Combinatorics Lecture Notes

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Content and warnings

I taught this class in 2011, 2012, 2013, 2016, 2017, between 3 and 8 lectures per year. These notes contain several topics I have taught during these years, but not everything was taught every year.

These notes are not intended for publication, and contain many (small or big) errors.

There are, of course, many references that one may use instead of these notes. Among the most classical are

- *R. Stanley, Enumerative Combinatorics 1& 2.* Some of our material in Section 2 (in particular section 2.6) is directly inspired by EC2.
- *P. Flajolet and B. Sedgewick, Analytic Combinatorics.*
- *M. Aigner, Combinatorial Enumeration.*
- *M. Aigner and G.Ziegler, Proofs from the book.*

For some of the material I also used the survey paper

- *M. Bousquet-Mélou, Rational and Algebraic series in enumeration.*

The contents of Chapters 4,5 may less easily be found in textbooks, I suggest to look at the corresponding published papers for details on each result (they are not hard to find). However, all the material in these notes is self-contained.

⟶ These notes were written as support notes for the students who attended the lectures. I put them online at the request of some of the students and colleagues. But these notes are still in a draft form. In particular, in some places I give proofs or notions that only scratch the surface of some more conceptual things – that would naturally appeal for more developments or at least references, not given here.

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Chapter 1

Trees, rational and algebraic generating functions

1.1 Combinatorial classes and rational languages

In this section we describe a general setting that is well suited to the decompositions of combinatorial objects. The main notions are the ones of *combinatorial class* and *generating function*.

A *combinatorial class* is a set $\mathcal{C}$, equipped with a *size* function $|.| : \mathcal{C} \to \mathbb{N}$, such that for any $n \geq 0$ the set $\mathcal{C}_n := \{c \in \mathcal{C}, |c| = n\}$ formed by objects of size n is finite. The *generating function*, or *generating series* of $\mathcal{C}$ is the formal power series:

$$C(x) := \sum_{c \in \mathcal{C}} x^{|c|} = \sum_{n \geq 0} c_n x^n,$$

where $c_n := \#\mathcal{C}_n$ is the number of objects of size n.

Notation convention: In the rest of the text we will keep the same typographic conventions as in the definition. For example, if a combinatorial class is denoted by the curved letter $\mathcal{L}$, then its generating function will be denoted by L, the number of objects of size n by l_n, etc., without recalling this notation. We will also use the notation $[x^n]$ to denote the extraction of the coefficient of x^n in a formal power series, for example:

$$[x^n]L(x) = l_n.$$

Example: Let $\mathcal{C}$ be the set of all finite binary words, with size given by the length. Then $C(x) = \frac{1}{1-2x}$ and $c_n = [x^n]C(x) = 2^n$.

1.1.1 Constructions for combinatorial classes

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Let $\mathcal{A}$ and $\mathcal{B}$ be two combinatorial classes. Here are several ways of obtaining new combinatorial classes from $\mathcal{A}$ and $\mathcal{B}$.

- The *disjoint union* $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is defined by $\mathcal{C}_n = \mathcal{A}_n \uplus \mathcal{B}_n$. We have $c_n = a_n + b_n$ and $C(x) = A(x) + B(x)$.

- The *product* $\mathcal{C} = \mathcal{A} \times \mathcal{B}$, defined by $\mathcal{C} = \{(\alpha, \beta), \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$, with size function:

$$|\gamma| = |\alpha| + |\beta| \quad \text{if } \gamma = (\alpha, \beta).$$

An object in $\mathcal{C}_n$ is formed by an object in $\mathcal{A}_k$ and an object in $\mathcal{B}_{n-k}$ for some $k \in [\![0, n]\!]$, so we have $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, which implies that $C(x) = A(x)B(x)$. The best way to understand this formula is to expand the product directly on combinatorial objects:

$$\begin{aligned} C(x) = \sum_{c \in \mathcal{C}} z^{|c|} &= \sum_{a \in \mathcal{A}, b \in \mathcal{B}} z^{|a|+|b|} \\ &= \left(\sum_{a \in \mathcal{A}} z^{|a|} \right) \left(\sum_{b \in \mathcal{B}} z^{|b|} \right) = A(x)B(x). \end{aligned}$$

- The *powers* defined by $\mathcal{A}^k = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$ for $k \geq 1$, and $\mathcal{A}^0 = \mathbf{1} := \{\epsilon\}$, where $\mathbf{1}$ is the unity class, formed by a single object ϵ of size 0. Clearly, the generating function of $\mathbf{1}$ is 1, and more generally the generating function for $\mathcal{A}^k$ is $A(x)^k$.
- The *sequence* $\mathcal{C} = \mathrm{SEQ}(\mathcal{A})$, which is defined **only if** $\mathcal{A}_0 = \emptyset$, by:

$$\mathcal{C} := \bigcup_{k \geq 0} \mathcal{A}^k.$$

In other words, an element of $\mathcal{C}$ is a word $c = a_1 a_2 \ldots a_k$, where the a_i's are elements of $\mathcal{A}$, with size function

$$|c| = |a_1| + |a_2| + \cdots + |a_k|.$$

Applying previous constructions, the generating function is given by:

$$C(x) = \sum_{k \geq 0} A(x)^k = \frac{1}{1 - A(x)}.$$

Notice that each construction (union, product, sequence) corresponds to a simple operation on generating series. We are thus equipped with a **dictionnary** (the dictionnary of symbolic combinatorics as some people call it) that automatically translates a combinatorial construction into the world of generating functions. This dictionnary is the most powerful tool in enumerative combinatorics, as we will see.

(Silly) example: Let $\mathcal{C}$ be the class of binary words over $\{a, b\}$. We have $\mathcal{C} = \mathrm{SEQ}(\{\{a\} + \{b\})$. The generating function of the singleton class $\{a\}$ is x, and it is the same for $\{b\}$, so applying the dictionnary we have:

$$C(x) = \frac{1}{1 - (x + x)} = \frac{1}{1 - 2x},$$

a result that, of course, we already knew.

1.1.2 Special case: regular languages

Here we will see our first non trivial class of examples: regular languages. Recall that a regular language over a finite alphabet A is a language that can be defined by a *regular expression* involving $+$ (union), $\cdot$ (concatenation) and * (Kleene star=iteration). Equivalently, this is a language recognized by a finite automaton.

For counting purposes, we will be especially interested in non-ambiguous regular expressions. Recall that a regular expression is *non-ambiguous* if there is at most one way of producing each given word $w \in A^*$ from this expression. For example, the language L of words over $\{a, b\}$ containing exaclty one b, such that the b is either at the very end or at the very beginning is generated by either one of the two expressions:

$$L = a^*b + ba^* \qquad L = a^*b + baa^*.$$

The first expression is ambiguous (the word b can be generated in two ways) whereas the second is non-ambiguous. A nice feature of the theory of regular languages is that any regular language admits a non-ambiguous regular expression. Notice that a non-ambiguous regular expression can be seen as a specification of the language as a combinatorial class using the constructors $+, \times, \mathrm{SEQ}$. For example, we have:

$$(a + b)^* = \mathrm{SEQ}(\{a\} + \{b\}).$$

From the dictionnary of symbolic combinatorics, we obtain immediately:

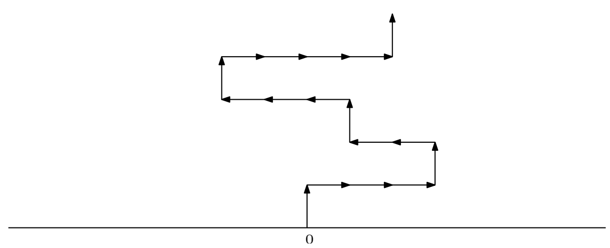
Theorem 1. *Let $\mathcal{L}$ be a regular language. Then its generating function is* rational, *i.e. there exists two polynomials P and Q such that:* $L(x) = \dfrac{P(x)}{Q(x)}$.

Remark: This theorem is fully *effective*, i.e. the rational function can really be computed in practice, provided we have a regular expression.
Cultural remark: Let $L(x) = P(x)/Q(x)$ be a rational function whose expansion as a power series has nonnegative integer coefficients. Does there exist a regular language whose generating function is L? The answer is not always yes, but there is a very nice neccessary and sufficient condition, involving the complex poles of L (TODO: find reference!!)

1.1.3 A combinatorial example: semi-directed self-avoiding paths

This example is taken from Gilles Schaeffer's lecture notes from previous years.

We consider semi-directed paths on the grid $\mathbb{Z}^2$, starting at $(0, 0)$, having only steps N, W, E, and *self-avoinding.* This last condition means that the path visits each vertex in $\mathbb{Z}^2$ at most once. Here is an example:



This path can equivalently represented by the word: $NE^3NW^2NW^3NE^4N$. It is clear that these paths are in bijections with words over $\{N, E, W\}$ avoiding the factors WE and EW. A non-ambiguous regular expression for this language is:

$$\{(\epsilon + WW^* + EE^*)N\}^*(\epsilon + WW^* + EE^*).$$

It follows that the generating function is:

$$\begin{aligned} L(x) &= \frac{1}{1 - \left(1 + \frac{x}{1-x} + \frac{x}{1-x}\right)x} \cdot \left(1 + \frac{x}{1-x} + \frac{x}{1-x}\right) \\ &= \frac{1+x}{1-2x-x^2}. \end{aligned}$$

This is already something. For example, if you want to know l_{25}, the number of paths of length 25, just enter this expression in Maple (or Mathematica, or whatever) and ask for the power series expansion at order 25. More interestingly, write the fractial partial expansion:

$$L(x) = \frac{c_1}{1 - xX_1} + \frac{c_2}{1 - xX_2}$$

where $1 - 2x - x^2 = (1 - xX_1)(1 - xX_2)$, so that $X_{1,2} = 1 \pm \sqrt{2}$, and $c_1 = c_2 = \frac{1}{2}$. Extracting the coefficient of x^n in this expression is straightforward:

$$l_n = [x^n]L(x) = \frac{1}{2}X_1^n + \frac{1}{2}X_2^n.$$

Since $|X_1| > |X_2|$ we have the asymptotic behaviour when $n \to \infty$:

$$l_n \sim \frac{1}{2}(1 + \sqrt{2})^n.$$

Exercise. Consider words over $\{a, b\}$ avoiding the factor $abaa$.

- write a non-ambiguous regular expression
- deduce the generating function
- deduce the coefficients in closed form, and find their asymptotics

1.1.4 Complements on rational functions

The world of rational functions is very well-behaved. Basically, everything we have done on the last example (partial fraction decomposition, asymptotics) is true in general. We have:

Theorem 2 (Basics of rational functions, cf Stanley, EC1)**.** *Let $Q(x) = 1 + \alpha_1 x + \cdots + \alpha_d x^d$ be a polynomial with complex coeffcients ($\alpha_d \neq 0$). Write $Q(x) = \prod_{i=1}^{k}(1 - \gamma_i x)^{d_i}$ with distinct γ_i's. Let $A(x) = \sum a_n x^n$ be a formal power series. The following are equivalent:*

1. *$A(x) = \frac{P(x)}{Q(x)}$ with $\deg P < d$.*
2. *$\forall n \geq 0 : a_{n+d} + \alpha_1 a_{n+d-1} + \cdots + \alpha_d a_n = 0$.*
3. *$\forall n \geq 0$ one has $a_n = R_1(n)\gamma_1^n + R_2(n)\gamma_2^n + \cdots + R_k(n)\gamma_k^n$, where $R_1, R_2, \ldots, R_k$ are polynomials, with $\deg R_i < d_i$.*

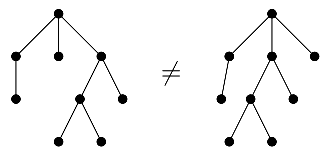
1.2 Counting trees

1.2.1 RPT and their passport

A *tree* is a connected graph without cycle. A *rooted plane tree*, or *RPT* in these notes (*APE* in French during the lectures), is a tree having some additional structure:

- there is a distinguished vertex, called the root;
- the tree is drawn "hanging from the root", so there is a natural genealogical structure, and in particular, a notion of *children* of a vertex;
- the chidren of every vertex are ordered from left to right.

The definition is better explained with a picture:



The *arity* of a vertex is its number of children. If $\mathfrak{t}$ is a RPT, its *passport* is the sequence $(r_0, r_1, \dots)$ where r_i is the number of vertices of arity i. Notice that the total number of vertices of the tree is

$$N = \sum_{i \geq 0} r_i$$

and the number of edges is

$$E = \sum_{i \geq 0} i \cdot r_i.$$

Since $\mathfrak{t}$ is a tree, we have $N = E + 1$ so we see that the passport is constrained by the relation:

$$\sum_{i \geq 0} (1 - i) r_i = 1. \tag{1.1}$$

A *plane forest with k connected components* is an ordered sequence of k rooted plane trees. Its passport is defined in a similar way. Note that for forests we have $N = E + k$ so that

$$\sum_{i \geq 0} (1 - i) r_i = k.$$

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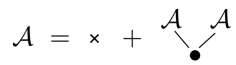
1.2.2 Binary trees, a bad (but instructive) method

A complete binary tree is a RPT in which all vertices have arity 0 or 2, i.e. $r_i = 0$ unless $i \in \{0, 2\}$. We let $n := r_2$ be the number of inner vertices. By (1.1), the number of leaves is $r_0 = n + 1$. We say that n is the *size* of the complete binary tree.

The following question is THE question that any combinatorics class must answer: **Question: What is the number a_n of complete binary trees of size n?**. To answer this question, our first method is to apply the dictionnary of symbolic combinatorics. By virtue of the decomposition:



the generating series $A(x)$ is solution of the equation

$$A(x) = 1 + xA(x)^2. \tag{1.2}$$

In the next sections, we will see several good ways of treating this equation. But, right now, and to illustrate the power of generating functions, we go on with bare hands, using only some elementary knowledge of power series expansions. In the rest of this paragraph, we consider $A(x)$ as a power series, absolutely convergent in a neighbourhood of 0. First, one can solve this equation (polynomial, degree 2 !) as $A(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$. Now, think of expanding this as a power series near $x = 0$, and you'll see that the only "valid" solution is

$$A(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

At this point, we have "explicitely" determined the generating function of the numbers a_n: this is already something. To go further, recall Newton's generalized binomial formula:

Proposition 3 (Newton's generalized binomial formula)**.** *Let $\alpha \in \mathbb{C}$ and x with $|x| < 1$. Then one has:*

$$(1+x)^\alpha = \sum_{n \geq 0} \binom{\alpha}{n} x^n,$$

where

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-n+1)}{n!}.$$

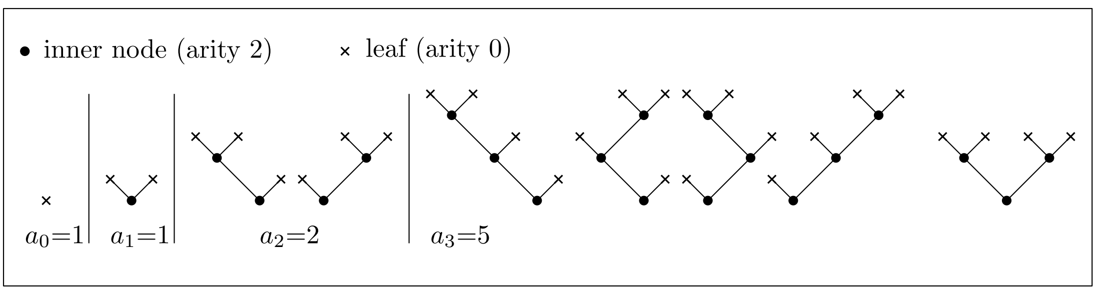


Figure 1.1: Binary trees of small sizes

Using Newton's formula we have:

$$
\begin{aligned}
a_n = [x^n]A(x) &= [x^n]\frac{1-\sqrt{1-4x}}{2x} \\
&= -\frac{1}{2}[x^{n+1}](1-4x)^{1/2} \\
&= (-1)^n 4^{n+1}\frac{1}{2}[x^{n+1}](1-x)^{1/2} \\
&= (-1)^n 4^{n+1}\frac{1}{2}\binom{1/2}{n+1} \\
&= (-1)^n 4^{n+1}\frac{1}{2}\frac{1/2(-1/2)(-3/2)\dots(1/2-n+1)}{(n+1)!} \\
&= \frac{(2n)!}{n!(n+1)!},
\end{aligned}
$$

and that's it, we have a nice formula for the number of binary trees of size n. The sequence of numbers we just found is one of the most important in all combinatorics, and it even has a name:

Definition 4. The number

$$\mathrm{Cat}(n) = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{2n+1}\binom{2n+1}{n}$$

is called the n-th Catalan number.

The formula for the Catalan number is very simple, much simpler than the intermediate formulas in our computations. At this point, it is natural to ask a few questions:

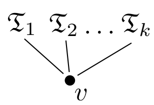
- We have been very lucky to be able to solve the equation for $A(x)$. What if we had had a polynomial equation of bigger degree that is not solvable by radicals ?
- Is there a way to go from Equation (1.2) to the formula for a_n directly, without explicitely solving for $A(x)$?
- The formula for a_n is very "combinatorial": basically, a binomial coefficient divided by something linear. Is there a combinatorial proof ?

The third question will be answered positively (in much greater generality) in the next section. The first two will be (partially) answered by the Lagrange Inversion Formula, in Section 1.2.4.

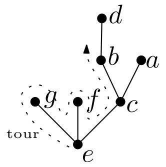
1.2.3 Lukaciewicz words and the cycle lemma

Let $\mathfrak{T}$ be a RPT. We define the *left prefix order* as the ordered list of the vertices of $\mathfrak{t}$ defined by:

if $\mathfrak{T} = \bullet_v$ then $l(\mathfrak{T}) = v$

if $\mathfrak{T} =$  then $l(\mathfrak{T}) = v, l(\mathfrak{T}_1), l(\mathfrak{T}_2) \dots, l(\mathfrak{T}_{\mathfrak{k}})$

In other words, $l\mathfrak{T}$ is the list of vertices of $\mathfrak{T}$ in the order of discovery when one makes the tour of the tree from left to right:



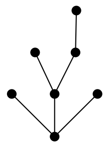
$\longrightarrow \quad l(\mathfrak{T}) = e, g, f, c, b, d, a$

We consider the (infinite) alphabet $\mathcal{A} := \{x_0, x_1, x_2, \dots\}$.

Definition 5. Let $\mathfrak{T}$ be a RPT, and let $l(\mathfrak{T}) = v_1, v_2, \dots, v_N$ be the list of its vertices in left prefix order. The *Lukaciewicz word* of $\mathfrak{T}$, $w(\mathfrak{T}) \in \mathcal{A}^*$ is defined by:

$$w(\mathfrak{T}) := x_{a(v_1)} x_{a(v_2)} \dots x_{a(v_N)},$$

where $a(v)$ is the arity of the vertex v. The Lukaciewicz word of a forest is defined as the concatenation of the Lukaciewicz words of its connected components.



$\longrightarrow \quad x_3 x_0 x_2 x_0 x_1 x_0 x_1$

Remark 1. One can reconstruct the tree starting with its Lukaciewicz word. Just observe that one can proceed "greedily", from left to right. At each step of the reconstruction, just insert a new vertex (corresponding to the last read letter) in the leftmost slot available. For forests, proceed in the same way, but, if at some point there is no more slots available, create a new connected component and go on.

Definition 6. For $k \geq 1$, the *Lukaciezicz language* $\mathcal{L}_{-k} \subset \mathcal{A}^*$ is the set of all Lukaciewicz words $w(\mathfrak{F})$, where $\mathfrak{F}$ is a plane forest with k connected components. In particular $\mathcal{L}_{-1}$ is in bijection with rooted plane trees.

We are now going to work with *paths* rather than *words*. First define the *weight function*

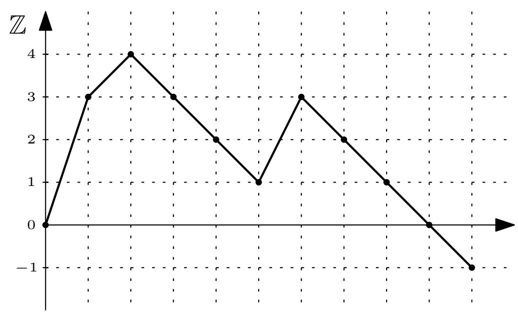
$$\begin{aligned} \delta \; : \; & \mathcal{A} \longrightarrow \mathbb{R} \\ & x_i \longmapsto i - 1. \end{aligned}$$

We can represent a word of length N by the sequence of its weights, or equivently by a walk of length N on $\mathbb{Z}$, that starts at 0, and whose increments are given by the weights:

word: $x_4 x_2 x_0 x_0 x_0 x_3 x_0 x_0 x_0 x_0$

weights: $-3, 1, -1, -1, -1, 2, -1, -1, -1, -1$

path:



Proposition 7. *A word $w = w_1 w_2 \dots w_n$ is an element of the Lukaciewicz language $\mathcal{L}_{-k}$ if and only if:*

$$\begin{cases} \delta(w_1) + \delta(w_2) + \dots + \delta(w_N) = -k \\ \forall i \in [\![0, N-1]\!], \delta(w_1) + \delta(w_2) + \dots + \delta(w_i) \geq -k + 1. \end{cases}$$

This condition is equivalent to say that the corresponding path stays strictly above the ordinate $-k$, except at its very last step where it reaches the ordinate $-k$.

Example: The last path drawn stays above -1 except at its very last step where it reaches -1. Therefore the corresponding word is the Lukaciewicz word of a rooted plane tree (draw it!).

Proof of Proposition 7. Start from a word, and try to reconstruct the tree (or forest) greedily, from left to right, as in Remark 1. Let A_i and R_i be respectively the number of slots available in the current connected component, and the number of connected components yet to create, after the i-th step of the execution of this algorithm (i.e., after the i-th vertex has been inserted). It is clear by induction (make a picture!) that we have:

$$A_i + R_i = \delta(w_1) + \delta(w_2) + \dots + \delta(w_i) + k.$$

Now, the algorithm succeeds in constructing a forest with k components if and only if there is always either a slot available or the possibility of creating a new component (i.e. $A_i + R_i \geq 1$), except at the very last step where we have exhausted all slots and all components (i.e. $A_N + R_N = 0$). These are exactly the conditions stated in Proposition 7. □

Here is the main theorem of this section:

Theorem 8. *Let $(r_0, r_1, \dots)$ be a finite sequence of integers such that $\sum_i r_i = N$ and $\sum (1 - i) r_i = k$. Then the number of Lukaciewicz words in $\mathcal{L}_{-k}$ having r_i letters x_i for all $i \geq 0$ is:*

$$\frac{k}{N} \binom{N}{r_0, r_1, r_2, \dots} = \frac{k(N-1)!}{r_0! r_1! r_2! \dots}. \tag{1.3}$$

This number is also the number of plane forests with k connected components having r_i vertices of arity i for all $i \geq 0$. In particular, the number of rooted plane trees of passport $r_0, r_1, r_2, \dots$ is given by:

$$\frac{1}{N} \binom{N}{r_0, r_1, r_2, \dots}.$$

We now proceed with the proof of the theorem, and its main ingredient, the so-called *Cycle lemma*[1]. In what follows, $(r_0, r_1, r_2, \dots)$ is a fixed sequence as in the statement of the theorem. First notice that the multinomial coefficient appearing in formula (1.3) has a clear interpretation in terms of paths. More precisely, let $\mathcal{B}$ be the set of (arbitrary) words over $\mathcal{A}$ having r_i letters x_i for all $i \geq 0$. Then the cardinality of $\mathcal{B}$ is:

$$\begin{aligned} |\mathcal{B}| &= \binom{N}{r_0} \binom{N - r_0}{r_1} \binom{N - r_0 - r_1}{r_2} \dots \\ &= \frac{N!}{r_0! r_1! r_2! \dots} = \binom{N}{r_0, r_1, r_2 \dots}. \end{aligned}$$

[1]English: cycle lemma. French: lemme cyclique.

Now, if $w = w_0 w_1 \dots w_{N-1}$ is an element of $\mathcal{B}$, and $i \in [\![0, N-1]\!]$, define its i-th conjugate as the word:

$$\sigma^i(w) := w_i w_{i+1} \dots w_{N-1} w_0 w_1 \dots w_{i-1}.$$

Theorem 8 is a direct consequence of the following (very important!) lemma:

Lemma 9 (Cycle lemma)**.** *Let w be a word in $\mathcal{B}$. Then exactly k conjugates of w are Lukaciewicz words, i.e.:*

$$\mathrm{Card}\{i \in [\![0, N-1]\!] \;:\; \sigma^i(w) \in \mathcal{L}_{-k}\} = k.$$

Proof that Lemma 9 implies Theorem 8. By the cycle lemma, we have a mapping:

$$\mathcal{B} \times \{1, 2, \dots, k\} \to \mathcal{B} \cap \mathcal{L}_{-k}$$

that sends a pair (w, j) to its conjugate $\sigma_{i_j}(w)$ where $i_1 \leq i_2 \leq \dots \leq i_k$ are the k conjugates of w in $\mathcal{L}_{-k}$. Moreover each word w in $\mathcal{B} \cap \mathcal{L}_{-k}$ has N preimages (all its conjugates). Therefore by "lemme des bergers" we have:

$$k \cdot \mathrm{Card}\mathcal{B} = N \cdot \mathrm{Card}\left(\mathcal{B} \cap \mathcal{L}_{-k}\right),$$

so that

$$\mathrm{Card}\left(\mathcal{B} \cap \mathcal{L}_{-k}\right) \frac{k}{N} \binom{N}{r_0, r_1, r_2, \dots}.$$

□

Examples:

For binary trees, we have, say $r_2 = n$, so that $r_0 = n + 1$ and $N = N + 1$. We obtain:

$$\#\{\text{ binary trees of size } n\} = \frac{1}{2n+1}\binom{2n+1}{n}.$$

So that's it: we have a bijective proof of the Catalan formula!!

Say you want to count 5-ary trees of size n, i.e. rooted plane trees in which $r_5 = n$ and $r_i = 0$ if $i \notin \{0, 5\}$. First note that $r_0 = 4n + 1$, so $N = 5n + 1$, and we have:

$$\#\{\text{ 5-ary trees of size } n\} = \frac{1}{5n+1}\binom{5n+1}{n}. \tag{1.4}$$

Exercise: do m-ary trees!

Returning to 5-ary trees, the generating function is solution of the equation

$$T(x) = 1 + xT(x)^5.$$

Could we find Formula (1.4) directly from this equation? Yes! Thanks to the Lagrange inversion formula, in the next section.

1.2.4 The Lagrange Inversion formula

We now present an important tool in combinatorial enumeration: the Lagrange inversion formula. Strictly speaking, it is equivalent to the cycle lemma (more precisely, to Theorem 8), and this is actually the way we will prove it. But in practice, it is a very efficient tool for the following reason: contrarily to bijective methods, it applies also to situations where the "combinatorics" of the problem is not clear, and where the best we can do is rely on generating functions, equations, and computations.

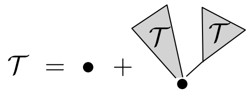
Theorem 10 (Lagrange Inversion Formula)**.** *Let $\Phi(y) = a_0 + a_1 y + a_2 y^2 + \dots$ be a formal power series, with $a_0 \neq 0$. Let $F(x)$ be a formal power series solution of the equation:*

$$F(x) = x \cdot \Phi\big(F(x)\big). \tag{1.5}$$

Then the coefficient of x^n in $F(x)^k$ is given by:

$$[x^n]\Big(F(x)^k\Big) = \frac{k}{n}[y^{n-k}]\Big(\Phi(y)^n\Big). \tag{1.6}$$

Example 1: rooted plane trees with n edges. Let us count now rooted plane trees (all of them, no degree restriction), by the number of edges. Considering the rightmost edge outgoing from the root, we have the decomposition:



which gives: $T(x) = 1 + xT(x)^2$, or equivalently, letting $S(x) = T(x) - 1$:

$$S(x) = x(1 + S(x))^2.$$

We have seen this equation several times already. Let's just remark that the LIF, applied with $\Phi = (1 + y)^2$, gives an immediate way to treat it.

$$\begin{aligned} [x^n]T(x) = [x^n]S(x) &= \frac{1}{n}[y^{n-1}](1+y)^{2n} \\ &= \frac{1}{n}\binom{2n}{n-1} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Example 2: m-ary trees of size n. Assume you want to count m-ary trees (all vertices of arity m or 0), by the number of inner nodes. The generating function satisfies:

$$T(x) = 1 + xT(x)^m.$$

Letting $S(x) = T(x) - 1$ be the generating function of 'non-trivial" m-ary trees, we thus have:

$$S(x) = x\big(1 + S(x)\big)^m.$$

We are in the domain of application of the Lagrange inversion formula, with $\Phi(y) = (1+y)^m$. Therefore the number of m-ary trees of size n is for $n \geq 1$:

$$\begin{aligned}[x^n]T(x) = [x^n]S(x) &= \frac{1}{n}[y^{n-1}](1+y)^{mn} \\ &= \frac{1}{n}\binom{mn}{n-1} \\ &= \frac{1}{(m-1)n+1}\binom{mn}{n}.\end{aligned}$$

If you have not done yet the previous exercise, do it now, and check that we find the same answer as with the cycle lemma!

Example 3: Cayley trees (2016) You have already counted Cayley trees in a previous lecture, but let us see how to rederive the formula from generating functions. We let t_n be the number of *Cayley trees*, that is, graph-theoretical trees on the vertex set $[1..n]$. We also let $a_n = nt_n$ be the number of Cayley trees on $[1..n]$ having a distinguished vertex (we call these trees rooted).

Then we have $a_1 = 1$, and for $n \geq 1$, we have:

$$a_{n+1} = (n+1)\sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{n_1,\ldots,n_k\geq 1\\ \sum n_i=n}}\binom{n}{n_1,n_2,\ldots,n_k}a_{n_1}a_{n_2}\ldots a_{n_k}.$$

To understand this equation observe that each rooted tree Cayley of size $(n+1)$ can be obtained as follows:

- choose the label of the root (there are $(n+1)$ choices);
- choose the degree $k \geq 1$ of the root;
- choose the sizes $n_1, n_2, \ldots, n_k \geq 1$ of the subtrees attached to the root; these size must add up to n;
- choose how to dispatch the numbers in $[1..n]$ among the different subtrees; this is accounted by the multinomial coefficient;
- choose one of the a_{n_1} subtrees of size n_1, one of the a_{n_2} subtrees of size n_2, etc.

Note that if we do that, we count each configuration where the root has degree k exactly $k!$ times (since in a Cayley tree the subtrees are not ordered), which is why we have to divide by $k!$ in the above formula.

Very nice, we obtained a recurrence formula. What to do next? Let's first remark that the recurrence can be rewritten in the very suggestive form:

$$\frac{a_{n+1}}{(n+1)!} = \sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{n_1,\ldots,n_k\geq 1\\ \sum n_i=n}}\frac{a_{n_1}}{n_1!}\frac{a_{n_2}}{n_2!}\cdots\frac{a_{n_k}}{n_k!}.$$

It is very natural to introduce the following generating function:

$$A(x) = \sum_{n\geq 1}\frac{a_n}{n!}x^n,$$

since the recurrence equation above is now simply equivalent to:

$$A(x) = x\sum_{k\geq 1}\frac{1}{k!}A(x)^k = xexp(A(x)).$$

We can now apply the Lagrange Inversion Formula. We directly get:

$$\frac{a_n}{n!} = [x^n]A(x) = \frac{1}{n}[u^{n-1}](exp(u))^n = \frac{1}{n}[u^{n-1}]e^{nu} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!}.$$

We thus get $a_n = n^{n-1}$ and $t_n = n^{n-2}$ which is Cayley's formula!
Note: what we just did, by hand, is an example of the SET construction for exponential generating functions. Roughly speaking, every time we have a family $\mathcal{A}$ of labelled objects and we form the generating function $A(x) = \sum_n \frac{a_n}{n!}x^n$, then the generating function $exp(A(x))$ can be interpreted as counting "sets" of objects in $\mathcal{A}$. The general proof is the same as the one we did here. Since a rooted Cayley tree naturally decomposes as a root plus a "set" of smaller Cayley trees, this proof of Cayley's formula is not just an accident and is, in fact, one of the most direct and natural ones. For more on classes of labelled objects, exponential generating functions, and the SET construction, see Flajolet and Sedgewick.
Another remark: it was important here to consider *rooted* trees: even if there is just a factor of n between rooted and unrooted trees in this case, it is necessary to deal with rooted trees in order to obtain a tractable combinatorial decomposition and recurrence formula. There exist some bijective proofs of Cayley's formula that do not require any further rooting, such as the proof using Prüfer encodings.

Exercise. The number of complete binary trees with n inner nodes and the number of rooted plane trees with n edges are both the n-th Catalan number. Find a direct bijection between the two.

We now give two proofs of the Lagrange Inversion Formula. The first one is combinatorial and shows well the equivalence with the cycle lemma. The second one is short and simple, but it requires (elementary) knowledge of complex analysis.

First proof of the Lagrange Inversion Formula (Theorem 10). Write $\Phi(y) = a_0 + a_1 y + a_2 y^2 + \dots$. Then, the series $F(x)$ appearing in Equation (1.5) is the generating function of rooted plane trees, where x counts the number of vertices, and where each vertex of arity i is counted with a weigth a_i. In other words:

$$F(x) = \sum_{\mathfrak{T}:\ RPT} x^{\#\text{vertices}(\mathfrak{T})} \prod_i a_i^{r_i(\mathfrak{T})}$$

where $(r_i(\mathfrak{T}))_{i\geq 0}$ is the passport of the tree $\mathfrak{T}$. Therefore, $F(x)^k$ is the generating function of plane forests with k connected components (with the same weighting). By Theorem 8, we thus have:

$$F(x)^k = \sum_{\substack{r_0, r_1, \dots \\ \sum (i-1) r_i = k}} \frac{k}{n} \binom{\sum r_i}{r_0, r_1, \dots} x^{\sum r_i} \prod_{i\geq 0} a_i^{r_i}.$$

Extracting the coefficient of x^n we obtain:

$$\begin{aligned} [x^n]F(x)^k &= \sum_{\substack{r_0 + r_1 + \dots = n \\ \sum (i-1) r_i = k}} \frac{k}{n} \binom{n}{r_0, r_1, \dots} \prod_i a_i^{r_i} \\ &= \frac{k}{n}[y^{n-k}]\big(a_0 + a_1 y + a_2 y^2 + \dots\big)^n. \end{aligned}$$

□

The second proof we present is analytic, and is taken from Flajolet and Sedgewick, appendix A. If you don't know complex analysis, you should skip that. If you think you may know, the only two things we will use are Cauchy's formula for the n-th coefficient of a series:

$$[x^n]f(x) = \frac{1}{2\pi i}\oint f(x)\frac{dx}{x^{n+1}},$$

and the change of variable formula.

Second proof of the Lagrange Inversion Formula (Theorem 10). This proof starts with a trick: first notice that for any power series $g(x)$ one has $[x^n]g(x) = \frac{1}{n}[x^{n-1}]g'(x)$, and apply this to $g(x) = F(x)^k$. This gives:

$$[x^n]F(x)^k \;=\; \frac{1}{n}[x^{n-1}]\frac{d}{dx}F(x)^k = \frac{k}{n}[x^{n-1}]F'(x)F(x)^{k-1}.$$

Now express this last coefficient with Cauchy's formula, and make the change of variable $y = F(x)$:

$$\begin{aligned}[x^n]F(x)^k &= \frac{k}{2i\pi n}\oint F'(x)F(x)^{k-1}\frac{dx}{x^n} \\ &= \frac{k}{2i\pi n}\oint y^{k-1-n}\Phi(y)^n dy,\end{aligned}$$

since by (1.5) we have $\frac{1}{x} = \Phi(y)/y$ and $dy = F'(x)dx$. Applying Cauchy's formula in the other direction, the last quantity is equal to

$$\frac{k}{n}[y^{n-k}]\Phi(y)^n.$$

□

1.3 Complement: some general facts on algebraic series

In this section we give several complements on algebraic series, without proof. We refer to Stanley's book or the the paper of Mireille Bousquet-Mélou mentioned in the introduction for more details and for references.

Definition 11. A formal power series $F(x)$ is *algebraic* if there exists a polynomial $P(x, y)$, different from the 0 polynomial, with coefficients in $\mathbb{Q}$, such that $P(x, F(x)) = 0$.

Example: For the generating function of Catalan numbers, take $P(x, y) = xy^2 - y + 1$.

Algebraic functions have several nice properties, the first ones being properties of closure under the usual operations:

Theorem 12. *If F and G are algebraic, then so are $F + G$, FG, and $\frac{d}{dx}F$.*

Remark 2. If $F(x) = \frac{P(x)}{Q(x)}$ then $Q(x)F(x) - P(x) = 0$, so rational series form a subset of algebraic series.

If $A(x) = \sum_{n\geq 0} a_n x^n$ is algebraic, then for n large enough, its coefficients satisfy a linear recurrence with polynomial coefficients, i.e.:

$$p_0(n)a_n + p_1(n)a_{n-1} + \cdots + p_k(n)a_{n-k} = 0,$$

where $p_0, p_1, \ldots, p_k$ are polynomials. In particular the coefficients (a_n) can be computed in linear time.

In applications coming from combinatorics, it is often the case that we do not obtain directly *one* polynomial equation for *one* unknown series, but a *system* of algebraic equations for *several* unknown series. For example during the lecture we considered an example of colored plane trees (with 2 colors of vertices, each color having certain restrictions on its arity) leading to the equations:

$$\begin{cases} T_\circ &= 1 + xT_\circ T_\bullet \\ T_\bullet &= 1 + x^2 T_\circ^2. \end{cases}$$

Eliminating by hand, we can obtain a non-trivial equation for $T_\circ$:

$$T_\circ = 1 + xT_\circ(1 + x^2 T_\circ^2),$$

showing that $T_\circ$, and thus also $T_\bullet$, are algebraic series.

This phenomenon is not isolated and it is very often the case that such elimination is possible. We have

Theorem 13 (Polynomial elimination)**.** *Let $F_1(x), F_2(x), \ldots, F_k(x)$ be formal power series solutions of a system of equations:*

$$\begin{cases} F_1(x) &= P_1(x, F_1(x), \ldots, F_k(x)) \\ \quad\cdots & \\ F_k(x) &= P_k(x, F_1(x), \ldots, F_k(x)) \end{cases}$$

Assume that the system is proper*, i.e. that for all $1 \leq i \leq k$ one has*

$$P_i(x, y_1, y_2, \ldots, y_k) = y_i - Q_i(x, y_1, y_2, \ldots, y_k),$$

where Q_i has no constant term and no term of the form $c_j y_j$ with $c_j \in \mathbb{Q}$.

Then the system has a unique solution $(F_1(x), F_2(x), \ldots, F_k(x))$ in the space of formal power series, and each series $F_i(x)$ is algebraic.

This theorem has an important application to context-free languages. A *context-free* grammar consists in a set $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ of *symbols*, a finite alphabet $\mathcal{A}$ of *letters*, disjoint from $\mathcal{S}$, and a set of *rewriting rules* of the form $S_i \to w$ where w is a non-empty word on $\mathcal{S} \cup \mathcal{A}$. The grammar is proper if it has no rule of the form $S_i \to S_j$. The language $\mathcal{L}$ recognised by the grammar is the set of words over $\mathcal{A}$ that can be obtained from S_1 by applying finitely many rewriting rules. A language is *context-free* if it is the language recognized by some context-free grammar (and we can always assume that this grammar is proper).

The grammar is *non-ambiguous* if for every word recognised by the grammar, there is a unique derivation tree that enables to recognize this word with the grammar. In this case we also say that the language is *non-ambiguous*.

Applying the dictionary of symbolic combinatorics, given any non-ambiguous context-free grammar, we can write a system of equations for the generating functions $F_1, F_2, \ldots, F_k$ of words recognized starting from the symbols $S_1, S_2, \ldots, S_k$. The previous theorem and the definitions of "proper system" and "proper grammar" clearly imply:

Theorem 14. *The generating function, by the length, of a context-free proper non-ambiguous language is algebraic.*

Chapter 2

Enumeration on graphs

2.1 Preliminary: the inclusion-exclusion formula

As a preliminary to the introduction of the chromatic polynomial, we recalled the inclusion-exclusion formula. This formula is (very) useful when you want to count objects that *do not* satisfy certain properties, but it is easier to count objects that *do* satisfy these properties.

Theorem 15 (Inclusion-Exclusion)**.** *Let S be a finite set, and let $A_1, A_2, \dots, A_r$ be* properties*, i.e., subsets of S. Then the number of elements in S that satisfy* none *of the properties $(A_i)_{1\leq i\leq r}$ is given by:*

$$|A_1^c \cap A_2^c \cap \cdots \cap A_r^c| = |S| + \sum_{k=1}^{r}(-1)^k \sum_{i_1<i_2<\cdots<i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

Remark 3. In the right-hand side, the quantity $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ is the number of elements that satisfy simultaneously the properties $A_{i_1}, A_{i_2}, \dots, A_{i_k}$, but we say nothing about the other properties (they could be satisfied or not).

Example: derangements. This is THE classical example. We want to determine the number d_n of permutations of $\{1, \dots, n\}$ having no fixed point (i.e. such that $\sigma(i) \neq i$ for all $1 \leq i \leq n$). Such permutations are called *derangements*. We do this by inclusion-exclusion. Let S be the set of all permutations of $\{1, 2, \dots, n\}$, and for $1 \leq i \leq n$ let A_i be the subset of permutations having i as a fixed point. By definition, a derangement is a permutation that satisfies none of the properties A_i, so we have by inclusion-exclusion:

$$d_n = n! + \sum_{k=1}^{r}(-1)^k \sum_{i_1<i_2<\cdots<i_k} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

Now, for any $i_1 < i_2 < \cdots < i_k$, there are $(n-k)!$ permutations having $i_1, i_2, \dots i_k$ as fixed points, i.e., $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!$. Moreover, there are $\binom{n}{k}$ different choices of the indices $i_1 < i_2 < \dots, i_k$ so we finally determine the number of derangements:

$$\begin{aligned} d_n &= n! + \sum_{k=1}^{n}(-1)^k \binom{n}{k}(n-k)! \\ &= \sum_{k=0}^{n}(-1)^k \frac{n!}{k!}. \end{aligned}$$

As an interesting application, notice that the probability that a permutation chosen uniformly at random is a derangement is equal to $\frac{d_n}{n!} = \sum_{k=0}^{n}(-1)^k \frac{1}{k!}$. In particular, this probability converges to $e^{-1} \approx 0,37$ as n tends to infinity.

Proof of the Inclusion-Exclusion formula. From the point of view of indicator functions, the formula is a mere consequence of distributivity. More precisely, for $s \in S$ and $1 \leq k \leq r$, let $\delta_{s,k} = \begin{cases} 1 & \text{if } s \in A_k \\ 0 & \text{else.} \end{cases}$. Then we have:

$$\begin{aligned} |A_1^c \cap A_2^c \cap \cdots \cap A_r^c| &= \sum_{s\in S}(1-\delta_{s,1})(1-\delta_{s,2})\dots(1-\delta_{s,r}) \\ &= \sum_{s\in S}\Big(1+\sum_{k=1}^{r}(-1)^k \sum_{i_1<i_2<\cdots<i_k} \delta_{s,i_1}\delta_{s,i_2}\dots\delta_{s,i_k}\Big) \\ &= |S| + \sum_{k=1}^{r}(-1)^k \sum_{i_1<i_2<\cdots<i_k} \underbrace{\sum_{s\in S}\delta_{s,i_1}\delta_{s,i_2}\dots\delta_{s,i_k}}_{=|A_{i_1}\cap A_{i_2}\cap\cdots\cap A_{i_k}|}. \end{aligned}$$

□

2.2 The chromatic polynomial

Let $G = (V, E)$ be a graph. We allow loops (edges linking a vertex to itself) and multiple edges. If q is a positive integer, a proper q-coloring of G is mapping $\phi : V \to \{1, 2, \dots, q\}$ such that for each edge $(x, e) \in E$ one has $\phi(x) \neq \phi(y)$. In other words, it is a coloring of the vertices of G with q colors, in such a way that no edge is monochromatic. We let $\chi_G(q)$ be the number of proper q-colorings of G.

Examples.

- If G contains a loop then $\chi_G(q) = 0$ for all $q \geq 1$.
- If G is planar then $\chi_g(4) \neq 0$. This is a way to state the (difficult!) 4-color theorem.
- If G consists of n isolated vertices and no edges, then $\chi_G(q) = q^n$. Indeed we can color each vertex independently and arbitrarily.
- If G is a tree, then $\chi_G(q) = q(q-1)^{|V|-1}$. Indeed, choose any vertex as the root of the tree: you have q choices to color it. Once this has been done, you can color the tree "from the root to the leaves", and you have $(q-1)$ choices for the color of each vertex (all color except the color of the father).
- Similarly, if G is a forest, then $\chi_G(q) = q^{\kappa(G)}(q-1)|V| - \kappa(G)$, where $\kappa(G)$ is the number of connected components of the graph G.

Theorem 16. *Let $G = (V, E)$ be a graph. Then the quantity $\chi_G(q)$ is a polynomial in q, called the* chromatic polynomial *of G. This polynomial is given by the explicit expression:*

$$\chi_G(q) = \sum_{F\subset E}(-1)^{|F|}q^{k(F)}, \tag{2.1}$$

where the sum runs over all subsets of edges $F \subset E$, and where $k(F)$ is the number of connected components of the graph (V, F).

Proof. We use the inclusion-exclusion formula. Let $S = \{1, 2, \ldots, q\}^V$ be the set of all colorings (proper or not) of the vertices of G with q colors. For each edge $e = (e_-, e_+) \in E$, consider the property A_e "the edge e is monochromatic":

$$A_e = \{\phi \in \{1, 2, \ldots, q\}^V, \quad \phi(e_-) = \phi(e_+)\}.$$

Then by definition, $\chi_G(q)$ is the number of colorings that satisfy none of the properties A_e, $e \in E$. Therefore we can apply the inclusion-exclusion formula. Letting $E = \{e_1, e_2, \ldots, e_r\}$ we have:

$$\begin{aligned} \chi_G(q) &= |A^c_{e_1} \cap A^c_{e_2} \cap \cdots \cap A^c_{e_r}| \\ &\overset{INC-EXC}{=} q^{|V|} + \sum_{k=1}^{r} (-1)^k \sum_{i_1 < i_2 < \cdots < i_k} |A_{e_{i_1}} \cap A_{e_{i_2}} \cap \cdots \cap A_{e_{i_k}}|. \end{aligned}$$

For each k-tuple of indices $i_1 < i_2 < \cdots < i_k$, the subset of edges $F = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ induces a subgraph of E with k edges. By definition, the quantity $|A_{e_{i_1}} \cap A_{e_{i_2}} \cap \cdots \cap A_{e_{i_k}}|$ is the number of colorings of this subgraph in which each edge is monochromatic, that is, the number of colorings of this subgraph in which all vertices in each connected component have the same color. Clearly, the number of such coloring is $q^{\kappa(F)}$, since we have one (free) choice of color for each connected component. Moreover, each subset of edges $F \subset E$ of size $k \geq 1$ can be written as $F = \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ for an appropriate choice of the indices. Therefore we have:

$$\chi_G(q) = q^{|V|} + \sum_{\substack{F \subset E \\ F \neq \emptyset}} (-1)^{|F|} q^{\kappa(F)}.$$

This coincides with (2.1) upon noticing that the contribution of the empty set in (2.1) is $(-1)^0 q^{\kappa(\emptyset)} = q^{|V|}$. □

2.3 The Tutte polynomial

To start with, we notice that the chromatic polynomial satisfies a so-called *deletion-contraction* recurrence.

Proposition 17 (deletion-contraction for the chromatic polynomial)**.** *Let G be a graph and let e be an edge of G. Then*

- *if e is a loop, then $\chi_G(q) = 0$*
- *if e is not a loop, then $\chi_G(q) = \chi_{G \setminus e}(q) - \chi_{G/e}(q)$,*

where $G \setminus e$ is the graph G where the edge e has been deleted, and G/e is the graph G where the edge e has been contracted.

Remark 4. Recall that we allow loops and multiple edges. If $e = (u, v)$ and if there are other edges than e between u and v, then these edges become loops in the graph G/e.

Proof of Proposition 17. Let e be a non-loop and write $e = (u, v)$. We partition the proper colorings ϕ of the graph $G \setminus e$ into two classes:

- Colorings such that $\phi(u) = \phi(v)$. Clearly, these colorings are in bijection with proper colorings of the graph G/e (just merge the vertices u and v, and keep the color $\phi(u) = \phi(v)$ for the vertex resulting of the merge).

- Colorings such that $\phi(u) \neq \phi(v)$. Clearly, these colorings are in bijection with proper colorings of the graph G (just put back the edge e, and since $\phi(u) \neq \phi(v)$, you have a proper coloring).

Therefore we have $\chi_{G\backslash e}(q) = \chi_G(q) + \chi_{G/e}(q)$. □

We now define the Tutte polynomial, which is a bivariate (far reaching) generalization of the chromatic polynomial. An edge e is an *isthmus* if removing e increases the number of connected components.

Definition 18. The *Tutte polynomial* $T_G(x,y)$ of a graph G is defined inductively by the following relations. If G has no edge, then $T_G(x,y) = 1$. Else, let e be an edge of G, then:

- if e is an isthmus, then $T_G(x,y) = x \cdot T_{G/e}(x,y)$.
- if e is a loop, then $T_G(x,y) = y \cdot T_{G\backslash e}(x,y)$.
- if e is neither an isthmus nor a loop, then $T_G(x,y) = T_{G\backslash e}(x,y) + T_{G/e}(x,y)$.

Remark 5. It is not obvious at all that the definition above makes sense. What is clear is that if the polynomial $T_G(x,y)$ exists, it is unique (since the definition gives an algorithm to compute it in finite time). What is not clear is that, if we make different choices of edges during the computation, we end up finding the same result. This is the subject of the next theorem.

Theorem 19. *The previous definition makes sense. More precisely, the Tutte polynomial is given by the following expression:*

$$T_G(x,y) = \sum_{F \subset E} (x-1)^{\kappa(F)-\kappa(E)}(y-1)^{\kappa(F)+|F|-|V|}. \tag{2.2}$$

Proof. The only thing to prove is that the polynomial defined by (2.2) is solution of the relations stated in Definition 18. This will prove the existence of the Tutte polynomial (as we already noticed, uniqueness is straightforward). In what follows, $T_G(x,y)$ is defined by (2.2) and we check the conditions of Definition 18 one by one.

If G has no edge, i.e. if $E = \emptyset$ then $T_G(x,y) = 1$ since only the empty set contributes to the sum. In the other cases, let e be an edge of E. We partition the subsets of F into two sets: those that do not contain e (that is, the subsets of $E \setminus \{e\}$) and those that contain the edge e. There is an obvious bijection between these two families of subsets given by:

$$F \longmapsto F \cup \{e\}.$$

In (2.2), we can therefore replace the sum over subsets F of E, by a sum over subsets F of $E \setminus \{e\}$, and for each of them consider separately the contribution of F and $F \cup \{e\}$. This gives:

$$\begin{aligned}
&T_G(x,y) \\
= &\sum_{F \subset E} (x-1)^{\kappa(F)-\kappa(E)}(y-1)^{\kappa(F)+|F|-|V|} \\
= &\sum_{F \subset E\backslash\{e\}} (x-1)^{\kappa(F)-\kappa(E)}(y-1)^{\kappa(F)+|F|-|V|} \\
&\quad + \sum_{F \subset E\backslash\{e\}} (x-1)^{\kappa(F\cup\{e\})-\kappa(E)}(y-1)^{\kappa(F\cup\{e\})+|F\cup\{e\}|-|V|} &(2.3)\\
= &\sum_{F \subset E\backslash\{e\}} (x-1)^{\kappa(F)-\kappa(E)}(y-1)^{\kappa(F)+|F|-|V|}\Big(1+(x-1)^{\delta_{e,F}}(y-1)^{1+\delta_{e,F}}\Big). &(2.4)
\end{aligned}$$

where $\delta_{e,F} = \kappa(F \cup \{e\}) - \kappa(F)$.

In what follows we denote by $\tilde{V}$ and $\tilde{E}$ the set of vertices and edges of G/e, respectively. Notice that $|\tilde{V}| = |V| - 1$ and $|\tilde{E}| = |E| - 1$. If $F \subset E \setminus \{e\}$ we denote by $\tilde{\kappa}(F)$ and $\kappa'(F)$ the number of connected components induced by F in the graphs G/e and $G \setminus e$, respectively.

We now proceed with the proof. As in Definition 18, we distinguish three cases (depending on the case we will use (2.3) or (2.4)).

- If e is a loop then for all $F \subset E \setminus \{e\}$ one has: $\delta_{e,F} = 0$, to that (2.4) gives:

$$T_G(x, y) = y \sum_{F \subset E \setminus \{e\}} (x-1)^{\kappa(F)-\kappa(E)} (y-1)^{\kappa(F)+|F|-|V|}.$$

  Now since e is a loop we have $\kappa(F) - \kappa(E) = \kappa'(F) - \kappa'(E)$, and $\kappa(F) - |V| = \kappa'(F) - |V|$. Therefore

$$T_G(x, y) = yT_{G \setminus e}(x, y).$$

- If e is an isthmus then for all $F \subset E \setminus \{e\}$ one has: $\delta_{e,F} = -1$, so that (2.4) gives:

$$T_G(x, y) = x \sum_{F \subset E \setminus \{e\}} (x-1)^{\kappa(F)-\kappa(E)-1} (y-1)^{\kappa(F)+|F|-|V|}.$$

  Now since e is an isthmus we have $\kappa(F) - \kappa(E) - 1 = (\kappa(F) - 1) - \kappa(E) = \tilde{\kappa}(F) - \tilde{\kappa}(\tilde{E})$, and $\kappa(F) - |V| = \tilde{\kappa}(F) + 1 - |V| = \tilde{\kappa}(F) - |\tilde{V}|$. Therefore

$$T_G(x, y) = xT_{G/e}(x, y).$$

- If e is neither an isthmus nor a loop, then we consider (2.3). For all $F \subset E \setminus \{e\}$ we have on the one hand $\kappa(F) = \kappa'(F)$, and on the other hand $\kappa(F \cup \{e\}) = \tilde{\kappa}(F)$. Moreover notice that $|F \cup \{e\}| - |V| = |F| + 1 - |V| = |F| - |\tilde{V}|$. Therefore the first sum in (2.4) is equal to $T_{G \setminus e}(x, y)$, and the second sum is equal to $T_{G/e}(x, y)$, so we have:

$$T_G(x, y) = T_{G \setminus e}(x, y) + T_{G/e}(x, y).$$ □

2.4 Euler tours and the BEST theorem

> *NOTE: Sections 2.4, 2.5 and 2.6 follow quite closely Stanley's reference book* Enumerative Combinatorics, Volume 2, Chapter 5 (except from the combinatorial proof of the Matrix-Tree theorem).

Let $D = (V, E)$ be a directed graph on p vertices and q edges, we let $\text{degin}(v)$ and $\text{degout}(v)$ be the *in*− and *out*− degree of the vertex $v \in V$.

Definition 20. A *tour* is a list of edge $e_1, e_2, \dots, e_r$ such that for $i \in \{1, \dots, r\}$ the origin of e_i is the extremity of e_{i+1} (indices being understood modulo r).

A tour is a *Eulerian* if each edge $e \in E$ appears exactly once (equivalently, if $r = q$).

A graph is *Eulerian* if it has an Eulerian tour.

We say that a digraph is *balanced* if for all $v \in V$, one has $\text{degin}(v) = \text{degout}(v)$.

Theorem 21 (Euler)**.** *A graph is Eulerian if and only if it is balanced and its underlying unoriented graph is connected.*

Proof. The left-to-right implication is straightforward, so let us focus on the other one. We first claim that for each $e \in E$ there exists a tour (non necessarily Eulerian) such that $e_1 = e$. If e_1 is a loop, this is clear. If not, there exists some "exit edge" e_2 starting from the endpoint of e_1. If the endpoint of e_2 is the origin of e_1, we are done. Else, we iterate the construction. At each step, either we have completed a tour, or we have entered the current vertex once more that we have left it, so by hypothesis there exists a vacant exit edge. Since the graph is finite, we are sure to complete a tour at some point.

We now prove the theorem. By the first claim there exists a tour $C = e_1, \ldots, e_r$ of some length $r \geq 1$. If $r = q$ we are done. Else consider the graph $\tilde{D} = D \setminus C$. Then in $\tilde{D}$, there is at least one connected component H that has a vertex in common with C. So applying the claim to H we can find a tour $\tilde{C}$ in H. We can now follow the tour C and "make a detour by $\tilde{C}$" to obtain another tour, strictly longer than C. Repeating the construction as many times as it is necessary we build up a tour of length $r = q$. □

Definition 22. Let $D = (V, E)$ be a directed graph and $v \in V$. A *spanning tree of D oriented towards v* is a subgraph T of D, such that there exists a unique directed path in T from any vertex of V to v.

The BEST Theorem (named after *de Bruijn, van Aardenne-Ehrenfest, Smith, Tutte* but due only to the first two of them) is the remarkable result:

Theorem 23 (BEST). *Let $D = (V, E)$ be a directed graph, balanced and with a connected underlying graph. Let e be any edge, and let v be the origin of e. Let $Euler(e)$ and $Tree(v)$ be respectively the number of Euler tours of D starting with e and the number of spanning trees of D oriented towards v. Then one has:*

$$Euler(e) = Tree(v) \times \prod_{u \in V} (\mathrm{degout}(u) - 1)! \tag{2.5}$$

Remark 6. The right hand-side is clearly independent of v, so we get the additional result that, *in an Eulerian directed graph*, the number $Tree(v)$ does not depend on the choice of the vertex v. This result, of course, is not true for any directed graph (Think of an oriented tree, for example).

Proof. The proof is based on the construction of the "exit-tree" of the Eulerian tour.
• Given an Euler tour of D starting from e, we construct the *exit-tree*, which is defined as follows. For each vertex $u \neq v$, we let e_u be the last edge leaving the vertex u during the tour. The exit-tree is defined as the union of all these exit edges.

It is easy to see that the "exit-tree" is indeed a tree: it has p vertices, $p - 1$ edges, and no cycles (indeed: define the *index* of a vertex as the index in the tour of its exit edge. Then an exit edge always goes from a vertex to either v or a vertex of a larger index: this prevents the existence of cycles. This also shows that this tree is oriented towards v)

In this way we can associate to the tour the pair $(T, (\sigma_u)_{u \in V})$ where T is the exit-tree and for each $u \in V$, σ_u is a permutation of the outgoing edges at u different from e_u (or from e if $u = v$) that remembers in which order these edges have been taken by the tour.
• Conversely, let T be a tree oriented towards v, let e_u be the unique edge of T leaving each vertex $u \in V$. Also define $e_v := e$. For each $u \in V$ fix a permutation σ_u of the outgoing edges at u different from e_u. Then one reconstructs an Eulerian tour in the following way:

- start with the edge e

- iterate the following procedure: when one reaches a vertex u, if there still exists an outgoing edge at u that is not e_u, then exit u with the first such edge, in the ordering given by the permutation σ_u. Else, call u *saturated*, and exit u with the edge e_u.

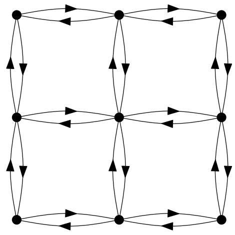
We claim that this procedure always reconstructs an Euler tour. First, when one reaches a new vertex u, then either $u = v$ or the vertex has been reached once more than it has been left, in which case, by the balance condition, there is an exit edge available, Therefore, the only vertex at which the procedure can stop is v. When this happens, we have constructed a tour of D: we still have to show that this is an Eulerian tour, that is, that all edges have been visited. To see that, we claim the following:

when a vertex u is declared "saturated", all its children in the tree T are saturated.

To see that the claim is true, let u be a saturated vertex and let u' be a child of u in T. Since u is saturated the edge of T $u' \to u$ has been taken by the tour: but since this edge belongs to T, by construction, this means that u' is saturated. By the claim, we deduce that when v is saturated, by induction all the vertices of T are saturated, so all edges have been visited and our tour is an Eulerian tour.

• In the previous construction it is clear that the tree T is the "exit-tree" of the tour we have reconstructed, so that we have a bijection between Eulerian tours starting from e and pairs $(T, (\sigma_u)_{u\in V})$. Now, the number of choices of the pair $(T, (\sigma_u)_{u\in V})$ is exactly the right-hand side of (2.5), so the BEST theorem is proved. □

Example 1. As an application (still following Stanley, EC2) we studied the example of a mailman operating on a four-block domain of the city (say New-York... but be careful that all the roads are two-ways):



We admitted (but this can be checked by hand in a few minutes) that this graph had 192 spanning trees rooted at the top-right vertex, so that by the BEST theorem its number of Euler tours starting from a given edge (say where the post office is) is $192 \times 1!^4 \times 2!^4 \times 3!^1 = 18432$. We concluded that the mailman can spend all its career taking a different path every day. Observe also that the theorem is no only enumerative but also effective: the proof we have given enables the mailman to try all the tours sequentially in a very easy way (for example he can first order once and for all the 192 spanning trees in some arbitrary way, and then decide on some lexicographical order to generate the permutations σ_u incrementally).

2.5 The matrix-tree theorem

NOTE: Sections 2.4, 2.5 and 2.6 follow quite closely Stanley's reference book Enumerative Combinatorics, Volume 2, Chapter 5 (except from the combinatorial proof of the Matrix-Tree theorem).

The main message of this section is that, sometimes, in combinatorics, the solution to some difficult enumeration problem is given by a determinant. And, sometimes, this determinant is easy (or: not so hard) to compute with some linear algebra. We are going to see that now with the matrix-tree theorem. Another example, in the next lecture, will be the "Lindström-Gessel-Viennot" path-counting formula.

In this section all graphs will be loopless, even if I forget to mention it: the reason for that is that, in dealing with spanning trees, loops are not very interesting...

Definition 24. Let $D = (V, E)$ be a loopless directed graph with p vertices, and note $V = \{v_1, v_2, \ldots, v_p\}$. The *Laplacian matrix of D* is the $p \times p$ matrix defined by:

$$L_{i,j} = \begin{cases} -m_{i,j} & \text{if } i \neq j, \\ \text{degout}(v_i) & \text{if } i = j, \end{cases}$$

where $m_{i,j}$ is the number of (oriented) edges from v_i to v_j.

Remark 7. One can also write this definition as:

$$L := \text{Diag}(\text{degout}(v_1), \ldots, \text{degout}(v_p)) - A$$

where A is the (oriented) adjacency matrix of D. This viewpoint will be especially interesting when the graph is k-outregular for some $k \geq 1$, in which case $L = kId - A$.

Here is the matrix-tree theorem:

Theorem 25 (Matrix-tree theorem)**.** *Let $k \in \{1, 2, \ldots, p\}$ and let L_0 be the matrix obtained by removing the k-th row and the k-th column from L. Then the number of spanning trees of D oriented towards v_k is given by the determinant of L_0:*

$$\det L_0 = Tree(v_k).$$

As an immediate corollary we can count the number of spanning trees of unoriented graphs:

Corollary 26. *Les G be a loopless graph (unoriented), with vertex set $V = \{v_1, v_2, \ldots, v_p\}$, and define its* Laplacian *as the symmetric matrix:*

$$L_{i,j} = \begin{cases} -m_{i,j} & \textit{if } i \neq j, \\ \deg(v_i) & \textit{if } i = j, \end{cases}$$

where $m_{i,j}$ is the number of edges between v_i and v_j. Let L_0 be obtained by erasing the k-th row and column of L. Then the number of spanning trees of G is equal to $\det L_0$.

Proof of the corollary from the theorem. Construct a directed graph D from G by duplicating each edge into two oriented edges (one in each direction), and fix k. Then (unoriented) spanning trees of G are bijection with spanning trees of D oriented towards v_k. □

Proof of the Theorem. During the lecture I gave two proofs: one by linear algebra, which was taken from Stanley, EC2, and another one, purely combinatorial, which I reproduce here (I don't know any reference for this proof, let's say it is folklore). Do not be discouraged by its relative length on paper: once you have the picture in mind, it is really easy.

For convenience in the notation we assume that $k = n$ (i.e. we erase the last row and last column of L). The general case is obtained simply by renaming vertices and conjugating the matrix. By definition of an $(n-1) \times (n-1)$ determinant we have:

$$\begin{aligned}\det L_0 &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \epsilon(\sigma)(L_0)_{1,\sigma_1}(L_0)_{2,\sigma_2} \dots (L_0)_{n-1,\sigma_{n-1}} \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}} \epsilon(\sigma) \left(\prod_{i:\ \sigma_i \neq i} (-m_{i,\sigma_i}) \right) \left(\prod_{i:\ \sigma_i = i} (\text{degout}(v_i)) \right), \qquad (2.6)\end{aligned}$$

by definition of the matrix L (in the product, we have separated fixed points of σ from other elements, because the definition of $L_{i,j}$ is split into two cases: $i = j$ and $i \neq j$).

We are now going to interpret this sum as a sum over subgraphs of D with two colors of edges. Given a permutation $\sigma \in \mathfrak{S}_{n-1}$, we choose for each fixed point i of σ one of the degout(v_i) outgoing edges of v_i, and we color it red; for each i which is not a fixed point, we chose one of the m_{i,σ_i} edges going from i to σ_i, and we color it blue. This leads us to introduce *red-blue configurations*: A red-blue configuration is an oriented subgraph of D, with red and blue edges, which has the following properties:

- the set of blue edges is a vertex-disjoint union of directed cycles on the set of vertices $\{v_1, v_2, \dots, v_{n-1}\}$ (not all these vertices necessarily belong to a cycle, some vertices can be left over).
- call a *red vertex* a vertex in $\{v_1, v_2, \dots, v_{n-1}\}$ which is not incident to any blue edge. Then each red vertex has exactly one outgoing red edge (this edge may be directed towards v_n, this is allowed).

Then (2.6) can be interpreted as a sum over all red-blue configurations: red vertices correspond to fixed points, the blue cycles correspond to the cycles of σ of length ≥ 2, the choice of one outgoing (red) edge for each fixed point corresponds to the factor degout(v_i), and if there are several edges between v_i and v_{σ_i}, the factor $m_{i\sigma_i}$ accounts for the choice of the corresponding blue edge.

In this sum, each configuration is counted with a weight equal to $(-1)^b$ where b is the number of blue cycles: to see that, note that in (2.6), each cycle of σ of length ≥ 2 (i.e.: each blue cycle) gives rise to a factor of $(-1)^{k-1}$ in the signature $\epsilon(\sigma)$, and to another factor of $(-1)^k$ because of the minus sign in the product: therefore each blue cycle gives rise to a factor of $(-1)^{k-1+k} = -1$, which gives a total weight of $(-1)^b$ in total for the red-blue configuration.

Once we have this interpretation, we are almost done. Indeed I claim that in the sum (2.6), the contribution of the red-blue configurations which have at least one cycle is equal to 0. Indeed, let K be a red-blue configuration having at least one cycle, and let c be the cycle of R which contains the vertex of minimum index among vertices belonging to cycles. Define K' to be the red-blue configuration which is the same as K expect that we exchange the color of the cycle c. On the one hand, it is clear that the mapping $K \longmapsto K'$ is an involution; on the other hand, one has $(-1)^{\text{blue cycles of } K} + (-1)^{\text{blue cycles of } K'} = 0$. This shows that in (2.6), only the red-blue configurations having no cycles have a positive contribution, as claimed. Such configurations have p vertices and $p-1$ edges so they are trees, and it is not difficult to see that they are oriented towards v_n. Finally, since they are formed of red vertices only, they are counted with weight 1. Thus (2.6) is simply the sum over all spanning trees oriented towards v_n of the quantity 1, which proves the theorem. □

As an application we gave a proof of Cayley's formula for the number of trees on the vertex set $\{1, 2, \ldots, n\}$. By definition, such a tree is nothing but a spanning tree of the complete graph K_n on $\{1, 2, \ldots, n\}$. By the undirected corollary of the matrix-tree theorem, the number of labelled trees is therefore the $(n-1) \times (n-1)$ determinant:

$$l_n = \begin{vmatrix} n-1 & -1 & -1 & -1 & \ldots & -1 \\ -1 & n-1 & -1 & -1 & \ldots & -1 \\ -1 & -1 & n-1 & -1 & \ldots & -1 \\ & & & \ddots & & \\ & & & & \ddots & \\ -1 & -1 & -1 & -1 & \ldots & n-1 \end{vmatrix}.$$

A very simple way to compute this determinant is to realize that the set of columns vectors $(x_1, x_2, \ldots, x_{n-1})$ such that $\sum x_i = 0$ is a eigenspace associated to the eigenvalue n: since this vector space has dimension $(n-1) - 1 = n-2$ this gives us contribution of n^{n-2} to the determinant. On the other hand the vector space generated by $(1, 1, \ldots, 1)$ (i.e. the orthogonal of the previous one) is an eigenspace of eigenvalue 1. Therefore we have found all the $(n-1)$ eigenvalues of our matrix, and the determinant is the product of them, i.e. $l_n = 1 \times n^{n-2} = n^{n-2}$.

Another way (there are many) to compute this determinant is to observe that the $(n-1) \times (n-1)$ matrix we have to take the determinant of is equal to $n\mathrm{Id} - J$ where J is the all-one matrix. But J has rank one so it has only one non-zero eigenvalue, equal to its trace, which is $n-1$. Therefore (think of a triangulating basis) the $(n-1)$ eigenvalues of $n\mathrm{Id} - J$ are $(n, n, \ldots, n; 1)$ and we are done.

In some applications the matrix L is much more comfortable to work with than the matrix L_0 because it possesses some symmetry properties inherited from the graph D (and one loses the symmetry when passing from L to L_0). For these cases, the so-called "eigenvalue version of the matrix-tree theorem" can be useful. It is based on the following lemma:

Lemma 27 (Linear algebra)**.** *Let M be a $n \times n$ matrix such that the sum of each row is zero, and the sum of each column is 0. Let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of M with $\mu_n = 0$.*

Let M_0 be obtained by removing the k-th row and column of M. Then one has:

$$\det(L_0) = \frac{1}{n} \mu_1 \mu_2 \ldots \mu_{n-1}.$$

Proof. Row and Column manipulations. □

Note that the Laplacian matrix of a directed graph D satisfies the hypotheses of the lemma if and only if D is such that $\mathrm{degout}(v) = \mathrm{degin}(v)$ at each vertex (the row sums are always 0 in any Laplacian matrix, but the columns sums being 0 is equivalent to this hypothesis). In other words, if D is balanced.

Corollary 28 (Matrix-tree theorem, eigenvalue version, for balanced directed graphs)**.** *Let $D = (V, E)$ be a balanced directed graph, and let $\mu_1, \mu_2, \ldots, \mu_n$ be the eigenvalues of its Laplacian matrix with $\mu_n = 0$. Then the number of spanning trees of D oriented towards any given fixed vertex is equal to $\frac{1}{n} \mu_1 \mu_2 \ldots \mu_{n-1}$.*

Remark 8. In particular in the balanced case the number of spanning trees does not depend on the root vertex: we already observed that as a corollary of the BEST theorem.

Recall that we have obtained an undirected version of the matrix-tree theorem by "duplicating the edges": since in this process the balance condition is always fulfilled, we obtain the following:

Corollary 29 (Matrix-tree theorem, eigenvalue version, undirected graph)**.** *Let G be an unoriented graph and let $\mu_1, \mu_2, \dots, \mu_n$ be the eigenvalues of its Laplacian matrix with $\mu_n = 0$. Then the number of spanning trees of G is equal to $\frac{1}{n}\mu_1\mu_2 \dots \mu_{n-1}$.*

2.6 Illustration: The de Bruijn graph

> *NOTE: Sections 2.4, 2.5 and 2.6 follow quite closely Stanley's reference book* Enumerative Combinatorics, Volume 2, Chapter 5 (except from the combinatorial proof of the Matrix-Tree theorem).

We spent some time discussing the de Bruijn Graph B_n. Let $n \geq 1$ be an integer, and consider the graph with vertex set V being all the binary words of length $n-1$, and an oriented edge from $a_1a_2 \dots a_{n-1}$ to $b_1b_2 \dots b_{n-1}$ if and only if $a_2a_3 \dots a_{n-1} = b_1b_2 \dots b_{n-2}$. There are exactly two edges leaving each vertex, corresponding to the two choices of the letter b_{n-1} with the above notation.

A *de Bruijn sequence of degree* n is a binary sequence $u_1u_2 \dots u_k$ containing circularly each binary word of length n exactly once (by "circularly" we mean that the pattern can be of the form $u_{k-r} \dots u_k u_1 \dots u_{n-r-1}$: it can overlap the boundary). It is easy to see that one has necessarily $k = 2^n$, and that de Bruijn sequences beginning with the word 0^n are in bijection with Euler tours of the graph B_n starting with the loop-edge $0^{n-1} \to 0^{n-1}$. (to see that, just think that you are maintaining a buffer of the last $n-1$ letters you have read: which vertex you are in tells you the state of your buffer. Then observe that the edges of B_n are in bijection with the words of length n).

Therefore by the BEST theorem, we can count the number of de Bruijn sequences by counting the spanning trees of B_n oriented towards 0^{n-1}, and by the eigenvalue version of the matrix-tree theorem, we know that this number is equal to $\frac{1}{2^{n-1}}$ times the product of the eigenvalues of the Laplacian. Moreover, since the graph is 2-outregular, the Laplacian is given by:

$$L = 2\text{Id} - A$$

where A is the adjacency matrix of B_n (these are all $2^{n-1} \times 2^{n-1}$ matrices). To determine the eigenvalues of A, there is a trick: we observe that in B_n, there is a unique path of length $n-1$ between any pair of vertices: therefore $A^{n-1} = J$, the all-one matrix of size 2^{n-1}. This matrix has rank one, so it has $2^{n-1} - 1$ eigenvalues equal to 0, and the last one is equal to its trace, which is 2^{n-1}. Therefore the eigenvalues of A are 0 ($2^{n-1} - 1$ times) and 2 (once). Hence, the eigenvalues of $L = 2\text{Id} - A$ are 2 (with multiplicity $2^{n-1} - 1$) and 0 (with multiplicity 1, as expected...). Therefore the number of spanning trees of B_n oriented towards 0^{n-1} is:

$$\frac{1}{2^{n-1}} \times 2^{2^{n-1}-1} = 2^{2^{n-1}-n}.$$

By the BEST theorem, we obtain the number of Euler tours starting from the edge $0^{n-1} \to 0^{n-1}$ by multiplying this with $1!^{2^{n-1}} = 1$. Finally, to obtain all de Bruijns sequence, we must multiply this by the number of edges (possible starting edges of en Eulerian tour of B_n;

equivalently, number of possible starting words of length n for our sequence), which is 2^n. We obtain:

Theorem 30. *The number of de Bruijn sequences of degree n is equal to $2^{2^{n-1}}$.*

Chapter 3

Yet more "signed combinatorics"

3.1 The Lindström-Gessel-Viennot lemma.

3.1.1 Introduction: intervals in the Stanley lattice and the reflection principle

We consider the set $\mathcal{D}_n$ of Dyck paths of size n, i.e. paths $P = (x_0, x_1, \dots, x_{2n})$ on the nonnegative integers $\mathbb{N}$ starting from $x_0 = 0$ and ending at $x_{2n} = 0$, taking only steps -1 or $+1$.

There is a natural partial order on $\mathcal{D}_n$, the Stanley order: we say that two paths P and Q are such that $P \leq Q$ if for all i, one has $x_i \leq y_i$, where $P = (x_0, x_1, \dots, x_{2n})$ and $Q = (y_0, y_1, \dots, y_{2n})$. In other words, the path P is (weakly) below the path Q on the picture.

Our goal is two determine the number I_n of pairs (P, Q) such that $P \leq Q$, or equivalently, the number of *Intervals* $[P, Q]$ for the Stanley order (recall that the interval $[P, Q]$ is defined as the set $[P, Q] = \{R, P \leq R \leq Q\}$).

We first observe that one has $P \leq Q$ if and only if the path P is strictly under the path $Q + 2 := (y_0 + 2, y_1 + 2, \dots, y_{2n} + 2)$. One direction is immediate, for the other one observe that for each i the coordinates x_i and y_i have the same parity, so $x_i < y_i + 2$ implies $x_i \leq y_i$.

Now consider two paths $P = (x_0, x_1, \dots, x_{2n})$ and $Q' = (y'_0, y'_1, \dots, y'_{2n})$ on $\mathbb{N}$, with only steps -1 and $+1$, with $x_0 = x_{2n} = 0$ but this time $y_0 = y_{2n} = 2$. We say that these paths are non intersecting if there is no i such that $x_i = y'_i$. We make the following observation: if P and Q are non intersecting, then the path Q stays ≥ 2 all along. Indeed, if they are non-intersecting, in particular one has $x_i < y'_i$ for all i (otherwise they would cross somewhere), so $2 \leq y'_i$ because P stays nonnegative and because x_i and y'_i have the same parity. Therefore we have:

Lemma 31. *Pairs of Dyck paths (P, Q) such that $P \leq Q$ are in bijection with non-intersecting pairs of nonnegative paths $P = (x_0, x_1, \dots, x_{2n})$ and $Q' = (y'_0, y'_1, \dots, y'_{2n})$, with only steps -1 and $+1$, with $x_0 = x_{2n} = 0$ and $y_0 = y_{2n} = 2$.*

For all $i, j \leq 0$ we let $N_{2n}(i, j)$ be the number of paths of length $2n$ that go from ordinate i to ordinate j in $2n$ steps and stay nonnegative. So the the total number of pairs (P, Q') such that P goes from 0 to 0 and Q' goes from 2 to 2 is:

$$N_{2n}(0, 0) \times N_{2n}(2, 2).$$

Now we have:

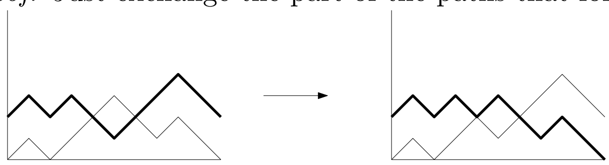
$$\Big(\text{non intersecting pairs}\Big) = \Big(\text{all pairs}\Big) - \Big(\text{intersecting pairs}\Big).$$

The following exchange lemma will be our first version of Linström-Gessel-Viennot:

Lemma 32 (exchange)**.** *Pairs of nonnegative paths (P, Q') of length $2n$ such that P goes from 0 to 0 and Q' goes from 2 to 2, and such that P and Q' intersect each other are counted by:*

$$N_{2n}(0,2) \times N_{2n}(2,0).$$

Proof. Just exchange the part of the paths that follow the first intersection:



You obtain two new (nonnegative) paths, going respectively from 0 to 2 and from 2 to 0. This is easily seen to be a bijection, since two paths going respectively from 0 to 2 and from 2 to 0 necessarily intersect each other. □

We thus have:

$$I_n = N_{2n}(2,2)N_{2n}(0,0) - N_{2n}(2,0)N_{2n}(0,2).$$

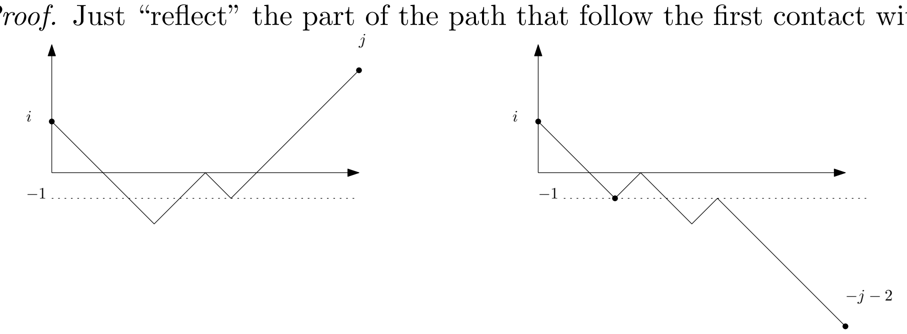
To conclude, it remains to determine the number $N_{2n}(i,j)$ of nonnegative paths going from i to j in $2n$ steps. We are going to use the same idea: extend the set of paths we consider, and remove the “bad paths” that we don’t want to count. The number of paths going from i to j in $2n$ steps but which are not necessarily nonnegative is simply given by the binomial:

$$\binom{2n}{n + \frac{j-i}{2}} \qquad (j - i \text{ even.}),$$

since $n + \frac{j-i}{2}$ is the number of $+1$ steps of such a path. To this number, in order to obtain $N_{2n}(i,j)$ we should remove the number of paths going from i to j that reach at least once the ordinate -1. Now comes the famous and elegant **reflexion principle**:

Lemma 33 (Reflexion principle)**.** *Paths going from i to j that reach the ordinate -1 are in bijection with paths going from i to $-j-2$.*

Proof. Just “reflect” the part of the path that follow the first contact with the ordinate -1:



□

From the reflexion principle, we immediately have (for $i, j \geq 0$ and $j - i$ even):

$$N_{2n}(i,j) = \binom{2n}{n + \frac{j-i}{2}} - \binom{2n}{n + \frac{i+j+2}{2}}.$$

Note in particular that $N_{2n}(0,0)$ gives you back the n-th Catalan number (this is maybe the easiest proof of that Dyck paths are coutned by Catalan numbers).

Putting everything together, and doing the algebra, we have finally proved:

Theorem 34. *The number of intervals in the Stanley lattice is equal to:*

$$\begin{aligned} I_n &= 12 \frac{(2n)!(2n+1)!}{(n+3)!(n+2)!(n+1)!^2} \\ &= \frac{6}{(n+3)(n+2)} \cdot \mathrm{Cat}(n)\mathrm{Cat}(n+1). \end{aligned}$$

3.1.2 The Lindström-Gessel-Viennot lemma

Let $D = (V, E)$ be a directed graph without any directed cycle (one can have cycles in the undirected underlying graph, but no directed cycles). We assume that D is simple (no multiplicities) but we put a *weight* $w(e)$ on each edge $e \in E$. The weight is extended multiplicatively to paths and subgraphs, for example the weight of a path $P = (e_1, e_2, \dots, e_r)$ is the product $w(e_1)w(e_2)\dots w(e_r)$.

We fix a number $n \geq 1$ and two n-tuples of vertices $(v_1, v_2, \dots, v_n)$ and $(w_1, w_2, \dots w_n)$, that we imagine respectively as the "starting points" and the "ending points".

We define:

$$M_{i,j} = \sum_{P : v_i \overset{*}{\to} w_j} w(P)$$

where the sum is taken over all directed paths from v_i to w_j, and $w(P)$ is the weight of P. For example if $w \equiv 1$ then $M_{i,j}$ is just the number of directed paths from v_i to w_j.

Definition 35. A *path-system* is a tuple $(\sigma; P_1, P_2, \dots, P_n)$ where σ is an element of $\mathfrak{S}_n$ and for all $i \in \{1, 2, \dots, n\}$, P_i is a directed path from v_i to w_{σ_i}. The signature of S is $\epsilon(S) := \epsilon(\sigma)$, and its weight is $w(S) = w(P_1)w(P_2)\dots w(P_n)$.

The path system S is *non-intersecting* if for all $i \neq j$ the paths P_i and P_j have no vertex in common. The path system is *intersecting* if it is not non-interesting (i.e. if there exists at least one vertex which is shared by at least two paths).

We now state the Lindström-Gessel-Viennot lemma. If the general form does not look very appealing to you, look at the corollary right after, which is more directly interesting:

Theorem 36 (LGV lemma)**.** *With the hypotheses and notation above, one has:*

$$\det\left(\left(M_{i,j}\right)_{1 \leq i,j \leq n}\right) = \sum_S \epsilon(S) w(S)$$

where the sum is taken on all non-intersecting paths systems.

Corollary 37 (Simplest form of LGV lemma). *Assume that the directed graph D and the points $\{v_1, v_2, \dots, v_n\}$ and $\{w_1, w_2, \dots, w_n\}$ are such that the only non-intersecting paths systems have positive signature. Assume moreover that $w \equiv 1$. Then* $\det\Big((M_{i,j})_{1\le i,j\le n}\Big)$ *is equal to the number of non-intersecting paths systems.*

Proof. The proof is very easy: expand the determinant, and realize that all intersecting path systems disappear in the expansion. That's all!

More precisely, write:

$$\begin{aligned}\det M &= \sum_{\sigma} \epsilon(\sigma) M_{1,\sigma_1} M_{2,\sigma_2} \dots M_{n,\sigma_n} \\ &= \sum_{\sigma} \epsilon(\sigma) \sum_{P_1 : v_1 \to w_{\sigma_1}} w(P_1) \sum_{P_2 : v_2 \to w_{\sigma_2}} w(P_2) \cdots \sum_{P_n : v_n \to w_{\sigma_n}} w(P_n) \\ &= \sum_{S} \epsilon(S) w(S), \end{aligned} \tag{3.1}$$

where the sum is taken on *all* path-systems S (intersecting or not). We want to show that in (3.1), if we restrict the sum to the path-systems which are intersecting we get 0 (then we will have proved the theorem). To do that, we are going to group the intersecting path-systems in pairs, in such a way that the sum of the two contributions in each pair is 0.

Let $S = (P_1, P_2, \dots, P_n)$ be an intersecting path-system. We consider the smallest i such that P_i intersects another path P_j for $j > i$, we let x be the first vertex of P_i which is common to another path, and we consider the smallest $j > i$ such that P_j contains x. We then define two new paths $\tilde{P}_i$ and $\tilde{P}_j$ by taking P_i and P_j and exchanging the part of the paths which follow the contact with x. This defines a new path system $\tilde{S} = (P_1, \dots, \tilde{P}_i, \dots \tilde{P}_j, \dots P_n)$ (we leave the paths P_k unchanged for $k \notin \{i, j\}$). This path system is such that $w(\tilde{S}) = w(S)$ since it is made of the same set of edges as S, and it is such that $\epsilon(\tilde{S}) = -\epsilon(S)$ since the two corresponding permutations differ only by the transposition $(i\ j)$. Therefore we have:

$$\epsilon(S)w(S) + \epsilon(\tilde{S})w(\tilde{S}) = 0.$$

Finally, the mapping $S \longmapsto \tilde{S}$ is clearly an involution. Therefore we can group the intersecting path-systems in pairs $(S, \tilde{S})$ of total zero contribution, and only the non-intersecting path-systems contribute to (3.1). This is exactly what the theorem states, so the proof is done. □

Note that there is nothing to prove to obtain the Corollary, which is a simple specialisation.

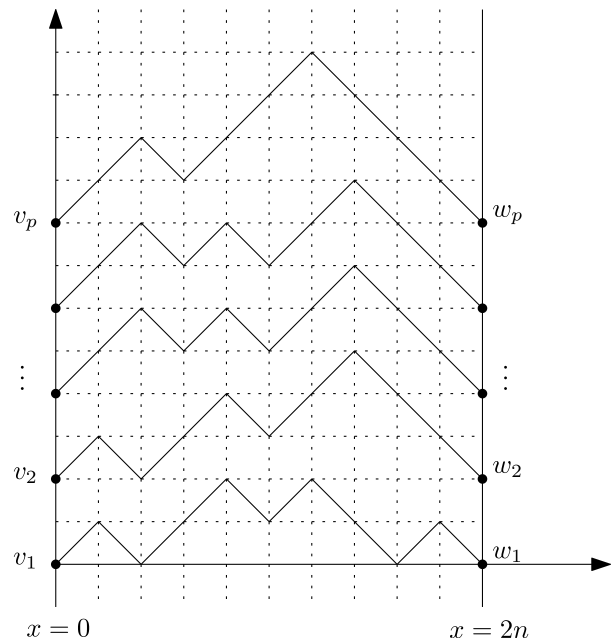
Remark 9. In the proof we *have* used the hypothesis that D is acyclic. First, if one has directed cycles, there can be infinitely many paths between i and j, so the quantity $M_{i,j}$ can be infinite: this objection is not very serious since we could for example have restricted our study to the case where all-edge weights are close enough from 0, in which case the quantity $M_{i,j}$ is still finite even if it is defined as a sum over infinitely many paths. More importantly, we have used the hypothesis when we said *"we exchange the parts of P_i and P_j that follow x"*: indeed we implicitly used the fact that each path touches x no more than once. Otherwise, if there are several contacts of the same path with x, it would be impossible to define the "exchange" procedure in way which is non-ambiguous and involutive. As a matter of fact, the LGV lemma is FALSE in general for graphs having directed cycles.

3.1.3 Non intersecting Catalan paths, a.k.a. watermelons

As an application of the LGV lemma, we will generalize our previous result on the number of intervals in the Stanley lattice. Instead of 2 paths, we will consider an arbitrary number p of paths one under the other. More precisely, consider the points in $\mathbb{Z}^2$:

$$v_i = (0, 2i-2) \text{ and } w_i = (2n, 2i-2), \text{ for } 1 \le i \le p.$$

Between these points, we consider paths taking only steps $(1,+1)$ or $(1,-1)$, and whose ordinate stays nonnegative. For example, paths between v_1 and w_1 are nothing but Dyck paths of length $2n$, and are counted by the Catalan number $\mathrm{Cat}(n)$. We want to count non-intersecting paths systems going from $\{v_1, v_2, \dots, v_n\}$ to $\{w_1, w_2, \dots, w_n\}$. Such systems are called *positive watermelons*, and here is an example:



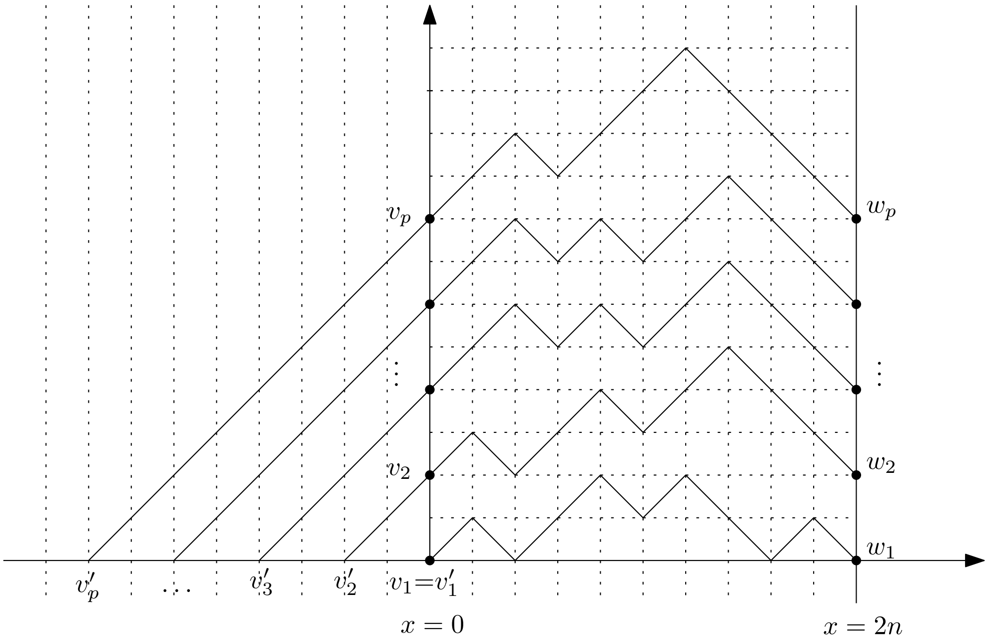
Since, by planarity, such systems are such that v_i is connected to w_i for all i, we can apply the simple corollary to the LGV lemma, and the number $W_p(n)$ of positive watermelons of width $2n$ made of p paths is given by the determinant:

$$W_p(n) = \det\Big(N_{2n}(2i-2, 2j-2)\Big)_{1 \le i,j \le p}.$$

There are several ways to compute this determinant (unfortunately, none of them is straightforward). The way we choose here starts with a combinatorial trick that transforms this determinant into another one, which be by easier to evaluate. Consider the points:

$$v'_i = (-2i, 0) \text{ for } 1 \le i \le p.$$

Then any system of non-intersecting paths from $\{v_1, v_2, \dots, v_p\}$ to $\{w_1, w_2, \dots, w_p\}$ can be transformed into a system of non-intersecting paths from $\{v'_1, v'_2, \dots, v'_p\}$ to $\{w_1, w_2, \dots, w_p\}$ as follows:



Since we consider *non-intersecting systems of paths*, this is easily seen to be a bijection (observe that in any non-intersecting system from $\{v'_1, v'_2, \ldots, v'_p\}$ to $\{w_1, w_2, \ldots, w_p\}$, the left-part of the drawing is "frozen" – there is not enough space for paths to do something else that linking straightly v'_i to v_i). Therefore we have:

$$W_p(n) = \det\left(N_{2n+2i}(0, 2j-2)\right)_{1\le i,j\le p}. \tag{3.2}$$

Remark 10. Using the same trick on the right part of the picture, we could introduce the points $w_j = (2n + 2j - 2, 0)$, and rewrite our determinant as:

$$W_p(n) = \det\left(\mathrm{Cat}(n+i+j-2)\right)_{1\le i,j\le p}.$$

But, we will choose here to work with the expression (3.2) to do the computation. We have:

Theorem 38. *The number of positive watermelons of with $2n$ with p paths is given by:*

$$W_p(n) = \prod_{i=0}^{p-1} \frac{(2i+1)!\mathrm{Cat}(n+i)}{(n+p+i)(n+p+i-1)\ldots(n+i+2)}.$$

Remark 11. This is a beautiful formula!!!

Proof. If you have ever computed a Vandermonde determinant in your life, then you know that to compute a determinant, it helps to have *polynomial* entries in your matrix. So, in (3.2), we start by factoring out all the greatest denominators on each line and column, to obtain a determinant with polynomial entries:

$$\begin{aligned}(3.2) &= \det\left(\frac{2j-1}{2n+2i-1}\binom{2n+2i-1}{n+j}\right)_{1\le i,j\le p} \\ &= \prod_{i=1}^{p}\frac{(2j-1)(2n+2i-2)!}{(n+p-i)!(n+p+i-1)!} \\ &\quad \times \det\left(\left[(n+p-i)\ldots(n+i-j+1)\right]\left[(n+i+p-1)\ldots(n+i+j)\right]\right)_{1\le i,j\le p}\end{aligned}$$

In each entry of the determinant above, we will group factors by pairs, using:

$$(n+i+t-1)(n-i+t) = \left(n+t-\frac{1}{2}\right)^2 - \left(i-\frac{1}{2}\right)^2.$$

This determinant thus rewrites as:

$$\det\left(\left[\left(n+p-\frac{1}{2}\right)^2 - \left(i-\frac{1}{2}\right)^2\right]\dots\left[\left(n+j+1-\frac{1}{2}\right)^2 - \left(i-\frac{1}{2}\right)^2\right]\right)_{1\le i,j\le p}$$

Using column manipulations, this determinant is equal to:

$$\det\left(\left(i-\frac{1}{2}\right)^{2(p-j}\right)_{1\le i,j\le p},$$

which, as a Vandermonde (!) determinant, evaluates as:

$$\prod_{1\le i<j\le p}\left((i-1/2)^2-(j-1/2)^2\right) = \prod_{1\le i<j\le p}(i-j)(i+j-1) = \prod_{j=1}^{p}(2j-2)!.$$

Putting everything together, we obtain:

$$(3.2) \quad = \quad \prod_{i=1}^{p}\frac{(2j-1)(2n+2i-2)!}{(n+p-i)!(n+p+i-1)!}\prod_{j=1}^{p}(2j-2)!,$$

which is easily rewritten as $\displaystyle\prod_{i=0}^{p-1}\frac{(2i+1)!\mathrm{Cat}(n+i)}{(n+p+i)(n+p+i-1)\dots(n+i+2)}$. □

3.2 Plane Partitions in a box and MacMahon's formula

As an application of the LGV lemma, we have sketched the proof of one of the most remarkable formulas of combinatorics: MacMahon's formula for plane partitions in a box.

A plane parition is a finite array of integers, weakly decreasing along rows and columns. When we write a plane partition we do not write the entries equal to zero, so we obtain a finite array looking like that:

7	7	3	3	1
5	5	3	2	
4	3	3		
2	1			

Here, we say that the partition has width 5, height 4 and largest part 7 (this terminology is not standard). We can imagine plane partitions as $3D$ objects, by imagining that on each entry of the matrix we put a pile of cube, the number of cubes being given by the value of the entry. Following this representation we say that the plane partition *fits inside a box* of dimensions $a \times b \times c$ if its width, height, and largest part are respectively at most a, b, and c. We have:

Theorem 39 (MacMahon's formula, 1915)**.** *The number $M(a,b,c)$ of plane partitions fitting in a box of dimensions $a \times b \times c$ is given by the explicit formula:*

$$M(a,b,c) = \prod_{i=1}^{a}\prod_{j=1}^{b}\prod_{k=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{3.3}$$

Remark 12. There is a beautiful symmetry in a, b, c in this formula, which is not totally obvious if one thinks of our first definition of plane partitions. However, if we think of plane partitions as "piles of cubes", this symmetry is clear: it just corresponds to exchanging the axes of the $3D$ coordinate system.

Sketch of the proof. The proof of MacMahon's formula given here consists of four steps:
1. Put plane partitions in bijection with certain hexagon tilings;
2. Put those tilings in bijection with certain systems of non-intersecting paths;
3. Use the LGV lemma to express the result as a big determinant;
4. Well... compute the big determinant.

From now on I am going to rely extensively on the Figure I handed out during the lecture, reproduced as the last page of this document.

Step 1: plane partitions and tilings. We consider an infinite triangular lattice (i.e. a tiling of the plane made by equilateral triangles with sides equal to 1: there are several ways to orient this lattice on a sheet of paper but we'll take the convention that each triangle contains a vertical segment). On this lattice, draw an hexagone $H_{a,b,c}$ of side lengths equal to (a, b, c, a, b, c) in counterclockwise order as on Figure 1.

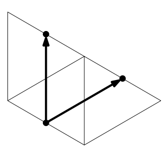
We consider tilings of this hexagon by rhombi[1], each rhombus being made of two triangles sharing an edge. An example of tiling is given on Figure 2. Observe that there are three types of rhombi, depending on the orientation of the middle edge shared by the two triangles: these three types of rhombi are represented to the right of Figure 3.

We can view any hexagon tiling in 3 dimensions by thinking of the drawing as the stereographic projection of a pile of cube, see Figure 3. More precisely, each rhombi corresponds to a face of a cube or to a part of the axial planes which is "visible" from the exterior (i.e., which is not shared with another cube nor with the axial planes), and as exemplified by the colors on Figure 3, the three types of rhombi corresponds to the three possible orientations of a face (parallel to one of the three axial planes).

So we *see* (yet we have not formally proved) that we obtain a plane partition (Figure 4). Clearly, this plane partition fits in a box of size $a \times b \times c$. And, conversely, the "3D-drawing" of such a partition clearly gives a rhombus tiling of our hexagon $H_{a,b,c}$. As a result we obtain that $M(a, b, c)$ is the same as the number of rhombus tilings of $H_{a,b,c}$: this concludes step 1.

Step 2: paths systems. We are going to encode rhombus tilings of $H_{a,b,c}$ by non intersecting paths systems on some appropriate graph.

We introduce some auxiliary directed graph D as follows. Consider the hexagon $H_{a,b,c}$ and all the triangles inside it as on Figure 1. The vertices of D are the middlepoints of the triangle-edges oriented "to the bottom and to the right": on Figure 5 (right), the triangulated hexagon is drawn in dotted lines, and those middle points are represented as black dots. Now add two outgoing edges at each middlepoint as follows:



[1]English: rhombus, rhombi; Français: losange, losanges.

Finally, call, $v_1, \dots v_a$ the middle points located on the lower boundary of length a of $H_{a,b,c}$, from bottom to top. Call similarly $w_1, w_2, \dots, w_n$ the middle points located on the top boundary of length a (Figure 5, right).

Then from our plane partition can build a system of non intersecting paths going from $(v_1, \dots, v_n)$ to $(w_1, \dots, w_n)$: each path represents a "layer" of the plane partition parallel to the plane $x = 0$: see Figure 5. If you think about it you will be convinced that this is actually a bijective correspondence (there are some local checks to do here to be sure that starting from a non-intersecting system of paths you can always reconstruct a unique plane partition – the edges contained in the paths enable you to place all the "green" and "yellow" rhombi, and you have to check that the remaining space is uniquely filled with white ones.).

So, as a conclusion to step 2, we see that $M(a, b, c)$ equals the number of non intersecting paths systems from $(v_1, \dots, v_n)$ to $(w_1, \dots, w_n)$ in D.

Step 3: apply LGV. Observe that by planarity, any path system from $(v_1, \dots, v_n)$ to $(w_1, \dots, w_n)$ in D corresponds to the identity permutation, so we are in the domain of application of the simple corollary of LGV. We thus have:

$$M(a, b, c) = \det M$$

Where M is an $a \times a$ matrix and for all i, j the entry $M_{i,j}$ is the number of oriented paths from v_i to w_j in D.

Now, rotate the picture by 60 degrees, as on Figure 6. We see that all the vertices of D lie on a square grid of length sides $1/2$ (vertical) and $\sqrt{3}/2$ (horizontal). More precisely the paths from v_i to w_j in D are in bijection with paths with increments ± 1 on $\mathbb{Z}$ going from $2i + b$ to $2j + c$ in $b + c$ steps. We obtain:

$$M_{i,j} = \binom{b+c}{\frac{b+c}{2} - \frac{2j-2i+c-b}{2}} = \binom{b+c}{c+j-i},$$

and by the LGV lemma we obtain an expression of $M(a, b, c)$ as an $a \times a$ determinant:

$$M(a, b, c) = \det \left(\binom{b+c}{c+j-i} \right)_{1 \le i,j, \le a}. \tag{3.4}$$

Step 4: computing the determinant The last step of the proof is quite far from our topic, but let us sketch it here briefly: it is worth it, as we will then have understood the proof of MacMahon's formula.

One of the very basic tricks to compute a determinant such as (3.4) is to replace in the formula the line index i with some new indeterminate X_i, and try to think in terms of polynomials of the new variables X_i: if you have ever computed a Vandermonde determinant, then you have already used that trick. Here, we are going to use this idea, but we first need to get rid of some factors in order to put all the i's on the numerator. We have:

$$\begin{aligned} M(a, b, c) = \quad & \prod_{i=1}^{a} \frac{(b+c)!}{(c+a-i)!(b+i-1)!} \\ \times \quad & \det \Big(\big(c - i + a\big) \dots \big(c - i + (j+1)\big) \big(b + i - (j-1)\big) \dots \big(b + i - 1\big) \Big)_{1 \le i,j, \le a} \end{aligned} \tag{3.5}$$

where the only thing we have done is to factor out $\frac{(b+c)!}{(c+a-i)!(b+i-1)!}$ for each line i of the matrix. Now we can use the following lemma due to Krattenthaler:

Lemma 40. *Let $X_1, X_2, \dots X_n; B_2, \dots B_n; C_2, \dots C_n$ be indeterminates. Then one has:*

$$\det\Big((X_i + A_n)\dots(X_i + A_{j+1})(X_i + B_j)\dots(X_i + B_2)\Big)_{1\le i,j\le n}$$
$$= \prod_{1\le i<j\le n} (X_i - X_j) \prod_{2\le i\le j\le n} (B_i - A_j). \tag{3.6}$$

Admitting the lemma, the proof of MacMahon's formula is almost complete: indeed we can evaluate (3.5) by using the Lemma with $n = a$, $X_i = i$, $A_j = -c - j$, and $B_j = b - j + 1$. One obtains a completely factorized product formula, and it just a matter of factor-rearrangement to obtain MacMahon's formula[2]. Now in order to keep no mystery let us say a few words about the proof of the Lemma:

Proof. We substract the $(r-1)$th column to the rth, the $(r-2)$th to the $(r-1)$th, etc., the first to the second. The determinant rewrites as:

$$(B_r - A_r)(B_{r-1} - A_{r-1})\dots(B_2 - A_2)\det\Big((X_i + A_n)\dots(X_i + A_{j+1})(X_i + B_{j-1})\dots(X_i + B_2)\Big)_{1\le i,j\le n}.$$

(note that we have one less factor in the matrix coefficient). Now, we substract the $(r-1)$th column to the rth, the $(r-2)$th to the $(r-1)$th, etc., the second to the third. The determinant in the previous equation (without the multiplicative factor) rewrites as:

$$(B_{r-1} - A_r)(B_{r-2} - A_{r-1})\dots(B_2 - A_3)\det\Big((X_i + A_n)\dots(X_i + A_{j+1})(X_i + B_{j-2})\dots(X_i + B_2)\Big)_{1\le i,j\le n}.$$

Iterating this operation we express the determinant in (3.6) as

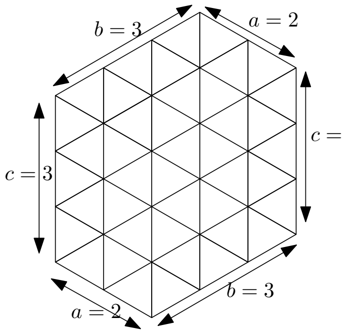
$$\left(\prod_{2\le i\le j\le n} (B_i - A_j)\right)\det\Big((X_i + A_n)\dots(X_i + A_{j+1})\Big)_{1\le i,j\le n}.$$

This last determinant is a polynomial in the variables X_i's, of degree $n(n-1)/2$, and it vanishes if $X_i = X_j$ (two equal lines), so it is a multiple of $\prod_{i<j}(X_i - X_j)$. The multiplicative constant is easily seen to equal 1. □

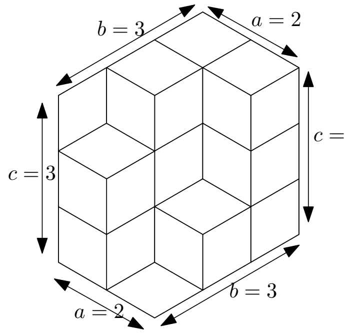
Remark 13. If someday in your life you need to compute a big determinant (this can happen, not only in enumerative combinatorics), or if you like that kind of things (this can happen too...) look for the paper *Advanced Determinant Calculus* by Christian Krattenthaler (that contains all you want to know about determinant calculus, including several different ways of computing the determinant (3.4)).

[2]It is just a matter of rearrangement, but this can get technical if not done in the proper way. A simple way to check that the expression of $M(a, b, c)$ obtained from the lemma equals the one in MacMahon's formula is to proceed by induction on a: just check that the ratios $\frac{M(a+1,b,c)}{M(a,b,c)}$ are the same. *Hint: in both cases this ratio equals* $\frac{(a+b+c-1)!(a-1)!}{(a+c-1)!(a+b-1)!}$.

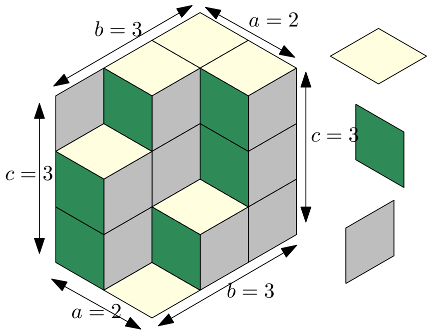
3.2.1 The figure I handed out in class: plane partitions, hexagons, paths systems



1. Hexagone $a \times b \times c$



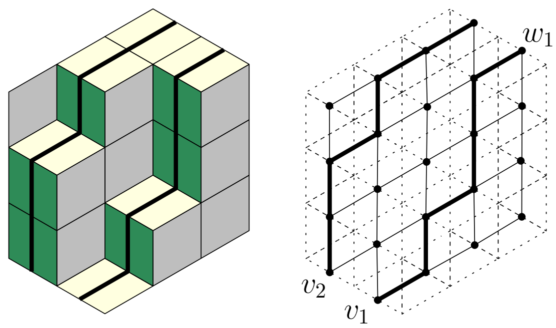
2. Un pavage par lozanges de cet hexagone



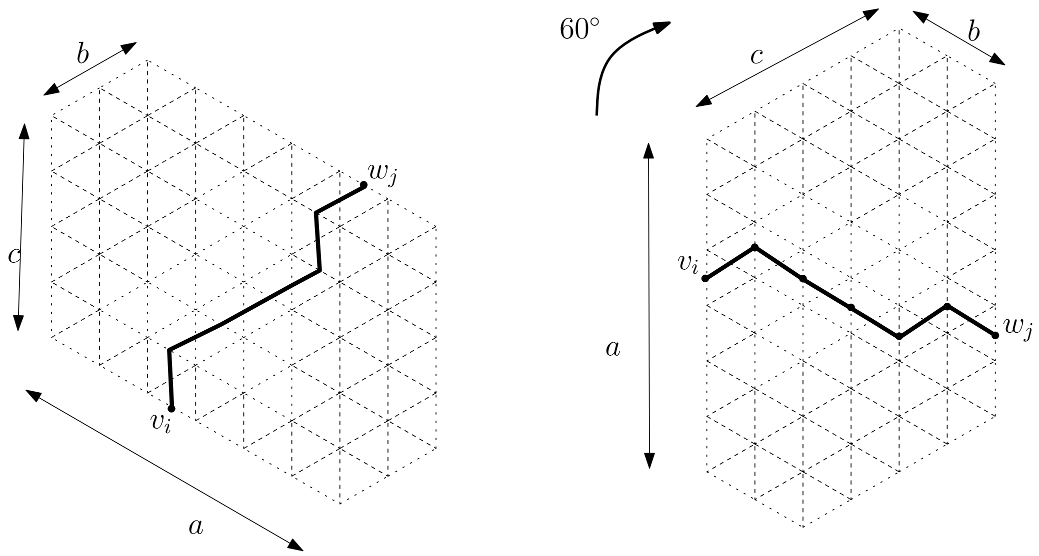
3. Le même pavage avec éclairage

3 3
3 1
2

4. Le même vu comme partition plane



5. Pour appliquer Lindström-Gessel-Viennot



6. Après rotation, un chemin de v_i à w_j peut se voir comme un chemin sur $\mathbb{Z}$ allant de $b + 2i - 1$ à $c + 2j - 1$.

G. Chapuy 2012

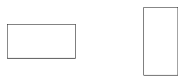
Chapter 4

Two nice formulas, with nice proofs!

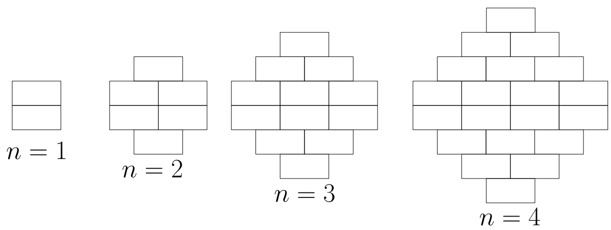
4.1 Tilings of the Aztec diamond via non-interesecting Schröder paths

4.1.1 The amazing power of two

In this lecture we consider regions on the plane made by unions of unit boxes with unit coordinates. A *domino* is a rectangular region of size 2×1 (horizontal domino) or 1×2 (vertical domino):

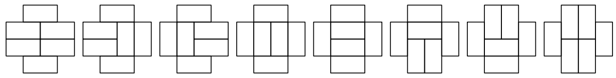


For $n \geq 1$, we define the Aztec diamond of size n as follows: we place one horizontal domino, then a line of two horizontal dominos below it, then a line of 3 dominos under it, etc. and we do this n times. We place the dominoes so that the picture has a vertical axis of symmetry. Then we repeat the process in the other direction: we place a line of n, then a line of $n-1$, etc.. We arrive the *Aztec diamond of size* n, illustrated for small values of n in the following figure:



The Aztec diamond of size n comes, by definition, equipped with a tiling by dominos in which all dominos are horizontal. But there are other ways to tile the same region, for example there are two when $n = 1$:

and there are eight when $n = 2$:



Definition 41. We let AZ_n be the number of tilings by dominoes of the Aztec diamond of size n.

You can try to draw all configurations of size $n = 3$, but beware, there are many! In fact we have:

Theorem 42 (Elkies, Kohn, Larsen, Propp 1992)**.** *The number of tilings by dominoes of the Aztec diamond of size n is given by the closed formula:*

$$AZ_n = 2^{\frac{n(n+1)}{2}}.$$

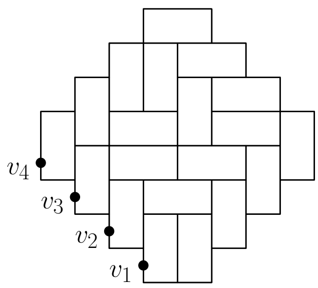
Remark 14. As I said during the class, this seemingly anecdotic statement has in fact played a very important role in many aspects of combinatorics, probability, representation theory, since its discovery. The very nice closed formula is just the tip of the iceberg that points at very interesting mathematics! Many proofs are known today, and here we will just see one proof of this result, due to Eu and Fu (2004), which is both very smart and very elegant.

Remark 15. These numbers grow faster than exponentially, as $2^{\Theta(n^2)}$, you may not be used to this. But this is in fact very natural here: the Aztec diamond of size n has *area* $\Theta(n^2)$ and it is natural to expect a positive amount of entropy per portion of area.

4.1.2 Schröder paths

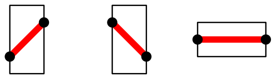
From now on my lecture consisted more on drawings than text (I should even say, time-evolving drawings which is even worse). If you didn't attend the lecture the following section may be a bit hard to read.

I will now explain how to associate to a tiling of an Aztec diamond of size n, a n-tuple of non-intersecting paths. The vertices of these paths will always be placed in the middle of a vertical unit edge belonging to some domino (this edge can be the left of right boundary of a horizontal domino, or a portion of the left or right boundary of a vertical domino). The n starting points of my paths are the middlepoints of the vertical unit segments lying at the Bottom-Left boundary of the Aztec diamond. I label then $v_1, v_2, \ldots, v_n$ as on the following picture (on which $n = 4$)



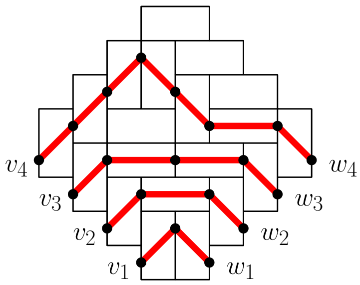
Now, from the v_i's I start drawing paths according to the following rule:

> *Each time a path touches the left boundary of a domino, we continue the path by adding a step going to the right boundary of the same domino according to the following rule:*



(∗∗)

Here is an example:



Call $w_1, w_2, \ldots, w_n$ the middlepoints lying at the center of vertical unit segments lying on the Bottom-Right boundary of the Aztec diamond. Then what appears to be true in the previous example is, in fact, always true:

Lemma 43. *The n-paths thus defined never touch nor cross each other. Moreover, for each $i \in [1..n]$, the path starting at the vertex v_i reaches the right boundary of the Aztec diamond at the vertex w_i.*

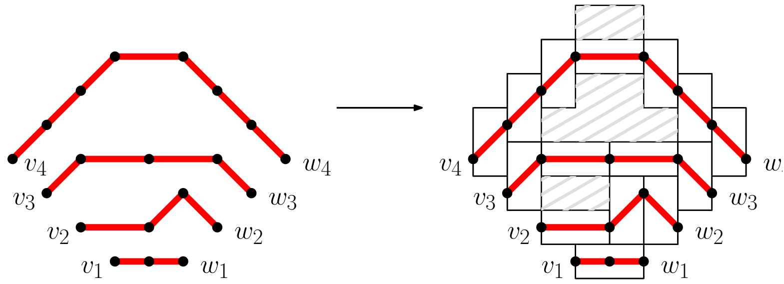
Proof. The proof is a simple observation on parity. Color the vertices of the lattice (more precisely, the shifted lattice made by middle points of vertical unit segments of the lattice) according to their parity: the vertex at coordinates $(i, j + \frac{1}{2})$ is colored $i + j \mod 2$.

Remark that all the v_i's have the same parity (say 0). Moreover remark that, by construction, our paths can only take steps of the form $(1, 1)$, $(1, -1)$, or $(2, 0)$, therefore they keep the parity constant. This implies that all the vertices visited by all the paths have parity 0. This has two helpful consequences. The first one is that two paths never reach at the same time the left boundary of a vertical domino: from this it easily follows that they never cross. Moreover, since the paths always go right, they have to hit the right boundary at some point. But the only vertices of parity 0 on the right boundary are the ones from the Bottom-Right boundary (the ones from the Top-Right have parity 1). These vertices are precisely $w_1, \ldots, w_n$ and since the paths are noncrossing, the one starting at v_i necessarily ends at w_i for each i. □

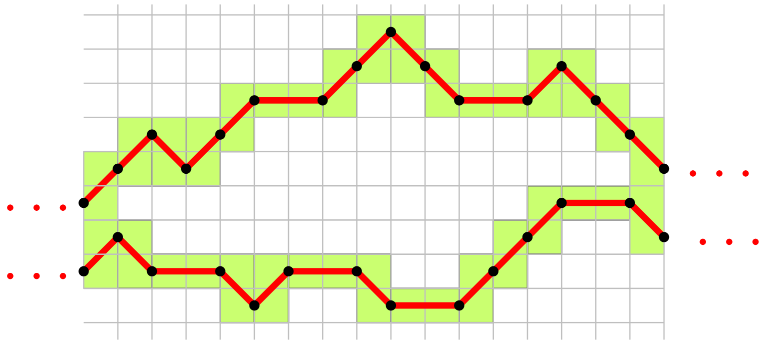
In fact, this construction is a bijection:

Proposition 44. *There is a bijection between tilings of the Aztec diamond of size n and non-intersecting n-tuples of paths going from $\{v_1, v_2, \ldots, v_n\}$ to $\{w_1, w_2, \ldots, w_n\}$, where paths are allowed to take steps $(1, 1)$, $(1, -1)$, $(2, 0)$.*

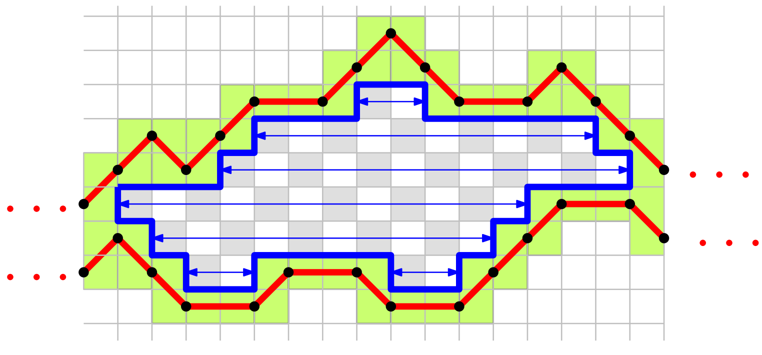
Proof. We have already seen how to form a system of paths from a tiling of the Aztec diamond. Conversely, starting from a system of paths, let us try to reconstruct a tiling. We start by placing the dominoes that fit directly "below" the steps: each step $(1, 1)$ or $(1, -1)$ gives rise to a vertical domino and each step $(2, 0)$ to a horizontal one via the rule (∗∗). Clearly these dominoes are non-overlapping, so we obtain a "partial" tiling of the Aztec Diamond region, *i.e.* a tiling with empty "holes" that remain to be tiled, such as on this example:



We now claim, and this will complete the proof, that each of these holes can be tiled by dominoes, and moreover that they can be tiled in a *unique* way (in fact we will prove that the unique tiling for each hole is the one where all dominoes are horizontal and have a given parity). A "hole" in the tiling consists in a region delimited by two Schröder paths between a time where they become "separated" and a time where they join again, here is an example:



The "hole" region is a union of horizontal strips. One easily checks that these horizontal strips have the following properties: 1. they are all of even length; 2. if we color the squares of the lattice in a chessboard fashion, then the leftmost square of each region is always of the same color. These two properties easily follow from parity considerations on the Schröder paths (exercise: do it!). In our running example we have seven such strips:



The first property shows that it is possible to tile this region with the "all horizontal" tiling. To see that this is the unique possible tiling, we argue by induction on the area of the hole. Consider a square in the hole having minimal abscissa. Then by property 2, the squares above and below it cannot be part of the hole, so this square can only be covered by a horizontal domino. After placing this domino, if the region yet to be covered is not empty, it still satisfies properties 1. and 2. By induction we conclude that the "all horizontal" tiling is the only tiling of the region, and the proof is done. □

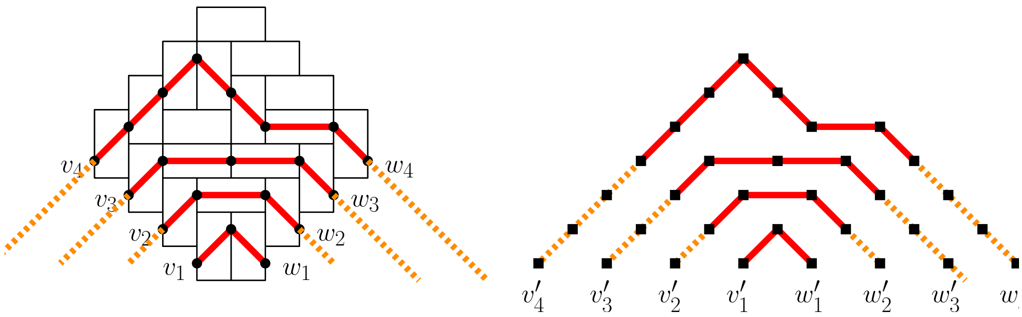
We are now close to be able to apply the LGV lemma. For this we will need one more definition:

Definition 45. A *Schröder path* is a path in $\mathbb{Z} \times \mathbb{N}$ taking steps $(1,1)$, $(1,-1)$ or $(2,0)$, starting and ending on the x-axis. For $n \geq 1$, let s_n be the number of Schröder paths from $(0,0)$ to $(2n,0)$.

We let the reader check that $s_1 = 2$, $s_2 = 6$. We have:

Proposition 46. *Tilings of the Aztec diamond of size n are bijection with non-intersecting systems of Schröder paths going from $\{v'_1, v'_2, \ldots, v'_n\}$ to $\{w'_1, w'_2, \ldots, w'_n\}$ where $v'_i = (-1-2i, 0)$ and $w'_j = (1+2j, 0)$ for $1 \leq i, j \leq n$.*

Proof. To obtain a system of non-intersecting Schröder paths from the previous pictures, we just extend all the paths with diagonal portions starting from $v_1, \ldots, v_n$ on the left and from $w_1, \ldots, w_n$ on the right:



Up to a proper shift of the coordinate axes, we obtain a noninteresecting system of Schröder paths from the v'_i to the w'_j, as claimed.

Conversely, consider a system of n nonintersecting Schröder paths from the v'_i to the w'_j. First we observe that by planarity, the vertex v'_i is necessarily linked to w'_i for each $1 \leq i \leq n$ (indeed paths cannot cross at vertices, and they cannot cross at edges since again all vertices have the same parity so an edge $(1,1)$ and an edge $(1,-1)$ are never in a position that they can intersect). Now we observe that the path going from v'_i to w'_i necessarily starts with at least $i-1$ up steps and ends with at least $i-1$ down steps: this is easily proved by induction on i, using that the paths are noncrossing (try to see why for $i = 2$, and you'll understand). Therefore, up to a shift of coordinate axes, these systems are in bijection with noninteresting systems of paths going from the v_i to the w_j. □

Remark 16. It is *not* true that *general* systems of paths from $\{v_1, \ldots, v_n\}$ to $\{w_1, \ldots, w_n\}$ are in bijection with systems of paths from $\{v'_1, \ldots, v'_n\}$ to $\{w'_1, \ldots, w'_n\}$. This is true only for *non-intersecting* systems of paths (and indeed the proof consists in showing that the "orange parts" of the paths are frozen, but this clearly uses the hypothesis of nonintersection).

We thus obtain our first enumerative result for AZ_n:

Corollary 47. *The number AZ_n of domino tilings of the Aztec diamond of size n is given by the determinant:*

$$AZ_n = \det\left(s_{i+j-1}\right)_{1 \leq i,j \leq n}.$$

Proof. This follows from the corollary of the LGV lemma, and from the fact that there are s_{i+j-1} Schröder paths from v'_i to w'_j. □

How to compute such a determinant? We could look into Krattenthaler's masterpiece (*Advanced determinant calculus, 1999*) for general strategies to compute Hankel determinants. Instead we will use the method of Eu and Fu (2004) based on the properties of small Schöder paths.

Definition 48. A Schröder path is *small* if it has no horizontal step on the x-axis. We let r_n be the number of small Schröder paths from $(0,0)$ to $(2n,0)$.

We easily verify that $r_1 = 1, r_2 = 3$. More generally we have:

Lemma 49. *For any $n \geq 1$ we have:*

$$s_n = 2r_n.$$

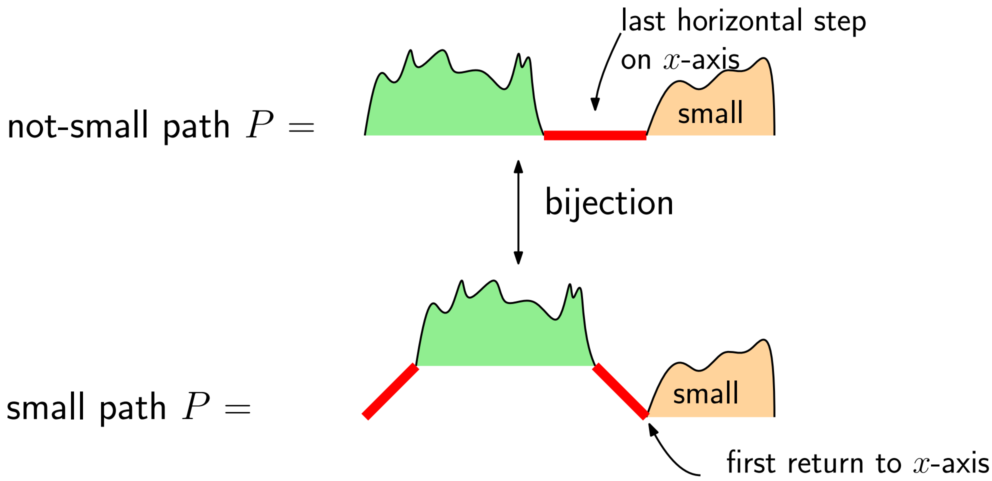
In other words, exactly half of Schröder paths are small.

Proof. We give a bijection between Schröder paths that are small and Schröder paths that are not small, all of total width $2n$. To do that, we show that both sets are in bijection with the set of pairs of paths (P_1, P_2) of total width $2n-2$ where P_1 is arbitrary and P_2 is small.

Let P be a Schröder path that is not small. Then P has a horizontal step on the x-axis, and let us write $P = P_1(FLAT)P_2$ where $(FLAT)$ is the *last* such step. Then by construction, P_2 is small. On the other hand, P_1 is an arbitrary (small or not) Schröder path.

Let Q be a Schröder path that is small. Then Q necessarily starts with an up step, and we can write $Q = (UP)Q_1(DOWN)Q_2$ where $(DOWN)$ is the *first* downstep returning to the x-axis since the beginning of the path. Then by construction, Q_2 is small (since Q is). On the other hand Q_1 can be small or not.

Combining these two decompositions gives the desired bijection. In pictures, it can be summarized as follows:

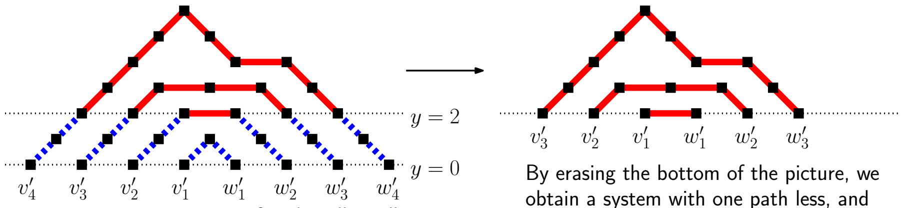


□

From the lemma it directly follows that:

$$AZ_n = det(s_{i+j-1})_{1\leq i,j\leq n} = 2^n \det(r_{i+j-1})_{1\leq i,j\leq n}.$$

Now, using the LGV lemma again (but in the opposite direction!) the determinant $\det(r_{i+j-1})_{1\le i,j\le n}$ is equal to the number of non intersecting systems of Schröder paths from $v'_1,\dots,v'_n$ to $w'_1,\dots,w'_n$ in which all paths are small. In particular, the path from v'_1 to w'_1 cannot be a horizontal step, so it has to be an up and a down step. This implies that the path starting at v'_2 has to start with TWO up steps, and end with TWO up steps, and by induction the same is true for the path from v'_i to w'_i for each $i \ge 2$. Thus if we remove the first TWO layers of the paths (i.e. we "cut" the part of the picture below the line $y = 2$), we see that those configurations are in bijection with configurations of noninteresting paths going from $\{v_1,\dots,v_{n-1}\}$ to $\{w_1,\dots,w_{n-1}\}$. To understand this, draw a picture;



non-intersecting system of paths, all small.
The fact that paths are small imposes that the first two layers are necessarily as on the picture (blue, dashed).

By erasing the bottom of the picture, we obtain a system with one path less, and now the paths can be small or not.

We thus obtain:

$$\det(r_{i+j-1})_{1\le i,j\le n} = AZ_{n-1}.$$

We can now conclude the proof of the main theorem. From the last two results we get:

$$AZ_n = 2^n AZ_{n-1},$$

which implies

$$\begin{aligned} AZ_n &= 2^{n+(n-1)+\cdots+2} AZ_1 \\ &= 2^{n+(n-1)+\cdots+2+1} \\ &= 2^{\frac{n(n+1)}{2}}. \end{aligned}$$

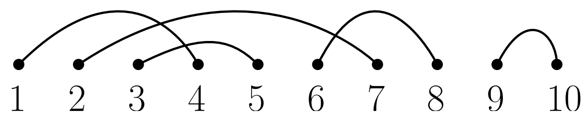
4.2 Matchings, gluings of polygons and the Harer Zagier formula

4.2.1 Matchings

The main objects of this lecture are matchings:

Definition 50. A *matching* of size $n \geq 1$ is a partition of the set $[1..2n]$ into pairs.

One often represents a matching by a diagram of arcs, for example here is the matching $\{\{1,4\};\{2,7\};\{3,5\};\{6,8\};\{9,10\}\}$ (of size $n = 5$)



If you understand why there are $n!$ permutations, then you should directly understand the following result:

Proposition 51. *The number of matchings of size n is equal to the product of all odd numbers from 1 to $(2n-1)$. This number is denoted by $(2n-1)!!$ and can also be expressed as:*

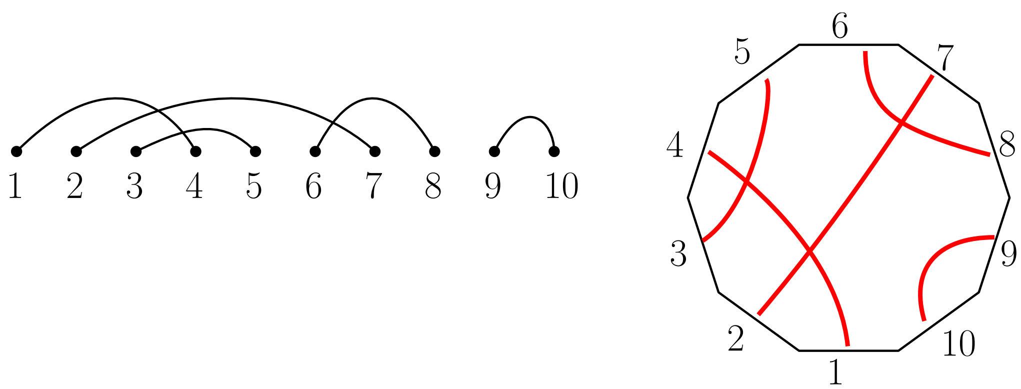
$$(2n-1)!! = (2n-1)(2n-3)(2n-5)\dots 1 = \frac{(2n)!}{2^n n!}.$$

Proof. We can construct all matchings of size n as follows: A- choose the number that is matched with 1; B- look for the smallest number that is not matched so far, and choose the number to match it with; repeat B- until all elements are matched. There are $(2n-1)$ choices at step A, and then at B there are $(2n-3)$ choices, then $(2n-5)$ choices, etc. The last expression given follows from the fact that the product of all odd numbers from 1 to $(2n-1)$ is equal to $(2n)!$ divided by the product of even numbers from 2 to $2n$, which is $2^n n!$ (factor one "two" in each factor). □

In particular, there are many matchings, more than exponentially many since (by Stirling's formula)

$$(2n-1)!! \sim \left(\frac{n}{e}\right)^n 2^{n+\frac{1}{2}}.$$

In this lecture we will prefer to represent matchings in a circular way rather than on a line, as on the following figure (right). We thus view a matching of size n as pairing the set of *edges* of a $2n$-gon, where by convention the edges are labelled from 1 to 2n in clockwise order.



In fact, we have already studied some very particular cases of matchings.

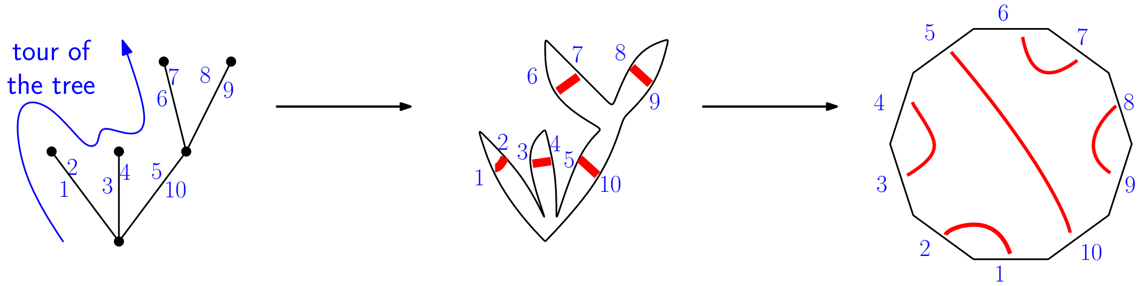
Definition 52. A matching of size n is *noncrossing* if there are no $i < j < k < \ell$ such that $\{i, k\}$ and $\{j, \ell\}$ are pairs in the matching.

In other words, a matching is noncrossing if on can draw it (circularly) without edge crossing. We have already seen these objects in disguised form before, indeed there is a very natural way to obtain a noncrossing matching from a rooted plane tree:

From trees to matchings

- start with a rooted plane tree with n edges;
- walk along the tour of the tree clockwise; you visit each side of each edge exactly once; number the edge-sides from 1 to $2n$ in the order you visit them;
- for each edge of tree you obtain a pair $\{i, j\}$ where i,j are the labels of its two sides. This defines the wanted matching of size n.

One can think of this construction as follows: with a (very sharp) knife, cut each edge of the tree longitudinally, and then unfold the drawing. The "interior" of the figure can be unfolded to a $2n$-gon, whose sides are the edge-sides of the original tree, where pairs in the matching remember which side-gluings have to be made to reconstruct the tree. Here is an example:



We directly see on this viewpoint that, starting from a noncrossing matching and gluing edge-sides, we obtain a rooted plane tree. We thus have:

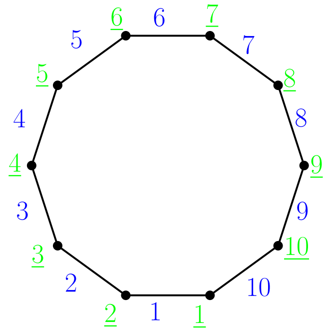
Proposition 53. *There is a bijection between noncrossing matchings of size n and rooted plane trees with n edges.*

It follows that there are $\mathrm{Cat}(n)$ noncrossing matchings, which is much less than $(2n-1)!!$ (note: $Cat(n) \leq 4^n$). So very few matchings are noncrossing. How could we try to "understand", or "classify", all the missing ones? We will try to do that in the next sections, by defining the genus of a matching, and enumerate matchings by genus.

4.2.2 Polygon gluings and the genus of a matching

We have just seen that, starting from a $2n$-gon, and doing some edge-identifications in a noncrossing way, we obtain a matching. But nothing prevents us from trying the same construction on an arbitrary matching! Let us do this, now.

Notational conventions. From now on, n will be fixed and we will work with matchings of size n. The underlying $2n$-gon has its edges numbered from 1 to $2n$ in clockwise order as before. We also number the vertices of the $2n$-gon, and in order to prevent confusion we will number those vertices with underlined numbers, from $\underline{1}$ to $\underline{2n}$, with the convention that $\underline{1}$ precedes the edge 1 in clockwise order:



We note $\mathcal{E} = [1..2n]$ and $\mathcal{V} = [\underline{1}..\underline{2n}]$ the set of edges and vertices of the $2n$-gon, respectively.

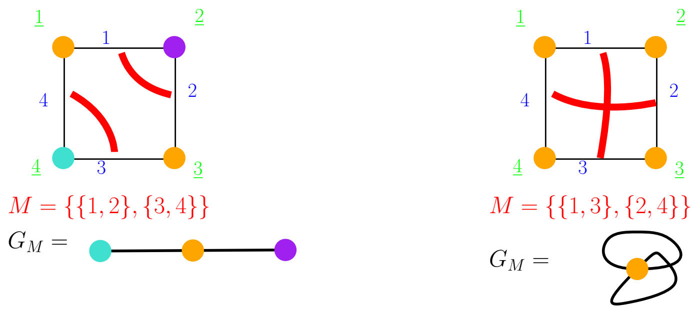
We will now generalize the gluing construction to general matchings. The idea is that, when we perform some side-gluings, some of the vertices in $\mathcal{V}$ become identified. To capture this, let us define an equivalence relation on $\mathcal{V}$ as follows[1]:

Definition 54. Given a matching M, we define $\equiv_M$ as the smallest equivalence relation on $\mathcal{V}$ with the following property. For each pair $\{i, j\}$ in M, we have $\underline{i} \equiv_M \underline{j+1}$ and $\underline{i+1} \equiv_M \underline{j}$ (with the convention that $\underline{0} = \underline{2n}$).

We let $V_M := \mathcal{V}/\equiv_M$ be the set of equivalence classes of this relation.

The set V_M is naturally equipped with a structure of multigraph with n edges (loops allowed), and we let G_M be this graph. Namely, each pair $\{i, j\}$ in M defines an edge between the equivalence class of $\underline{i}$ and the class of $\underline{i+1}$ (or equivalently between the class of $\underline{j}$ and the one $\underline{j+1}$). In other words, each pair in M corresponds to an edge in G_M.

This abstract definition exactly encapsulates the "gluing" construction that we used for noncrossing matchings: we identify sides of our polygon together, and in the process vertices become identified as well. Each pair in M gives rise to an edge in the tree (in the general case, the "tree" is not necessarily a tree anymore, it is our multigraph G_M). Here are two examples for $n = 2$:



We have the following important formula, which is a version of the famous Euler formula:

Proposition 55. *The number of vertices of the graph G_M, or equivalently the cardinality of the set V_M, is of the form*

$$|V_M| = n + 1 - 2g \tag{4.1}$$

where $g \geq 0$ is an integer. This integer is called the genus *of the matching M.*

[1] depending on the convention you choose for numbering $\mathcal{V}$ and $\mathcal{E}$ (clockwise, counterclockwise, and if i appears before or after $\underline{i}$ in the tour) you get different conventions for all the rest, including the equivalence relation; I am not sure to use the same conventions here as I did in the lectures...

Proof. The graph G_M is clearly connected, since the $2n$-gon is connected and any walk along the $2n$-gon projects to a walk in G_M. Since it has n edges, it has at most $n+1$ vertices and the quantity g defined by (4.1) is indeed nonnegative. So we only have to show that $(n+1) - |V_M|$ is an even number.

This property is a bit subtle, and we give a proof that uses the concept of signature of a permutation. This proof is not "accidental", but is rather a glimpse at the very deep link between polygon gluings, permutation products, and graphs embedded on surfaces (call me for more references). We define three permutations σ, ϕ, α acting on the set $[1..2n]$. The permutation ϕ is just defined as the cyclic permutation $\phi = (1, 2, \dots, 2n)$ – we interpret it as "turning one step around the $2n$-gon". The permutation α is a permutation with only cycles of length two. If we write $M = \{\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_n, j_n\}\}$ then $\alpha = (i_1, j_1)(i_2, j_2)\dots(i_n, j_n)$. Thus α is the involution that exchanges a number with the one it is matched with in M. We now define the permutation $\sigma = \alpha\phi$. We note that if $\{i, j\}$ belongs to the matching then $\sigma(i-1) = \alpha(i) = j$ and $\sigma(j-1) = \alpha(j) = i$. Comparing with the definition of $\equiv_M$, we conclude that *cycles of the permutation sigma are in bijection with equivalence classes for* $\equiv_M$. Therefore the signature of σ is $\epsilon(\sigma) = (-1)^{2n-|V_M|}$. But $\epsilon(\sigma) = \epsilon(\alpha)\epsilon(\phi)$ which is equal to $(-1)^{n+2n-1}$. We thus get $(-1)^{|V_M|} = (-1)^{n-1}$ and the proof is complete. □

Remark 17. The integer g is *indeed* the genus (=number of handles) of the topological surface obtained by gluing the sides of the polygon according to the matching. For example, in the left example of the last figure, by gluing opposite sides of a square we get a *torus*, of genus 1. If we glue according to a noncrossing matching, the surface obtained is a sphere and the genus is equal 0 (equivalently, G_M is a tree). Recall the pictures during the lectures!

4.2.3 The Harer-Zagier formula

We are now going to enumerate matchings by genus.

Definition 56. For $n \geq 1$ and $g \geq 0$, we let $\epsilon_g(n)$ be the number of matchings of size n and genus g.

The main result of this lecture is:

Theorem 57 (Harer and Zagier, 1986)**.** *For any $n \geq 1$ we have:*

$$\sum_{g \geq 0} \epsilon_g(n) X^{n+1-2g} = \sum_{k=1}^{n+1} \binom{X}{k} (2n-1)!! 2^{k-1} \binom{n}{k-1}. \tag{4.2}$$

where the equality is between polynomials of the variable X.

We remark that the first sum is a finite sum, since a matching of size n has genus at most $(n+1)/2$ by Euler's formula, so both sides are indeed polynomials of X. Before proving the formula, we remark that it *really* solves the question of enumerating matchings by genus. By picking the coefficient of X^{n+1} in (4.2), we obtain (note that only $k = n+1$ contributes to this coefficient in the RHS):

$$\epsilon_0(n) = \frac{1}{(n+1)!}(2n-1)!! 2^n = \frac{(2n)!}{n!(n+1)!} = \mathrm{Cat}(n),$$

which proves enumeratively that genus 0 matchings are exactly noncrossing matchings (it would have been easy to prove this directly, though). Similarly, extracting the coefficient of $\epsilon_1(n)$ (for which we have to sum the contributions of $k \geq n-1$ in the RHS) we obtain after simplification:

$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12}\mathrm{Cat}(n).$$

Exercise. More generally, deduce from the Harer-Zagier formula that for any $g \geq 0$ the number of matchings of genus g and size n has the form

$$\epsilon_g(n) = R_g(n)\mathrm{Cat}(n),$$

where R_g is a (computable) polynomial in n of degree $3g$ with rational coefficients.

4.2.4 Proof of Theorem 57, according to Bernardi (2010)

Many proofs of the Harer-Zagier formula are known, but here we will focus on an elegant combinatorial proof, due to Olivier Bernardi (former student of this class... although I was not teaching it at that time!), improving previous ideas of Bodo Lass. This proof was first given in 2010.

Step 1: coloring G_M, using all colors or not.

We first remark that it is enough to prove (4.2) when X is a positive integer: indeed both sides are polynomials in X and two polynomials that coincide on positive integers are equal. Therefore from now on, we assume that $X \geq 1$ is an integer. Now, recall that if M is a matching of genus g and size n, the graph G_M has $n+1-2g$ vertices, so X^{n+1-2g} is the number of ways to color the vertices of G_M with the colors $\{1, 2, \dots, X\}$ (each vertex gets a color, independently of others).

Therefore the LHS of (4.2) can be interpreted as the total number of matchings of size n (of any genus), in which the vertices of the graph G_M have been colored, arbitrarily, with the colors $1, 2, \dots, X$. Note that in this interpretation, not all colors in $[1..X]$ have necessarily been used: some colors can be forgotten. We exclude this behaviour in the following definition:

Definition 58. For $k \geq 1$, we let $\mu_k(n)$ be the total number of matchings M of size n in which the vertices of G_M have been colored with colors in $[1..k]$, in such a way that each color is used at least once.

Since every coloring with colors in $[1..X]$ can be obtained by choosing first the set of (say k) colors that will be used at least once, and then choosing a coloring in these colors, we have the relation:

$$\sum_{g \geq 0} \epsilon_g(n) X^{n+1-2g} = \sum_{k \geq 1} \binom{X}{k} \mu_k(n).$$

We thus conclude:

Lemma 59. *To prove Theorem 57, it is enough to show that for any $k, n \geq 1$ we have*

$$\mu_k(n) = (2n-1)!!2^{k-1}\binom{n}{k-1} = \frac{(2n)!}{2^{n-k+1}(k-1)!(n-k+1)!}. \tag{4.3}$$

Step 2: bi-eulerian tours

We now introduce the notion of *bi-eulerian tour*:

Definition 60. A *bi-eulerian* tour of a multigraph $\tilde{G} = (\tilde{V}, \tilde{E})$ is a closed walk on this graph, in which each edge is taken exactly twice, once in each direction (loops are also taken twice but the direction constraint doesn't apply to them, since direction is not defined for loops).

We remark that a bi-eulerian tour of a multigraph $\tilde{G}$ is the same as an Eulerian tour of the directed graph $\vec{\tilde{G}}$ obtained by replacing each edge (including loops) by two directed edges in opposite directions. This is important, because we the whole idea of this proof will be to apply the BEST theorem in the end!

We now state a subtle remark due to Bodo Lass:

Proposition 61. *There is a bijection between the set of matchings M of size n whose vertices are colored using* all *colors in $[1..k]$, and pairs $(\tilde{G}, \ell)$ where $\tilde{G}$ is a connected multigraph with n edges on the vertex set $[1..k]$ and ℓ is a bi-eulerian tour on $\tilde{G}$.*

Proof. Let M be a matching and assume that the vertices of G_M are colored using *all* colors in $[1..k]$. We naturally obtain a (multi)graph $\tilde{G}_M$ on $[1..k]$ by identifying together vertices of the same color. Moreover, we remark that the closed walk

$$\underline{1}, \underline{2}, \underline{3}, \ldots, \underline{2n}, \underline{1}$$

around the tour of the $2n$-gon induces a bi-eulerian tour on G_M (each side of the polygon is taken once clockwise, thus each edge of G_M is taken once in each direction), and therefore also a bi-eulerian tour on $\tilde{G}_M$.

Conversely, consider a graph $\tilde{G}$ on $[1..k]$ with n edges, and a bi-eulerian tour

$$\vec{e_1}, \vec{e_2}, \ldots, \vec{e_{2n}}$$

of $\tilde{G}$ (here we specify the tour as a list of directed edges). We define an equivalence relation $\equiv$ on $\mathcal{V}$ by saying that $\underline{i} \equiv \underline{j}$ if and only if the bieulerian tour on $\tilde{G}$ visits the same vertex of $\tilde{G}$ at times i and j. There is one equivalence class of $\equiv$ for each vertex of $\tilde{G}$, so equivalence classes of $\equiv$ are naturally labelled by $[1..k]$.

Now for each edge e of $\tilde{G}$, there are two directed edges $\vec{e_i}$ and $\vec{e_j}$ in the tour that correspond to the two times the edge e has been taken by the tour. We now define a matching on $[1..2n]$ by pairing i with j (and we do this for every edge e). Let M be the matching thus obtained. We claim that the equivalence relation $\equiv_M$ is finer than $\equiv$. This is clear by construction of M and by definition of $\equiv$ (if $\{i, j\}$ is in M then $\vec{e_i}$ and $\vec{e_j}$ are two opposite versions of the same edge in $\tilde{G}$, thus the in-vertex of one coincides with the out-vertex of the other; by comparing with the definition of $\equiv_M$ this shows that $\equiv$ satisfies the axioms required by $\equiv_M$, even if $\equiv$ could be coarser). Therefore, each equivalence class of $\equiv$ is a union of some equivalence classes of $\equiv_M$, and we can think of $\equiv$ as a coloring of the vertices of G_M with the colors in $[1..k]$.

Finally, the two constructions are clearly inverse one of each other since in all cases the matching is identified by pairing directed edges together according to the bieulerian tour. □

Step 3: Counting!

From the previous discussions, we would like to enumerate pairs $(\tilde{G}, \ell)$, where $\tilde{G}$ is a connected multigraph on $[1..k]$ and ℓ is a bieulerian tour. Consider such a pair and let $(d_1, \ldots, d_k)$ be the vertex degrees in $\tilde{G}$. By the BEST theorem (applied to the directed multigraph $\vec{\tilde{G}}$), the number of bieulerian tours of $\tilde{G}$ is equal to the number of spanning trees of $\tilde{G}$ times $\prod_i (d_i - 1)!$. Note that $(d_i - 1)!$ is the number of cyclic orderings of a set with d_i elements, so we can view $\prod_i (d_i - 1)!$ as the number of a cyclic orderings of the d_i edges incident to the vertex i in $\tilde{G}$.

Therefore the configurations we would like to enumerate are in bijection with tuples consisting of a graph $\tilde{G}$ on $[1..k]$, a rooted spanning tree τ of G, and a cyclic ordering of edges around each vertex in $\tilde{G}$.

Note that, because the edges around each vertex of $\tilde{G}$ are cyclically ordered, this is true in particular for the edges in the tree τ (just by restriction). So we can view τ as a *PLANE* tree (and in fact, a rooted plane tree, since trees in the BEST theorem are rooted). We will now explain how to reconstruct all such configurations. We will first choose the tree, and then add the edges not in the tree; in order to add the $(n - (k - 1))$ edges that are not in the tree, we will first add $2(n - (k - 1))$ "half-edges" (pictured as little blossoms attached to vertices) and we will then group the half-edges by pairs, in order to form the true edges of the graph.

More precisely ,we can reconstruct all such configurations as follows:

a) start with an arbitrary rooted *plane* tree, with vertices labelled from 1 to k; this tree has $k - 1$ edges;

b) dispatch $2(n - (k - 1))$ blossoms (=hanging half-edges) among the k vertices of the tree; The blossoms are cyclically ordered around the vertices, in other words the pair $(tree, blossoms)$ is viewed as an object that is drawn in the plane;

c) choose an arbitrary matching of the $2(n - k + 1)$ blossoms, thus reconstructing the $(n - k + 1)$ missings edges of $\tilde{G}$.

We now count the number of ways we have to do that:

- At step a), we have $\mathrm{Cat}(k - 1)$ ways to choose a rooted plane tree and $k!$ ways to label its vertices;
- At step b), we have to dispatch $2n - 2k + 2$ blossoms among the tour of the tree; the tree has $k - 1$ edges and thus its tour has $2k - 2$ sides. Thus, equivalently, we have to choose an arbitrary word of length $2n$ on the alphabet $\{tree, blossom\}$ having $2n - 2k + 2$ letters "blossom" and $2k - 2$ letters "tree". The number of choices is $\binom{2n}{2k-2}$.
- At step c), we choose an arbitrary matching on $2(n - k + 1)$ elements, we thus have $(2n - 2k + 1)!!$ choices.

Putting everything together, we obtain:

$$\begin{aligned} \mu_k(n) &= \mathrm{Cat}(k-1)k!\binom{2n}{2k-2}(2n-2k+1)!! \\ &= \frac{(2k-2)!k!(2n)!(2n-2k+2)!}{k!(k-1)!(2k-2)!(2n-2k+2)!(n-k+1)!2^{n-k+1}} \\ &= \frac{(2n)!}{(k-1)!(n-k+1)!2^{n-k+1}}, \end{aligned}$$

and in view of Lemma 59, the Harer-Zagier formula is proved!

4.3 Viennot's theory of heaps (2011)

I followed Gilles Schaeffer's lecture notes from previous years.

Chapter 5

Plane partitions again, vertex operators

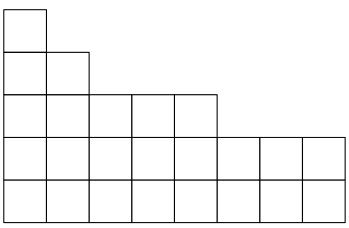
In this chapter, we will introduce the so-called *vertex operators* and use them to prove the formula for unbounded plane partitions in a ”computation free” way. .

5.1 Integer partitions, plane partitions and MacMahon’s formula

5.1.1 Integer partitions

Let $n \geq 0$ be an integer. An *integer partition* of n, or *partition* of n, is a sequence $\lambda_1 \geq \lambda_2 \geq \dots \lambda_\ell > 0$ of non-increasing positive integers summing to n. We write $\lambda \vdash n$ for “λ is a partition of n”. We also say that n is the *size* of λ. The integers $\lambda_1, \dots, \lambda_\ell$ are called the *parts* of λ and the number of parts is denoted by $l(\lambda) := \ell$. We often see a partition as infinite sequence $\lambda_1 \geq \lambda_2 \geq \dots$ by completing it with an infinite number of zeros.

A partition is often represented by its *Ferrer’s diagram*. An example is better than any definition, so here is the Ferrer’s diagram of the partition $\lambda = [8, 8, 5, 2, 1]$ of 24:



$\lambda = (8, 8, 5, 2, 1)$

We let $\mathcal{P}$ be the set of all integer partitions and for $n \geq 0$ we let $\mathcal{P}_n$ be the subset made by partitions of n. For $n \geq 0$, we let $p(n) = \#\mathcal{P}_n$ be the number of partitions of the integer n. By convention, the empty parition $\emptyset$ is a partition of 0, so that $p(0) = 1$. The first values of

$p(n)$ can be found by listing all the partitions of small sizes:

$p(0) = 1$	$0 =$ empty sum
$p(1) = 1$	$1 = 1$
$p(2) = 2$	$2 = 2$ $2 = 1 + 1$
$p(3) = 3$	$3 = 3$ $3 = 2 + 1$ $3 = 1 + 1 + 1$
$p(4) = 5$	$4 = 4$ $4 = 3 + 1$ $4 = 2 + 2$ $4 = 2 + 1 + 1$ $4 = 1 + 1 + 1 + 1$
$p(5) = 7$	(check it!)

Here are more values of the sequence $(p(n))_{n\geq 0}$:

$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231, 297, 385, 490, 627, 792, 1002, 1255, 1575, \ldots$

For partitions, it turns out that the numbers don't have a nice closed expression, but the generating function does. We have the easy theorem:

Theorem 62 (Euler). *The generating function of partitions is given by:*

$$P(q) := \sum_{n\geq 0} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{5.1}$$

Proof. The set of partitions having all parts $\leq k$ is specified as a combinatorial class by:

$$\mathrm{SEQ}(\square) \times \mathrm{SEQ}(\square\square) \times \mathrm{SEQ}(\square\square\square) \times \cdots \times \mathrm{SEQ}(\underbrace{\square\square\ldots\square}_{k \text{ boxes}}),$$

so its generating function is $\frac{1}{1-q} \times \frac{1}{1-q^2} \times \cdots \times \frac{1}{1-q^k}$. Now let k tend to infinity (we are considering formal power series here, so it is clear that taking the limit makes sense – since for any $l \geq 0$ the coefficient of q^l in the last expression is stationary k large enough). □

5.1.2 Plane partitions and MacMahon's formula

A *plane partition* is a finite array of integers, weakly decreasing along rows and columns. When we write a plane partition we do not write the entries equal to zero, so we obtain a finite array looking like that:

7	7	3	3	1
5	5	3	2	
4	3	3		
2	1			

The *size* of a plane partition is the sum of its entries. In our example, the size is thus 49. We let $\mathcal{R}_n$ be the set of plane partitions of size n, and we let $\mathcal{R} = \cup_{n\geq 0}\mathcal{R}_n$ be the set of all plane partitions. We let $r(n) = \#\mathcal{R}_n$ be the number of plane partitions of size n. Note that there is a unique plane partition of size 0 (the array full of zeros), so $r(0) = 1$.

Plane partitions are much more complicated objects than integer partitions. However, their generating function has an amazingly simple form:

Theorem 63 (MacMahon – main formula of this chapter). *The generating function of plane partitions is given by the explicit formula:*

$$R(q) := \sum_{n\geq 0} r(n) q^n = \prod_{i=1}^{\infty} \left(\frac{1}{1-q^i}\right)^i. \tag{5.2}$$

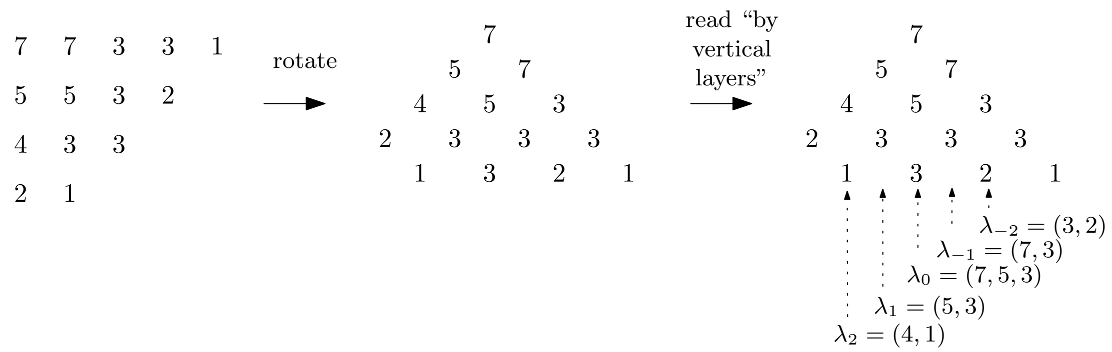
Remark 18. This formula looks very similar to Euler's formula for the generating function of integer partitions. However it is *much* more complicated to prove!

Remark 19. Using this formula one can easily determine the first numbers of plane partitions:

$$R(q) = 1 + q + 3\,q^2 + 6\,q^3 + 13\,q^4 + 24\,q^5 + 48\,q^6 + 86\,q^7 + 160\,q^8 + 282\,q^9 + 500\,q^{10} + \dots$$

Remark 20. It is natural to guess that similar formulas will exist for variants of partitions defined similarly in higher dimensions (3D-partitions, 4D-partitions, etc.). However this intuition is false: no such simple formula seems to exist in dimensions others than 1 and 2.

In order to prove MacMahon's formula, we will first rephrase the result in terms of sequences of *integer* partitions. Take the plane partition of the previous example, and rotate it by 45 degrees. Then, in "each vertical layer", you read an integer partition. This defines a bi-infinite sequence $(\dots, \lambda_{-1}, \lambda_0, \lambda_1, \dots)$ of integer partitions as in the following figure:



Since the original array of numbers was a plane partition, this sequence of partitions is not arbitrary. It satisfies constraints that are better expressed using the notion of *interlacing*.

Definition 64. Let λ and μ be two partitions. We write $\lambda \succ \mu$, and we say "λ and μ are interlaced", if for all $i \geq 1$ one has $\lambda_i \geq \mu_i \geq \lambda_{i+1}$, i.e.:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \dots$$

Looking at the previous picture, it is clear that $\lambda_0 \succ \lambda_{-1} \succ \lambda_{-2}$ and that $\lambda_0 \succ \lambda_1 \succ \lambda_2$. More generally, it is clear that these constraints express nothing but the fact that the original array of numbers (before rotation of 45 degrees) was non-decreasing along rows and columns. This viewpoint will actually be easier to work with:

Definition 65 (Plane partition – alternate definition). A *plane partition* is a bi-infinite sequence of partitions $(\dots, \lambda_{-2}, \lambda_{-1}, \lambda_0, \lambda_1, \lambda_2, \dots)$, such that $\lambda_i = \lambda_{-i} = \emptyset$ for i large enough, and satisfying the interlacing conditions:

$$\dots \prec \lambda_{-3} \prec \lambda_{-2} \prec \lambda_{-1} \prec \lambda_0 \succ \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \dots \tag{5.3}$$

5.2 A vector space, and two operators

We let V be the vector space made by formal linear combinations of partitions over $\mathbb{Q}$ (or $\mathbb{C}$ if you prefer). For example:

$$3.5 \cdot \emptyset - \frac{3}{7} \cdot (7,6,4) + \frac{11}{2} \cdot (8,8,1,1,1)$$

is an element of V. Sometimes we will also consider formal linear combinations with coefficients being formal power series in some variable (usually t, or t' or q). We will also sometimes consider *infinite* linear combinations. Typically we will consider elements of the form

$$\sum_{n \geq 0} q^n v_n$$

where for all $n \geq 0$, v_n is an element of V. The important thing will be that for any partition λ and for any power q^k of q, the coefficient of $q^k \lambda$ is well defined and finite. I don't give any more details here, and I don't specify properly over which field we are working (as you will see everything will be pretty obvious). Let's just say somehow inprecisely that we denote by Λ the vector space of all "infinite linear combinations of partitions with coefficients which are formal series in the parameters q, t, t'".

We equip the vector space Λ with the scalar product defined by:

$$(\lambda|\mu) = \mathbf{1}_{\lambda=\mu}.$$

In particular, for any $v \in \Lambda$ and $\lambda \in \mathcal{P}$, $(\lambda|v)$ is "the coefficient of λ in v".

We now define the two *vertex operators*. $\Gamma_+(t)$ and $\Gamma_-(t')$. They respectively interlace "upwards" and "downwards" the partitions on which they operate:

Definition 66 (Vertex operators). The operator $\Gamma_+(t)$ and $\Gamma_-(t')$ are the linear operators on Λ defined by:

$$\Gamma_+(t)\lambda = \sum_{\substack{\mu \in \mathcal{P} \\ \mu \succ \lambda}} t^{|\mu|-|\lambda|} \cdot \mu.$$

$$\Gamma_-(t')\lambda = \sum_{\substack{\mu \in \mathcal{P} \\ \mu \prec \lambda}} t'^{|\lambda|-|\mu|} \cdot \mu.$$

In concrete terms, $\Gamma_+(t)$ has the effect of "interlacing upwards" a partition in all possible ways, and the exponent of t remembers the extra-size added by the interlacing. $\Gamma_-(t')$ has a similar effect, but interlaces "downwards".

Using Definition 65, the generating function of plane partitions can be easily rewritten in terms of the vertex operators:

Proposition 67. *Let $R_{\leq k}(q)$ be the generating function of plane partitions that fit in a rectangle of size $k \times k$. Equivalently, let $R_{\leq k}(q)$ be the generating function of sequences:*

$$\emptyset = \lambda^{(k)} \prec \lambda^{(k-1)} \prec \cdots \prec \lambda^{(1)} \prec \lambda^{(0)} \succ \lambda^{(-1)} \succ \cdots \succ \lambda^{(k-1)} \succ \lambda^{(-k)} = \emptyset.$$

Then $R_{\leq k}$ is given by the scalar product:

$$R_{\leq k}(q) = \Big(\emptyset \ \Big| \underbrace{\Gamma_-(q^{-1})\Gamma_-(q^{-2}) \dots \Gamma_-(q^{-k})}_{k \text{ operators}} \underbrace{\Gamma_+(q^{k+1}) \dots \Gamma_+(q^{2k-1})\Gamma_+(q^{2k})}_{k \text{ operators}} \cdot \emptyset\Big). \tag{5.4}$$

Proof. The idea is that each sequence of the form

$$\lambda^{(k)} \prec \lambda^{(k-1)} \prec \cdots \prec \lambda^{(1)} \prec \lambda^{(0)} \succ \lambda^{(-1)} \succ \cdots \succ \lambda^{(k-1)} \succ \lambda^{(-k)} = \emptyset$$

can be constructed, from right to left, by starting with $\lambda^{(-k)} = \emptyset$, interlacing k times upwards, and interlacing k times downwards. Algebraically, by definition of the operators Γ_+, Γ_-, we have:

$$\begin{aligned} &\Gamma_-(q^{-1})\Gamma_-(q^{-2})\dots\Gamma_-(q^{-k})\Gamma_+(q^{k+1})\dots\Gamma_+(q^{2k-1})\Gamma_+(q^{2k})\cdot\emptyset \\ = &\sum_{\lambda^{(k)}\prec\cdots\prec\lambda^{(1)}\prec\lambda^{(0)}\succ\lambda^{(-1)}\succ\cdots\succ\lambda^{(-k)}=\emptyset} q^{\sum_{i=1}^{k}(i-1-k)(|\lambda_{i-1}|-|\lambda_i|)+\sum_{i=1-k}^{0}(k+1-i)(|\lambda_i|-|\lambda_{i-1}|)}\cdot\lambda^{(k)}. \end{aligned}$$

Taking the scalar product of this element with $\emptyset$ keeps only the sequences such that $\lambda^{(k)} = \emptyset$, which are exactly the sequences we want to count in the series $R_{\leq k}(q)$. So the only thing left to check is that the exponent of q in the last expression is indeed the total size of the sequence of partitions.

For $(1-k) \leq i \leq k$, write $u_i = |\lambda^{(i)}| - |\lambda^{(i-1)}|$ the difference of size between the two consecutive partitions. Note that $u_i \geq 0$ if $i \leq 0$ and $u_i \leq 0$ else. We have, for $-k \leq p \leq k$:

$$|\lambda^{(p)}| = \sum_{i=1-k}^{p} u_i,$$

so that the total size of the sequence is:

$$\sum_{p=-k}^{k} |\lambda^{(p)}| = \sum_{i=1-k}^{k} (k+1-i)u_i.$$

This shows that the exponent of q in the last formula indeed counts the total size. □

5.3 The commutation relation

We now are going to prove MacMahon's formula by evaluating (5.4). How? First, notice that, if we had taken the operators Γ_+ and Γ_- in the other direction, the product would have been easily evaluated. Indeed, for any $t_1, t_2, \dots, t_{2k}$ one has:

$$\left(\emptyset \,\middle|\, \Gamma_+(t_1)\dots\Gamma_+(t_2)\Gamma_+(t_k)\Gamma_-(t_{k+1})\Gamma_-(t_{k+2})\dots\Gamma_-(t_{2k})\cdot\emptyset\right) = 1. \quad (5.5)$$

Indeed, since the only partition μ such that $\emptyset \succ \mu$ is $\mu = \emptyset$, we have: $\Gamma_-(t_{k+1})\Gamma_-(t_{k+2})\dots\Gamma_-(t_{2k})\emptyset = \emptyset$. For the same reason, it is clear that the coefficient of $\emptyset$ in $\Gamma_+(t_1)\Gamma_+(t_2)\dots\Gamma_+(t_k)\emptyset$ is equal to 1.

Now, how do we go from (5.4) to something of the form (5.5)? The answer is clear: we commute operators! The only thing we will need is:

Proposition 68 (Commutation relation). *The operators Γ_- and Γ_+ satisfy the following "quasi-commutation" relation:*

$$\Gamma_-(t')\Gamma_+(t) = \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t'). \quad (5.6)$$

Remark 21. The relation says that, for any partition α, one has:

$$\Gamma_-(t')\Gamma_+(t)\alpha = \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha,$$

that is, that for any partitions α and β one has:

$$\Big(\beta \;\Big|\; \Gamma_-(t')\Gamma_+(t)\alpha\Big) = \Big(\beta \;\Big|\; \frac{1}{1-tt'}\Gamma_+(t)\Gamma_-(t')\alpha\Big).$$

By definition of the operators Γ_-, Γ_+, this is equivalent to saying that for any partitions α and β one has:

$$\sum_{\substack{\lambda \text{ such that} \\ \beta\prec\lambda\succ\alpha}} t'^{|\lambda|-|\beta|}t^{|\lambda|-|\alpha|} = \frac{1}{1-tt'} \cdot \sum_{\substack{\mu \text{ such that} \\ \beta\succ\mu\prec\alpha}} t^{|\beta|-|\mu|}t'^{|\alpha|-|\mu|}. \tag{5.7}$$

To prove the commutation relation, we are going to prove this last formula. More, we will prove it in a *bijective* way.

Bijective proof of (5.7)*, hence of Proposition 68.* Let $\alpha \succ \lambda \prec \beta$ and $k \geq 0$. Define $(t_i)_{i\geq 0}$ by $t_0 = k$ and $t_i = \min(\alpha_i, \beta_i) - \lambda_i$ for $i \geq 1$.

Then set $\mu_i = \max(\alpha_i, \beta_i) + t_{i-1}$. It is easy to check that μ is a partition such that $\alpha \prec \mu \succ \lambda$, and that $|\mu| = |\alpha| + |\beta| - |\lambda| + k$. Indeed one has:

- We have $\mu_i \geq \alpha_i$ since $\mu_i - \alpha_i = (\max(\alpha_i, \beta_i) - \alpha_i) + (\min(\alpha_{i-1}, \beta_{i-1}) - \lambda_{i-1})$, and both terms are ≥ 0 (for the second term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_{i-1} \leq \alpha_{i-1}$ and idem for β).
- We have $\mu_{i+1} \leq \alpha_i$ since $\mu_{i+1} - \alpha_i = (\max(\alpha_{i+1}, \beta_{i+1}) - \lambda_i) + (\min(\alpha_i, \beta_i) - \alpha_i)$, and both terms are ≤ 0 (for the first term, this is because $\lambda \prec \alpha$ and $\lambda \prec \beta$, so $\lambda_i \geq \alpha_{i+1}$ and idem for β).

Moreover we have $|\mu| = \sum_{i\geq 1}\mu_i = k + \sum_{i\geq 1}(\min(\alpha_i, \beta_i) + \max(\alpha_i, \beta_i)) = k + |\alpha| + |\beta|$.

We leave the reciprocal bijection as an exercise.

□

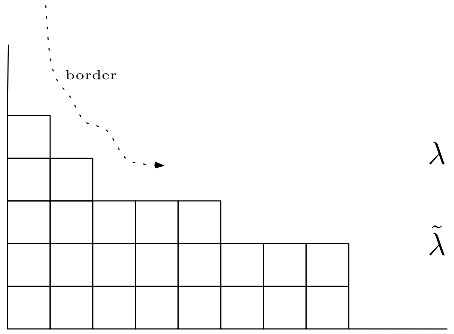
5.3.1 Another bijective proof of (5.7), with pictures

Although the proof above is very short, it is good to have an idea of what it does on a picture. This is a natural thing to do on a blackboard, it will be a bit longer on paper.

In order to represent the interlacing of partitions more clearly, we introduce the *fermionic representation* of a partition. Given a partition $\lambda_1 \geq \lambda_2 \geq \dots$, that we see as an infinite sequence by completing it with infinitely many parts equal to 0, define the sequence:

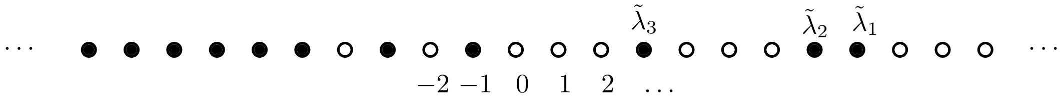
$$\tilde{\lambda}_i = \lambda_i + 1 - i \text{ for } i \geq 1.$$

Note that, since (λ_i) is non-decreasing, $(\tilde{\lambda}_i)$ is strictly decreasing. Now consider an infinite line of "positions", indexed by $\mathbb{Z}$, and place a *particle* at position $\tilde{\lambda}_i$ for all $i \geq 1$. One obtains the fermionic representation of λ:

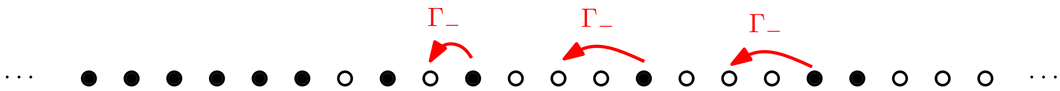


$$\lambda = (8, 8, 5, 2, 1, 0, 0, 0, \ldots)$$

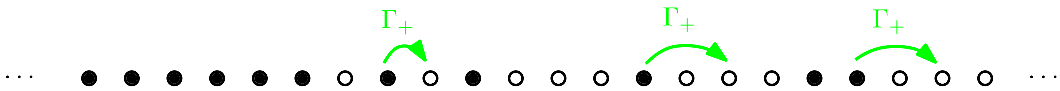
$$\tilde{\lambda} = (8, 7, 3, -1, -3, -5, -6, -7, \ldots)$$



Equivalently, the fermionic representation is the word obtained by reading the *border* of the partition from left to right, and writing a letter • for each vertical step, and a letter ∘ for each horizontal step. The effect of the operator $\Gamma_-(t')$ on fermions is to make *some* particles jump to the left:



The only constraint is that a particle cannot jump further than the original position (before jump) of the particle on its left. Note that $\Gamma_-(t)$ is the same, but makes particle jump to their *right*.

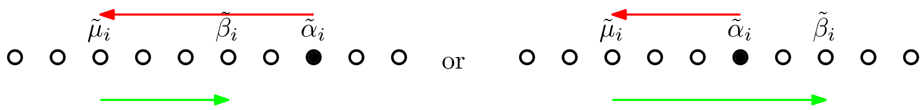


Note finally that in both cases the exponent of t (or t') counts the total displacement of particules.

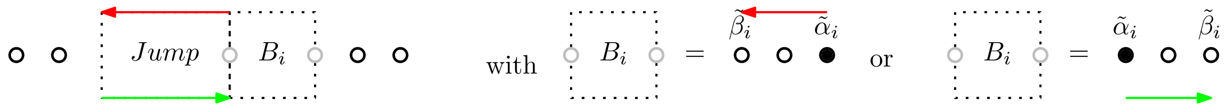
We now proceed with the proof. Keeping Remark 21 in mind, and in the sake of proving (5.7), we *fix* two partitions α and β. We are now going to describe, on the fermionic representation:
(a) the set of partitions μ such that $\beta \succ \mu \prec \alpha$,
(b) the set of partitions λ such that $\beta \prec \lambda \succ \alpha$,
and we will see that there is a simple connection between them.

Description of the set (a)

Let α, β, μ be three partitions such that $\beta \succ \mu \prec \alpha$, and look at the fermionic representations. When passing from α to μ, then to β, the particle in position $\tilde{\alpha}_i$ jumps to the left to $\tilde{\mu}_i$, then to the right to $\tilde{\beta}_i$. We represent the left jump with a red arrow, and the right jump with a green arrow, so the displacement of the particle looks like one of these two pictures:

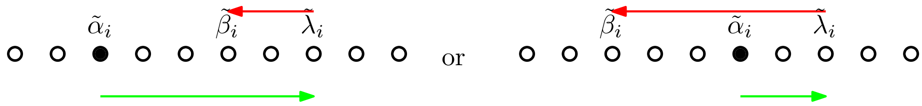


with the convention that the up arrow is taken first. This situation can be summarized by:

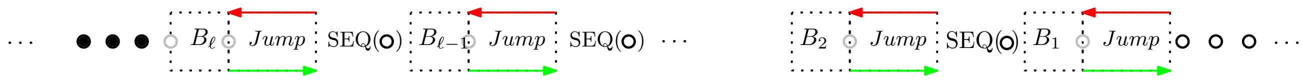


So in short, a triple $\beta \succ \mu \prec \alpha$ can be represented as a sequence of Jumps, Blocks, and empty positions as follows (**)

Description of the set (b)
Similarly, let α, λ, μ be three partitions such that $\beta \prec \lambda \succ \alpha$, and look at the fermionic representations. When passing from α to λ, then to β, the particle in position $\tilde{\alpha}_i$ jumps to the right to $\tilde{\lambda}_i$, then to the left to $\tilde{\beta}_i$. Therefore the displacement of the particle looks like one of these two pictures:



where in this case we use the convention that the bottom arrow is taken first. So a triple $\beta \prec \lambda \succ \alpha$ can be represented as a sequence of Jumps, Blocks, and empty positions as follows (***)



Conclusion of the proof of the commutation relation. Once the pictures are done, finding the bijection is obvious. Clearly, to go from a sequence of the form (**) to one of the form (***), one must do two things:

- push all the Jumps to the left (*i.e.* echange each jump with the $SEQ(\circ)$ on its left, leaving blocks in place)
- add an extra Jump at the very right.

The extra-jump at the very right can have any length $k \in [\![0, \infty)$, so the generating function contribution of this extra jump is:

$$\sum_{k=0}^{\infty} (tt')^k = \frac{1}{1 - tt'}.$$

The construction is clearly reversible and we have proved (5.7). Note that this pictorial bijection is in fact the same as the one described in a few lines in the previous section.

5.3.2 End of the proof of MacMahon's formula

Recall Equation (5.4):

$$R_{\leq k}(q) = \Big(\emptyset \,\Big|\, \underbrace{\Gamma_-(q^{-1})\Gamma_-(q^{-2})\dots\Gamma_-(q^{-k})}_{k \text{ operators}} \underbrace{\Gamma_+(q^{k+1})\dots\Gamma_+(q^{2k-1})\Gamma_+(q^{2k})}_{k \text{ operators}} \cdot \emptyset\Big).$$

In this formula, take the operator $\Gamma_+(q^{k+1})$ and send it to the left. To do that, you must "make it commute" with the operators $\Gamma_-(q^{-k}), \Gamma_-(q^{1-k}), \dots, \Gamma_-(q^{-1})$. For each of them, you must apply the commutation relation (Proposition 68, leading in total to a factor:

$$\frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \dots \cdot \frac{1}{1-q^k} = \prod_{i=1}^{k} \frac{1}{1-q^i}.$$

Now, do the same with the operator $\Gamma_+(q^{k+1})$. Again, you have to "make it commute" with the operators $\Gamma_-(q^{-k}), \Gamma_-(q^{1-k}), \dots, \Gamma_-(q^{-1})$. Apply the commutation relation (Proposition 68, this leads to a factor:

$$\frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdot \dots \cdot \frac{1}{1-q^{k+1}} = \prod_{i=1}^{k} \frac{1}{1-q^{i+1}}.$$

Similarly, for all $1 \leq j \leq k$ passing the $\Gamma_+(q^{k+j})$ operator to the left, the multiplicative factor comming from the commutation relations is:

$$\prod_{i=1}^{k} \frac{1}{1-q^{i+j-1}}.$$

When all the Γ_+ operators have been passed to the left, we can apply (5.5). The remaining scalar product is 1. Therefore we have proved:

Proposition 69. *The generating function of plane partitions fitting in a rectangle of size $k \times k$ is given by:*

$$R_{\leq k}(q) = \prod_{i=1}^{k} \prod_{i=1}^{k} \frac{1}{1-q^{i+j-1}}.$$

Now, let k tend to infinity, so that $R_{\leq k}(q)$ tends (coefficientwise) to the generating function $R(q)$ of all plane partitions. We obtain tMacMahon's formula:

$$R(q) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1-q^{i+j-1}} = \prod_{i=1}^{\infty} \left(\frac{1}{1-q^i} \right)^i.$$