the orthogonal projection on the plane $x+y+z=0$ induces an homeomorphism from the discrete surface to the plane. As done for stepped planes, one provides any discrete surface with a coding as a two-dimensional word over a three-letter alphabet. In the present paper, we call discrete surfaces *stepped surfaces*, since such objects are not discrete, in the sense, that they are not subsets of $\mathbb{Z}^3$.

Let us recall that a substitution is a non-erasing morphism of the free monoid. A notion of multidimensional substitution of non-constant length acting on multidimensional words is studied in [AI01,AIS01,ABI02,ABS04], inspired by the geometrical formalism of [IO93,IO94]. These multidimensional substitutions are proved to be well-defined on multidimensional Sturmian words. Such a multidimensional substitution can be associated to any usual Pisot unimodular substitution. The aim of the present paper is to explore the domain of definition of such generalized substitutions. For the sake of clarity, we have chosen to work out in full details the example of [ABS04]. We prove that the image of a stepped surface under the action of this multidimensional substitution is well-defined. Our proofs will be based on a geometrical approach. We then use the functionality and the projection on the plane $x+y+z=0$ along $(1,1,1)$ to recover the corresponding results for multidimensional words.

We work here in the three-dimensional space for clarity issues but all the results and methods presented extend in a natural way to $\mathbb{R}^n$.

2 Basic notions

2.1 One-dimensional substitutions

Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^\star$ be the set of finite words over $\mathcal{A}$. The empty word is denoted by ε. A *substitution* is an endomorphism of the free-monoid $\mathcal{A}^\star$ such that the image of every letter of $\mathcal{A}$ is non-empty. Such a definition naturally extends to infinite or biinfinite words in $\mathcal{A}^\mathbb{N}$ and $\mathcal{A}^\mathbb{Z}$.

We assume $\mathcal{A}=\{1,\ldots,d\}$. Let σ be a substitution over $\mathcal{A}$. The *incidence matrix* of σ, denoted $M_\sigma=(m_{i,j})_{(i,j)\in\{1,\ldots,d\}^2}$, is defined by:

$$M_\sigma = (|\sigma(j)|_i)_{(i,j)\in\{1,\ldots,d\}^2},$$

where $|\sigma(j)|_i$ is the number of occurrences of i in $\sigma(j)$.

Let $\psi:\ \mathcal{A}^\star\to\mathbb{N}^d$, $w\mapsto(|w|_i)_{i\in\{1,\cdots,d\}}$ be the Parikh mapping, that is, the homomorphism obtained by abelianization of the free monoid. One has for every $w\in\mathcal{A}^\star$, $\psi(\sigma(w))=M_\sigma\psi(w)$.

Example 1. Let $\sigma:\{1,2,3\}\longrightarrow\{1,2,3\}^\star$ be the substitution defined by $\sigma: 1\mapsto 13,\ 2\mapsto 1,\ 3\mapsto 2$. Then,

$$M_\sigma = \begin{pmatrix} 1 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}.$$

A substitution σ is said to be a *Pisot substitution* if the characteristic polynomial of its incidence matrix M_σ admits a dominant eigenvalue $\lambda > 1$ such that all its conjugates α satisfy $0 < |\alpha| < 1$. The incidence matrix of a Pisot substitution is primitive [CS01], that is, it admits a power with positive entries.

Finally, a substitution is said to be *unimodular* if $\det M_\sigma = \pm 1$.

From now on, let σ denote a unimodular Pisot substitution over the three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$.

2.2 Stepped planes

There are several ways to approximate planes by integer points [BCK04]. Usually, these methods consist in selecting integer points within a bounded distance from the considered plane. Such objects are called *discrete planes*.

Let $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ be the canonical basis of $\mathbb{R}^3$. We call *unit cube* any translate of the *fundamental unit cube* with integral vertices, that is, any set $\mathbf{x} + \mathcal{C}$ where $\mathbf{x} \in \mathbb{Z}^3$ and $\mathcal{C}$ is the fundamental unit cube:

$$\mathcal{C} = \left\{ \lambda_1 \mathbf{e_1} + \lambda_3 \mathbf{e_3} + \lambda_3 \mathbf{e_3} \mid (\lambda_1, \lambda_2, \lambda_3) \in [0,1]^3 \right\}.$$

Let $\mathcal{P} : \ \langle \mathbf{v}, \mathbf{x} \rangle + \mu = 0$, with $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$. The *stepped plane* $\mathfrak{P}_\mathcal{P}$ associated to $\mathcal{P}$, also called discrete plane in [ABS04], is defined as the union of the faces of the integral cubes that connect the set $\{\mathbf{x} \in \mathbb{Z}^3 \mid 0 \leq \langle \mathbf{v}, \mathbf{x} \rangle + \mu < \|\mathbf{v}\|_1 = \sum v_i\}$. More precisely:

Definition 1. *[IO93,IO94] We consider the plane $\mathcal{P} : \ \langle \mathbf{v}, \mathbf{x} \rangle + \mu = 0$, with $\mathbf{v} \in \mathbb{R}^3_+$ and $\mu \in \mathbb{R}$. Let $\mathcal{C}_\mathcal{P}$ be the union of the unit cubes intersecting the open half-space of equation $\langle \mathbf{v}, \mathbf{x} \rangle + \mu < 0$. The* stepped plane *$\mathfrak{P}_\mathcal{P}$ associated to $\mathcal{P}$ is defined by: $\mathfrak{P}_\mathcal{P} = \overline{\mathcal{C}_\mathcal{P}} \setminus \mathring{\mathcal{C}}_\mathcal{P}$, where $\overline{\mathcal{C}_\mathcal{P}}$ (resp. $\mathring{\mathcal{C}}_\mathcal{P}$) is the closure (resp. the interior) of the set $\mathcal{C}_\mathcal{P}$ in $\mathbb{R}^3$, provided with its usual topology. The vector $\mathbf{v}$ (resp. μ) is called the* normal vector *(resp. the* translation parameter*) of the stepped plane $\mathfrak{P}_\mathcal{P}$.*

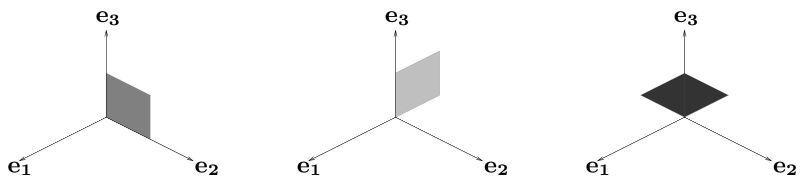
It is clear, by construction, that a stepped plane is connected and is a union of faces of unit cubes. In fact, by introducing a suitable definition of faces, we can describe the stepped plane as a partition of such faces, as detailed below.

Let E_1, E_2 and E_3 be the three following *fundamental faces* (see Figure 1):

$$\begin{aligned} E_1 &= \left\{ \lambda \mathbf{e_2} + \mu \mathbf{e_3} \mid (\lambda, \mu) \in [0,1[^2 \right\}, \\ E_2 &= \left\{ -\lambda \mathbf{e_1} + \mu \mathbf{e_3} \mid (\lambda, \mu) \in [0,1[^2 \right\}, \\ E_3 &= \left\{ -\lambda \mathbf{e_1} - \mu \mathbf{e_2} \mid (\lambda, \mu) \in [0,1[^2 \right\}. \end{aligned}$$

For $\mathbf{x} \in \mathbb{Z}^3$ and $i \in \{1, 2, 3\}$, the *face of type i pointed on $\mathbf{x} \in \mathbb{Z}^3$* is the set $\mathbf{x} + E_i$. Let us notice that each pointed face includes exactly one integer point, namely, its *distinguished vertex*. As mentioned above we obtain:

Theorem 1. *[BV00,ABI02] A stepped plane $\mathfrak{P}$ is partitioned by its pointed faces.*



(a) E_1. (b) E_2. (c) E_3.

Fig. 1. The three fundamental faces.

Finally, an easy way to characterize the type of a pointed face included in a stepped plane is given by:

Theorem 2. *Let* $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3_+$ *and* $\mu \in \mathbb{R}$. *Let* $\mathfrak{P} = \mathfrak{P}(\mathbf{v}, \mu)$ *be the stepped plane with normal vector* $\mathbf{v}$ *and translation parameter* μ. *Let* $I_1 = [0, v_1[$, $I_2 = [v_1, v_1 + v_2[$ *and* $I_3 = [v_1 + v_2, v_1 + v_2 + v_3[$. *Then,*

$$\forall k \in \{1,2,3\}, \forall \mathbf{x} \in \mathfrak{P},\ x + E_k \subset \mathfrak{P} \iff \langle \mathbf{x}, \mathbf{v} \rangle + \mu \in I_k.$$

Let $\mathcal{P}_0$ be the diagonal plane of equation $x+y+z=0$ and let π be the projection on $\mathcal{P}_0$ along $(1,1,1)$.

Theorem 3. *[ABI02] Let* $\mathfrak{P}$ *be a stepped plane. The restriction* $\pi_{\mathfrak{P}}$ *of* π *from* $\mathfrak{P}$ *onto* $\mathcal{P}_0$ *is a bijection. Furthermore, the set of points of* $\mathfrak{P}$ *with integer coordinates is in one-to-one correspondance with the lattice* $\mathbb{Z}\pi(\mathbf{e_1}) + \mathbb{Z}\pi(\mathbf{e_2})$.

This theorem allows us to code a stepped plane $\mathfrak{P}$ as a two-dimensional word $u \in \{\psi, 2, 3\}^{\mathbb{Z}^2}$ as follows: for all $(m,n) \in \mathbb{Z}^2$, for $i = 1,2,3$, then

$$u(m,n) = i \iff \pi_{\mathfrak{P}}^{-1}(m\pi(\mathbf{e_1}) + n\pi(\mathbf{e_2})) + E_i \subset \mathfrak{P}.$$

2.3 Stepped surfaces

It is thus natural to try to extend the previous definitions and results to more general objects:

Definition 2. *[Jam04] A connected union* $\mathfrak{S}$ *of pointed faces* $\mathbf{x} + E_k$, *where* $\mathbf{x} \in \mathbb{Z}^3$ *and* $i \in \{1,2,3\}$, *is called a* stepped surface *if the restriction* $\pi_{\mathfrak{S}} : \mathfrak{S} \longrightarrow \mathcal{P}_0$ *of* π *is a bijection.*

A two-dimensional word $u \in \{1,2,3\}^{\mathbb{Z}^2}$ *is said to be a coding of the stepped surface* $\mathfrak{S}$ *if for all* $(m,n) \in \mathbb{Z}^2$, *for* $i = 1,2,3$, *then*

$$u(m,n) = i \iff \pi_{\mathfrak{S}}^{-1}(m\pi(\mathbf{e_1}) + n\pi(\mathbf{e_2})) + E_i \subset \mathfrak{S}.$$

In particular, a stepped plane is a stepped surface, according to what precedes.

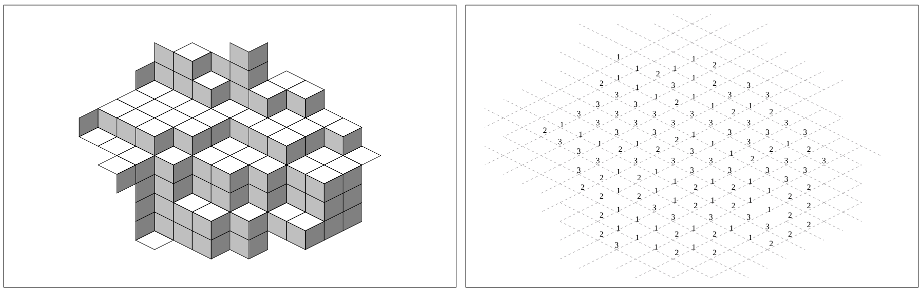


Fig. 2. A piece of a stepped surface and its two-dimensional coding.

3 Generalized substitutions acting on faces of a stepped plane

The aim of this section is to recall the notion of *generalized substitution* acting on faces of a stepped plane [AI01,AIS01,Pyt02].

Let σ denote a unimodular Pisot substitution over the three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$. Let M_σ be its incidence matrix, and let $\alpha, \lambda_1, \lambda_2$ denote its eigenvalues with $\alpha > 1 > |\lambda_1| \geq |\lambda_2| > 0$. Let $\mathfrak{P}$ be the *contracting plane* of M_σ, that is, the real plane generated by the eigenvectors associated to λ_1, λ_2.

Since the incidence matrix of a Pisot substitution is primitive [CS01], then, according to Perron-Frobenius Theorem, the eigenvalue α admits a positive eigenvector $\mathbf{v}$. Let us denote by $\mathfrak{P}_\sigma$ the stepped plane with normal vector $\mathbf{v}$ and translation parameter $\mu = 0$.

Example 2. We continue Example 1. The characteristic polynomial of M_σ is $x^3 - x^2 - 1$; it admits one eigenvalue $\alpha > 1$ (which is known as the second smallest Pisot number), and two complex conjugate eigenvalues of modulus strictly smaller than 1. The contracting plane of M_σ is the plane with equation $\alpha^2 x + \alpha y + z0$.

Definition 3. *[IO93,IO94,ABI02,ABS04] Let σ be a unimodular substitution over the three-letter alphabet $\mathcal{A} = \{1, 2, 3\}$. Let $\mathfrak{P}_\sigma$ be the stepped plane associated to σ. The* generalized substitution Σ_σ *associated to σ is defined as follows:*

$$\Sigma_\sigma(\mathbf{x} + E_i) = \bigcup_{k=1}^{3} \bigcup_{\substack{P \\ \sigma(k)=PiS}} \left(M_\sigma^{-1} \left[\mathbf{x} - \psi(P) - \sum_{j=1}^{i} \mathbf{e_j} \right] \right) + \sum_{j=1}^{k} \mathbf{e_j} + E_k$$

Example 3. Let $\sigma : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$. Then,

$$\begin{aligned} \Sigma_\sigma : \mathbf{x} + E_1 &\mapsto \left(M_\sigma^{-1}\mathbf{x} + \mathbf{e_1} - \mathbf{e_2} + E_1\right) \cup \left(M_\sigma^{-1}\mathbf{x} + \mathbf{e_1} + E_2\right), \\ \mathbf{x} + E_2 &\mapsto M_\sigma^{-1}\mathbf{x} + \mathbf{e_1} + E_3, \\ \mathbf{x} + E_3 &\mapsto M_\sigma^{-1}\mathbf{x} - \mathbf{e_2} - \mathbf{e_3} + E_1. \end{aligned}$$

In combinatorial terms, Σ_σ can be coded as

$$1 \mapsto \begin{matrix} 2 \\ 1 \end{matrix} \qquad 2 \mapsto 3 \qquad 3 \mapsto 1.$$

Let $r = r(m,n) = -\lceil (\alpha^2 m + \alpha n)/(\alpha^2 + \alpha + 1) \rceil + 1$. One has:

$$\begin{array}{l} ((m,n),1) \mapsto ((1-n, m-n-r(m,n)-1),1) + ((1-n, m-n-r(m,n)),2) \\ ((m,n),2) \mapsto ((1-n, m-n-r(m,n)),3) \\ ((m,n),3) \mapsto ((1-n, m-n-r(m,n)),1). \end{array}$$

Theorem 4. *[AI01] Let σ be a unimodular Pisot substitution over the three-letter alphabet $\mathcal{A} = \{1,2,3\}$, let $\mathfrak{P}_\sigma$ be the stepped plane associated to σ and let Σ_σ be the generalized substitution associated to σ.*

i) Two distinct faces have disjoint images under Σ_σ.
ii) The generalized substitution Σ_σ maps any pattern of $\mathfrak{P}_\sigma$ (that is, any finite union of faces of $\mathfrak{P}_\sigma$) on a pattern of $\mathfrak{P}_\sigma$.
iii) $\Sigma_\sigma(\mathfrak{P}_\sigma) \subseteq \mathfrak{P}_\sigma$.

Since Σ_σ is well-defined on $\mathfrak{P}_\sigma$ (according to Theorem 4 i)), and since $\mathfrak{P}_\sigma$ is invariant under the action of Σ_σ, it is natural to investigate the action of Σ_σ on any stepped plane. More precisely, given a stepped plane $\mathfrak{P}(\mathbf{v},\mu)$, can we extend the domain of definition of the generalized substitution Σ_σ to the patterns of $\mathfrak{P}(\mathbf{v},\mu)$? In fact:

Theorem 5. *Let σ be a unimodular Pisot substitution, let M_σ be its incidence matrix, and let Σ_σ be the generalized substitution associated to σ.*

For any stepped plane $\mathfrak{P}(\mathbf{v},\mu)$ with $\mathbf{v} \in \mathbb{R}^3_+$, one has:

i) The images of two distinct pointed faces of $\mathfrak{P}(\mathbf{v},\mu)$ by Σ_σ are disjoint.
ii) The image of $\mathfrak{P}(\mathbf{v},\mu)$ is included in the stepped plane $\mathfrak{P}({}^tM \cdot \mathbf{v},\mu)$:

$$\Sigma_\sigma(\mathfrak{P}(\mathbf{v},\mu)) \subseteq \mathfrak{P}({}^tM \cdot \mathbf{v},\mu)$$

Proof (Sketch). The proof is based on the same ideas as in the proof of Lemma 2 and 3 in [AI01]. It mainly uses the following geometric interpretation of Theorem 2: a pointed face $\mathbf{x} + E_i$ is included in $\mathfrak{P}(\mathbf{v},\mu)$ if and only the point $\mathbf{x} + \sum_{k=1}^{i-1} \mathbf{e_k}$ is above the plane $\langle \mathbf{v},\mathbf{x}\rangle + \mu = 0$ while the point $\mathbf{x} + \sum_{k=1}^{i} \mathbf{e_k}$ is below the latter. ■

4 Generalized substitutions acting on faces of a stepped surface

4.1 The general case

Since the image of a stepped plane by a generalized substitution is a subset of a stepped plane, it is interesting to investigate the action of generalized substitutions over a more general class of stepped objets, namely, the stepped surfaces. In fact,

Theorem 6. *Let $\mathfrak{S}$ be a stepped surface. Let σ be a unimodular Pisot substitution over the three-letter alphabet $\{1,2,3\}$ and let Σ_σ be the associated generalized substitution. Then, the image of two distinct pointed faces of $\mathfrak{S}$ are disjoint. Furthermore, the restriction $\pi_{\Sigma_\sigma(\mathfrak{S})} : \Sigma_\sigma(\mathfrak{S}) \longrightarrow \mathcal{P}_0$ is 1-1.*

Proof (Sketch). We first notice that given two faces $\mathbf{x}+E_i$ and $\mathbf{y}+E_j$, then there exists a stepped plane $\mathfrak{P}$ with positive normal vector containing simultaneously $\mathbf{x}+E_i$, $\mathbf{y}+E_j$ and $\mathbf{z}+E_k$. We then apply Theorem 5. ∎

In other words, it remains to prove that $\pi_{\Sigma_\sigma(\mathfrak{S})}$ is onto and that $\Sigma_\sigma(\mathfrak{S})$ is a connected union of faces to deduce that $\Sigma_\sigma(\mathfrak{S})$ is a stepped surface according to Definition 2. Let us investigate this problem in the particular case of the generalized substitution Σ_σ associated to the substitution $\sigma : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$.

4.2 The particular case of $\sigma : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$.

In the present section, σ denotes the substitution $\sigma : 1 \mapsto 13,\ 2 \mapsto 1,\ 3 \mapsto 2$ whereas Σ_σ is the generalized substitition associated to σ:

$$\begin{aligned}\Sigma_\sigma : \mathbf{x}+E_1 &\mapsto \left(M_\sigma^{-1}\mathbf{x}+\mathbf{e_1}-\mathbf{e_2}+E_1\right) \cup \left(M_\sigma^{-1}\mathbf{x}+\mathbf{e_1}+E_2\right),\\ \mathbf{x}+E_2 &\mapsto M_\sigma^{-1}\mathbf{x}+\mathbf{e_1}+E_3,\\ \mathbf{x}+E_3 &\mapsto M_\sigma^{-1}\mathbf{x}-\mathbf{e_2}-\mathbf{e_3}+E_1.\end{aligned}$$

Let us show that for this substitution, then the image of a stepped surface is still a stepped surface. First, given a two-dimensional word $u \in \{1,2,3\}^{\mathbb{Z}^2}$, we call *hook-word* a factor of u with the following shape (see Fig. 3):

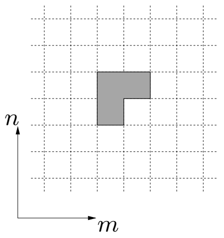


Fig. 3. Hook-shape.

The set of hook-words of u with a hook-shape is called the hook-language of u. In [Jam04,JP04], the authors reduced the recognition problem of the two-dimensional words coding discrete surfaces to a hook recognition problem. More precisely,

Theorem 7. [Jam04,JP04] *Let $u \in \{1,2,3\}^{\mathbb{Z}^2}$. Then u is a coding of a discrete surface in the sense of Definition 2 if and only if the hook-language of u is included in the following set of patterns (see Fig. 4).*

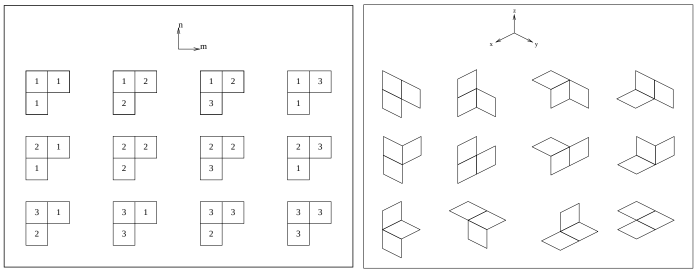


Fig. 4. Left: The permitted hook-words. Right: The 3-dimensional representation of the permitted hook-words.

We conversely associate to each permitted hook-word its 3-dimensional representation as a connected union of faces as depicted in Figure 4: the coding of any occurrence of this 3-dimensional representation in a stepped surface is equal to the corresponding hook-word.

Proposition 1. *The image by Σ_σ of all the 3-dimensional representations of the permitted hooks (see Fig. 5) are connected in $\mathbb{R}^3$.*

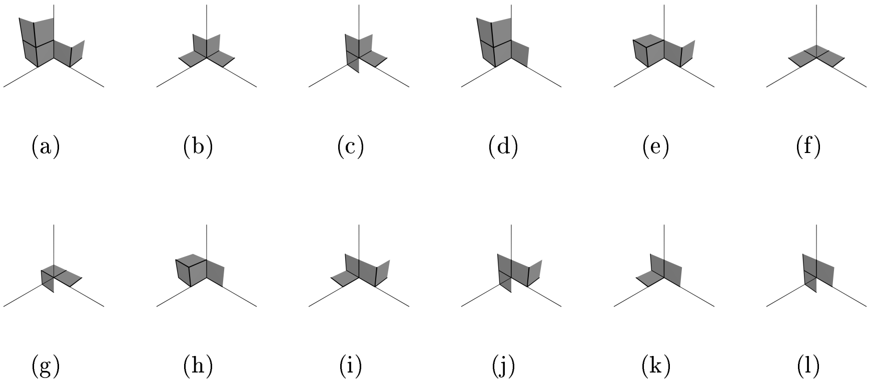
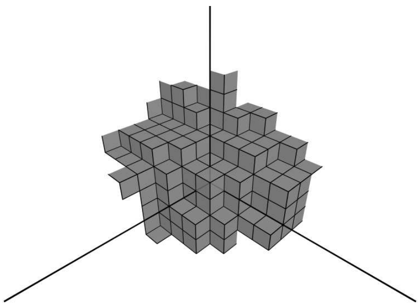


Fig. 5. The image of the permitted hooks by Σ_σ.

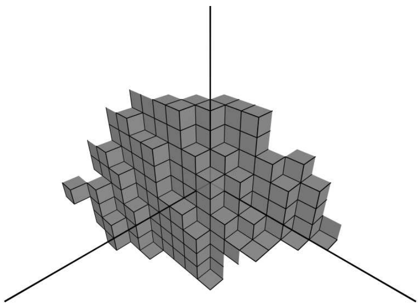
We then deduce that:

Theorem 8. *The image of a stepped surface $\mathfrak{S}$ by Σ_σ is connected and the restriction of the projection map π to the latter is injective. Furthermore, all the hook-words occurring in the coding with respect to the injective projection $\pi_{\Sigma_\sigma(\mathfrak{S})}$ (see Theorem 6) are permitted hook-words.*

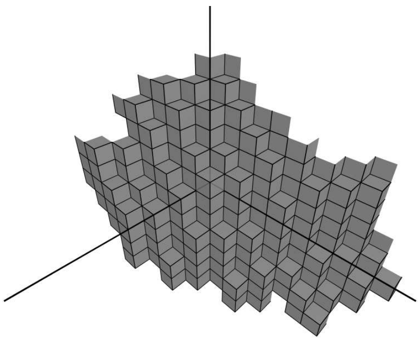
Proof (Sketch). According to Theorem 6, the image of a stepped surface by Σ_σ is well-defined. The connectedness follows from Proposition 1. Consider now a union H of three faces whose coding according to the injective projection $\pi_{\Sigma_\sigma}(\mathfrak{S})$ (see Theorem 6) is a hook-word U_H. There exist (at most) three faces of which the union of the images by Σ_σ contains H. One checks that the distance (defined as $d(\mathbf{v}, \mathbf{w}) = |\mathbf{w} - \mathbf{v}|_1$) between the distinguished vertices of those faces is uniformly bounded. By performing a finite case study, one checks that the hook-word U_H is permitted. ■



(a) $\mathfrak{S}$



(b) $\Sigma_\sigma(\mathfrak{S})$.



(c) $\Sigma_\sigma^2(\mathfrak{S})$.

Fig. 6. A piece of a non-planar stepped surface $\mathfrak{S}$ and 2 iterations by Σ_σ.

Remarks. Given a stepped surface $\mathfrak{S}$ containing the unit cube $\{\mathbf{e_1} + E_1,\, \mathbf{e_1} + \mathbf{e_2} + E_2,\, \mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3} + E_3, \}$, then the sequence of stepped surfaces $(\Sigma_\sigma^n(\mathfrak{S}))_{n\in\mathbb{N}}$ seems to converge towards the stepped plane $\mathcal{P}_\sigma$ (see Fig. 6); to be more precise, the limit points of the sequence $(\Sigma_\sigma^n(\mathfrak{S}))_{n\in\mathbb{N}}$ are subsets of $\mathcal{P}_\sigma$. We will investigate these convergence results and more generally, the possibility of extension of the domain of definition of these multidimensional substitutions to any stepped surface in a subsequent paper. Let us note that this study can also be applied to obtain an efficient generation methods of stepped planes and surfaces.

References

[AI01] P. Arnoux and S. Ito. Pisot substitutions and Rauzy fractals. *Bull. Belg. Math. Soc. Simon Stevin*, 8(2):181–207, 2001. Journées Montoises d'Informatique Théorique (Marne-la-Vallée, 2000).

[ABI02] P. Arnoux, V. Berthé, and S. Ito. Discrete planes, $\mathbb{Z}^2$-actions, Jacobi-Perron algorithm and substitutions. *Ann. Inst. Fourier (Grenoble)*, 52(2):305–349, 2002.

[ABS04] P. Arnoux, V. Berthé, and A. Siegel. Two-dimensional iterated morphisms and discrete planes. *Theor. Comput. Sci.*, 319(1-3):145–176, 2004.

[AIS01] P. Arnoux, S. Ito, and Y. Sano. Higher dimensional extensions of substitutions and their dual maps. *J. Anal. Math.*, 83:183–206, 2001.

[BCK04] V. Brimkov, D. Coeurjolly, and R. Klette. Digital planarity - a review. Technical Report RR2004-24, Laboratoire LIRIS - Université Lumière Lyon 2, may 2004.

[BV00] V. Berthé and L. Vuillon. Tilings and rotations on the torus: a two-dimensional generalization of sturmian sequences. *Discrete Math.*, 223(1-3):27–53, 2000.

[CS01] V. Canterini and A. Siegel. Geometric representation of substitutions of Pisot type. *Trans. Amer. Math. Soc.*, 353(12):5121–5144, 2001.

[IO93] S. Ito and M. Ohtsuki. Modified Jacobi-Perron algorithm and generating markov partitions for special hyperbolic toral automorphisms. *Tokyo J. Math.*, 16:441–472, 1993.

[IO94] S. Ito and M. Ohtsuki. Parallelogram tilings and Jacobi-Perron algorithm. *Tokyo J. Math.*, 17:33–58, 1994.

[Jam04] D. Jamet. On the Language of Discrete Planes and Surfaces. In *Proceedings of the Tenth International Workshop on Combinatorial Image Analysis*, pages 227-241. Springer-Verlag, 2004.

[JP04] D. Jamet, G. Paquin. Discrete surfaces and infinite smooth words *FPSAC'05.*

[KR04] R. Klette, A . Rosenfeld, *Digital straightness-A review*, Discrete Applied Mathematics, 139:197–230, 2004.

[LOTH02] N. Lothaire. Algebraic combinatorics on words. *Cambridge University Press*, 2002.

[Pyt02] N. Pytheas Fogg. *Substitutions in Dynamics, Arithmetics and Combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer Verlag, 2002.

[Rev91] J.-P. Reveillès. *Géométrie discrète, calcul en nombres entiers et algorithmique*. Thèse de Doctorat, Université Louis Pasteur, Strasbourg, 1991.