Relational Graph Models, Taylor Expansion and Extensionality

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Abstract

We define the class of relational graph models and study the induced order- and equational- theories. Using the Taylor expansion, we show that all λ-terms with the same Böhm tree are equated in any relational graph model. If the model is moreover extensional and satisfies a technical condition, then its order-theory coincides with Morris's observational pre-order. Finally, we introduce an extensional version of the Taylor expansion, then prove that two λ-terms have the same extensional Taylor expansion exactly when they are equivalent in Morris's sense.

Keywords: lambda calculus, linear logic, differential nets, extensional Böhm trees, Taylor expansion.

Introduction

An important problem in the theory of programming languages is to determine when two programs are equivalent. For λ-calculus, it has become standard to regard two programs M and N as equivalent when they are *contextually equivalent* with respect to some fixed set $\mathcal{O}$ of *observables*. This means that we can plug either M or N into any context $C(-)$, i.e. any program with a *hole*, without noticing any difference in the global behaviour: $C(M)$ reduces to an observable in $\mathcal{O}$ exactly when $C(N)$ does.

Two notable examples are $\equiv^{\mathrm{hnf}}$ and Morris's equivalence $\equiv^{\mathrm{nf}}$ [17] obtained by taking as observables the head normal forms and the β-normal forms, respectively. Working with these definitions is difficult because of the quantification over all possible contexts. However, researchers have found alternative characterisations of these program equivalences based on syntactic trees or denotational models.

For instance, two programs are equivalent with respect to $\equiv^{\mathrm{hnf}}$ whenever they have the same Nakajima tree [18] or, equivalently, when their interpretations coincide in Scott's model $\mathcal{D}_\infty$ [21]. Similarly, $\equiv^{\mathrm{nf}}$ is captured by extensional Böhm trees [14] and Coppo, Dezani and Zacchi's filter model $\mathcal{D}_{\mathrm{cdz}}$ [6].

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The idea behind Böhm trees, and their extensional versions, is to extract the computational content of a program by representing its output as a possibly infinite tree — the continuity of this representation allows to infer properties of the whole tree by studying its finite approximants. For this reason Böhm-like trees and continuous models relied to them via approximation theorems constituted for over forty years the main tools to reason about the behaviour of a program. A limitation of these methods is that they abstract away from the execution process and overlook quantitative aspects such as the time, space, or energy consumed by a computation.

The present paper fits in a wider research programme whose aim is to rebuild the traditional theory of program approximations, by replacing it with a mathematical model of resource consumption. The starting point is [9], where Ehrhard and Regnier propose to analyse the behaviour of a program via its *Taylor expansion*, which is a generally infinite series of "resource approximants". Such approximants are terms of a *resource calculus* corresponding to a finitary fragment of the differential λ-calculus [7]. Each resource approximant t of a λ-term M captures a particular choice of the number of times M must call its sub-routines during its execution.

Both the differential λ-calculus and the Taylor expansion can be naturally interpreted in the relational semantics of linear logic [16]. The first author *et al.* built a relational model $\mathcal{D}_\omega$ living in such a semantics [5] and proved, using standard techniques, that the induced equality is exactly $\equiv^{\mathrm{hnf}}$ [15], just like for Scott's model $\mathcal{D}_\infty$ [12]. In this paper we provide syntactical and denotational methods based on Taylor expansion that allow to characterise Morris's equivalence $\equiv^{\mathrm{nf}}$.

First, we introduce the class of *relational graph models* (rgms) of λ-calculus, which are the relational analogous of graph models [3], and describe them as non-idempotent intersection type systems [19]. This class is general enough to encompass all relational models individually introduced in the literature [5,13], including $\mathcal{D}_\omega$ (while Scott's $\mathcal{D}_\infty$ cannot be a graph model since it is extensional). We then show that: (*i*) all rgms satisfy an approximation theorem for resource approximants (Theorem 3.10); (*ii*) in any rgm preserving the polarities of its "empty type" ω, β-normalisable λ-terms can be easily characterized (Lemma 4.3). As a consequence, we get that all extensional rgms preserving ω-polarities induce as order-theory Morris's observational pre-order, and hence $\equiv^{\mathrm{nf}}$ as equality (Corollary 4.6). As an instance, we provide the rgm $\mathcal{D}_\star$ generated by $\star \to \star \simeq \star$ where $\star$ is the only atom. It should be compared with the aforementioned filter model $\mathcal{D}_{\mathrm{cdz}}$, which has the same theory but is more complicated since it has two non-trivially ordered atoms $\varphi_\top \leq \varphi_\star$ and is generated by two equations $\varphi_\top \simeq \varphi_\star \to \varphi_\top$ and $\varphi_\star \simeq \varphi_\top \to \varphi_\star$.

Finally, we provide a notion of *extensional Taylor expansion* characterising, like extensional Böhm trees, Morris's equivalence while keeping the quantitative information. Intuitively, the extensional Taylor expansion of a λ-term is the η-normal form of its resource approximants. The definition is tricky because the η-reduction is meaningless on a *single* resource approximant — one should look at the whole series of approximants to decide whether an element should reduce or not. Our solution is to define a labeling as a global operation on the series of approximants, and then a local η-reduction on labeled terms. Two programs are then $\equiv^{\mathrm{nf}}$-equivalent exactly when they have the same extensional Taylor expansion (Theorem 5.17). We leave for future works a characterisation of Morris's preorder based on Taylor expansion.

Basic notations and conventions. We let $\mathbf{N}$ denote the set of natural numbers. Given a set A, $\mathcal{P}(A)$ (resp. $\mathcal{P}_{\mathrm{f}}(A)$) is the set of all (resp. finite) subsets of A and $\mathcal{M}_{\mathrm{f}}(A)$ is the set of all finite multisets over A. Finite multisets are represented as unordered lists $m = [\alpha_1, \dots, \alpha_n]$ with repetitions, $[]$ being the empty multiset.

Given a reduction $\to_{\mathtt{r}}$ we write $\twoheadrightarrow_{\mathtt{r}}$ ($=_{\mathtt{r}}$) for its transitive and reflexive (and symmetric) closure. A term t has an $\mathtt{r}$-normal form $\mathrm{nf}_{\mathtt{r}}(t)$, if $t \twoheadrightarrow_{\mathtt{r}} \mathrm{nf}_{\mathtt{r}}(t) \not\to_{\mathtt{r}}$.
N.B. Unless otherwise stated, throughout the paper we suppose that all operators $F : A \to B$ are extended to $\mathcal{P}(A)$ in the natural way: $F(a) = \{F(\alpha) \mid \alpha \in a\}$.

1 Lambda Calculus and Böhm Trees

We will generally use the notation of Barendregt's classic work [2] for λ-calculus. Let us fix an infinite set Var of variables. The set Λ of λ-*terms* is defined by:

$$\Lambda : \quad M, N, P ::= x \mid \lambda x.M \mid MN \qquad \text{for all } x \in \mathrm{Var}.$$

The set $\mathrm{fv}(M)$ of *free variables* of M and the α-conversion are defined as usual, see [2, Ch. 1§2]. A λ-term M is *closed* if $\mathrm{fv}(M) = \emptyset$. We denote by Λ^o the set of closed λ-terms. From now on, λ-terms will be considered up to α-conversion.

Given two λ-terms M, N we denote by $M\{N/x\}$ the capture-free substitution of N for all free occurrences of x in M. The β- and η-reductions are given for granted.

Concerning specific λ-terms, we fix the identity $\mathbf{I} = \lambda x.x$, its η-expansion $\mathbf{1} = \lambda xy.xy$, the paradigmatic looping term $\Omega = \Delta\Delta$ where $\Delta = \lambda x.xx$, Turing's fixpoint combinator $\Theta = \lambda f.\Theta_f\Theta_f$ where $\Theta_f = \lambda x.f(xx)$ and $\mathbf{J} = \Theta(\lambda zxy.x(zy))$ a term reducing to an infinite η-expansion of $\mathbf{I}$.

A λ-term M is called *solvable* if it has a *head normal form* (*hnf*, for short), that is if $M \twoheadrightarrow_\beta \lambda x_1 \dots x_n.yN_1 \cdots N_k$ (for $n, k \geq 0$); otherwise M is called *unsolvable*.

Given a *context* $C(-)$, i.e. a λ-term with a *hole* denoted by $(-)$, we write $C(M)$ for the λ-term obtained from C by substituting M for the hole possibly with capture of free variables in M. Given $\mathcal{O} \subseteq \Lambda$, the $\mathcal{O}$-*observational pre-order* is defined by:

$$M \sqsubseteq^{\mathcal{O}} N \iff \forall C(-)\ .\ C(M) \twoheadrightarrow_\beta M' \in \mathcal{O} \text{ entails } C(N) \twoheadrightarrow_\beta N' \in \mathcal{O}.$$

The induced *equivalence* $M \equiv^{\mathcal{O}} N$ is defined as $M \sqsubseteq^{\mathcal{O}} N$ and $N \sqsubseteq^{\mathcal{O}} M$. To obtain *Morris's pre-order* $\sqsubseteq^{\mathrm{nf}}$ and *equivalence* $\equiv^{\mathrm{nf}}$ just take as $\mathcal{O}$ the set of β-nfs [17].

The *Böhm tree* $\mathrm{BT}(M)$ of a λ-term M is defined coinductively: if M is unsolvable then $\mathrm{BT}(M) = \bot$; if M is solvable, then $M \twoheadrightarrow_\beta \lambda x_1 \dots x_n.yN_1 \cdots N_k$ and

$$\mathrm{BT}(M) = \begin{array}{ccc} & \lambda x_1 \dots x_n.y & \\ \swarrow & & \searrow \\ \mathrm{BT}(N_1) & \cdots & \mathrm{BT}(N_k) \end{array}$$

Such a definition is sound in the sense that $M =_\beta N$ entails $\mathrm{BT}(M) = \mathrm{BT}(N)$. Examples of Böhm trees are: $\mathrm{BT}(\mathbf{I}) = \mathbf{I}$, $\mathrm{BT}(\mathbf{1}) = \mathbf{1}$, $\mathrm{BT}(\Delta) = \Delta$, $\mathrm{BT}(\Omega) = \bot$,

$$\mathrm{BT}(\lambda x.y\Omega) = \begin{array}{c} \lambda x.y \\ | \\ \bot \end{array} \qquad \mathrm{BT}(\mathbf{J}) = \begin{array}{c} \lambda x z_0.x \\ | \\ \lambda z_1.z_0 \\ | \\ \lambda z_2.z_1 \\ \vdots \end{array} \qquad \mathrm{BT}(\Theta) = \begin{array}{c} \lambda f.f \\ | \\ f \\ | \\ f \\ \vdots \end{array}$$

Given two Böhm trees T, T' we set $T \leq_\perp T'$ if and only if T results from T' by replacing some subtrees with $\perp$. The set $\mathcal{N}$ of *finite approximants* is the set of λ-terms possibly containing $\perp$ inductively defined as follows: $\perp \in \mathcal{N}$; if $a_i \in \mathcal{N}$ for $i = 1, \dots, n$ then $\lambda\vec{x}.ya_1 \cdots a_n \in \mathcal{N}$. Hereafter we will confuse finite Böhm trees with normal approximants. Notice that the set of all finite approximants of a Böhm tree T, given by $T^* = \{a \in \mathcal{N} \mid a \leq_\perp T\}$, is an ideal with respect to $\leq_\perp$ [1, §2.3].

A λ-*theory* is any congruence on Λ containing $=_\beta$. A λ-theory is: *extensional* if it contains $=_\eta$; *sensible* if it equates all unsolvables. We denote by: $\lambda\beta\eta$ the least extensional λ-theory; $\mathcal{B}$ the λ-theory equating all λ-terms having the same Böhm tree; $\mathcal{B}\eta$ the least λ-theory containing $\mathcal{B}$ and $\lambda\beta\eta$; $\mathcal{H}^+$ (resp. $\mathcal{H}^*$) the λ-theory characterizing $\equiv^{\mathrm{nf}}$ (resp. $\equiv^{\mathrm{hnf}}$). From [2, Thm. 17.4.16] we get $\mathcal{B} \subsetneq \mathcal{B}\eta \subsetneq \mathcal{H}^+ \subsetneq \mathcal{H}^*$.

2 Resource Calculus and Taylor Expansion

We briefly recall Ehrhard's *resource calculus* [8], using the syntax proposed by Tranquilli in [22]. We are considering here the promotion-free fragment of [22].

Syntax. The set Λ^r of *resource terms* and the set Λ^b of *bags* are defined by:

$$\Lambda^r: \quad s, t ::= \ x \ \mid \lambda x.t \ \mid tb \qquad\qquad \Lambda^b: \quad b ::= \ [s_1, \dots, s_n] \text{ where } n \geq 0. \tag{1}$$

Resource terms are in functional position, while bags are in argument position and represent unordered lists of resource terms. Intuitively, in a term of shape $t[s_1, \dots, s_n]$ each s_i is a linear resource, that is t cannot duplicate nor erase it.

We will deal with bags as if they were multisets presented in multiplicative notation: 1 is the empty bag and $b_1 \cdot b_2$ is the multiset union of b_1 and b_2.

We use the power notation $[s^k]$ for the bag $[s, \dots, s]$ containing k copies of s.

The α-equivalence and the set $\mathrm{fv}(t)$ of free variables of t are defined as for the ordinary λ-calculus. Resource terms and bags are considered up to α-equivalence.

As a syntactic sugar, we extend all the constructors of the grammar (1) as pointwise operations on (possibly infinite) sets of resource terms or bags. That is, given $\mathbb{T} \subseteq \Lambda^r$ and $\mathbb{B}, \mathbb{B}' \subseteq \Lambda^b$ we use the following notations: $\lambda x.\mathbb{T} = \{\lambda x.t \mid t \in \mathbb{T}\}$, $\mathbb{T}\mathbb{B} = \{tb \mid t \in \mathbb{T}, b \in \mathbb{B}\}$, $[\mathbb{T}] = \{[t] \mid t \in \mathbb{T}\}$ and $\mathbb{B} \cdot \mathbb{B}' = \{b \cdot b' \mid b \in \mathbb{B},\ b' \in \mathbb{B}'\}$.

Observe that, in the particular case of empty set, we get $\lambda x.\emptyset = \emptyset$, $t\emptyset = \emptyset$, $\emptyset b = \emptyset$, $[\emptyset] = \emptyset$ and $\emptyset \cdot b = \emptyset$. Hence, $\emptyset$ annihilates any resource term or bag.

This kind of meta-syntactic notation is discussed thoroughly in [9].

Reductions. Given a relation $\to_{\mathtt{r}} \subseteq \Lambda^r \times \mathcal{P}_{\mathrm{f}}(\Lambda^r)$ its *context closure* is the least relation in $\mathcal{P}_{\mathrm{f}}(\Lambda^r) \times \mathcal{P}_{\mathrm{f}}(\Lambda^r)$ such that, when $t \to_{\mathtt{r}} \mathbb{T}$, we have:

$$\lambda x.t \to_{\mathtt{r}} \lambda x.\mathbb{T}, \quad tb \to_{\mathtt{r}} \mathbb{T}b, \quad s([t] \cdot b) \to_{\mathtt{r}} s([\mathbb{T}] \cdot b), \quad \{t\} \cup \mathbb{S} \to_{\mathtt{r}} \mathbb{T} \cup \mathbb{S}.$$

We say that $t \in \Lambda^r$ *is in* $\mathtt{r}$*-normal form* if there is no $\mathbb{T}$ such that $t \to_{\mathtt{r}} \mathbb{T}$. When $\to_{\mathtt{r}}$ is confluent, $\mathrm{nf}_{\mathtt{r}}(t) \in \mathcal{P}_{\mathrm{f}}(\Lambda^r)$ denotes the unique $\mathtt{r}$-normal form of t, if it exists.

The *degree of* x *in* t, written $\deg_x(t)$, is the number of free occurrences of x in t. A β*-redex* is a resource term of the shape $(\lambda x.t)[s_1, \dots, s_k]$ and its *contractum* is a finite set of resource terms: when $\deg_x(t) = k$, it is the set of all possible resource terms obtained by linearly replacing each free occurrence of x in t by exactly one of the s_i's; otherwise, when $\deg_x(t) \neq k$, it is just $\emptyset$.

Formally, we define $\to_\beta$ as the context closure of:

$$(\lambda x.t)[s_1,\dots,s_k] \to_\beta \begin{cases} \bigcup_{p\in\mathfrak{S}_k} t\{s_{p(1)}/x_1,\dots,s_{p(k)}/x_k\} & \text{if } \deg_x(t)=k,\\ \emptyset & \text{otherwise.}\end{cases}$$

where $\mathfrak{S}_k$ is the group of permutations of $\{1,\dots,k\}$ and $x_1,\dots,x_n$ is an arbitrary enumeration of the free occurrences of x in t. Note that β-reduction is strongly normalizing (SN, for short) on $\mathcal{P}_{\mathrm{f}}(\Lambda^r)$, since whenever $t \to_\beta \mathbb{T}$ the size of t is strictly bigger than the size of each resource term in $\mathbb{T}$. Moreover, β-reduction is weakly confluent, and therefore confluent by Newman's lemma.

Theorem 2.1 *The β-reduction is strongly normalizing and confluent on $\mathcal{P}_{\mathrm{f}}(\Lambda^r)$.*

In the resource calculus there is no sensible notion of η-reduction on $\mathcal{P}_{\mathrm{f}}(\Lambda^r)$.

Taylor expansion. The *Taylor expansion* of a λ-term, as defined in [7,9], is a translation developing every λ-calculus application as an infinite series of resource applications with rational coefficients. For our purpose it is enough to consider a simplified version $\mathcal{T}(-) : \Lambda \to \mathcal{P}(\Lambda^r)$ corresponding to the support [4] of the actual Taylor expansion; that is, we consider possibly infinite sets of resource λ-terms.

Definition 2.2 The *Taylor expansion* $\mathcal{T}(M) \subseteq \Lambda^r$ of a λ-term M is defined by:

$$\mathcal{T}(x) = x, \qquad \mathcal{T}(\lambda x.M) = \lambda x.\mathcal{T}(M), \qquad \mathcal{T}(MN) = \mathcal{T}(M)\mathcal{M}_{\mathrm{f}}(\mathcal{T}(N)).$$

The Taylor expansion is extended to finite approximants in $\mathcal{N}$ by setting $\mathcal{T}(\bot) = \emptyset$, and to Böhm trees T by setting $\mathcal{T}(T) = \bigcup\{\mathcal{T}(a) \mid a \in T^*\}$.

Some examples of Taylor expansions of ordinary λ-terms are:

$$\mathcal{T}(\mathbf{I}) = \{\mathbf{I}\},\ \ \mathcal{T}(\Delta) = \{\lambda x.x[x^n] \mid n \geq 0\},\ \mathcal{T}(\lambda y.xyy) = \{\lambda y.x[y^n][y^k] \mid n,k \geq 0\},$$

$$\mathcal{T}(\Omega) = \{(\lambda x.x[x^{n_0}])[\lambda x.x[x^{n_1}],\dots,\lambda x.x[x^{n_k}]] \mid k, n_0,\dots,n_k \geq 0\},$$

$$\begin{aligned}\mathcal{T}(\Theta) = \{&\lambda f.(\lambda x.f[x[x^{n_1}],\dots,x[x^{n_k}]])[\lambda x.f[x[x^{n_{1,1}}],\dots,x[x^{n_{1,k_1}}]],\dots,\\ &\lambda x.f[x[x^{n_{h,1}}],\dots,x[x^{n_{h,k_h}}]]] \mid k,n_i,h,n_{i,j} \geq 0\},\end{aligned}$$

$$\begin{aligned}\mathcal{T}(\mathbf{J}) = \{&t[\lambda zxy.x[z[y^{n_{1,1}}],\dots,z[y^{n_{1,k_1}}]],\dots,\\ &\lambda zxy.x[z[y^{n_{h,1}}],\dots,z[y^{n_{h,k_h}}]]] \mid t \in \mathcal{T}(\Theta),\ h,k_i,n_{i,j} \geq 0\}.\end{aligned}$$

From the examples above it is clear that if a λ-term M has a β-redex, then there are resource terms $t \in \mathcal{T}(M)$ having β-redexes too. However, by Theorem 2.1, each t has a unique β-nf and we can always compute $\mathrm{nf}_\beta(\mathcal{T}(M)) = \bigcup\{\mathrm{nf}_\beta(t) \mid t \in \mathcal{T}(M)\}$. For instance: $\mathcal{T}(\mathbf{I})$, $\mathcal{T}(\Delta)$ and $\mathcal{T}(\lambda y.xyy)$ are already β-normal, while $\mathrm{nf}_\beta(\mathcal{T}(\Omega)) = \emptyset$.

Lemma 2.3 *Let $a \in \mathcal{N}$ and $M \in \Lambda$, then $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}(M))$ entails $a \in \mathrm{BT}(M)^*$.* → proof in tech. app.

The following results proved in [8] show the strong relationship between the Böhm tree of a λ-term, and its Taylor expansion.

[4] I.e., the set of those resource terms appearing in the series with a non-zero coefficient.

Theorem 2.4 *For every* λ*-term* M*,* $\mathrm{nf}_\beta(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$*.*

Corollary 2.5 *For all* $M, N \in \Lambda$*,* $\mathrm{BT}(M) = \mathrm{BT}(N)$ *iff* $\mathrm{nf}_\beta(\mathcal{T}(M)) = \mathrm{nf}_\beta(\mathcal{T}(N))$*.*

Using Theorem 2.4, we can easily calculate further examples:

$\mathrm{nf}_\beta(\mathcal{T}(\Theta)) = \{\lambda f.f1, \lambda f.f[(f1)^n], \lambda f.f[f[(f1)^{n_1}], \dots, f[(f1)^{n_k}]], \dots\}$,
$\mathrm{nf}_\beta(\mathcal{T}(\mathbf{J})) = \{\lambda xz_0.x1, \lambda xz_0.x[(\lambda z_1.z_01)^n], \dots\}$.

3 Relational Graph Models and Intersection Types

In this section we introduce the class of *relational graph models* (rgm, for short); some examples of such models were individually studied in [13].

3.1 Relational Graph Models

We call rgms *relational* because they are (linear) reflexive objects in the ccc $\mathbf{MRel}$ [5], the Kleisli category of $\mathbf{Rel}$ with respect to the comonad $\mathcal{M}_\mathrm{f}(-)$. In $\mathbf{MRel}$ the objects are all the sets, a morphism $f \in \mathbf{MRel}(A, B)$ is any relation between $\mathcal{M}_\mathrm{f}(A)$ and B, and the exponential object $A \Rightarrow B$ is given by $\mathcal{M}_\mathrm{f}(A) \times B$. Any function $f : A \to B$ can be sent to $f^\dagger \in \mathbf{MRel}(A, B)$ by setting $f^\dagger = \{([a], f(a)) \mid a \in A\}$.

Definition 3.1 A *relational graph model* $\mathcal{D} = (D, i)$ is given by an infinite set D and a total injection $i : \mathcal{M}_\mathrm{f}(D) \times D \to D$. $\mathcal{D}$ is *extensional* when i is bijective.

Every rgm $\mathcal{D} = (D, i)$ induces a reflexive object $(D, i^\dagger, (i^{-1})^\dagger)$, i.e. $D \Rightarrow D \lhd D$ since $i^\dagger; (i^{-1})^\dagger = \mathrm{Id}_{D \Rightarrow D}$. When $\mathcal{D}$ is moreover extensional we also have $(i^{-1})^\dagger ; i^\dagger = \mathrm{id}_D$. These reflexive objects are all *linear* in the sense of [16] and live in a differential ccc, they are therefore sound models of the resource calculus as well (Theorem 3.8).

Rgms, just like the regular ones [3], can be built by performing the free completion of a partial pair. A *partial pair* $\mathcal{A}$ is a pair (A, j) where A is a non-empty set of elements (called *atoms*) and $j : \mathcal{M}_\mathrm{f}(A) \times A \to A$ is a partial injection. We say that $\mathcal{A}$ is *extensional* when j is a bijection between $\mathrm{dom}(j)$ and A. Wlog., we will only consider partial pairs $\mathcal{A}$ whose underlying set A does not contain any pair.

Definition 3.2 The *completion* $\overline{\mathcal{A}}$ of a partial pair $\mathcal{A}$ is the pair $(\overline{A}, \overline{j})$ defined as: $\overline{A} = \bigcup_{n \in \mathbf{N}} A_n$, where $A_0 = A$ and $A_{n+1} = ((\mathcal{M}_\mathrm{f}(A_n) \times A_n) - \mathrm{dom}(j)) \cup A$; the function $\overline{j}$ is given by $\overline{j}(a, \alpha) = j(a, \alpha)$ if $(a, \alpha) \in \mathrm{dom}(j)$, $\overline{j}(a, \alpha) = (a, \alpha)$ otherwise.

Note that, for every rgm $\mathcal{D}$ we have $\overline{\mathcal{D}} = \mathcal{D}$ (up to isomorphism).

Proposition 3.3 *If* $\mathcal{A}$ *is a partial pair, then* $\overline{\mathcal{A}}$ *is an* rgm*. When* $\mathcal{A}$ *is extensional, also* $\overline{\mathcal{A}}$ *is extensional.*

Proof The proof of the fact that $\overline{\mathcal{A}}$ is an rgm is analogous to the one for regular graph models [3]. It is easy to check that when j is bijective, also $\overline{j}$ is. □

Example 3.4 We define the relational analogues of:

- Engeler's model [10]: $\mathcal{E} = \overline{(\mathbf{N}, \emptyset)}$, first defined in [13],
- Scott's model [21]: $\mathcal{D}_\omega = \overline{(\{\varepsilon\}, \{([], \varepsilon) \mapsto \varepsilon\})}$, first defined (up to iso) in [5],
- Coppo, Dezani and Zacchi's model [6]: $\mathcal{D}_\star = \overline{(\{\star\}, \{([\star], \star) \mapsto \star\})}$.

Notice that $\mathcal{D}_\omega$ and $\mathcal{D}_\star$ are extensional, while $\mathcal{E}$ is not.

$$\frac{}{x:\sigma\vdash^{\mathcal{D}} x:\sigma}\ \texttt{var} \qquad \frac{\Gamma, x:\mu\vdash^{\mathcal{D}} M:\sigma}{\Gamma\vdash^{\mathcal{D}} \lambda x.M:\mu\to\sigma}\ \texttt{lam} \qquad \frac{\Gamma\vdash^{\mathcal{D}} M:\tau \quad \sigma\simeq^{\mathcal{D}}\tau}{\Gamma\vdash^{\mathcal{D}} M:\sigma}\ \texttt{eq}$$

$$\frac{\Gamma_0\vdash^{\mathcal{D}} M:\wedge_{i=1}^{n}\sigma_i\to\tau \quad \Gamma_i\vdash^{\mathcal{D}} N:\sigma_i \quad \text{for } i=1,\dots,n}{\Gamma_0\wedge(\wedge_{i=1}^{n}\Gamma_i)\vdash^{\mathcal{D}} MN:\tau}\ \texttt{app}$$

(a) Non-idempotent intersection type system for Λ and $\mathcal{N}$.

$$\frac{\Gamma_0\vdash^{\mathcal{D}} t:\wedge_{i=1}^{n}\sigma_i\to\tau \quad \Gamma_i\vdash^{\mathcal{D}} s_i:\sigma_i \quad \text{for } i=1,\dots,n}{\Gamma_0\wedge(\wedge_{i=1}^{n}\Gamma_i)\vdash^{\mathcal{D}} t[s_1,\dots,s_n]:\tau}\ \texttt{app}'$$

(b) Non-idempotent intersection type system for Λ^r.

Figure 1: The intersection type systems for Λ, $\mathcal{N}$ and Λ^r. The other rules for typing Λ^r are analogous to (`var`), (`lam`), (`eq`) of Figure 1(a) and are omitted.

3.2 Non-Idempotent Intersection Type Systems

As discussed thoroughly in [19], the choice of presenting a relational model as a reflexive object or as a non-idempotent intersection type system is more a matter of taste rather than a technical decision. Here we provide the latter presentation.

Let $\mathcal{A}$ be a partial pair and $\mathcal{D}$ be its completion. The set $\mathsf{T}_{\mathcal{D}}$ of *types* and the set $\mathsf{I}_{\mathcal{D}}$ of *non-idempotent intersections* are defined by mutual induction (for $\alpha\in A$):

$$\mathsf{T}_{\mathcal{D}}:\quad \sigma,\tau\ ::=\ \alpha\ \mid\ \mu\to\sigma \qquad\qquad \mathsf{I}_{\mathcal{D}}:\quad \mu,\nu\ ::=\ \omega\ \mid\ \sigma\ \mid\ \sigma\wedge\mu$$

Note that types are (unary) intersections while the converse does not hold; indeed intersections may only appear at the left-hand side of an arrow. Thus ω is not a type, it denotes the empty intersection and is therefore its neutral element ($\mu\wedge\omega=\mu$). Accordingly, we write $\wedge_{i=1}^{n}\sigma_n$ for $\sigma_1\wedge\cdots\wedge\sigma_n$ when $n\geq 1$, and for ω when $n=0$. Types will be considered up to associativity and commutativity of $\wedge$ and neutrality of ω, while we assume that the intersection is *not* idempotent, that is $\sigma\wedge\sigma\neq\sigma$.

Every $\sigma\in\mathsf{T}_{\mathcal{D}}$ ($\mu\in\mathsf{I}_{\mathcal{D}}$) corresponds to an element $\sigma^\bullet$ of D ($\mu^\bullet$ of $\mathcal{M}_{\mathrm{f}}(D)$) defined as $\alpha^\bullet=\alpha$, $(\mu\to\tau)^\bullet=i(\mu^\bullet,\tau^\bullet)$ and $(\sigma_1\wedge\cdots\wedge\sigma_n)^\bullet=[\sigma_1^\bullet,\dots,\sigma_n^\bullet]$. Hence, the model $\mathcal{D}$ induces a congruence on the intersection types: $\sigma\simeq^{\mathcal{D}}\tau$ if and only if $\sigma^\bullet=\tau^\bullet$.

An *environment* is a map $\Gamma:\mathrm{Var}\to\mathsf{I}_{\mathcal{D}}$ such that $\mathrm{dom}(\Gamma)=\{x\mid\Gamma(x)\neq\omega\}$ is finite. We write $x_1:\mu_1,\dots,x_n:\mu_n$ for the environment Γ such that $\Gamma(x_i)=\mu_i$ and $\Gamma(y)=\omega$ for all $y\notin\vec{x}$. The environment mapping all variables to ω is denoted by $\emptyset$, or just omitted as in Example 3.6. The intersection $\Gamma_1\wedge\Gamma_2$ and the equivalence $\Gamma_1\simeq^{\mathcal{D}}\Gamma_2$ of two environments are defined pointwise; note that $\Gamma\wedge\emptyset=\Gamma$.

Definition 3.5 The interpretation of $M\in\Lambda$ (or $M\in\mathcal{N}$) in $\mathcal{D}$ is defined as:

$$[\![M]\!]^{\mathcal{D}}=\{(\Gamma,\sigma)\mid\Gamma\vdash^{\mathcal{D}} M:\sigma\},\ \text{where the type system}\ \vdash^{\mathcal{D}}\ \text{is given in Fig. 1(a).}$$

The definition of $[\![t]\!]^{\mathcal{D}}$ for $t\in\Lambda^r$ is analogous, using the rules of Fig. 1(b). Note that $\vdash^{\mathcal{D}}$ also works for terms in $\mathcal{N}$: $\bot$ is not typable, but e.g. $\vdash^{\mathcal{D}}\lambda x.x\bot:(\omega\to\tau)\to\tau$.

Example 3.6 Let $\mathcal{D}$ be any rgm. Then we have: $[\![\mathbf{I}]\!]^{\mathcal{D}}=\{\sigma\mid\sigma\simeq\tau\to\tau,\ \tau\in\mathsf{T}_{\mathcal{D}}\}$, $[\![\mathbf{1}]\!]^{\mathcal{D}}=\{\sigma\mid\sigma\simeq(\mu\to\tau)\to\mu\to\tau,\ \tau\in\mathsf{T}_{\mathcal{D}},\mu\in\mathsf{I}_{\mathcal{D}}\}$, $[\![\mathbf{J}]\!]^{\mathcal{D}}=\{\sigma\mid\sigma\simeq(\omega\to\tau)\to\omega\to\tau,\ \tau\in\mathsf{T}_{\mathcal{D}}\}$, $[\![\lambda x.x\Omega]\!]^{\mathcal{D}}=\{\sigma\mid\sigma\simeq(\omega\to\tau)\to\tau,\ \tau\in\mathsf{T}_{\mathcal{D}}\}$, $[\![\Omega]\!]^{\mathcal{D}}=\emptyset$. It follows that $[\![\mathbf{I}]\!]=[\![\mathbf{1}]\!]$ in both $\mathcal{D}_\omega$ and $\mathcal{D}_\star$, but $[\![\mathbf{I}]\!]^{\mathcal{D}_\omega}=[\![\mathbf{J}]\!]^{\mathcal{D}_\omega}$, while $\star\in[\![\mathbf{I}]\!]^{\mathcal{D}_\star}-[\![\mathbf{J}]\!]^{\mathcal{D}_\star}$.

When $\mathcal{D}$ is clear from the context we simply write $\simeq$, $\vdash$ and $[\![-]\!]$. Note that $\Gamma \vdash M : \sigma$ implies $\mathrm{dom}(\Gamma) \subseteq \mathrm{fv}(M)$ and $\Gamma' \vdash M : \sigma'$ for $\Gamma \simeq \Gamma'$ and $\sigma \simeq \sigma'$ [19].

Theorem 3.7 (Inversion Lemma, cf. [19]) *Let $\mathcal{D}$ be an rgm.*

(i) $\Gamma \vdash x : \sigma$ *entails* $\Gamma = x : \tau$ *for* $\tau \simeq \sigma$,

(ii) $\Gamma \vdash \lambda x.M : \sigma$ *if and only if* $\Gamma, x : \mu \vdash M : \tau$ *for some* $\mu \to \tau \simeq \sigma$,

(iii) $\Gamma \vdash MN : \sigma$ *entails that* $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^n \Gamma_i)$ *for some* $n \geq 0$, $\Gamma_0 \vdash M : \wedge_{i=1}^n \sigma_i \to \sigma$ *and* $\Gamma_i \vdash N : \sigma_i$.

For resource λ-terms an analogous statement holds, where (iii) is replaced with:

(iii') $\Gamma \vdash t[s_1, \dots, s_n] : \sigma$ *entails* $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^n \Gamma_i)$, $\Gamma_0 \vdash t : \wedge_{i=1}^n \sigma_i \to \sigma$ *and* $\Gamma_i \vdash s_i : \sigma_i$.

Theorem 3.8 *Let $\mathcal{D}$ be an rgm, then for Λ and Λ^r:*

(i) Substitution lemma, subject reduction and subject expansion hold in $\vdash^{\mathcal{D}}$.

(ii) The interpretation $[\![-]\!]^{\mathcal{D}}$ *is* sound *with respect to* $=_\beta$.

Proof (i) is proved in [19] for Λ and in [16] for relational models of Λ^r.
(ii) follows from (i). □

The λ-*theory* and the *order theory* induced by $\mathcal{D}$ are given by $\mathrm{Th}(\mathcal{D}) = \{(M, N) \mid [\![M]\!] = [\![N]\!]\}$ and $\mathrm{Th}_{\leq}(\mathcal{D}) = \{(M, N) \mid [\![M]\!] \subseteq [\![N]\!]\}$, respectively. We write $\mathcal{D} \models M = N$ if $(M, N) \in \mathrm{Th}(\mathcal{D})$, and $\mathcal{D} \models M \leq N$ if $(M, N) \in \mathrm{Th}_{\leq}(\mathcal{D})$. A model $\mathcal{D}$ is $\mathcal{O}$-*inequationally fully abstract* when $\mathcal{D} \models M \leq N$ if and only if $M \sqsubseteq^{\mathcal{O}} N$, and $\mathcal{O}$-*fully abstract* when $\mathcal{D} \models M = N$ if and only if $M \equiv^{\mathcal{O}} N$.

Lemma 3.9 *If $\mathcal{D}$ is an extensional rgm, then* $\lambda\beta\eta \subseteq \mathrm{Th}(\mathcal{D})$.

Proof The equivalence between $\Gamma \vdash M : \sigma$ and $\Gamma \vdash \lambda x.Mx : \sigma$ when $x \notin \mathrm{fv}(M)$ follows by induction on σ using the fact that $\alpha \simeq \mu \to \tau$ for every atomic type α. □

As a consequence, the λ-theories induced by rgms and by regular graph models are different, since no graph model is extensional. For instance, the λ-theory of $\mathcal{D}_\omega$, the relational analogue of Scott's $\mathcal{D}_\infty$, is $\mathcal{H}^\star$ [15]. That is $\mathcal{D}_\omega$ is hnf-fully abstract.

While approximation theorems for Böhm trees and idempotent intersection type systems are usually proved through reducibility techniques, the following one for Taylor expansion and rgms can be proved by induction on the type derivation using the subject reduction (Theorem 3.8) and the SN of Λ^r (Theorem 2.1).

Theorem 3.10 (Approximation Theorem) *Let M be a λ-term. Then* → proof in tech. app.

$$\Gamma \vdash M : \sigma \text{ if and only if there exists } t \in \mathcal{T}(M) \text{ such that } \Gamma \vdash t : \sigma.$$

Therefore $[\![M]\!] = [\![\mathcal{T}(M)]\!]$.

Corollary 3.11 *For all rgms $\mathcal{D}$ we have that* $\mathcal{B} \subseteq \mathrm{Th}(\mathcal{D})$. *In particular* $\mathrm{Th}(\mathcal{D})$ *is sensible and* $[\![M]\!]^{\mathcal{D}} = \emptyset$ *for all unsolvable λ-terms M.*

Proof From Theorem 3.10 we have $[\![M]\!] = [\![\mathcal{T}(M)]\!] = \bigcup_{t \in \mathcal{T}(M)} [\![t]\!]$. By subject reduction for Λ^r (Theorem 3.8) this is equal to $\bigcup_{t \in \mathcal{T}(M)} [\![\mathrm{nf}_\beta(t)]\!]$, which is equal to $\bigcup_{t \in \mathcal{T}(\mathrm{BT}(M))} [\![t]\!] = [\![\mathcal{T}(\mathrm{BT}(M))]\!]$, by Theorem 2.4. Therefore, whenever $\mathrm{BT}(M) = \mathrm{BT}(N)$ we get $[\![M]\!] = [\![\mathcal{T}(\mathrm{BT}(M))]\!] = [\![\mathcal{T}(\mathrm{BT}(N))]\!] = [\![N]\!]$. □

4 Full Abstraction for Morris's Observational Preorder

This section is devoted to show that every extensional rgm $\mathcal{D}$ satisfying the condition of Definition 4.1 — in particular $\mathcal{D}_\star$ — is (inequationally) fully abstract with respect to Morris's pre-order $\sqsubseteq^{\mathrm{nf}}$. Rather than working directly with $\sqsubseteq^{\mathrm{nf}}$, and building separating contexts, we use Levy's notion of *extensional Böhm tree*

$$\mathrm{BT}^e(M) = \{\mathrm{nf}_\eta(a) \mid a \in \mathrm{BT}(M')^*,\ M' \twoheadrightarrow_\eta M\}.$$

Indeed, it is well known that $M \sqsubseteq^{\mathrm{nf}} N$ exactly when $\mathrm{BT}^e(M) \subseteq \mathrm{BT}^e(N)$ [11] and that two λ-terms have the same extensional Böhm tree when their Böhm trees are equal up to (possibly infinitely many) η-expansions of *finite depth*. These trees are therefore different from Nakajima trees: for instance $\mathbf{I} \in \mathrm{BT}^e(\mathbf{I}) - \mathrm{BT}^e(\mathbf{J})$.

Examples of extensional Böhm trees are: $\mathrm{BT}^e(\mathbf{1}) = \mathrm{BT}^e(\mathbf{I})$,
$\mathrm{BT}^e(\mathbf{I}) = \{\bot, \mathbf{I}, \lambda x z_0.x\bot, \lambda x z_0.x(\lambda z_1.z_0(\lambda z_2.z_1\bot)), \dots\}$, $\mathrm{BT}^e(\mathbf{J}) = \mathrm{BT}^e(\mathbf{I}) - \{\mathbf{I}\}$,
$\mathrm{BT}^e(\lambda y.xyy) = \{\bot, x\bot, \lambda y.xyy, \lambda y.xy\bot, \dots\}, \mathrm{BT}^e(x\Omega) = \mathrm{BT}^e(\lambda y.xyy) - \{\lambda y.xyy\}$.

Given a polarity $p \in \{+,-\}$, we define inductively for all types σ the relations $\omega \in^p \sigma$ and $\omega \in^{\neg p} \sigma$, where $\neg p$ is the opposite polarity, as: (i) $\omega \in^- \mu \to \tau$ if $\mu = \omega$; (ii) if $\omega \in^p \tau$ then $\omega \in^p \mu \to \tau$; (iii) if $\omega \in^{\neg p} \tau$ then $\omega \in^p \tau \wedge \mu \to \tau'$. When $\omega \in^+ \sigma$ ($\omega \in^- \sigma$) we say that ω *occurs positively* (*negatively*) in σ. We write $\omega \notin^+ \sigma$ ($\omega \notin^- \sigma$) if ω does not occur positively (negatively) in σ. These notions extend to intersections in the obvious way, for instance $\omega \in^p \sigma_1 \wedge \dots \wedge \sigma_n$ if $\omega \in^p \sigma_i$ for some i.

Definition 4.1 An rgm $\mathcal{D}$ *preserves ω-polarities* whenever $\omega \in^p \sigma$ and $\sigma \simeq \tau$ entail $\omega \in^p \tau$, for all $\sigma, \tau \in \mathsf{T}_{\mathcal{D}}$ and $p \in \{+,-\}$.

For instance $\mathcal{E}$ and $\mathcal{D}_\star$ preserve ω-polarities, while $\mathcal{D}_\omega$ does not because $\omega \in^+ (\omega \to \varepsilon) \to \varepsilon \simeq \varepsilon \to \varepsilon$ but $\omega \notin^+ \varepsilon \to \varepsilon$. Note that, if an rgm $\mathcal{D}$ preserve ω-polarities, then we also have that $\omega \notin^p \sigma$ and $\sigma \simeq \tau$ entail $\omega \notin^p \tau$ (where $p \in \{+,-\}$).

Proposition 4.2 *Let $\mathcal{A}$ be a partial pair such that, for all $m \in \mathcal{M}_{\mathrm{f}}(A)$ and $\alpha \in A$, $(m,\alpha) \in \mathrm{dom}(j)$ entails that $m \neq []$. Then $\overline{\mathcal{A}}$ preserves ω-polarities.* → proof in tech. app.

Lemma 4.3 *Let $M \in \Lambda$. The following are equivalent:* → proof in tech. app.

(i) *M has a normal form,*

(ii) *there is $a \in \mathrm{BT}(M)^*$ that does not contain $\bot$,*

(iii) *there is $t \in \mathrm{nf}_\beta(\mathcal{T}(M))$ that does not contain the empty bag 1,*

(iv) *in every rgm $\mathcal{D}$ preserving ω-polarities, $\Gamma \vdash^{\mathcal{D}} M : \sigma$ for some environment Γ and type σ such that $\omega \notin^+ \sigma$ and $\omega \notin^- \Gamma$ (that is $\omega \notin^- \Gamma(x)$ for all $x \in \mathrm{Var}$).*

Proof [Sketch] ($i \iff ii$) is trivial and ($ii \iff iii$) follows from Theorem 2.4.

($iii \Rightarrow iv$) One proves by induction on the β-normal t that $\Gamma \vdash t : \sigma$ holds for some Γ, σ such that $\omega \notin^- \Gamma$ and $\omega \notin^+ \sigma$. Then one concludes by subject expansion for Λ^r and the approximation theorem (Theorem 3.10).

($iv \Rightarrow iii$) By the approximation theorem and subject reduction for Λ^r there is $t \in \mathrm{nf}_\beta\mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$ is derivable for some Γ, σ satisfying $\omega \notin^- \Gamma$ and $\omega \notin^+ \sigma$. Then, using Theorem 3.7 and the preservation of ω-polarities, one proves by induction on the structure of normal form of t that it does not contain 1. □

Notice that in the model $\mathcal{D}_\omega$, which does not preserve ω-polarities, the above lemma does not hold. For instance, $\omega \notin^+ \varepsilon \to \varepsilon \in [\![\mathbf{J}]\!]^{\mathcal{D}_\omega}$, but $\mathbf{J}$ is not normalizing.

In Coppo, Dezani and Zacchi's model $\mathcal{D}_{\rm cdz}$ presented in [6], there is an atomic type $\varphi_\star$ (resp. $\varphi_\top$) characterizing the terms having a β-nf (resp. persistent β-nf).

In the model $\mathcal{D}_\star$ the type $\star$ captures those λ-terms $M \in \Lambda^o$ having a normal form that is "linear". A λ-term M is called *linear* whenever: (i) every $y \in \mathrm{fv}(M)$ occurs once in M; (ii) every subterm $\lambda x.N$ of M is such that x occurs once in N.

Lemma 4.4 *Let $M \in \Lambda$ and $\Gamma = x_1 : \star, \dots, x_n : \star$. Then $\Gamma \vdash^{\mathcal{D}_\star} M : \star$ if and only if M has a linear β-normal form and $\mathrm{fv}(\mathrm{nf}_\beta(M)) = \mathrm{dom}(\Gamma)$.* → `proof in tech. app.`

We now prove the main results of the section.

Theorem 4.5 *Let $\mathcal{D}$ be an extensional* rgm *preserving ω-polarities. The following are equivalent (for $M, N \in \Lambda^o$):*

(i) $\mathcal{D} \models M \leq N$,

(ii) $M \sqsubseteq^{\rm nf} N$,

(iii) $\mathrm{BT}^e(M) \subseteq \mathrm{BT}^e(N)$.

Proof ($i \Rightarrow ii$) Suppose $[\![M]\!] \subseteq [\![N]\!]$ and consider a context $C(-)$ such that $C(M)$ has a normal form. By Lemma 4.3 there is $\sigma \in [\![C(M)]\!]$ such that $\omega \notin^+ \sigma$. Since $[\![-]\!]$ is contextual we have $[\![C(M)]\!] \subseteq [\![C(N)]\!]$, therefore $\sigma \in [\![C(N)]\!]$ and, by applying Lemma 4.3 again, we conclude that $C(N)$ has a normal form.

($ii \iff iii$) See Hyland's original paper [11], or [20] for a cleaner proof.

($iii \Rightarrow i$) We have:

$$
\begin{aligned}
[\![M]\!] &= \cup_{M' \twoheadrightarrow_\eta M} [\![M']\!] && \text{by Lemma 3.9}\\
&= \cup_{M' \twoheadrightarrow_\eta M} [\![\mathcal{T}(M')]\!] && \text{by Theorem 3.10}\\
&= \cup_{M' \twoheadrightarrow_\eta M} [\![\mathrm{nf}_\beta \mathcal{T}(M')]\!] && \text{by Theorem 3.8}(ii) \text{ for } \Lambda^r\\
&= \cup_{M' \twoheadrightarrow_\eta M} [\![\mathrm{BT}(M')^*]\!] && \text{by Theorem 2.4}\\
&= \cup_{M' \twoheadrightarrow_\eta M} [\![\mathrm{nf}_\eta \mathrm{BT}(M')^*]\!] && \text{by Lemma 3.9}\\
&= [\![\mathrm{BT}^e(M)]\!] && \text{by definition of } \mathrm{BT}^e(M).
\end{aligned}
$$

Thus $\mathrm{BT}^e(M) \subseteq \mathrm{BT}^e(N)$ entails $[\![M]\!] = [\![\mathrm{BT}^e(M)]\!] \subseteq [\![\mathrm{BT}^e(N)]\!] = [\![N]\!]$. □

Corollary 4.6 (Full abstraction) *Every extensional* rgm *$\mathcal{D}$ respecting ω-polarities has order-theory $\mathrm{Th}_\leq(\mathcal{D}) = \{(M,N) \mid M \sqsubseteq^{\rm nf} N\}$ and λ-theory $\mathrm{Th}(\mathcal{D}) = \mathcal{H}^+$.*

5 Extensional Taylor Expansion and η-Trees

We introduce the notion of *extensional Taylor expansion $\mathcal{T}^\eta(M)$ of a λ-term M* and prove that it is equal to the Taylor expansion of the extensional Böhm tree of M (Theorem 5.15). This result is the analogue of Theorem 2.4. As a byproduct, we obtain a new syntactical characterization of $\equiv^{\rm nf}$ (Corollary 5.17).

For technical reasons, we work with an alternative notion of extensional Böhm tree of M, that will be denoted by $\mathrm{BT}^\eta(M)$. Rather than producing a set of η-normal approximants, $\mathrm{BT}^\eta(-)$ gives an actual (possibly infinite) η-normal tree.

The *η-normal form $\eta(T)$ of a Böhm tree T* is defined coinductively: $\eta(\bot) = \bot$ and

$$\eta\begin{pmatrix} \lambda x_1 \ldots x_n.y \\ T_1 \quad \cdots \quad T_m \end{pmatrix} = \begin{cases} \eta\begin{pmatrix} \lambda x_1 \ldots x_{n-1}.y \\ T_1 \quad \cdots \quad T_{m-1} \end{pmatrix} & \text{If } x_n \notin \mathrm{fv}(yT_1\cdots T_{m-1}) \\ & T_m \in \mathcal{N}\text{, i.e. it is finite} \\ & \text{and } T_m \twoheadrightarrow_\eta x_n, \\ \begin{matrix} \lambda x_1 \ldots x_n.y \\ \eta(T_1) \quad \cdots \quad \eta(T_m) \end{matrix} & \text{otherwise.} \end{cases}$$

Therefore, we define the *Böhm η-tree* $\mathrm{BT}^\eta(M)$ *of a λ-term M* as $\eta(\mathrm{BT}(M))$.

Examples of Böhm η-trees are: $\mathrm{BT}^\eta(\mathbf{J}) = \mathrm{BT}(\mathbf{J})$, $\mathrm{BT}^\eta(\lambda y.xyy) = \lambda y.xyy$, $\mathrm{BT}^\eta(\lambda xy_1y_2.x(\lambda z_1.y_1(\lambda z_2.z_1(\lambda z_3.z_2z_3))y_2) = \mathrm{BT}^\eta(\mathbf{I}) = \mathbf{I}$, and $\mathrm{BT}^\eta(\lambda y.x\bot y) = x\bot$.

The notions of $\mathrm{BT}^\eta(-)$ and $\mathrm{BT}^e(-)$ are equivalent in the sense that, for all $M, N \in \Lambda$, $\mathrm{BT}^e(M) = \mathrm{BT}^e(N)$ if and only if $\mathrm{BT}^\eta(M) = \mathrm{BT}^\eta(N)$ [23,14]. On the other hand, $\mathrm{BT}^e(M) \subseteq \mathrm{BT}^e(N)$ is not equivalent to $\mathrm{BT}^\eta(M) \leq_\bot \mathrm{BT}^\eta(N)$. E.g. $\mathrm{BT}^e(x\bot) \subseteq \mathrm{BT}^e(\lambda y.xyy)$ but $\mathrm{BT}^\eta(x\bot) = x\bot \not\leq_\bot \lambda y.xyy = \mathrm{BT}^\eta(\lambda y.xyy)$.

5.1 Extensional Taylor Expansion

In order to obtain the analogue of Ehrhard and Regnier's Theorem 2.4 in the extensional setting, the extensional Taylor expansion of M should be the η-normal form of $\mathrm{nf}_\beta\mathcal{T}(M)$, just like $\mathrm{BT}^\eta(M)$ is the η-normal form of $\mathrm{BT}(M)$.

The problem is that defining an η-reduction on $\mathcal{P}(\mathrm{nf}_\beta(\Lambda^r))$ is no easy task. Consider for instance the *naïve* definition $\to_\eta = \cup_{k\geq 0}(\to_{\eta\mathrm{k}})$ where $\lambda x.t[x^k] \to_{\eta\mathrm{k}} t$ if $x \notin \mathrm{fv}(t)$. This correctly reduces $\mathcal{T}(\lambda y.xy) = \{\lambda y.x[y^k] \mid k \geq 0\}$ to $\{x\}$, but the fact that $\lambda y.x1 \to_{\eta_0} x$ is a problem, since $\lambda y.x1$ also belongs to $\mathcal{T}(\lambda y.x\Omega)$, whereas $x \notin \mathcal{T}(\mathrm{nf}_\eta(\lambda y.x\Omega)) = \{\lambda y.x1\}$. Similarly, $\lambda y.x1[y]$ as an element of $\mathcal{T}(\lambda y.xzy)$ is supposed to η-reduce to $x1$, while as an element of $\mathcal{T}(\lambda y.xyy)$ should be η-normal.

These examples reveal that, while the β-reduction of $\mathcal{T}(M)$ can be performed *locally* by reducing each term individually, the η-reduction of $\mathrm{nf}_\beta\mathcal{T}(M)$ must be a *global* operation, that considers the whole set of terms before deciding whether a term should reduce or not. Rather than defining an infinitary rewriting system handling countably many terms, we prefer to divide the problem of computing the η-normal form of $\mathcal{T}(M)$ into two phases:

(i) we first define a labeling $\mathcal{L}(-)$ on the terms $t \in \mathcal{T}(M)$ as a global operation annotating on the empty bags 1 occurring in t:

- whether they "come from" a finite η-expansion of some variable y; for instance $\lambda y.x1 \in \mathcal{T}(\lambda y.x(\lambda z.yz))$ should be labeled as $\lambda y.x1_{\eta(y)}$,
- the set of free variables that were forgotten by taking 1 in the Taylor expansion; for instance $\lambda y.x1[y] \in \mathcal{T}(\lambda y.xyy)$ should be labeled as $\lambda y.x1^y[y]$.

(ii) We then define a local reduction $\to_{\eta^\ell}$ on $\mathcal{L}(\mathrm{nf}_\beta\mathcal{T}(M))$ that exploits this extra-information annotated to perform the η-reduction only when it is safe.

The definition of the labeling $\mathcal{L}$ (Definition 5.1) relies on a certain homogeneity exhibited by the structure of the resource terms in $\mathrm{nf}_\beta\mathcal{T}(M)$. As shown in [4], this homogeneity relies on a *definedness relation* $\preceq$ between normal resource terms:

$$\lambda x_1 \dots x_n.y \preceq \lambda x_1 \dots x_n.y \qquad \frac{t \preceq t' \quad b \preceq b'}{tb \preceq t'b'} \qquad 1 \preceq b \qquad \frac{\exists t' \in b'\ \forall t \in b\,,\ t \preceq t'}{b \preceq b'}$$

The relation $\preceq$ is not a preorder since it is transitive, but not reflexive. For instance, $x[y1[y], y[y]1] \not\preceq x[y1[y], y[y]1]$, since $y1[y] \not\preceq y[y]1$ and $y[y]1 \not\preceq y1[y]$. See the discussion after Definition 9 in [4] for more properties of this relation, and examples. Notice that all singletons $\{\lambda x_1 \dots x_n.y\}$ (for $n \geq 0$) are ideals with respect to $\preceq$.

By Lemma 12 in [4], every ideal $\mathbb{S}$ has one of the following shapes: $\{x\}$, $\lambda x.\mathbb{T}$, $\mathbb{T}\mathbb{B}$ for some ideals $\mathbb{T}$ and $\mathbb{B}$. Therefore, the following definition is sound.

Definition 5.1 Let $\mathbb{S} \subseteq \mathrm{nf}_\beta(\Lambda^r)$ be an ideal with respect to $\preceq$ and $t \in \mathbb{S}$. The labeled term $\mathcal{L}(t, \mathbb{S})$ is defined as follows:

$$\mathcal{L}(x, \{x\}) = x, \quad \mathcal{L}(\lambda x.t, \lambda x.\mathbb{T}) = \lambda x.\mathcal{L}(t, \mathbb{T}), \quad \mathcal{L}(tb, \mathbb{T}\mathbb{B}) = \mathcal{L}(t, \mathbb{T})\mathcal{L}(b, \mathbb{B}),$$

$$\mathcal{L}([t_1, \dots, t_k], \mathbb{B}) = [\mathcal{L}(t_1, \textstyle\bigcup \mathbb{B}), \dots, \mathcal{L}(t_k, \textstyle\bigcup \mathbb{B})], \text{ for } k > 0$$

$$\mathcal{L}(1, \mathbb{B}) = \begin{cases} 1^x_{\eta(x)} & \text{if there exists } t' \in \bigcup \mathbb{B} \text{ such that } t' \twoheadrightarrow_{\eta'} x, \\ 1^{\mathrm{fv}(\mathbb{B})} & \text{otherwise.} \end{cases} \qquad (\bullet)$$

where $\to_{\eta'}$ is $\lambda x.t[x^{k+1}] \to_{\eta'} t$ when $x \notin \mathrm{fv}(t)$. We set $\mathcal{L}(\mathbb{S}) = \{\mathcal{L}(t, \mathbb{S}) \mid t \in \mathbb{S}\}$. Given a labelled term t, we write $\ulcorner t \urcorner$ for the term obtained by erasing all its labels.

The labeling $\mathcal{L}(-)$ can be always applied to $\mathrm{nf}_\beta\mathcal{T}(M)$ thanks to the following.

Proposition 5.2 *[4, Lemma 23] Let $M \in \Lambda$. Then $\mathrm{nf}_\beta\mathcal{T}(M)$ is an ideal w.r.t. $\preceq$.*

Remark 5.3 The definition of $\mathcal{L}(t, \mathbb{S})$ will be only used when $\mathbb{S}$ is the β-normal of a Taylor expansion. Under this hypothesis, the case $\mathcal{L}(1, \mathbb{B})$ is applied when $\bigcup \mathbb{B} = \mathcal{T}(M)$ for some β-normal $M \in \Lambda$ and Condition $(\bullet)$ becomes "there is $t \in \mathcal{T}(M)$ such that $t \twoheadrightarrow_{\eta'} x$" which holds exactly when $M \twoheadrightarrow_\eta x$.

For example, for $t = \lambda y.x11$ and $\mathbb{S} = \mathrm{nf}_\beta\mathcal{T}(\lambda y.x\Omega y) = \{\lambda y.x1[y^n] \mid n \geq 0\}$ we have $\mathcal{L}(t, \mathbb{S}) = \lambda y.\mathcal{L}(x, \{x\})\mathcal{L}(1, \{1\})\mathcal{L}(1, \{[y^k] \mid k \geq 0\}) = \lambda y.x1^{\emptyset}1^y_{\eta(y)}$. While $\mathcal{L}(\lambda y.x11, \mathrm{nf}_\beta\mathcal{T}(\lambda y.xyy)) = \lambda y.x1^y_{\eta(y)}1^y_{\eta(y)}$. Thus $\mathcal{L}(\mathcal{T}(\lambda y.xyy)) = \{\lambda y.x1^y_{\eta(y)}1^y_{\eta(y)}\}$ $\cup\{\lambda y.x1^y_{\eta(y)}[y^{n+1}] \mid n \geq 0\} \cup \{\lambda y.x[y^{k+1}]1^y_{\eta(y)} \mid k \geq 0\} \cup \{\lambda y.x[y^{k+1}][y^{n+1}] \mid n, k \geq 0\}$.

The definition of the set $\widetilde{\mathrm{fv}}(t)$ of *free variables of a labeled term t* is analogous to the one of $\mathrm{fv}(t)$, except for the clauses $\widetilde{\mathrm{fv}}(1^x_{\eta(x)}) = \{x\}$ and $\widetilde{\mathrm{fv}}(1^{\vec{x}}) = \{\vec{x}\}$.

Remark 5.4 Given $T = \mathrm{BT}(M)$, $x \in \mathrm{fv}(T)$ iff $x \in \widetilde{\mathrm{fv}}(t)$ for every $t \in \mathcal{L}(\mathcal{T}(T))$.

Definition 5.5 The reduction $\to_{\eta^\ell}$ on labelled β-normal resource terms, is the contextual closure of $\cup_{n \in \mathbb{N}}(\to_{\eta^\ell_n})$ where $\to_{\eta^\ell_n}$ is defined as follows:

$$(\eta^\ell_0)\ \lambda x.t1^x_{\eta(x)} \to_{\eta^\ell_0} t, \text{ if } x \notin \widetilde{\mathrm{fv}}(t), \qquad (\eta^\ell_{n+1})\ \lambda x.t[x^{n+1}] \to_{\eta^\ell_{n+1}} t, \text{ if } x \notin \widetilde{\mathrm{fv}}(t).$$

For example, we have $\mathcal{L}(\lambda y.x1[y], \mathrm{nf}_\beta\mathcal{T}(\lambda y.xzy)) = \lambda y.x1^z_{\eta(z)}[y] \to_{\eta^\ell} x1^z_{\eta(z)}$, while $\mathcal{L}(\lambda y.x1[y], \mathrm{nf}_\beta\mathcal{T}(\lambda y.xyy)) = \lambda y.x1^y_{\eta(y)}y$, which is already η^ℓ-normal.

Lemma 5.6 *The reduction $\to_{\eta^\ell}$ is SN and confluent.*

Proof The reduction $\to_{\eta^\ell}$ is SN since the size of the term decreases. It is moreover weakly confluent, and therefore confluent by Newman's lemma. □

Definition 5.7 The *extensional Taylor expansion* of a λ-term M is given by:

$$\mathcal{T}^\eta(M) = \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathrm{nf}_\beta \mathcal{T}(M))\urcorner$$

In the definition above, β- and η^ℓ-reductions are separated because the reduction $\beta \cup \eta^\ell$ is not confluent: for instance $\lambda x.\mathbf{I}[x,x] \to_{\eta^\ell} \mathbf{I}$ while $\lambda x.\mathbf{I}[x,x] \to_\beta \emptyset$.

5.2 Eta-Reduction on Böhm Approximants

We now provide the technical tools that will be used to prove Theorem 5.15. By Theorem 2.4, it is enough to prove that $\mathcal{T}(\mathrm{BT}^\eta(M))$ is equal to $\ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner$. The difficulty lies in that $\mathrm{BT}^\eta(M)$, which is the η-normal form of $\mathrm{BT}(M)$, is defined coinductively on $\mathrm{BT}(M)$, while the η^ℓ-reduction of $\mathcal{T}(\mathrm{BT}(M))$ works on a set of (labeled) resource terms coming from the finite approximants in $\mathrm{BT}(M)^*$. Therefore, as an intermediate step, we define the η-normal form of the set $\mathrm{BT}(M)^*$ mimicking what we did in Subsection 5.1 for sets of resource terms. In particular, even in this framework the η-reduction must be a global operation; therefore, we introduce a labeling on finite approximants in the spirit of Definition 5.1.

Given $\mathbb{M} \subseteq \mathcal{N}$, $\mathbb{M}\!\downarrow$ denotes its downward closure $\{a \in \mathcal{N} \mid \exists b \in \mathbb{M}, a \leq_\perp b\}$. When $\mathbb{M}$ is an ideal, we have that $\mathbb{M} = \mathbb{M}\!\downarrow$ and all its elements have a similar syntactic structure, except for $\perp$. We adopt for sets $\mathbb{M}$ of approximants the same syntactic sugar we used for $\mathcal{P}(\Lambda^r)$, by extending all the constructors of the grammar of $\mathcal{N}$ as pointwise operations on $\mathcal{P}(\mathcal{N})$. For instance the ideal $\mathrm{BT}(\mathbf{J}x)^*$ can be written as $\{\lambda z_0.x(\mathrm{BT}(\mathbf{J}z_0)^*)\}\!\downarrow = \lambda z_0.x(\mathrm{BT}(\mathbf{J}z_0)^*) \cup \{\perp\}$.

Definition 5.8 Let $\mathbb{M} \subseteq \mathcal{N}$ be an ideal w.r.t. $\leq_\perp$ and $a \in \mathbb{M}$. Define $\mathcal{E}(a, \mathbb{M})$ as:

$$\mathcal{E}(x, \{x\}\!\downarrow) = x, \qquad \mathcal{E}(\lambda x.a, (\lambda x.\mathbb{M})\!\downarrow) = \lambda x.\mathcal{E}(a, \mathbb{M}\!\downarrow),$$

$$\mathcal{E}(ac, (\mathbb{M}\mathbb{N})\!\downarrow) = \mathcal{E}(a, \mathbb{M}\!\downarrow)\mathcal{E}(c, \mathbb{N}),$$

$$\mathcal{E}(\perp, \mathbb{M}) = \begin{cases} \perp^x_{\eta(x)} & \text{if there exists a } \perp\text{-free } a \in \mathbb{M} \text{ such that } a \twoheadrightarrow_\eta x, \quad (\circ) \\ \perp^{\mathrm{fv}(\mathbb{M})} & \text{otherwise.} \end{cases}$$

We extend the definition to $\mathbb{M}$ by setting $\mathcal{E}(\mathbb{M}) = \{\mathcal{E}(a, \mathbb{M}) \mid a \in \mathbb{M}\}$.

Notice that in the case $(\mathbb{M}\mathbb{N})\!\downarrow$ above, the set $\mathbb{N}$ is already downward closed. As $\mathrm{BT}(M)^*$ is an ideal for every $M \in \Lambda$, we can always compute $\mathcal{L}(\mathrm{BT}(M)^*)$. Condition $(\circ)$ is then equivalent to check that $\mathbb{M} = \mathrm{BT}(M')^*$ for some $M' \twoheadrightarrow_\eta x$.

As we did for resource terms, we speak of *labeled approximants* a, we define the set $\widetilde{\mathrm{fv}}(a)$ by adding the clauses $\widetilde{\mathrm{fv}}(\perp^x_{\eta(x)}) = \{x\}$ and $\widetilde{\mathrm{fv}}(\perp^{\vec{x}}) = \{\vec{x}\}$, and we write $\ulcorner a \urcorner$ for the term obtained from a by erasing all its labels.

Remark 5.9 Given $T = \mathrm{BT}(M)$, $x \in \mathrm{fv}(T)$ iff $x \in \widetilde{\mathrm{fv}}(t)$ for every $t \in \mathcal{E}(T^*)$.

Definition 5.10 The reduction $\to_{\eta^e}$ on labeled approximants is defined as:

$$\lambda x.a\perp^x_{\eta(x)} \to_{\eta^e} a, \text{ if } x \notin \widetilde{\mathrm{fv}}(a), \qquad \lambda x.ax \to_{\eta^e} a, \text{ if } x \notin \widetilde{\mathrm{fv}}(a).$$

It is easy to check that also $\to_{\eta^e}$ is strongly normalizing and confluent.

After a technical lemma, we show that the η^e-reduction on $\mathcal{E}(\mathrm{BT}(M))$ computes exactly the finite approximants of the co-inductively defined tree $\mathrm{BT}^\eta(M)$. Given two sets of terms $\mathbb{X}, \mathbb{Y}$ and a reduction $\to_{\mathtt{r}}$ we write $\mathbb{X} \Rightarrow_{\mathtt{r}} \mathbb{Y}$ if for all $t_1 \in \mathbb{X}$ there is $t_2 \in \mathbb{Y}$ such that $t_1 \twoheadrightarrow_{\mathtt{r}} t_2$ and for all $t_2 \in \mathbb{Y}$ there is $t_1 \in \mathbb{X}$ such that $t_1 \twoheadrightarrow_{\mathtt{r}} t_2$.

Lemma 5.11 *Let $T = \lambda\vec{x}y.zT_1\cdots T_{k+1}$ be a Böhm tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(zT_1\cdots T_k)$. Then $\mathcal{E}(T^*) \Rightarrow_{\eta^e} \mathcal{E}((\lambda\vec{x}.zT_1^*\cdots T_k^*)\downarrow)$.* $\to$ proof in tech. app.

Proposition 5.12 *For all $M \in \Lambda$, we have $\mathrm{BT}^\eta(M)^* = \ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner$.* $\to$ proof in tech. app.

Proof [Sketch] One proceeds by co-induction on $\mathrm{BT}(M)$ using Lemma 5.11. □

5.3 A Taylor-Based Characterization of Morris's Equivalence

Now that the technical tools for proving the main result of the section are finally in place, we are able to prove that the extensional Taylor expansion of a λ-term M, actually captures the Taylor expansion of $\mathrm{BT}^\eta(M)$.

We first need the following technical results, then we show a sort of commutation between the η^ℓ-normalization and the Taylor expansion.

Lemma 5.13 *Let $T = \lambda\vec{x}y.zT_1\cdots T_{k+1}$ be a Böhm tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(zT_1\cdots T_k)$. Then $\mathcal{L}(\mathcal{T}(T)) \Rightarrow_{\eta^\ell} \mathcal{L}(\mathcal{T}(\lambda\vec{x}.zT_1\cdots T_k))$.* $\to$ proof in tech. app.

Proposition 5.14 *For all $M \in \Lambda$, $\mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner) = \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner$.* $\to$ proof in tech. app.

Proof [Sketch] By coinduction on $\mathrm{BT}(M)$, applying Lemma 5.13. □

We can finally prove the main result of the section.

Theorem 5.15 *For every λ-term M, $\mathcal{T}^\eta(M) = \mathcal{T}(\mathrm{BT}^\eta(M))$.*

Proof Collecting the results above, we have the following chain of equalities:

$$\begin{aligned}
\mathcal{T}^\eta(M) &= \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathrm{nf}_\beta\mathcal{T}(M))\urcorner && \text{by Definition 5.7}\\
&= \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner && \text{by Theorem 2.4}\\
&= \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^e}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner) && \text{by Prop. 5.14}\\
&= \mathcal{T}(\mathrm{BT}^\eta(M)^*) && \text{by Prop. 5.12}
\end{aligned}$$
□

Corollary 5.16 *For all $M, N \in \Lambda$, we have $\mathrm{BT}^\eta(M)^* \subseteq \mathrm{BT}^\eta(N)^*$ if and only if $\mathcal{T}^\eta(M) \subseteq \mathcal{T}^\eta(N)$.*

Proof ($\Rightarrow$) Let $t \in \mathcal{T}^\eta(M)$. Then there is $a \in \mathrm{BT}^\eta(M)^*$ such that $t \in \mathcal{T}(a)$. Since $\mathrm{BT}^\eta(M)^* \subseteq \mathrm{BT}^\eta(N)^*$, we have that $a \in \mathrm{BT}^\eta(N)^*$. So $t \in \mathcal{T}(\mathrm{BT}^\eta(N))$ and we get from Theorem 5.15 that $t \in \mathcal{T}^\eta(N)$.

($\Leftarrow$) Let $a \in \mathrm{BT}^\eta(M)^*$. Then by Theorem 5.15 $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}^\eta(M)) = \mathcal{T}^\eta(M) \subseteq \mathcal{T}^\eta(N)$. Since $\mathcal{T}^\eta(N) = \mathcal{T}(\mathrm{BT}^\eta(N))$ holds still by Theorem 5.15, we have that $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}^\eta(N))$. From Lemma 2.3 we conclude that $a \in \mathrm{BT}^\eta(N)^*$. □

A further corollary is that the notion of extensional Taylor expansion provides an alternative characterization of Morris's equivalence.

Corollary 5.17 *For $M, N \in \Lambda$, we have $M \equiv^{\mathrm{nf}} N$ if and only if $\mathcal{T}^\eta(M) = \mathcal{T}^\eta(N)$.*

Proof We have the following chain of equivalences: By [23] $M \equiv^{\mathrm{nf}} N$ if and only if $\mathrm{BT}^\eta(M) = \mathrm{BT}^\eta(N)$, that is $\mathrm{BT}^\eta(M)^* = \mathrm{BT}^\eta(N)^*$. By Corollary 5.16 this holds if and only if $\mathcal{T}^\eta(M) = \mathcal{T}^\eta(N)$ does. □

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A Technical Appendix

This technical appendix is devoted to provide some proofs that were omitted or just sketched in the article.

A.1 Omitted proofs of Section 2

Lemma 2.3 *Let $a \in \mathcal{N}$ and $M \in \Lambda$, then $\mathcal{T}(a) \subseteq \mathcal{T}(\mathrm{BT}(M))$ entails $a \in \mathrm{BT}(M)^*$.*

Proof By structural induction on a.

Case $a = \bot$ and $\mathcal{T}(a) = \emptyset \subseteq \mathcal{T}(\mathrm{BT}(M))$. Then it is trivial since $\bot \in \mathrm{BT}(M)$.

Case $a = \lambda\vec{x}.ya_1 \cdots a_k$ and $\mathcal{T}(a) = \bigcup_{n_1,\dots,n_k \geq 0} \lambda\vec{x}.y[\mathcal{T}(a_1)^{n_1}] \cdots [\mathcal{T}(a_k)^{n_k}] \subseteq \mathcal{T}(\mathrm{BT}(M))$. Then $M \twoheadrightarrow_\beta \lambda\vec{x}.yN_1 \cdots N_k$ for some $N_1, \dots, N_k \in \Lambda$ such that $\mathcal{T}(a_i) \subseteq \mathcal{T}(\mathrm{BT}(N_i))$. By induction hypothesis $a_i \in \mathrm{BT}(N_i)^*$ for all $1 \leq i \leq k$ and we conclude that $\lambda\vec{x}.ya_1 \cdots a_k \in \mathrm{BT}(M)^*$. □

A.2 Omitted proofs of Section 3

Theorem 3.10 (Approximation Theorem) *Let M be a λ-term. Then $\Gamma \vdash M : \sigma$ if and only if there exists $t \in \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$.*

Proof ($\Rightarrow$) The proof is by induction on a derivation of $\Gamma \vdash M : \sigma$. We proceed by case analysis on the last rule applied in the derivation.

Case `var`. We have $x : \sigma \vdash x : \sigma$ using the rule (`var`). This case is trivial since $\mathcal{T}(x) = \{x\}$.

Case `lam`. We have $\Gamma \vdash \lambda x.N : \sigma$ using the rule (`lam`). By Theorem 3.7(ii), we have that $\Gamma, x : \mu \vdash N : \tau$ for some $\mu \to \tau \simeq \sigma$. By IH, there exists $t' \in \mathcal{T}(N)$ such that $\Gamma, x : \mu \vdash t' : \tau$. Therefore, $\lambda x.t' \in \mathcal{T}(\lambda x.N)$ and

$$\dfrac{\dfrac{\Gamma, x : \mu \vdash t' : \tau}{\Gamma \vdash \lambda x.t' : \mu \to \tau}\,(\mathtt{lam}) \qquad \mu \to \tau \simeq \sigma}{\Gamma \vdash \lambda x.t' : \sigma}\,(\mathtt{eq})$$

Case `app`. We have $\Gamma \vdash NP : \sigma$ using the rule (`app`). By Theorem 3.7(iii), there is a decomposition $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^n \Gamma_i)$ for some $n \geq 0$, such that $\Gamma_0 \vdash N : \wedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash P : \sigma_i$. By IH, there exists $s \in \mathcal{T}(N)$ such that $\Gamma_0 \vdash s : \wedge_{i=1}^n \sigma_i \to \sigma$, and there exist $t_1, \dots, t_n \in \mathcal{T}(P)$ such that $\Gamma_i \vdash t_i : \sigma_i$.

Therefore we have that $s[t_1, \dots, t_n] \in \mathcal{T}(NP)$ and:

$$\dfrac{\Gamma_0 \vdash s : \wedge_{i=1}^n \sigma_i \to \sigma \qquad \Gamma_i \vdash t_i : \sigma_i \quad \forall i \in \{1, \dots, n\}}{\Gamma \vdash s[t_1, \dots, t_n] : \sigma}$$

Case `eq`. Let $\Gamma \vdash M : \sigma$ using the rule (`eq`). Then $\Gamma \vdash M : \tau$ for some $\tau \simeq \sigma$. By IH there exists $t \in \mathcal{T}(M)$ such that $\Gamma \vdash t : \tau$. By applying (`eq`) we derive $\Gamma \vdash t : \sigma$.

This concludes the left-to-right implication.

($\Leftarrow$) Let $t \in \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$. We proceed by induction on the derivation for such a type assignment.

Case var. We have $x : \sigma \vdash x : \sigma$ and $x \in \mathcal{T}(M)$ which entails $M = x$ by definition of the Taylor expansion. This case is therefore trivial.

Case app. We have $s[t_1, \dots, t_n] \in \mathcal{T}(M)$ such that $\Gamma \vdash s[t_1, \dots, t_n] : \sigma$. By Theorem 3.7(iii)', we get the decomposition $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^n \Gamma_i)$ and the typing assignments $\Gamma_0 \vdash t : \wedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash s_i : \sigma_i$. By definition of Taylor expansion, if $s[t_1, \dots, t_n] \in \mathcal{T}(M)$ then $M = NP$ for some $N, P \in \Lambda$ such that $s \in \mathcal{T}(N)$ and $t_1, \dots, t_n \in \mathcal{T}(P)$. By IH, $\Gamma_0 \vdash N : \wedge_{i=1}^n \sigma_i \to \sigma$ and $\Gamma_i \vdash P : \sigma_i$ for all $i \in \{1, \dots, n\}$. Therefore we derive:

$$\frac{\Gamma_0 \vdash N : \wedge_{i=1}^n \sigma_i \to \sigma \quad \Gamma_i \vdash P : \sigma_i \quad \forall i \in \{1, \dots, n\}}{\Gamma \vdash NP : \sigma} \ (\texttt{app})$$

Case lam. We have $\lambda x.t \in \mathcal{T}(M)$ such that $\Gamma \vdash \lambda x.t : \sigma$. By definition of Taylor expansion, $\lambda x.t \in \mathcal{T}(M)$ entails $M = \lambda x.N$ for some $N \in \Lambda$ such that $t \in \mathcal{T}(N)$. By Theorem 3.7(ii), one gets $\Gamma, x : \mu \vdash t : \tau$ for some $\mu \to \tau \simeq \sigma$. By IH, we have $\Gamma, x : \mu \vdash N : \tau$. Therefore, we can derive

$$\frac{\dfrac{\Gamma, x : \mu \vdash N : \tau}{\Gamma \vdash \lambda x.N : \mu \to \tau} \ (\texttt{lam}) \quad \mu \to \tau \simeq \sigma}{\Gamma \vdash \lambda x.N : \sigma} \ (\texttt{eq})$$

Case eq. Let $t \in \mathcal{T}(M)$ and suppose $\Gamma \vdash t : \sigma$ comes from $\Gamma \vdash t : \tau$ by $(\texttt{eq})$. By IH, we have $\Gamma \vdash M : \tau$. By applying $(\texttt{eq})$ we derive $\Gamma \vdash N : \sigma$. □

A.3 Omitted proofs of Section 4

We recall the definition of "ω occurs positively/negatively in a type σ".

Definition A.1 The relations $\omega \in^p \sigma$ for $p \in \{+, -\}$ are defined as follows:

(i) $\omega \in^- \mu \to \sigma$ for any type σ and intersection μ such that $\mu = \omega$;

(ii) if $\omega \in^p \sigma$ then $\omega \in^p \mu \to \sigma$ for any intersection μ;

(iii) if $\omega \in^p \sigma$ then $\omega \in^{\neg p} \sigma \wedge \mu \to \tau$ for any types σ, τ and intersection μ.

Remark that the condition $\mu = \omega$ in (i) is non-trivial, since equality $=$ between types includes the neutrality of ω. For instance $\omega \in^- \omega \wedge \omega \to \sigma$ as $\omega \wedge \omega = \omega$.

Proposition 4.2 *Let $\mathcal{A}$ be a partial pair such that, for all $m \in \mathcal{M}_{\mathrm{f}}(A)$ and $\alpha \in A$, $(m, \alpha) \in \mathrm{dom}(j)$ entails that $m \neq []$. Then $\overline{\mathcal{A}}$ preserves ω-polarities.*

Proof We perform an induction loading and prove that, for all type $\sigma, \tau \in \mathsf{T}_{\overline{\mathcal{A}}}$ and $p \in \{+, -\}$: if $\omega \in^p \sigma$ and $\tau \simeq^{\overline{\mathcal{A}}} \sigma$ then $\tau^\bullet \notin A$ and $\omega \in^p \tau$. In the rest of the proof we will just write $\simeq$ for $\simeq^{\overline{\mathcal{A}}}$.

We proceed by induction on the definition of $\omega \in^p \sigma$.

Case (i). Suppose that $\omega \in^p \sigma$ because $p = -$ and $\sigma = \omega \to \gamma$, then we need to prove that $\omega \in^- \tau$, for any τ such that $\tau \simeq \sigma$, that is such that $\tau^\bullet = \sigma^\bullet$. By definition, we have:

$$\sigma^\bullet = (\omega \to \gamma)^\bullet = \overline{\jmath}([], \gamma^\bullet) = ([], \gamma^\bullet)$$

where the last equality follows from Definition 3.2 and the hypothesis that $([], \gamma^\bullet) \notin \mathrm{dom}(j)$. From $\tau^\bullet = ([], \gamma^\bullet)$ we get that $\tau^\bullet \notin A$ since A does not contain any pair, and this entails that also τ cannot be atomic.

Suppose therefore $\tau = \mu \to \delta$, then we have $\tau^\bullet = (\mu \to \delta)^\bullet = \bar{j}(\mu^\bullet, \delta^\bullet) = \bar{j}([], \gamma^\bullet) = \sigma^\bullet$. From the injectivity of $\bar{j}$, we get that $\mu^\bullet = []$ and $\delta^\bullet = \gamma^\bullet$, so $\tau = \omega \to \delta$ and $\omega \in^- \tau$.

Case (ii). Suppose that $\omega \in^p \sigma$ because $\sigma = \mu \to \gamma$ and $\omega \in^p \gamma$. Then

$$\sigma^\bullet = (\mu \to \gamma)^\bullet = \bar{j}(\mu^\bullet, \gamma^\bullet) = \tau^\bullet.$$

From $\omega \in^p \gamma$, $\gamma \simeq \gamma$ and the induction hypothesis, we get that $\gamma^\bullet \notin A$ and therefore $(\mu^\bullet, \gamma^\bullet) \notin \mathrm{dom}(j)$. By Definition 3.2, we have that $\bar{j}(\mu^\bullet, \gamma^\bullet) = (\mu^\bullet, \gamma^\bullet)$, and since this is equal to $\tau^\bullet$, we get $\tau = \nu \to \delta$ for some ν, δ. From $\bar{j}(\mu^\bullet, \gamma^\bullet) = \bar{j}(\nu^\bullet, \delta^\bullet)$ and the injectivity of $\bar{j}$ we get that $\mu^\bullet = \nu^\bullet$ and $\gamma^\bullet = \delta^\bullet$.

From $\omega \in^p \gamma$ and $\delta \simeq \gamma$ we get, by induction hypothesis, that $\omega \in^p \delta$ and therefore $\omega \in^p \nu \to \delta = \tau$.

Case (iii). Suppose that $\omega \in^p \sigma$ because $\sigma = \gamma_1 \wedge \mu \to \gamma_2$ and $\omega \in^{\neg p} \gamma_1$. From $\gamma_1 \wedge \mu \to \gamma_2 \simeq \tau$, we get

$$(\gamma_1 \wedge \mu \to \gamma_2)^\bullet = \bar{j}([\gamma_1^\bullet] + \mu^\bullet, \gamma_2^\bullet) = \tau^\bullet.$$

Suppose, by the way of contradiction, that τ is an atomic type α. Then, we have $\bar{j}([\gamma_1^\bullet] + \mu^\bullet, \gamma_2^\bullet) = \alpha$ which implies, by Definition 3.2, that $([\gamma_1^\bullet] + \mu^\bullet, \gamma_2^\bullet) \in \mathrm{dom}(j) \subseteq \mathcal{M}_{\mathrm{f}}(A) \times A$. In particular, we get $\gamma_1^\bullet \in A$, which is impossible since $\omega \in^{\neg p} \gamma_1$ and $\gamma_1 \simeq \gamma_1$, so by the induction hypothesis we conclude that $\gamma_1^\bullet$ is not atomic.

So, $\tau = \nu \to \delta_2$, and $(\nu \to \delta_2)^\bullet = \bar{j}(\nu^\bullet, \delta_2^\bullet) = \bar{j}([\gamma_1^\bullet] + \mu^\bullet, \gamma_2^\bullet)$. Since $\bar{j}$ is injective, $\nu^\bullet = [\gamma_1^\bullet] + \mu^\bullet$ and $\delta_2^\bullet = \gamma_2^\bullet$. Therefore, $\nu = \delta_1 \wedge \nu'$ such that $\delta_1^\bullet = \gamma_1^\bullet$ and $\nu'^\bullet = \mu^\bullet$. Since $\omega \in^{\neg p} \gamma_1$ and $\gamma_1 \simeq \delta_1$, by IH we get $\omega \in^{\neg p} \gamma_1$ and we conclude that $\omega \in^p \tau$. □

For convenience, we present Lemma 4.3 with an additional equivalent sentence (iii-bis), which is an intermediate step between (iii) and (iv).

Lemma 4.3 *Let $M \in \Lambda$. The following are equivalent:*

(i) *M has a normal form,*

(ii) *there is $a \in \mathrm{BT}(M)^*$ that does not contain $\bot$,*

(iii) *there is $t \in \mathrm{nf}_\beta(\mathcal{T}(M))$ that does not contain the empty bag 1,*

(iii-bis) *in every **rgm** $\mathcal{D}$ preserving ω-polarities, $\Gamma \vdash^{\mathcal{D}} t : \sigma$ for some $t \in \mathrm{nf}_\beta \mathcal{T}(M)$, environment Γ and type σ such that $\omega \notin^+ \sigma$ and $\omega \notin^- \Gamma$, that is $\omega \notin^- \Gamma(x)$ for all $x \in \mathrm{Var}$.*

(iv) *in every **rgm** $\mathcal{D}$ preserving ω-polarities, $\Gamma \vdash^{\mathcal{D}} M : \sigma$ for some environment Γ and type σ such that $\omega \notin^+ \sigma$ and $\omega \notin^- \Gamma$, that is $\omega \notin^- \Gamma(x)$ for all $x \in \mathrm{Var}$.*

Proof (*i* $\iff$ *ii*) is trivial.

(*ii* $\iff$ *iii*) follows from Theorem 2.4.

(iii ⇒ iii-bis) We prove that this implication holds more generally for any β-normal form t that does not contain 1 (regardless the fact that t belongs to a Taylor expansion). We proceed by structural induction on t.

Case $t = \lambda x.t'$ where t' is β-normal. By induction hypothesis, $\Gamma' \vdash t' : \tau$ holds for some context Γ' and type τ such that $\omega \notin^- \Gamma'$ and $\omega \notin^+ \tau$. Note that Γ' can be written as $\Gamma, x : \mu$ for some Γ and μ, therefore we can derive:

$$\frac{\Gamma, x : \mu \vdash t' : \tau}{\Gamma \vdash \lambda x.t' : \mu \to \tau}\ (\texttt{lam})$$

From $\omega \notin^- \Gamma'$ we get that $\omega \notin^- \Gamma$ and $\omega \notin^- \mu$, which entails $\omega \notin^+ \mu \to \tau$.

Case $t = yb_1 \cdots b_k$, for some $k \geq 0$, and each $b_i = [s_{i,1}, \ldots, s_{i,n_i}]$ (for $n_i \geq 0$) only contains β-normal terms. By induction hypothesis, there are environments Γ_{ij}, and types τ_{ij}, such that $\omega \notin^- \Gamma_{ij}$ and $\omega \notin^+ \tau_{ij}$ and $\Gamma_{ij} \vdash s_{ij} : \tau_{ij}$ holds. Then we can derive:

$$\frac{\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_k \to \alpha \quad \Gamma_{ij} \vdash s_{ij} : \tau_{ij} \quad i \in \{1, \ldots, k\},\ j \in \{1, \ldots, n_i\}}{\Gamma \vdash yb_1 \cdots b_k : \alpha}$$

where $\mu_i = \wedge_{j=1}^{n_i} \tau_{ij}$, $\Gamma_0 = y : \mu_1 \to \cdots \to \mu_k \to \alpha$ and $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^{k} \wedge_{j=1}^{n_i} \Gamma_{ij})$. As $\omega \notin^+ \tau_{ij}$ we also have $\omega \notin^- \mu_i$ and therefore $\omega \notin^- \Gamma_0$. From this, and the hypotheses that $\omega \notin^- \Gamma_{ij}$ we get that $\omega \notin^- \Gamma$. Of course $\omega \notin^- \alpha$ because α is an atom.

(iii-bis ⇒ iii) Consider $t \in \mathrm{nf}_\beta \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$ where Γ and σ satisfy the hypotheses of (iii-bis). We proceed by induction on the structure of the β-normal t.

Case $t = \lambda x.t'$ where t' is β-normal. By applying Theorem 3.7(ii) we have that $\Gamma, x : \mu \vdash t' : \tau$ holds for $\mu \in \mathsf{I}_\mathcal{D}$ and $\tau \in \mathsf{T}_\mathcal{D}$ such that $\sigma \simeq \mu \to \tau$. Since $\mathcal{D}$ preserves ω-polarities, $\omega \notin^+ \sigma$ entails $\omega \notin^+ \mu \to \tau$. As neither Γ nor μ has negative occurrences of ω, we have $\omega \notin^- (\Gamma, x : \mu)$ and $\omega \notin^+ \tau$, so, by the induction hypothesis, we get that t' does not have occurrences of 1. Therefore 1 does not occur in $\lambda x.t'$ either.

Case $t = yb_1 \cdots b_k$, for some $k \geq 0$, and each $b_i = [s_{i,1}, \ldots, s_{i,n_i}]$ (for $n_i \geq 0$) only contains β-normal terms. If $k = 0$ we are done, as y does not contain 1. Consider then the case $k > 0$. By iterating Theorem 3.7(iii') we know that there is a decomposition $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^{k} \wedge_{j=1}^{n_i} \Gamma_{ij})$ such that (setting $\mu_i = \wedge_{j=1}^{n_i} \tau_{ij}$):

$$\frac{\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_k \to \sigma \quad \Gamma_{ij} \vdash s_{ij} : \tau_{ij} \quad \text{for } i = 1, \ldots, k \quad j = 1, \ldots, n_i}{\Gamma \vdash yb_1 \cdots b_k : \sigma}$$

By Theorem 3.7(i), we get that $\Gamma_0 = y : \tau$ for some $\tau \simeq \mu_1 \to \cdots \to \mu_k \to \sigma$. From this, it follows that $\Gamma(y) = \tau \wedge \mu$ for some μ, so $\omega \notin^- \Gamma$ entails that $\omega \notin^- \tau$ and, as $\mathcal{D}$ preserves ω-polarities, we get that $\omega \notin^- \mu_1 \to \cdots \to \mu_k \to \sigma$. From this, on the one side we get that each μ_i is different from ω (that is, $n_i > 0$, so $b_i \neq 1$) and on the other side that $\omega \notin^+ \tau_{ij}$ holds for $1 \leq i \leq k$ and $1 \leq j \leq n_i$. We can therefore apply the induction hypothesis to each derivation $\Gamma_{ij} \vdash s_{ij} : \tau_{ij}$ and conclude that the terms s_{ij} do not contain 1, so neither does the term $yb_1 \cdots b_k$.

(*iii-bis* $\iff$ *iv*) Let us suppose (iv). By Theorem 3.10, we have that $\Gamma \vdash M : \sigma$ holds if and only if there exists $s \in \mathcal{T}(M)$ such that $\Gamma \vdash s : \sigma$. By Theorem 2.1 (strong normalization of Λ^r) and Theorem 3.8(i) (subject reduction), the latter is equivalent to the existence of $t \in \mathrm{nf}_\beta \mathcal{T}(M)$ such that $\Gamma \vdash t : \sigma$. Therefore, (iv) is equivalent to (iii-bis). □

For proving Lemma 4.4, we need the following remark and technical lemma.

Remark A.2 In the model $\mathcal{D}_\star$, we have that $\sigma \simeq \star$ holds if and only if σ is generated by the following grammar:

$$\gamma ::= \star \mid \gamma \to \gamma$$

In particular, $\mu \to \sigma \simeq \star$ entails that $\mu = \tau$ for some $\tau \simeq \star$ and $\sigma \simeq \star$.

Lemma A.3 *Let $N \in \Lambda$ be a β-normal form. If $\Gamma \vdash N : \sigma$, for some Γ and σ such that $\Gamma(x) \simeq \star$ for all $x \in \mathrm{dom}(\Gamma)$ and $\sigma \simeq \star$, then N is linear and $\mathrm{dom}(\Gamma) = \mathrm{fv}(N)$.*

Proof We proceed by structural induction on N.

Case $N = \lambda x.N'$ where N' is β-normal. From $\Gamma \vdash \lambda x.N' : \sigma$ we get, by Theorem 3.7(ii), that $\Gamma, x : \mu \vdash N' : \tau$ for some μ, τ such that $\mu \to \tau \simeq \sigma$ and, by transitivity of $\simeq$, we get that $\mu \to \tau \simeq \star$ holds. By Remark A.2 this entails $\mu = \gamma$ for some $\gamma \simeq \star$ and $\tau \simeq \star$, therefore we can apply the induction hypothesis and get that N' is linear and $\mathrm{dom}(\Gamma, x : \gamma) = \mathrm{fv}(N')$. Thus, $\lambda x.N'$ is also linear and $\mathrm{dom}(\Gamma) = \mathrm{fv}(N') - \{x\} = \mathrm{fv}(\lambda x.N')$ which is what we are meant to prove.

Case $N = yN_1 \cdots N_k$ such that $N_1, \dots, N_k$ are β-normal. By Theorem 3.7(iii) and there is a decomposition $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^{k} \wedge_{j=1}^{n_i} \Gamma_{ij})$ such that $\Gamma_0 \vdash y : \mu_1 \to \cdots \to \mu_k \to \sigma$ holds for some $\mu_i = \tau_{i1} \wedge \cdots \wedge \tau_{in_i}$ and $\Gamma_{ij} \vdash N_i : \tau_{ij}$ is derivable for all $1 \leq i \leq k$ and $1 \leq j \leq n_i$. By Theorem 3.7(i), $\Gamma_0 = y : \gamma$, for a type $\gamma \simeq \mu_1 \to \cdots \to \mu_k \to \sigma$. As $\Gamma_0(y) = \Gamma(y) = \gamma \simeq \star$ we also have by transitivity of $\simeq$ that $\mu_1 \to \cdots \to \mu_k \to \sigma \simeq \star$ which entails by Remark A.2 that $\mu_i = \tau_i$ (i.e. $n_i = 1$) and $\tau_i \simeq \star$. Therefore we have $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^{k} \Gamma_i)$ and $\Gamma_i \vdash N_i : \tau_i$ for some Γ_i such that $\Gamma_i(x) \simeq \star$ for all $x \in \mathrm{dom}(\Gamma_i)$ and $\tau_i \simeq \star$.

By the induction hypothesis we get that each N_i is linear and $\mathrm{dom}(\Gamma_i) = \mathrm{fv}(N_i)$. We conclude that $yN_1 \cdots N_k$ is linear and $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma_0) \cup (\bigcup_{i=1}^{k} \mathrm{dom}(\Gamma_i)) = \mathrm{fv}(yN_1 \cdots N_k)$. □

Lemma 4.4 *Let $M \in \Lambda$ and $\Gamma = x_1 : \star, \dots, x_n : \star$. Then $\Gamma \vdash^{\mathcal{D}_\star} M : \star$ if and only if M has a linear β-normal form and $\mathrm{dom}(\Gamma) = \mathrm{fv}(\mathrm{nf}_\beta(M))$.*

Proof ($\Rightarrow$) By Theorem 4.2, the `rgm` $\mathcal{D}_\star$ preserves ω-polarities. As ω does not occur positively nor negatively in $\star$, we can deduce by Lemma 4.3 that M has a β-normal form. By subject reduction, we derive $\Gamma \vdash \mathrm{nf}_\beta(M) : \star$ and, by Lemma A.3, we conclude that $\mathrm{nf}_\beta(M)$ is linear.

($\Leftarrow$) Suppose that $M \in \Lambda$ has a linear β-normal form and that the environment $\Gamma = x_1 : \star, \dots, x_n : \star$ is such that $\mathrm{dom}(\Gamma) = \mathrm{fv}(\mathrm{nf}_\beta(M))$. It is enough to prove that $\Gamma \vdash \mathrm{nf}_\beta(M) : \star$ is derivable, then one concludes by subject expansion (Theorem 3.8(i)) that $\Gamma \vdash M : \star$ holds. We proceed by induction on $\mathrm{nf}_\beta(M)$.

Case $\mathrm{nf}_\beta(M) = \lambda x.N'$ where N' is β-normal. Obviously, N' is linear and $\mathrm{dom}(\Gamma, x : \star) = \mathrm{fv}(N')$, so by the induction hypothesis

$$\dfrac{\dfrac{\Gamma, x : \star \vdash N' : \star}{\Gamma \vdash \lambda x.N' : \star \to \star}\,(\mathtt{lam}) \qquad \star \simeq \star \to \star}{\Gamma \vdash \lambda x.N' : \star}\,(\mathtt{eq})$$

is also derivable.

Case $\mathrm{nf}_\beta(M) = yN_1 \cdots N_k$ such that $N_1, \ldots, N_k$ are β-normal. We let Γ_i to be the environment such that $\Gamma_i(x) = \star$ if $x \in \mathrm{fv}(N_i)$ and $\Gamma_i(x) = \omega$ otherwise. As the N_i's are linear, we derive $\Gamma_i \vdash N_i : \star$ by the induction hypothesis. Then we can derive (for $\Gamma_0 = y : \star$):

$$\dfrac{\dfrac{\dfrac{}{\Gamma_0 \vdash y : \star}\,(\mathtt{var}) \qquad \star \simeq \star \to \cdots \to \star \to \star}{\Gamma_0 \vdash y : \star \to \cdots \to \star \to \star}\,(\mathtt{eq}) \qquad \Gamma_i \vdash N_i : \star \quad 1 \le i \le k}{\Gamma_0 \wedge (\wedge_{i=1}^k \Gamma_i) \vdash yN_1 \cdots N_k : \star}\,(\mathtt{lam})$$

To conclude, it is enough to check that $\Gamma = \Gamma_0 \wedge (\wedge_{i=1}^k \Gamma_i)$. □

A.4 Omitted proofs of Section 5

Lemma A.4 *Let $M \in \Lambda$ be a β-normal form such that $M \twoheadrightarrow_\eta x$. For all $a \in M^*$, we have that either $\mathcal{E}(a, M^*) = \bot^x_{\eta(x)}$ or $\mathcal{E}(a, M^*) \twoheadrightarrow_{\eta^e} x$.*

Proof Since M is β-normal, it has the shape $\lambda x_1 \ldots x_n.xN_1 \cdots N_k$. As $M \twoheadrightarrow_\eta x$ we get that $n = m$, $x \neq x_i$ and $N_i \twoheadrightarrow_\beta x_i$ for all $i \in \{1, \ldots, n\}$.

We proceed by induction on a.

Case $a = \bot$. Then $\mathcal{E}(a, M^*) = \bot^x_{\eta(x)}$ by Definition 5.8.

Case $a = \lambda x_1 \ldots x_n.xa_1 \cdots a_n$ with $a_i \in N_i^*$ for all $i \in \{1, \ldots, n\}$. By induction hypothesis, either $\mathcal{E}(a_i, N_i^*) \twoheadrightarrow_{\eta^e} x_i$ or $\mathcal{E}(a_i, N_i^*) \twoheadrightarrow_{\eta^e} \bot^{x_i}_{\eta(x_i)}$ so $\mathcal{E}(a, M^*) = \lambda x_1 \ldots x_n.x\mathcal{E}(a_1, N_1^*) \cdots \mathcal{E}(a_n, N_n^*) \twoheadrightarrow_{\eta^e} x$. □

Lemma 5.11 *Let $T = \lambda\vec{x}y.zT_1 \cdots T_{k+1}$ be a Böhm tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(zT_1 \cdots T_k)$. Then $\mathcal{E}(T^*) \Rightarrow_{\eta^e} \mathcal{E}((\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow)$.*

Proof We first prove that, given $a \in T^*$, there exists $a' \in (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow$ such that $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \mathcal{E}(a', (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow)$. We split into cases depending on a.

Case $a = \bot$. Then $\mathcal{E}(a, T^*) = \mathcal{E}(\bot, T^*) = \mathcal{E}(\bot, (\lambda\vec{x}y.zT_1^* \cdots T_k^* T_{k+1}^*)\!\downarrow)$. From the fact that T_{k+1} is finite, we get that $T_{k+1} \in \mathcal{N}$ and since $T_{k+1} \twoheadrightarrow_\eta y$ we have that T_{k+1} is $\bot$-free. As $y \notin \mathrm{fv}(zT_1 \cdots T_k)$, there is a $\bot$-free $c_1 \in T^*$ such that $c_1 \twoheadrightarrow_\eta z$ if and only if there exists a $\bot$-free $c_2 \in (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow$ such that $c_2 \twoheadrightarrow_\eta z$. Therefore $\mathcal{E}(\bot, (\lambda\vec{x}y.zT_1^* \cdots T_k^* T_{k+1}^*)\!\downarrow) = \mathcal{E}(\bot, (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow)$, so $a' = \bot$.

Case $a = \lambda\vec{x}y.za_1 \cdots a_{k+1}$, with $a_i \in T_i^*$ for $1 \le i \le k+1$. By definition, we have $\mathcal{E}(a, T^*) = \lambda\vec{x}y.z\mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*)\mathcal{E}(a_{k+1}, T_{k+1}^*)$. By hypothesis, T_{k+1} is actually a λ-term (i.e., finite and $\bot$-free) such that $T_{k+1} \twoheadrightarrow_\eta y$ so, by Lemma 5.11, either $\mathcal{E}(a_{k+1}, T_{k+1}) \twoheadrightarrow_{\eta^e} \bot^y_{\eta(y)}$ or $\mathcal{E}(a_{k+1}, T_{k+1}) \twoheadrightarrow_{\eta^e} y$. By Remark 5.9 $y \notin \mathrm{fv}(zT_1 \cdots T_k)$ entails $y \notin \widetilde{\mathrm{fv}}(z\mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*))$, hence in both cases we get $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \lambda\vec{x}.z\mathcal{E}(a_1, T_1^*) \cdots \mathcal{E}(a_k, T_k^*) \in \mathcal{E}((\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow)$. Therefore the a' we were looking for is just $\lambda\vec{x}.za_1 \cdots a_k$.

Second, we prove that for every $a' \in (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow$ there is $a \in T^*$ such that $\mathcal{E}(a, T^*) \twoheadrightarrow_{\eta^e} \mathcal{E}(a', (\lambda\vec{x}.zT_1^* \cdots T_k^*)\!\downarrow)$. Again, we split into cases depending on a'.

Case $a' = \bot$. It is enough to take $a' = \bot$ and reason as above.

Case $a' = \lambda\vec{x}.za'_1 \cdots a'_k$ with $a'_i \in T_i^*$ for all $1 \le i \le k$. Clearly, $\bot \in T_{k+1}^*$ and $\mathcal{E}(\bot, T_{k+1}^*) = \bot^y_{\eta(y)}$, since by hypothesis T_{k+1} is finite and $T_{k+1} \twoheadrightarrow_\eta y$. Therefore, for $a = \lambda\vec{x}y.za'_1 \cdots a'_k\bot \in T^*$ we have

$$\begin{array}{rcll} \mathcal{E}(a,T^*) & = & \lambda\vec{x}y.z\mathcal{E}(a'_1,T^*_1)\cdots\mathcal{E}(a'_k,T^*_k)\mathcal{E}(a'_{k+1},T^*_{k+1}) & \\ & = & \lambda\vec{x}y.z\mathcal{E}(a'_1,T^*_1)\cdots\mathcal{E}(a'_k,T^*_k)\bot^{y}_{\eta(y)} & \\ & \to_{\eta^e} & \lambda\vec{x}.z\mathcal{E}(a'_1,T^*_1)\cdots\mathcal{E}(a'_k,T^*_k) & \text{using Remark 5.9} \\ & = & \mathcal{E}(a',\lambda\vec{x}.zT^*_1\cdots T^*_k). & \end{array}$$

We conclude as $\mathcal{E}(a,T^*)\in\mathcal{E}(T^*)$. □

Lemma A.5 *For all Böhm trees T, we have $\eta(T)^* = \ulcorner \mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*))\urcorner$.*

Proof We proceed by co-induction on T.

If $T=\bot$, then $\eta(T)^* = \{\bot\} = \{\ulcorner\bot^{\emptyset}\urcorner\} = \{\ulcorner\mathcal{E}(\bot,\bot)\urcorner\} = \ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*))\urcorner$.

Otherwise, the Böhm tree T can be written in a unique way as $T = \lambda x_1\ldots x_n y_1\ldots y_m.zT_1\cdots T_kT'_1\cdots T'_m$ (for some $n,m,k\geq 0$) such that:

- $y_i\notin\mathrm{fv}(zT_1\cdots T_k)$, T'_i is finite and $T'_i\twoheadrightarrow_\eta y_i$ for all $i\in\{1,\ldots,m\}$,
- $x_n\in\mathrm{fv}(zT_1\cdots T_k)$ or T_k is infinite, or T_k is finite but does not η-reduce to x_n.

The following equalities hold:

$$\begin{array}{rll} \eta(T)^* & = \lambda\vec{x}.z\eta(T_1)^*\cdots\eta(T_k)^*\cup\{\bot\} & \text{by def. of } \eta(-) \\ & = \lambda\vec{x}.z\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_1))\urcorner\cdots\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_k))\urcorner\cup\{\bot\} & \text{by co-IH} \\ & = \ulcorner\lambda\vec{x}.z\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_1))\cdots\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_k))\urcorner & \\ & \quad\cup\ulcorner\{\mathcal{E}(\bot,(\lambda\vec{x}.zT^*_1\cdots T^*_k)\downarrow)\}\urcorner & \text{by def. of } \ulcorner\cdot\urcorner \\ & = \ulcorner\lambda\vec{x}.z\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_1))\cdots\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*_k)) & \text{by def. of } \ulcorner\cdot\urcorner \\ & \quad\cup\{\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(\bot,(\lambda\vec{x}.zT^*_1\cdots T^*_k)\downarrow)\})\urcorner & \text{and of } \mathrm{nf}_\eta(-) \\ & = \ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}\big(\lambda\vec{x}.z\mathcal{E}(T^*_1)\cdots\mathcal{E}(T^*_n)) & \\ & \quad\cup\{\mathcal{E}(\bot,(\lambda\vec{x}.zT^*_1\cdots T^*_k)\downarrow)\}\big)\urcorner & \text{by def. of } \mathrm{nf}_\eta(-) \\ & = \ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(\lambda\vec{x}.zT^*_1\cdots T^*_k)\downarrow)\urcorner & \text{by def. of } \mathcal{E}(-) \\ & = \ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T^*))\urcorner & \text{by Lemma 5.11.} \end{array}$$ □

Proposition 5.12 *For all $M\in\Lambda$, we have $\mathrm{BT}^\eta(M)^* = \ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner$.*

Proof Since $\mathrm{BT}^\eta(M) = \eta(\mathrm{BT}(M))$, the result follows directly by Lemma A.5. □

Lemma A.6 *Let $M\in\Lambda$ be a β-normal form such that $M\twoheadrightarrow_\eta x$. Then for all $t\in\mathcal{T}(M)$, we have $\mathcal{L}(t,\mathcal{T}(M))\twoheadrightarrow_{\eta^\ell} x$.*

Proof By hypothesis, M has the shape $\lambda x_1\ldots x_n.xM_1\cdots M_n$ (for some $n\geq 0$) such that, for all $i\in\{1,\ldots,n\}$, $x\neq x_i$ and M_i is a β-normal form such that $M_i\twoheadrightarrow_\eta x_i$. We proceed by induction on t. Since $t\in\mathcal{T}(M)$, we have $t=\lambda x_1\ldots x_n.xb_1\cdots b_n$ such that $b_i\in\mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_i))$ for every $1\leq i\leq n$.

If $n = 0$ we are done. Otherwise, by Definition 5.1 we have $\mathcal{L}(t, \mathcal{T}(M)) = \lambda x_1 \dots x_n.x\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_1))) \cdots \mathcal{L}(b_n, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_n)))$.

Suppose $b_n = [t_1, \dots, t_k]$ with $t_j \in \mathcal{T}(M_n)$ for all $j \in \{1, \dots, k\}$.

If $k = 0$ then, by Definition 5.1, $\mathcal{L}(b_n, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_n))) = 1^{x_n}_{\eta(x_n)}$ because $M_n \twoheadrightarrow_\eta x_n$ entails that there is $s \in \bigcup \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_n)) = \mathcal{T}(M_n)$ such that $s \twoheadrightarrow_{\eta'} x_n$. Therefore:

$$\begin{aligned}\mathcal{L}(t, \mathcal{T}(M_n)) &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \dots x_n.x\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_1))) \cdots \mathcal{L}(b_{n-1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_{n-1})))1^{x_n}_{\eta(x_n)} \\ &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \dots x_{n-1}.x\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_1))) \cdots \mathcal{L}(b_{n-1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_{n-1})))\end{aligned}$$

If $k > 0$, then by induction hypothesis $\mathcal{L}(t_{nj}, \mathcal{T}(M_n)) \twoheadrightarrow_{\eta^\ell} x_n$. Therefore,

$$\begin{aligned}\mathcal{L}(t, \mathcal{T}(M_n)) &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \dots x_n.x\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_1))) \cdots \mathcal{L}(b_{n-1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_{n-1})))[x_n^k] \\ &\twoheadrightarrow_{\eta^\ell} \lambda x_1 \dots x_{n-1}.x\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_1))) \cdots \mathcal{L}(b_{n-1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(M_{n-1})))\end{aligned}$$

By iterating this reasoning on $b_1, \dots, b_{n-1}$ we conclude that $\mathcal{L}(t, \mathcal{T}(M)) \twoheadrightarrow_{\eta^\ell} x$. □

Lemma 5.13 *Let $T = \lambda \vec{x}y.zT_1 \cdots T_{k+1}$ be a Böhm tree such that T_{k+1} is finite, $T_{k+1} \twoheadrightarrow_\eta y$ and $y \notin \mathrm{fv}(zT_1 \cdots T_k)$. Then $\mathcal{L}(\mathcal{T}(T)) \Rightarrow_{\eta^\ell} \mathcal{L}(\mathcal{T}(\lambda \vec{x}.zT_1 \cdots T_k))$.*

Proof We first take $t \in \mathcal{T}(T)$, that is $t = \lambda \vec{x}y.zb_1 \cdots b_{k+1}$ with $b_i \in \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_i))$, and show that $\mathcal{L}(t, \mathcal{T}(T)) \twoheadrightarrow_{\eta^\ell} \mathcal{L}(t', \mathcal{T}(\lambda \vec{x}.zT_1 \cdots T_k))$ holds for $t' = \lambda \vec{x}.zb_1 \cdots b_k \in \mathcal{L}(\mathcal{T}(\lambda \vec{x}.zT_1 \cdots T_k))$. By definition of the labeling $\mathcal{L}(-)$, we have $\mathcal{L}(t, \mathcal{T}(T)) = \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_{k+1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1})))$. By Remark 5.4 we have that $y \notin \mathrm{fv}(zT_1 \cdots T_k)$ implies $y \notin \widetilde{\mathrm{fv}}(z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_k, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))$.

Suppose that $b_{k+1} = [t_1, \dots, t_n]$, we split into cases depending on n.

Case $n = 0$. As $T_{k+1} \twoheadrightarrow_\eta y$, then T_{k+1} is $\perp$-free finite tree, and therefore there exists an $s \in \mathcal{T}(T_{k+1})$ without empty bags such that $s \twoheadrightarrow_{\eta'} y$. Hence $\mathcal{L}(b_{k+1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) = \mathcal{L}(1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) = 1^y_{\eta(y)}$ since $s \in \bigcup \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1})) = \mathcal{T}(T_{k+1})$. Therefore we have:

$$\begin{aligned}\mathcal{L}(t, \mathcal{T}(T)) &= \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_{k+1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) \\ &= \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_k, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))1^y_{\eta(y)} \\ &\to_{\eta^\ell} \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_k, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k))) \\ &= \mathcal{L}(\lambda \vec{x}y.zb_1 \cdots b_k, \mathcal{T}(\lambda \vec{x}y.zT_1 \cdots T_k)).\end{aligned}$$

Case $n > 0$. Then $t_i \in \mathcal{T}(T_{k+1})$ for $1 \leq i \leq n$, and $\mathcal{L}(b_{k+1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) = [\mathcal{L}(t_1, \mathcal{T}(T_{k+1})), \dots, \mathcal{L}(t_n, \mathcal{T}(T_{k+1}))]$. Since $T_{k+1} \twoheadrightarrow_\eta y$, then T_{k+1} is a $\perp$-free finite tree (that is a β-normal λ-term), so by Lemma A.6 we have $\mathcal{L}(t_i, \mathcal{T}(T_{k+1})) \twoheadrightarrow_{\eta^\ell} y$ for every $1 \leq i \leq n$. Therefore:

$$\begin{aligned}\mathcal{L}(t, \mathcal{T}(T)) &= \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_{k+1}, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) \\ &\twoheadrightarrow_{\eta^\ell} \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_k, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))[y^n] \\ &\to_{\eta^\ell} \lambda \vec{x}y.z\mathcal{L}(b_1, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))) \cdots \mathcal{L}(b_k, \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))\end{aligned}$$

Second, we take $s \in \mathcal{T}(\lambda\vec{x}.zT_1\cdots T_k)$, i.e. $s = \lambda\vec{x}.zb_1\cdots b_k$ with $b_i \in \mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_i))$, and show that $\mathcal{L}(t,\mathcal{T}(T)) \twoheadrightarrow_{\eta^\ell} \mathcal{L}(s,\mathcal{T}(\lambda\vec{x}.zT_1\cdots T_k))$ for $t = \lambda\vec{x}y.zb_1\cdots b_k 1 \in \mathcal{T}(T)$.

As $T_{k+1} \twoheadrightarrow_\eta y$, then T_{k+1} is $\bot$-free finite tree, and therefore there exists an $s \in \mathcal{T}(T_{k+1})$ without empty bags such that $s \twoheadrightarrow_{\eta'} y$. Thus $\mathcal{L}(1,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) = 1^y_{\eta(y)}$ since $s \in \bigcup\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1})) = \mathcal{T}(T_{k+1})$. Hence, we have:

$$\begin{array}{rcl}
\mathcal{L}(t,\mathcal{T}(T)) & = & \lambda\vec{x}y.z\mathcal{L}(b_1,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1)))\cdots\mathcal{L}(b'_k,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))\mathcal{L}(1,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_{k+1}))) \\
& = & \lambda\vec{x}y.z\mathcal{L}(b_1,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1)))\cdots\mathcal{L}(b_k,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))1^y_{\eta(y)} \\
& \to_{\eta^\ell} & \lambda\vec{x}.z\mathcal{L}(b_1,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1)))\cdots\mathcal{L}(b_k,\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k))) \\
& = & \mathcal{L}(s,\mathcal{T}(\lambda\vec{x}.zT_1\cdots T_k)).
\end{array}$$

This completes the proof. □

Lemma A.7 *For all Böhm tree T the following equality holds:*

$$\mathcal{T}(\ulcorner \mathrm{nf}_{\eta^{\mathsf{e}}}\mathcal{E}(T^*)\urcorner) = \ulcorner \mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(T))\urcorner$$

Proof We proceed by co-induction on T.

If $T = \bot$, then the equality follows because $\mathcal{T}(\bot) = \emptyset$.

Otherwise, the Böhm tree T can be written in a unique way as $T = \lambda x_1\dots x_n y_1\dots y_m.zT_1\cdots T_k T'_1\cdots T'_m$ (for some $n,m,k \geq 0$) such that:

- $y_i \notin \mathrm{fv}(zT_1\cdots T_k)$, T'_i is finite and $T'_i \twoheadrightarrow_\eta y_i$ for all $i \in \{1,\dots,m\}$,
- $x_n \in \mathrm{fv}(zT_1\cdots T_k)$ or T_k is infinite, or T_k is finite but does not η-reduce to x_n.

Therefore, the following equalities hold:

$$\begin{array}{ll}
\mathcal{T}(\ulcorner \mathrm{nf}_{\eta^{\mathsf{e}}}\mathcal{E}(T^*)\urcorner) = \mathcal{T}(\ulcorner \mathrm{nf}_{\eta^{\mathsf{e}}}\mathcal{E}((\lambda x_1\dots x_n.zT_1\cdots T_k)\downarrow)\urcorner) & \text{by Lemma 5.11} \\
= \mathcal{T}(\ulcorner\lambda\vec{x}.z\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_k^*))\urcorner \cup \{\ulcorner\mathcal{E}(\bot,\lambda\vec{x}.zT_1^*\cdots T_k^*)\urcorner\}) & \text{by def. of } \mathcal{E}(-) \\
= \mathcal{T}(\ulcorner\lambda\vec{x}.z\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_1^*))\cdots\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_k^*))\urcorner) \cup \mathcal{T}(\ulcorner\mathcal{E}(\bot,\lambda\vec{x}.zT_1^*\cdots T_k^*)\urcorner) & \text{by def. of } \mathcal{T}(-) \\
= \lambda\vec{x}.z\mathcal{M}_{\mathrm{f}}(\mathcal{T}(\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_1^*))\urcorner))\cdots\mathcal{M}_{\mathrm{f}}(\mathcal{T}(\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_k^*))\urcorner)) \cup \mathcal{T}(\bot) & \text{by def. of } \mathcal{T}(-) \\
= \lambda\vec{x}.z\mathcal{M}_{\mathrm{f}}(\mathcal{T}(\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_1^*))\urcorner))\cdots\mathcal{M}_{\mathrm{f}}(\mathcal{T}(\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}(\mathcal{E}(T_k^*))\urcorner)) & \text{since } \mathcal{T}(\bot) = \emptyset \\
= \lambda\vec{x}.z\mathcal{M}_{\mathrm{f}}(\ulcorner\mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_1)))\urcorner)\cdots\mathcal{M}_{\mathrm{f}}(\ulcorner\mathrm{nf}_{\eta^\ell}(\mathcal{L}(\mathcal{T}(T_k)))\urcorner) & \text{by co-IH} \\
= \ulcorner\mathrm{nf}_{\eta^\ell}(\lambda\vec{x}.z\mathcal{M}_{\mathrm{f}}(\mathcal{L}(\mathcal{T}(T_1)))\cdots\mathcal{M}_{\mathrm{f}}(\mathcal{L}(\mathcal{T}(T_k))))\urcorner & \text{by def. of } \mathrm{nf}_{\eta^\ell} \\
= \ulcorner\mathrm{nf}_{\eta^\ell}\mathcal{L}(\lambda\vec{x}.z\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_1))\cdots\mathcal{M}_{\mathrm{f}}(\mathcal{T}(T_k)))\urcorner & \text{by def. of } \mathcal{L}(-) \\
= \ulcorner\mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\lambda\vec{x}.zT_1\cdots T_k))\urcorner & \text{by def. of } \mathcal{T}(-) \\
= \ulcorner\mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(T))\urcorner & \text{by Lemma 5.13.} \square
\end{array}$$

Proposition 5.14 *For all $M \in \Lambda$, $\mathcal{T}(\ulcorner\mathrm{nf}_{\eta^{\mathsf{e}}}\mathcal{E}(\mathrm{BT}(M)^*)\urcorner) = \ulcorner\mathrm{nf}_{\eta^\ell}\mathcal{L}(\mathcal{T}(\mathrm{BT}(M)))\urcorner$.*

Proof It follows directly from Lemma A.7. □