# The Bang Calculus Revisited 

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#### Abstract

Call-by-Push-Value (CBPV) is a programming paradigm subsuming both Call-by-Name (CBN) and Call-by-Value (CBV) semantics. The essence of this paradigm is captured by the Bang Calculus, a (concise) term language connecting CBPV and Linear Logic.

This paper presents a revisited version of the Bang Calculus, called $\lambda$ !, enjoying some important properties missing in the original formulation. Indeed, the new calculus integrates permutative conversions to unblock value redexes while being confluent at the same time. A second contribution is related to nonidempotent types. We provide a quantitative type system for our $\lambda$ !-calculus, and we show that the length of the (weak) reduction of a typed term to its normal form plus the size of this normal form is bounded by the size of its type derivation. We also explore the properties of this type system with respect to CBN/CBV translations. We keep the original CBN translation from $\lambda$-calculus to the Bang Calculus, which preserves normal forms and is sound and complete with respect to the (quantitative) type system for CBN. However, in the case of CBV, we reformulate both the translation and the type system to restore two main properties: preservation of normal forms and completeness. Last but not least, the quantitative system is refined to a tight one, which transforms the previous upper bound on the length of reduction to normal form plus its size into two independent exact measures for them.


Keywords: Call-by-Push-Value, Bang Calculus, Intersection Types

## 1. Introduction

Call-by-Push-Value. The Call-by-Push-Value (CBPV) paradigm, introduced by P.B. Levy [41, 42], distinguishes between values and computations under the slogan "a value is, a computation does". It subsumes the $\lambda$-calculus

[^0]by adding some primitives that allow to capture both the Call-by-Name (CBN) and Call-by-Value (CBV) semantics. CBN is a lazy strategy that consumes arguments without any preliminary evaluation, potentially duplicating work, while CBV is greedy, always computing arguments disregarding whether they are used or not, which may prevent a normalising term from terminating, e.g. $(\lambda x . I) \Omega$, where $I=\lambda x . x$ and $\Omega=(\lambda x . x x)(\lambda x . x x)$.

Essentially, CBPV introduces unary primitives thunk and force. The former freezes the execution of a term (i.e. it is not allowed to compute under a thunk) while the latter fires again a frozen term. Informally, force (thunk $t$ ) is semantically equivalent to $t$. Resorting to the paradigm slogan, thunk turns a computation into a value, while force does the opposite. Thus, CBN and CBV are captured by conveniently labelling a $\lambda$-term using force and thunk to pause/resume the evaluation of a subterm depending on whether it is an argument (CBN) or a function (CBV). In doing so, CBPV provides a unique formalism capturing two distinct $\lambda$-calculi strategies, thus allowing to study operational and denotational semantics of CBN and CBV in a unified framework.

Bang calculus. T. Ehrhard [26] introduced a typed calculus, that can be seen as a variation of CBPV, to establish a relation between this paradigm and Linear Logic (LL). A simplified version of this formalism is later dubbed Bang calculus [27], showing in particular how CBPV captures the CBN and CBV semantics of $\lambda$-calculus via Girard's translations of intuitionistic logic into LL. A further step in this direction 16] uses Taylor expansion [28] in the Bang Calculus to approximate terms in CBPV. The Bang calculus is essentially an extension of $\lambda$-calculus with two new constructors, namely bang (!) and dereliction (der), together with the reduction rule der $(!t) \mapsto t$. There are two notions of reduction for the Bang calculus, depending on whether it is allowed to reduce under a bang constructor or not. They are called strong and weak reduction respectively. Indeed, it is weak reduction that makes bang/dereliction play the role of the primitives thunk/force. Hence, these modalities are essential to capture the essence behind the CBN-CBV duality. A similar approach appears in [46], studying (simply typed) CBN and CBV translations into a fragment of IS4, recast as a very simple $\lambda$-calculus equipped with an indeterminate lax monoidal comonad.

Non-Idempotent Types. Intersection types, pioneered by [17, 18], can be seen as a syntactical tool to denote programs. They are invariant under the equality generated by the evaluation rules, and type all and only all normalising terms. They were originally defined as idempotent types, so that the equation $\sigma \cap \sigma=\sigma$ holds, thus preventing any use of the intersection constructor to count resources. On the other hand, non-idempotent types, pioneered by [29], are inspired by LL and can be seen as a syntactical formulation of its relational model 31, 12]. This connection suggests a quantitative typing tool, being able to specify properties related to the consumption of resources, a remarkable investigation pioneered by de Carvalho's seminal PhD thesis [20] (see also [22]). Non-idempotent types have also been used to provide characterisations of complexity classes [10]. Several papers explore the qualitative and quantitative aspects of non-idempotent types for different higher order lan-
guages, as for example Call-by-Name, Call-by-Need and Call-by-Value $\lambda$-calculi, as well as extensions to Classical Logic. Some references are [14, 25, 4, 3, 38]. Other relational models were directly defined in the more general context of LL, rather than in the $\lambda$-calculus 21, 34, 24, 23].

An interesting recent research topic concerns the use of non-idempotent types to provide bounds of reduction lengths. More precisely, the size of type derivations has often been used as an upper bound to the length of different evaluation strategies [44, 25, 36, 14, 37, 38]. A key notion behind these works is that when $t$ evaluates to $t^{\prime}$, then the size of the type derivation of $t^{\prime}$ is smaller than the one of $t$, thus the size of type derivations provides an upper bound for the length of the reduction to a normal form as well as for the size of this normal form.

A crucial point to obtain exact bounds, instead of upper bounds, is to consider only minimal type derivations, as the ones in [20, 11, 24]. Another approach was taken in [1], which uses an appropriate notion of tightness to implement minimality, a technical tool adapted to Call-by-Value 32, 3, 40], Call-by-Need 4], pattern-matching languages [7], and control operators 39].

### 1.1. Contributions and Related Works

This article presents a reformulation of the untyped Bang calculus, and proposes a quantitative study of it by means of non-idempotent types.

The Untyped Reduction. The Bang calculus in [26] suffers from the absence of permutative conversions [45, 15], making some redexes syntactically blocked when open terms are considered. A consequence of this approach is that there are some normal forms that are semantically equivalent to non-terminating programs, a situation which is clearly unsound. This is repaired in [27] by adding permutative conversions specified by means of $\sigma$-reduction rules, which are crucial to unveil hidden (value) redexes. However, this approach presents a major drawback since the resulting combined reduction relation is not confluent (Page 6 in 27] or Example 2.3 below).

Our revisited Bang calculus, called $\lambda$ !, fixes these two problems at the same time. Indeed, the syntax is enriched with explicit substitutions, and $\sigma$ equivalence is integrated in the primary reduction system by using the distance paradigm [5], without any need to unveil hidden redexes by means of an independent relation. This approach restores confluence.

The Untyped CBN and CBV Encodings. CBN and CBV (untyped) translations are extensively studied in [33, 19, 43], where the authors establish two encodings $c b n$ and $c b v$, from untyped $\lambda$-terms into untyped terms of the Bang calculus, such that when $t$ reduces to $u$ in CBN (resp. CBV), cbn $(t)$ reduces to $c b n(u)($ resp. $c b v(t)$ reduces to $c b v(u))$ in the Bang calculus. However, CBV normal forms in $\lambda$-calculus are not necessarily translated to normal forms in the Bang calculus.

We extend to explicit substitutions the original CBN translation from $\lambda$ calculus to the Bang calculus, which preserves normal forms, and we reformulate the CBV one in such a way that, in contrast to [33], our CBV translation does preserve normal forms. In order to achieve the preservation of normal forms,
we use a non-compositional CBV translation based on superdevelopments 35, 48], which performs reduction of created redexes during the translation. Our revisited notion of reduction inside the Bang calculus naturally encodes head CBN, i.e. reduction does not take place in arguments of applications, as well as open CBV, i.e. reduction does not take place inside abstractions. More precisely, the $\lambda!$-calculus encodes head CBN and open CBV specified by means of explicit substitutions (see for example [6]). These two notions are dual: head CBN forbids reduction inside arguments, which are translated to bang terms, while open CBV forbids reduction under $\lambda$-abstractions, also translated to bang terms.

The Typed System. Starting from the relational model for the Bang calculus proposed in [33], we propose a type system for the $\lambda!$-calculus, called $\mathcal{U}$, based on non-idempotent intersection types. System $\mathcal{U}$ is able to fully characterise normalisation, in the sense that a term $t$ is $\mathcal{U}$-typable if and only if $t$ is normalising. More interestingly, we show that system $\mathcal{U}$ has also a quantitative flavour, in the sense that the length of any reduction sequence from $t$ to normal form plus the size of this normal form is bounded by the size of the type derivation of $t$. We show that system $\mathcal{U}$ also captures the standard non-idempotent intersection type system $\mathcal{N}$ for CBN, in the sense that a $\lambda$-term $t$ is typable in $\mathcal{N}$ if and only if its translation $\operatorname{cbn}(t)$ is typable in $\mathcal{U}$ with the same type and context. Concerning CBV, we define a new type system $\mathcal{V}$ and we show that our CBV translation enjoys the same property. System $\mathcal{V}$ characterises termination of open CBV, in the sense that $t$ is typable in $\mathcal{V}$ if and only if $t$ is terminating in open CBV. This can be seen as another (collateral) contribution of this article. Moreover, the CBV embedding in 33] is not complete with respect to their type system for CBV. System $\mathcal{V}$ recovers completeness (left as an open question in [33]). Finally, an alternative CBV encoding of typed terms is proposed. This encoding is not only sound and complete, but now enjoys preservation of normal-forms.

A Refinement of the Type System Based on Tightness. A major observation concerning $\beta$-reduction in $\lambda$-calculus (and therefore in the Bang calculus) is that the size of normal forms can be exponentially bigger than the number of steps to these normal forms. This means that bounding the sum of these two integers at the same time is too rough, not very relevant from a quantitative point of view. Following ideas in [20, 11, 1], we go beyond upper bounds. Indeed, another major contribution of this article is the refinement of the nonidempotent type system $\mathcal{U}$ to another type system $\mathcal{E}$, equipped with constants and counters, together with an appropriate notion of tightness (i.e. minimality). This new formulation fully exploits the quantitative aspect of the system, in such a way that upper bounds provided by system $\mathcal{U}$ are refined now into independent exact bounds for time and space. More precisely, we show that a term $t$ admits a tight type derivation with counters $(b, e, s)$ if and only if $t$ is normalisable in $(b+e)$-steps and its normal form has size $s$. Therefore, exact measures concerning the dynamic behaviour of $t$, are extracted from a static (tight) typing property of $t$.

This is a revised and extended version of the authors' article [13].

Road-map. Sec. 2 introduces the $\lambda$ !-calculus. Sec. 3 presents the sound and complete type system $\mathcal{U}$. Sec. 4 discusses (untyped and typed) CBN/CBV translations. In Sec. 5 we refine system $\mathcal{U}$ into system $\mathcal{E}$, and we prove soundness and completeness. Conclusions and future work are discussed in Sec. 6

## 2. The Bang Calculus Revisited

This section presents a revisited (conservative) extension of the original Bang calculi [26, 27], called $\lambda$ !. From a syntactical point of view, we just add explicit substitution operators. From an operational point of view, we use reduction at a distance [5], thus integrating permutative conversions without jeopardising confluence (see the discussion below).

Given a countably infinite set $\mathcal{X}$ of variables $x, y, z, \ldots$ we consider the following grammar for terms, denoted by $\mathcal{T}$, and contexts:

```
            (Terms) \(\quad t, u \quad::=\quad x \in \mathcal{X}|t u| \lambda x . t|!t| \operatorname{der} t \mid t[x \backslash u]\)
    (List Contexts) \(\quad \mathrm{L} \quad::=\square \mid \mathrm{L}[x \backslash t]\)
    (Contexts) \(\quad \mathrm{C} \quad::=\square|\mathrm{C} t| t \mathrm{C}|\lambda x . \mathrm{C}| \mathrm{C}|\operatorname{der} \mathrm{C}| \mathrm{C}[x \backslash u] \mid t[x \backslash \mathrm{C}]\)
(Weak Contexts) \(\mathrm{W}::=\square|\mathrm{W} t| t \mathrm{~W}|\lambda x . \mathrm{W}| \operatorname{der} \mathrm{W}|\mathrm{W}[x \backslash u]| t[x \backslash \mathrm{~W}]\)
```

Terms of the form $t[x \backslash u]$ are closures, and $[x \backslash u]$ is called an explicit substitution (ES). Special terms are $\mathrm{I}=\lambda z . z, \mathrm{~K}=\lambda x \cdot \lambda y \cdot x, \Delta=\lambda x \cdot x!x$, and $\Omega=\Delta!\Delta$. Weak contexts do not allow the symbol $\square$ to occur inside the bang construct. This is similar to weak contexts in $\lambda$-calculus, where $\square$ cannot occur inside $\lambda$-abstractions. We will see in Sec. 4 that weak reduction in the $\lambda!$-calculus perfectly captures head reduction in CBN, disallowing reduction inside arguments, as well as open CBV, disallowing reduction inside abstractions. We use $\mathrm{C}\langle t\rangle$ (resp. $\mathrm{W}\langle t\rangle$ and $\mathrm{L}\langle t\rangle$ ) for the term obtained by replacing the hole $\square$ of $C$ (resp. W and L) by $t$. In order to increase readability we use the following notational conventions: the application is left associative and has higher priority than the $\lambda$-abstraction, so that for instance we may write $\lambda x$.t ur for $\lambda x .((t u) r)$. The unary operators have higher priority than the binary ones, so that for instance $!t u$ reads $(!t) u$. Also, the explicit substitution operator has higher priority than the other binary operators. Nevertheless, we use these notations with parcimony and we add parenthesis only whenever they could be misleading. The notions of free and bound variables are defined as expected, in particular, $\mathrm{fv}(t[x \backslash u]) \stackrel{\text { def }}{=} \mathrm{fv}(t) \backslash\{x\} \cup \mathrm{fv}(u), \mathrm{fv}(\lambda x . t) \stackrel{\text { def }}{=} \mathrm{fv}(t) \backslash\{x\}$, $\mathrm{bv}(t[x \backslash u]) \stackrel{\text { def }}{=} \mathrm{bv}(t) \cup\{x\} \cup \mathrm{bv}(u)$ and $\mathrm{bv}(\lambda x . t) \stackrel{\text { def }}{=} \mathrm{bv}(t) \cup\{x\}$. We extend the standard notion of $\alpha$-conversion [9] to ES, as expected, so that bound variables can always be renamed. Thus e.g. $\lambda x . x={ }_{\alpha} \lambda y . y$ and $x[x \backslash z]={ }_{\alpha} y[y \backslash z]$. We use $t\{x \backslash u\}$ to denote the meta-level substitution operation, i.e. all the free occurrences of the variable $x$ in the term $t$ are replaced by $u$. This operation is defined, as usual, modulo $\alpha$-conversion. We use two special predicates to distinguish abstractions and bang terms possibly affected by a list of explicit substitutions. Indeed, abs $(t)$ holds iff $t=\mathrm{L}\left\langle\lambda x . t^{\prime}\right\rangle$ for some L and $\operatorname{bang}(t)$ holds
iff $t=\mathrm{L}\left\langle!t^{\prime}\right\rangle$ for some L. Finally, we define the following notion of size for terms of the $\lambda$ !-calculus:

Definition 2.1. The w-size of $t \in \mathcal{T}$ is inductively defined as follows:

$$
\begin{array}{llll}
|x|_{\mathrm{W}} & \stackrel{\text { def }}{=} 0 & \left.|t|\right|_{\mathrm{W}} & \stackrel{\text { def }}{\text { def }} 0 \\
|t u|_{\mathrm{W}} & \stackrel{\text { def }}{=} & 1+|t|_{\mathrm{W}}+|u|_{\mathrm{W}} & |\operatorname{der} t|_{\mathrm{W}} \\
\stackrel{\text { def }}{=} 1+|t|_{\mathrm{W}} \\
|\lambda x . t|_{\mathrm{W}} & \stackrel{\text { def }}{=} 1+|t|_{\mathrm{W}} & \mid t\left[\left.x \backslash u\right|_{\mathrm{W}}\right. & \stackrel{\text { def }}{=} 1+|t|_{\mathrm{W}}+|u|_{\mathrm{W}}
\end{array}
$$

The $\lambda!$-calculus is given by the set of terms $\mathcal{T}$ and the (weak) reduction relation $\rightarrow_{\mathrm{w}}$, which is defined as the union of $\rightarrow_{\mathrm{dB}}$ (distant Beta), $\rightarrow_{\mathrm{s}}$ ! (substitute bang) and $\rightarrow_{\mathrm{d}!}$ (distant bang), defined respectively as the closure by weak contexts W of the following rewriting rules:

$$
\begin{array}{rll}
\mathrm{L}\langle\lambda x . t\rangle u & \mapsto_{\mathrm{dB}} & \mathrm{~L}\langle t[x \backslash u]\rangle \\
t[x \backslash \mathrm{~L}\langle!u\rangle] & \mapsto_{\mathrm{s}!} & \mathrm{L}\langle t\{x \backslash u\}\rangle \\
\operatorname{der}(\mathrm{L}\langle!t\rangle) & \mapsto_{\mathrm{d}!} & \mathrm{L}\langle t\rangle
\end{array}
$$

We assume that all these rules avoid capture of free variables.
Example 2.2. Let $t_{0}=\operatorname{der}(!\mathrm{K})(!\mathrm{I})(!\Omega)$. Then,

$$
t_{0} \rightarrow_{\mathrm{d}!} \mathrm{K}(!\mathrm{I})(!\Omega) \rightarrow_{\mathrm{dB}}(\lambda y . x)[x \backslash!\mathrm{I}](!\Omega) \rightarrow_{\mathrm{dB}} x[y \backslash!\Omega][x \backslash!\mathrm{I}] \rightarrow_{\mathrm{s}!} x[x \backslash!\mathrm{I}] \rightarrow_{\mathrm{s}!} \mathrm{I}
$$

Observe that the second dB-step uses action at a distance, where L is $\square[x \backslash!\mathrm{I}]$.
Given the translation of the Bang Calculus into LL proof-nets [26], we refer to dB -steps as m-steps (multiplicative) and $\{\mathrm{s}!$, $\mathrm{d}!\}$-steps as e-steps (exponential).

Observe that reduction is at a distance, in the sense that the list context L in the rewriting rules allows the main constructors involved in these rules to be separated by an arbitrary finite list of substitutions. This new formulation integrates permutative conversions inside the main (logical) reduction rules of the calculus, in contrast to [27] which treats these conversions by means of a set of independent $\sigma$-rewriting rules, thus inheriting many drawbacks. More precisely, in the first formulation of the Bang calculus [26], there are hidden (value) redexes that block reduction, thus creating a mismatch between normal terms that are semantically non-terminating. The second formulation in 27] recovers soundness, by integrating a notion of $\sigma$-equivalence which is crucial to unveil hidden redexes and ill-formed terms (called clashes) 1 . However, adding $\sigma$-reduction to the logical reduction rules does not preserve confluence. Our notion of reduction addresses these two issues at the same time ${ }^{2}$ : it integrates permutative conversions and is confluent (Theorem 2.8).

[^1]Example 2.3. The following example is a simplified version of the one in 27] showing that their calculus is not confluent. There are two permutative conversions involved in this example:

$$
\begin{array}{llll}
((\lambda x . s) r) u & \mapsto_{\sigma_{1}} & (\lambda x . s u) r & x \notin \mathrm{fv}(u) \\
(\lambda y . \lambda x . s) r & \mapsto_{\sigma_{2}} & \lambda x .(\lambda y . s) r & x \notin \mathrm{fv}(r) \cup\{y\}
\end{array}
$$

The term $t=(\lambda x \cdot \lambda y \cdot z)\left(\operatorname{der} x^{\prime}\right)\left(\operatorname{der} y^{\prime}\right)$ contains both a $\sigma_{1}$ and a $\sigma_{2}$ redex:

$$
\begin{aligned}
& t \rightarrow_{\sigma_{1}}\left(\lambda x \cdot(\lambda y \cdot z)\left(\operatorname{der} y^{\prime}\right)\right)\left(\operatorname{der} x^{\prime}\right)=t_{1} \\
& t \rightarrow_{\sigma_{2}}\left(\lambda y \cdot(\lambda x \cdot z)\left(\operatorname{der} x^{\prime}\right)\right)\left(\operatorname{der} y^{\prime}\right)=t_{2}
\end{aligned}
$$

where $\rightarrow_{\sigma_{i}}$ is the closure by weak contexts of $\mapsto_{\sigma_{i}}$ for $i=1,2$. The terms $t_{1}$ and $t_{2}$, which are different, are normal forms in [27], so that confluence is lost. In the $\lambda!$-calculus, however, we have:

$$
t \rightarrow_{\mathrm{dB}}(\lambda y . z)\left[x \backslash \operatorname{der} x^{\prime}\right]\left(\operatorname{der} y^{\prime}\right) \rightarrow_{\mathrm{dB}} z\left[y \backslash \operatorname{der} y^{\prime}\right]\left[x \backslash \operatorname{der} x^{\prime}\right]=t_{0}
$$

where $t_{0}$ is the (unique) normal form of $t$.
We write $\rightarrow_{\mathrm{w}}$ (resp. $\rightarrow_{\mathrm{w}}^{+}$) for the reflexive-transitive (resp. transitive) closure of $\rightarrow_{\mathrm{w}}$. We write $t \rightarrow_{\mathrm{w}}^{(b, e)} u$ if $t \rightarrow_{\mathrm{w}} u$ using $b \mathrm{~dB}$ steps and $e\{\mathrm{~s}!, \mathrm{d}!\}$-steps.

The reduction relation $\rightarrow_{\mathrm{w}}$ enjoys a kind of (weak) diamond property, i.e. onestep divergence can be closed in one step if the diverging terms are different, since $\rightarrow_{\mathrm{w}}$ is not reflexive. Otherwise stated, the reflexive closure of $\rightarrow_{\mathrm{w}}$ enjoys the strong diamond property.

Lemma 2.4. If $t \rightarrow_{p_{1}} t_{1}$ and $t \rightarrow_{p_{2}} t_{2}$ where $t_{1} \neq t_{2}$ and $p_{1}, p_{2} \in\{\mathrm{~dB}, \mathrm{~s}!, \mathrm{d}!\}$, then there exists $t_{3}$ such that $t_{1} \rightarrow_{p_{2}} t_{3}$ and $t_{2} \rightarrow_{p_{1}} t_{3}$.

Proof. The proof is by induction on $t$. To make the notations easier to read, we write $u \mathrm{~L}$ instead of $\mathrm{L}\langle u\rangle$, where $\mathbb{L}$ is the list of substitutions of the context L .

We only show the two interesting cases (root reduction with superposition of redexes), all the other ones being straightforward:

- $t=\left(\left(\lambda x^{\prime} . t^{\prime}\right) \mathbb{L}_{1}\left[x \backslash(!u) \mathbb{L}_{2}\right] \mathbb{L}_{3}\right) u^{\prime} \rightarrow_{\mathrm{dB}} t^{\prime}\left[x^{\prime} \backslash u^{\prime}\right] \mathbb{L}_{1}\left[x \backslash(!u) \mathbb{L}_{2}\right] \mathbb{L}_{3}=t_{1}$ and $t \rightarrow_{\mathrm{s}!}\left(\left(\left(\lambda x^{\prime} . t^{\prime}\right) \mathbb{L}_{1}\right)\{x \backslash u\} \mathbb{L}_{2} \mathbb{L}_{3}\right) u^{\prime}=\left(\left(\lambda x^{\prime} . t^{\prime}\{x \backslash u\}\right)\left(\mathbb{L}_{1}\{x \backslash u\}\right) \mathbb{L}_{2} \mathbb{L}_{3}\right) u^{\prime}=$ $t_{2}$. By $\alpha$-conversion we can assume $x \notin \mathrm{fv}\left(u^{\prime}\right)$ so that by defining

$$
t_{3}=t\{x \backslash u\}\left[x^{\prime} \backslash u^{\prime}\right]\left(\mathbb{L}_{1}\{x \backslash u\}\right) \mathbb{L}_{2} \mathbb{L}_{3}=t\left[x^{\prime} \backslash u^{\prime}\right]\{x \backslash u\}\left(\mathbb{L}_{1}\{x \backslash u\}\right) \mathbb{L}_{2} \mathbb{L}_{3}
$$

we can close the diagram as follows: $t_{1} \rightarrow_{\mathrm{s}}$ ! $t_{3}$ and $t_{2} \rightarrow_{\mathrm{dB}} t_{3}$.

- $t=\operatorname{der}\left(\left(!u^{\prime}\right) \mathbb{L}_{1}\left[x \backslash(!u) \mathbb{L}_{2}\right] \mathbb{L}_{3}\right) \rightarrow_{\mathrm{d}}!u^{\prime} \mathbb{L}_{1}\left[x \backslash(!u) \mathbb{L}_{2}\right] \mathbb{L}_{3}=t_{1}$ and also $t \rightarrow_{\mathrm{s}}$ ! $\operatorname{der}\left(\left(!u^{\prime}\right) \mathbb{L}_{1}\{x \backslash u\} \mathbb{L}_{2} \mathrm{~L}_{3}\right)=t_{2}$. We close the diagram with $t_{1} \rightarrow_{\mathrm{s}} t_{3}$ and $t_{2} \rightarrow_{\mathrm{d}!} t_{3}$, where $t_{3}=\left(u^{\prime} \mathbb{L}_{1}\right)\{x \backslash u\} \mathbb{L}_{2} \mathbb{L}_{3}=u^{\prime}\{x \backslash u\} \mathbb{L}_{1}\{x \backslash u\} \mathbb{L}_{2} \mathbb{L}_{3}$.

The result above does not hold if reductions are allowed inside arbitrary contexts. Consider for instance the term $t=(x!x)[x \backslash!(I!I)]$. We have $t \rightarrow_{s}$ ! (I!I)! (I! I) and, if we allow the reduction in $t$ of the dB-redex I! I appearing banged inside the explicit substitution, we get $t \rightarrow_{\mathrm{dB}}(x!x)[x \backslash!z[z \backslash!\mathrm{I}]$. Now, the term $(x!x)[x \backslash!z[z \backslash!\mathrm{I}]]$ s!-reduces to $z[z \backslash!\mathrm{I}]!(z[z \backslash!\mathrm{I}])$, whereas two dBreductions are needed in order to close the diamond, i.e. to rewrite (I! I) ! (I! I) into $z[z \backslash!I]!(z[z \backslash!\mathrm{I}])$.

It is possible to w-reduce two different redexes of a term in such a way that the same reduct (modulo $\alpha$-conversion) is obtained. For instance, if $t=$ $x[y \backslash!u][z \backslash!u]$ then $t \rightarrow_{\mathrm{w}} x[y \backslash!u]$ for the $W$-context $\square$, and $t \rightarrow_{\mathrm{w}} x[z \backslash!u]$ for the W-context $\square[z \backslash!u]$. Nevertheless, in such a case the reduction rules must be the same, as we shall establish in Lemma 2.6. As a preliminary result, we prove that no term can w-reduce to itself in one w-step.

Lemma 2.5. For all $t \in \mathcal{T}, t \not 力_{\mathrm{w}} t$.
Proof. We prove by induction on $t$ that if $t \rightarrow_{\mathrm{w}} t^{\prime}$ then $t \neq t^{\prime}$.

- If $t=x$ then the statement holds vacuously.
- If $t=\lambda x$.r or $t=!r$, then the reduction $t \rightarrow_{\mathrm{w}} t^{\prime}$ takes place inside $r$ and we conclude by the $i . h$.
- If $t=$ der $r$ and the reduction $t \rightarrow_{\mathrm{w}} t^{\prime}$ takes place inside $r$ then we conclude by the $i . h$. If the reduction $t \rightarrow_{\text {w }} t^{\prime}$ takes place at the root then it must be the case that $r=\mathrm{L}\langle!s\rangle$ and $t^{\prime}=\mathrm{L}\langle s\rangle$, so that $t \neq t^{\prime}$.
- If $t=r s$ then we reason as in the previous case.
- If $t=u[x \backslash s]$, then let us write $t=r\left[x_{1} \backslash s_{1}\right] \ldots\left[x_{n} \backslash s_{n}\right]$ in such a way that:
$-n \geq 1$
$-r$ is not an explicit substitution.
- for all $1 \leq i \leq n, x_{i} \notin \mathrm{fv}\left(s_{i}\right) \cup \mathrm{bv}\left(s_{i}\right)$.

As in the previous cases, the $i . h$. settles the cases in which the reduction $t \rightarrow_{\mathrm{w}} t^{\prime}$ takes place inside $r$ or inside $s_{i}$, for $1 \leq i \leq n$. The only remaining cases to consider are the s!-steps of the form

$$
\begin{array}{rll}
t & \rightarrow_{\mathrm{s}!} & \mathrm{L}\left\langle r\left[x_{1} \backslash s_{1}\right] \ldots\left[x_{j-1} \backslash s_{j-1}\right]\left\{x_{j} \backslash s^{\prime}\right\}\right\rangle\left[x_{j+1} \backslash s_{j+1}\right] \ldots\left[x_{n} \backslash s_{n}\right] \\
& = & r^{\prime}\left[y_{1} \backslash u_{1}\right] \ldots\left[y_{m} \backslash u_{m}\right]=t^{\prime}
\end{array}
$$

where $s_{j}=\mathrm{L}\left\langle!s^{\prime}\right\rangle$, for some $\mathrm{L}, s^{\prime}$ and $1 \leq j \leq n$, and where $r^{\prime}$ is not an explicit substitution. Let $l$ be the length of the list of explicit substitutions L. If $l>1$ then $m>n$, so that $t^{\prime} \neq t$.

- If $l=0$ and $r\left\{x_{j} \backslash s^{\prime}\right\}$ is not an explicit substitution, then $m=n-1$, so that $t^{\prime} \neq t$. If $l=0$ and $r\left\{x_{j} \backslash s^{\prime}\right\}$ is an explicit substitution, then it must be the case that $r=x_{j}$. In this case, $t$ starts with a free
occurrence of $x_{j}$ whereas $t^{\prime}$ starts with $s^{\prime}$, and by hypothesis $s^{\prime}$ does not contain free occurrences of $x_{j}$, being $s^{\prime}$ a subterm of $s_{j}$. Hence $t^{\prime} \neq t$.
- If $l=1$, then $s_{j}=\left(!s^{\prime}\right)[z \backslash u]$ for some $z, u$. If $m \neq n$ then $t^{\prime} \neq t$ and we are done, otherwise $u_{j}=u$, and, in order to have $t=t^{\prime}$ it should be the case that $u=s_{j}$. But $u$ is a proper subterm of $s_{j}$, so that $t \neq t^{\prime}$.

As a matter of fact, a weaker version of Lemma 2.5 stating that one-step cycles are impossible using dB or d ! only, is sufficient to prove Lemma 2.6, However, the stronger version presented here provides more insight on the $\lambda!$ calculus.

Lemma 2.6. If $t \rightarrow_{p_{1}} t^{\prime}$ and $t \rightarrow_{p_{2}} t^{\prime}$, then $p_{1}=p_{2}$, where $p_{1}, p_{2} \in\{\mathrm{~dB}, \mathrm{~s}!, \mathrm{d}!\}$.
Proof. We prove by induction on $t$ that if $t \rightarrow_{p_{1}} t_{1}, t \rightarrow_{p_{2}} t_{2}$ and $p_{1} \neq p_{2}$ then $t_{1} \neq t_{2}$.

- If $t=x$ then the statement holds vacuously.
- If $t=\lambda x$.r or $t=!r$ then both reductions must take place inside $r$, and the $i . h$. allows us to conclude.
- If $t=\operatorname{der} r$ and both reductions take place in $r$, then the $i . h$. allows us to conclude. Otherwise it must be the case that $r=\mathrm{L}\left\langle!r^{\prime}\right\rangle$, so that $t \rightarrow_{p_{1}} t_{1}$ is, say, $t \rightarrow_{\mathrm{d}!} \mathrm{L}\left\langle r^{\prime}\right\rangle$. If $\mathrm{L}\left\langle!r^{\prime}\right\rangle$ is not an explicit substitution, then no reduction $t \rightarrow p_{2} t_{2}$ with $p_{1} \neq p_{2}$ is possible since w-reductions do not take place inside a bang. Otherwise $t_{1}$ is an explicit substitution whereas $t_{2}=\operatorname{der} \mathrm{L}^{\prime}\left\langle!r^{\prime}\right\rangle$, for some $\mathrm{L}\left\langle!r^{\prime}\right\rangle \rightarrow_{p 2} \mathrm{~L}^{\prime}\left\langle!r^{\prime}\right\rangle$, is a dereliction so that $t_{1} \neq t_{2}$.
- If $t=r s$ and the reductions take place either both in $r$ or both in $s$, then the $i . h$. allows us to conclude. If $r s \rightarrow_{p_{1}} r^{\prime} s$ and $r s \rightarrow_{p_{2}} r s^{\prime}$, then Lemma 2.5 allows us to conclude. Otherwise it must be the case that $r=\mathrm{L}\left\langle\lambda x \cdot r^{\prime}\right\rangle$, so that $t \rightarrow_{p_{1}} t_{1}$ is, say, $t \rightarrow_{\mathrm{dB}} \mathrm{L}\left\langle r^{\prime}[x \backslash s]\right\rangle$, and $t_{1}$ is an explicit substitution. Now, the $p_{2}$-step must take place either in $r$ or in $s$, hence $t_{2}$ is an application, so that $t_{1} \neq t_{2}$.
- If $t=r[x \backslash s]$ and the reductions take place either both in $r$ or both in $s$, then the $i . h$. allows us to conclude. If $r[x \backslash s] \rightarrow_{p_{1}} r\left[x \backslash s^{\prime}\right]$ and $r[x \backslash s] \rightarrow_{p_{2}}$ $r^{\prime}[x \backslash s]$, then Lemma 2.5 allows us to conclude. Otherwise it must be the case that $s=\mathrm{L}\left\langle!s^{\prime}\right\rangle$, so that $t \rightarrow_{p_{1}} t_{1}$ is, say, $t \rightarrow_{s} \mathrm{~L}\left\langle r\left\{x \backslash s^{\prime}\right\}\right\rangle$. Let us write $t=r^{\prime}\left[x_{1} \backslash s_{1}\right] \ldots\left[x_{n} \backslash s_{n}\right][x \backslash s]$ in such a way that:
$-n \geq 0$
$-r^{\prime}$ is not an explicit substitution.
- for all $1 \leq i \leq n, x_{i} \notin \mathrm{fv}\left(s_{i}\right) \cup \mathrm{bv}\left(s_{i}\right)$.

Thus, $t_{1}=\mathrm{L}\left\langle r^{\prime}\left[x_{1} \backslash s_{1}\right] \ldots\left[x_{n} \backslash s_{n}\right]\left\{x \backslash s^{\prime}\right\}\right\rangle$. In the rest of the proof, we note $t \rightarrow p_{2} t_{2}$ the second w-reduction of $t$. Since $p_{1}=\mathrm{s}$ ! and $p_{1} \neq p_{2}, t \rightarrow p_{2} t_{2}$ is a dB-step or a d!-step taking place either in $r^{\prime}$, or in one of the $s_{i}$, $1 \leq i \leq n$, or in $s$. We observe first that if $r^{\prime}=x$ then $t_{2}$ starts with a free occurrence of $x$ and $t_{1}$ starts with $s^{\prime}$, where $x$ cannot occur free, so that $t_{1} \neq t_{2}$. It remains to consider the case $r^{\prime} \neq x$. As a matter of terminology, let us say that a term $u$ has exactly $n$ explicit substitutions if $u=u^{\prime}\left[z_{1} \backslash u_{1}\right] \ldots\left[z_{n} \backslash u_{n}\right]$ and $u^{\prime}$ is not an explicit substitution. In the case $r^{\prime} \neq x$ the term $t_{1}$ has exactly $n+m$ explicit substitutions, where $m$ is the length of L , and the term $t_{2}$ has at least $n+1$ explicit substitutions. Hence, if L is empty, then $t_{1} \neq t_{2}$. It remains to consider the case $\mathrm{L}=\mathrm{L}^{\prime}[w \backslash u]$ for some list of explicit substitutions $\mathrm{L}^{\prime}$, variable $w$ and term $u$. In this case the outermost explicit substitution of $t_{1}$ is $[w \backslash u]$. We conclude the proof by showing that in this case the outermost explicit substitution $\left[w^{\prime} \backslash u^{\prime}\right]$ of $t_{2}$ is such that $\|u\|<\left\|u^{\prime}\right\|$, where $\|t\|$ is the size of the syntactic tree of $t$ (all nodes count ons ${ }^{3}$ ). If the reduction $t \rightarrow_{p_{2}} t_{2}$ takes place outside the leftmost occurrence of $u$ then $u^{\prime}$ is of the form $\mathrm{L}_{1}\left\langle!s^{\prime}\right\rangle$ with $\mathrm{L}_{1}=\mathrm{L}_{2}[w \backslash u]$, hence $\|u\|<\left\|u^{\prime}\right\|$. If the reduction $t \rightarrow_{p_{2}} t_{2}$ takes place in the leftmost occurrence of $u$, then $u^{\prime}$ is of the form $\mathrm{L}_{1}\left\langle!s^{\prime}\right\rangle$ and $\mathrm{L}_{1}=\mathrm{L}^{\prime}\left[w \backslash u^{\prime \prime}\right]$, with $u \rightarrow_{\mathrm{dB}} u^{\prime \prime}$ or $u \rightarrow_{\mathrm{d}!} u^{\prime \prime}$. By remarking that $\left\|u^{\prime \prime}\right\|+2 \geq\|u\|$ (simple inspection of the rules), and that $\left\|u^{\prime}\right\| \geq\left\|u^{\prime \prime}\right\|+3$ we get $\|u\|<\left\|u^{\prime}\right\|$, in this case too, and we are done.

In the proof of the following lemma, pairs of natural numbers are used to decorate reduction sequences. More precisely, by writing $t \rightarrow{ }_{\mathrm{w}}^{(b, e)} u$ we mean that $t$ w-reduces to $u$ using $b$ multiplicative steps and $e$ exponential steps. We use the following order on pairs: $(a, b) \prec\left(a^{\prime}, b^{\prime}\right)$ iff $a<a^{\prime}$ and $b \leq b^{\prime}$, or $a \leq a^{\prime}$ and $b<b^{\prime}$. Moreover, we use the operation + on pairs to denote the pairwise addition.

Lemma 2.7. If $t \rightarrow{ }_{\mathrm{w}}^{c_{1}} u_{1}$ and $t \rightarrow{ }_{\mathrm{w}}^{c_{2}} u_{2}$, then there exists a term $t^{\prime}$ and pairs $\overline{c_{1}}, \overline{c_{2}}$ such that $u_{1} \rightarrow{ }_{\mathrm{w}}^{\overline{c_{2}}} t^{\prime}$, and $u_{2} \rightarrow{ }_{\mathrm{w}}^{\overline{c_{1}}} t^{\prime}$, where $\overline{c_{i}} \preceq c_{i}$ with $i=1,2$ and $c_{1}+\overline{c_{2}}=c_{2}+\overline{c_{1}}$.

Proof. In this proof we use the notation $q_{i}$ for pairs of the form $(1,0)$ or $(0,1)$.
The proof of the lemma is by induction on $c_{1}+c_{2}$. The cases $c_{1}=(0,0)$ or $c_{2}=(0,0)$ (including the base case) are all trivial. So let us suppose $c_{1} \succ(0,0)$
 with $i=1,2$.

[^2]If $t_{1}=t_{2}$, then $q_{1}=q_{2}$ by Lemma 2.6. We have $t_{1} \rightarrow_{\mathrm{w}}^{c_{1}^{\prime}} u_{1}$ and $t_{1} \rightarrow{ }_{\mathrm{w}}^{c_{2}^{\prime}} u_{2}$. Since $c_{1}^{\prime}+c_{2}^{\prime} \prec c_{1}+c_{2}$, then by the $i$.h. there exists $t^{\prime}$ such that $u_{1} \rightarrow{ }_{\mathrm{w}}^{c_{2}^{\prime}} t^{\prime}$, $u_{2} \rightarrow \frac{c_{1}^{\prime}}{\prime} t^{\prime}$ where $c_{1}^{\prime}+\overline{c_{2}^{\prime}}=c_{2}^{\prime}+\overline{c_{1}^{\prime}}$, and $\overline{c_{i}^{\prime}} \preceq c_{i}^{\prime}$ with $i=1,2$. Let us set $\overline{c_{i}}=\overline{c_{i}^{\prime}}$ with $i=\underline{1,2}$. We conclude since $c_{1}+\overline{c_{2}}=c_{1}^{\prime}+q_{1}+\overline{c_{2}^{\prime}}=q_{1}+c_{2}^{\prime}+\overline{c_{1}^{\prime}}=c_{2}+\overline{c_{1}}$ and $\overline{c_{i}}=\overline{c_{i}^{\prime}} \preceq c_{i}^{\prime} \prec c_{i}$ with $i=1,2$.

If $t_{1} \neq t_{2}$, then by Lemma 2.4 there exists $t_{3}$ such that $t_{1} \rightarrow_{\mathrm{w}}^{q_{2}} t_{3}$ and $t_{2} \rightarrow_{\mathrm{w}}^{q_{1}} t_{3}$. We now have $t_{2} \rightarrow_{\mathrm{w}}^{q_{1}} t_{3}$ and $t_{2} \rightarrow_{\mathrm{w}}^{c_{2}^{\prime}} u_{2}$. Since $c_{2}^{\prime}+q_{1} \prec c_{1}+c_{2}$, then the $i . h$. gives $t_{4}$ such that $u_{2} \rightarrow{ }_{\mathrm{W}}^{\overline{q_{1}}} t_{4}, t_{3} \rightarrow{ }_{\mathrm{w}}^{c_{2}^{\prime}} t_{4}, c_{2}^{\prime}+\overline{q_{1}}=q_{1}+\overline{c_{2}^{\prime}}, \overline{q_{1}} \preceq q_{1}$ and $\overline{c_{2}^{\prime}} \preceq c_{2}^{\prime}$. We now have $t_{1} \rightarrow \rightarrow_{\mathrm{w}}^{\prime} u_{1}$ and $t_{1} \rightarrow{ }_{\mathrm{w}}^{q_{2}}+\overline{c_{2}^{\prime}} t_{4}$. In order to apply the $i . h$. we need $c_{1}^{\prime}+q_{2}+\overline{c_{2}^{\prime}} \prec c_{1}+c_{2}$. Indeed, $c_{1}^{\prime}+q_{2}+\overline{c_{2}^{\prime}} \prec c_{1}+c_{2}=q_{1}+\underline{c_{1}^{\prime}+q_{2}}+c_{2}^{\prime}$
 $t_{4} \rightarrow{ }_{\mathrm{w}} \overline{c_{1}^{\prime}} t^{\prime}$, where $c_{1}^{\prime}+\overline{\overline{c_{2}^{\prime}}+q_{2}}=q_{2}+\overline{c_{2}^{\prime}}+\overline{c_{1}^{\prime}}, \overline{c_{1}^{\prime}} \preceq c_{1}^{\prime}$ and $\overline{\overline{c_{2}^{\prime}}+q_{2}} \preceq \overline{c_{2}^{\prime}}+q_{2}$. We can then conclude since $u_{2} \rightarrow{ }_{\mathrm{w}}^{\overline{q_{1}}+\overline{c_{1}^{\prime}}} t^{\prime}$. We have

- $\overline{\overline{c_{2}^{\prime}}+q_{2}} \preceq \overline{c_{2}^{\prime}}+q_{2} \preceq c_{2}^{\prime}+q_{2}$.
- $\overline{q_{1}}+\overline{c_{1}^{\prime}} \preceq q_{1}+c_{1}^{\prime}$.
- $q_{1}+c_{1}^{\prime}+\overline{\overline{c_{2}^{\prime}}+q_{2}}=q_{1}+q_{2}+\overline{c_{2}^{\prime}}+\overline{c_{1}^{\prime}}=q_{2}+\overline{c_{1}^{\prime}}+c_{2}^{\prime}+\overline{q_{1}}$.

This gives the following two major results.

## Theorem 2.8.

- The reduction relation $\rightarrow_{\mathrm{w}}$ is confluent.
- Any two different reduction paths to w-normal form have the same length.

Proof. The two statements follow from Lemma 2.7.
Example 2.9. Let us illustrate the previous result with $t=(x!x)\left[x \backslash \mathrm{I}^{\prime}\left(\mathrm{I}^{\prime}!\mathrm{I}\right)\right]$, where $\mathrm{I}^{\prime}=\lambda x .!x$. For that, let us first consider the following reduction

$$
\mathrm{I}^{\prime}!\mathrm{I} \rightarrow_{\mathrm{dB}}(!x)[x \backslash!\mathrm{I}] \rightarrow_{\mathrm{s}!}!\mathrm{I}
$$

Observe that the second step is a weak step, since the reduction takes place at the root of the term. On the contrary, notice that the step $!(x[x \backslash!\mathrm{I}]) \rightarrow_{\mathrm{s}!}!\mathrm{I}$ is not a weak step since the reduction takes place under a bang.

Similarly,

$$
\mathrm{I}!\mathrm{I} \rightarrow_{\mathrm{dB}} x[x \backslash!\mathrm{I}] \rightarrow_{\mathrm{s}!} \mathrm{I}
$$

Thus, $I^{\prime}!\mathrm{I} \rightarrow{ }_{\mathrm{w}}^{(1,1)}$ ! I and I! $\mathrm{I} \rightarrow \underset{\mathrm{w}}{(1,1)} \mathrm{I}$.

Coming back now to the original term $t$, the following essentially different weak reductions from $t$ to its normal form I have both length 7 :

$$
\begin{array}{rll}
t & \rightarrow_{\mathrm{w}}^{(1,1)} & (x!x)\left[x \backslash \mathrm{I}^{\prime}!\mathrm{I}\right] \rightarrow_{\mathrm{w}}^{(1,1)}(x!x)[x \backslash!\mathrm{I}] \rightarrow_{\mathrm{s}!} \mathrm{I}!\mathrm{I} \rightarrow_{\mathrm{w}}^{(1,1)} \mathrm{I} \\
t & \rightarrow_{\mathrm{dB}}^{\mathrm{dB}}(1,1) \\
\rightarrow_{\mathrm{w}}^{(1,1)} & (x!x)\left[x \backslash(!z)\left[z \backslash \mathrm{I}^{\prime}!\mathrm{I}\right]\right] \rightarrow_{\mathrm{s}!}(z!z)\left[z \backslash \mathrm{I}^{\prime}!\mathrm{I}\right] \\
(z!z)[\mathrm{I}] \rightarrow_{\mathrm{s}!} \mathrm{I}!\mathrm{I} \rightarrow_{\mathrm{w}}^{(1,1)} \mathrm{I}
\end{array}
$$

As explained above, the strong property expressed in the second item of Theorem 2.8 and illustrated in Example 2.9 relies essentially on the fact that reductions are disallowed under bangs. Observe the important role of the Lcontext $\square\left[z \backslash \mathrm{I}^{\prime}!I\right]$ in the second step of the last reduction sequence.

Normal forms and neutral terms. A term is said to be w-normal if there is no $t^{\prime}$ such that $t \rightarrow_{\mathrm{w}} t^{\prime}$, in which case we write $t \nrightarrow_{\mathrm{w}}$. This notion can be characterised by means of the following inductive grammars:

$$
\left.\begin{array}{rlll}
\text { (Neutral) } & \mathrm{ne}_{\mathrm{w}} & ::=x \in \mathcal{X}\left|\mathrm{na}_{\mathrm{w}} \mathrm{no}_{\mathrm{W}}\right| \operatorname{der}\left(\mathrm{nb}_{\mathrm{w}}\right) \mid \mathrm{ne}_{\mathrm{w}}\left[x \backslash \mathrm{nb}_{\mathrm{w}}\right] \\
\text { (Neutral-Abs) } & \mathrm{na}_{\mathrm{w}} & ::=\mathrm{l}=\mathrm{t}\left|\mathrm{ne}_{\mathrm{w}}\right| \mathrm{na}_{\mathrm{w}}\left[x \backslash \mathrm{nb}_{\mathrm{w}}\right] \\
\text { (Neutral-Bang) } & \mathrm{nb}_{\mathrm{w}} & ::= & \mathrm{ne} \\
\mathrm{w}
\end{array}\left|\lambda x . \mathrm{no}_{\mathrm{w}}\right| \mathrm{nb}_{\mathrm{w}}\left[x \backslash \mathrm{nb}_{\mathrm{w}}\right]\right\}
$$

As we shall see ( $c f$. Proposition (2.11), all these terms are w-normal. Moreover, neutral terms do not produce any kind of redexes when inserted into a context, while neutral-abs terms (resp. neutral-bang) may only produce s! or d! redexes (resp. dB redexes) when inserted into a context.

Remark 2.10. Some immediate properties of the sets of terms defined above are:

- $\mathrm{ne}_{\mathrm{w}}=\mathrm{na} \mathrm{w}_{\mathrm{w}} \cap \mathrm{nb}_{\mathrm{w}}$
- $\mathrm{no}_{\mathrm{w}}=n \mathrm{a}_{\mathrm{w}} \cup \mathrm{nb}_{\mathrm{w}}$
- for all terms $t, t \in \mathrm{na}_{\mathrm{w}}$ implies $t \in \mathrm{ne}_{\mathrm{w}}$ or bang $(t)$.
- for all terms $t, t \in \mathrm{nb}_{\mathrm{w}}$ implies $t \in \mathrm{ne}_{\mathrm{w}}$ or $\operatorname{abs}(t)$.
- for all terms $t, t \in \mathrm{na}_{\mathrm{w}}$ and $t \notin \mathrm{nb}_{\mathrm{w}}$ implies bang $(t)$.
- for all terms $t, t \in \mathrm{nb}_{\mathrm{w}}$ and $t \notin \mathrm{na}_{\mathrm{w}}$ implies abs $(t)$.

Proposition 2.11 (Normal Forms). For all $t \in \mathcal{T}, t \not 力_{\mathrm{w}}$ iff $t \in \mathrm{no}_{\mathrm{w}}$.
Proof. We prove simultaneously the following statements:
(a) $t \in \mathrm{ne}_{\mathrm{w}} \quad \Leftrightarrow \quad t \nrightarrow_{\mathrm{w}}$ and $\neg \operatorname{abs}(t)$ and $\neg \operatorname{bang}(t)$.
(b) $t \in \mathrm{na}_{\mathrm{w}} \quad \Leftrightarrow \quad t \not \not_{\mathrm{w}}$ and $\neg \mathrm{abs}(t)$.
(c) $t \in \mathrm{nb}_{\mathrm{w}} \quad \Leftrightarrow \quad t \not \nrightarrow \mathrm{w}$ and $\neg \operatorname{bang}(t)$.
(d) $t \in \mathrm{no}_{\mathrm{w}} \Leftrightarrow t \not \nrightarrow \mathrm{w}$.

The implication $\Rightarrow$ is proved by induction on the predicate $t \in \mathrm{no}_{\mathrm{w}}$. In all cases, it is sufficient to reason by a simple case analysis of the grammars defining the sets $n e_{w}, n a_{w}, n b_{w}$ and $n o_{w}$, using the suitable induction hypothesis.

The implication $\Leftarrow$ is proved by induction on $t$. As for $\Leftarrow$ (a): w-normal terms neither of the form $\mathrm{L}\left\langle!t^{\prime}\right\rangle$ nor $\mathrm{L}\left\langle\lambda x . t^{\prime}\right\rangle$ have necessarily one of the following shapes:

- $x$
- der $t$, where $t$ is normal and $\neg \operatorname{bang}(t)$.
- $t u$, where $t, u$ are normal and $\neg \operatorname{abs}(t)$.
- $t[x \backslash u]$, where $t, u$ are normal, $\neg \operatorname{abs}(t), \neg \operatorname{bang}(t)$ and $\neg \operatorname{bang}(u)$.

Each case is settled by using the corresponding case in the definition of ne $\mathrm{e}_{\mathrm{w}}$ and the suitable induction hypothesis. The same holds for $\Leftarrow$ (b), $\Leftarrow$ (c) and $\Leftarrow$ (d).

Clashes. Some ill-formed terms are not redexes but they don't represent a desired result for a computation either. They are called clashes (meta-variable c), and defined as follows:

$$
\mathrm{L}\langle!t\rangle u \quad t[y \backslash \mathrm{~L}\langle\lambda x . u\rangle] \quad \operatorname{der}(\mathrm{L}\langle\lambda x . u\rangle) \quad t(\mathrm{~L}\langle\lambda x . u\rangle)
$$

Observe that in the three first kind of clashes, replacing $\lambda x$. by !, and inversely, creates a (root) redex, namely $(\mathrm{L}\langle\lambda x . t\rangle) u, t[x \backslash \mathrm{~L}\langle!t\rangle]$ and $\operatorname{der}(\mathrm{L}\langle!t\rangle)$, respectively. In the fourth kind of clash, however, this is not the case since $t(\mathrm{~L}\langle!u\rangle)$ is not a redex in general.

A term is clash-free if it does not reduce to a term containing a clash, it is weakly clash-free, written wcf, if it does not reduce to a term containing a clash outside the scope of any constructor !. In other words, $t$ is not wcf if and only if there exist a weak context W and a clash $c$ such that $t \rightarrow{ }_{\mathrm{w}} \mathrm{W}\langle c\rangle$.

Weakly clash-free normal terms can be characterised as follows:

$$
\begin{aligned}
& \text { (Neutral wcf) } \quad \text { ne } \text { wcf } \quad:=x \in \mathcal{X}\left|\operatorname{ne}_{\text {wcf }} \mathrm{na}_{\text {wcf }}\right| \operatorname{der}\left(\mathrm{ne}_{\text {wcf }}\right) \mid \mathrm{ne}_{\text {wcf }}\left[x \backslash \mathrm{ne}_{\text {wcf }}\right] \\
& \text { (Neutral-Abs wcf) } n a_{\text {wcf }} \quad::=\quad!t\left|n e_{\text {wcf }}\right| n \operatorname{na}_{\text {wcf }}\left[x \backslash \text { ne }_{\text {wcf }}\right] \\
& \text { (Neutral-Bang wcf) } \quad n b_{\text {wcf }} \quad::=n e_{\text {wcf }} \mid \lambda x . \text { no }_{\text {wcf }} \mid n b_{\text {wcf }}\left[x \backslash \mathrm{ne}_{\text {wcf }}\right] \\
& \text { (Normal wcf) } \quad n 0_{\text {wcf }} \quad::=n a_{\text {wcf }} \mid n b_{\text {wcf }}
\end{aligned}
$$

Intuitively, $n o_{w c f}$ denotes $n o_{w} \cap{ }_{w c f}$ (respectively for $n e_{w c f}$, $n a_{w c f}$ and $n b_{w c f}$ ).
Proposition 2.12 (Clash-free normal forms). Let $t \in \mathcal{T}$. Then $t$ is a weakly clash-free normal form iff $t \in \mathrm{no}_{\text {wcf }}$.
Proof. Similar to Proposition 2.11.

## 3. The Type System $\mathcal{U}$

This section introduces a first type system $\mathcal{U}$ for our revisited version of the Bang calculus, which extends the one in 33] to explicit substitutions. We show in this paper that $\mathcal{U}$ does not only qualitatively characterise normalisation, but is also quantitative, in the sense that the length of the (weak) reduction of a typed term to its normal form plus the size of this normal form is bounded by the size of its type derivation. We also explore in Sec. 4 the properties of this type system with respect to the CBN and CBV translations.

Given a countable infinite set $\mathcal{T} \mathcal{V}$ of base types $\alpha, \beta, \gamma, \ldots$, we define the following sets of types:

$$
\begin{array}{rrll}
\text { (Types) } & \sigma, \tau & ::= & \alpha \in \mathcal{T V}|\mathrm{M}| \mathrm{M} \rightarrow \sigma \\
\text { (Multiset Types) } & \mathrm{M} & ::= & {\left[\sigma_{i}\right]_{i \in I} \text { where } I \text { is a finite set }}
\end{array}
$$

Multiset types will be indistinctly written as M or $\left[\sigma_{i}\right]_{i \in I}$, in both cases they denote a finite multiset of types. The empty multiset type is denoted by []. Also, $|\mathrm{M}|$ denotes the size of the multiset, thus if $\mathrm{M}=\left[\sigma_{i}\right]_{i \in I}$ then $|\mathrm{M}|=\#(I)$.

Two multiset types are identified if they are related by the relation $\equiv$, defined by induction on types as follows:

- $\alpha \equiv \alpha$,
- $\mathrm{M}_{1} \rightarrow \sigma_{1} \equiv \mathrm{M}_{2} \rightarrow \sigma_{2}$ if $\mathrm{M}_{1} \equiv \mathrm{M}_{2}$ and $\sigma_{1} \equiv \sigma_{2}$,
- $\left[\sigma_{i}\right]_{i \in I} \equiv\left[\tau_{j}\right]_{j \in J}$ if $|I|=|J|$ and there is a bijection function $f$ between $I$ and $J$ such that $\sigma_{i} \equiv \tau_{f(i)}$ for all $i \in I$.

Typing contexts (or just contexts), written $\Gamma, \Delta$, are functions from variables to multiset types, assigning the empty multiset to all but a finite set of variables. The support of $\Gamma$ is given by $\operatorname{supp}(\Gamma) \stackrel{\text { def }}{=}\{x \mid \Gamma(x) \neq[]\}$. The empty context is the context with an empty support. The union of contexts, written $\Gamma+\Delta$, is defined by $(\Gamma+\Delta)(x) \stackrel{\text { def }}{=} \Gamma(x) \sqcup \Delta(x)$, where $\sqcup$ denotes multiset union. An example is $(x:[\sigma], y:[\tau])+(x:[\sigma], z:[\tau])=(x:[\sigma, \sigma], y:[\tau], z:[\tau])$. This notion is extended to several contexts as expected, so that $+_{i \in I} \Gamma_{i}$ denotes a finite union of contexts (when $I=\emptyset$ the notation is to be understood as the empty context). We write $\Gamma \Downarrow x$ for the context $(\Gamma \rrbracket x)(x)=[]$ and $(\Gamma \backslash x)(y)=\Gamma(y)$ if $y \neq x$. Contexts can be compared as follows $\Gamma \subseteq \Delta$ iff $\Gamma(x) \sqsubseteq \Delta(x)$ for every variable $x$, where $\sqsubseteq$ is multiset inclusion.

Type judgements have the form $\Gamma \vdash t: \sigma$, where $\Gamma$ is a typing context, $t$ is a term and $\sigma$ is a type. The type system $\mathcal{U}$ for the $\lambda$ !-calculus is given in Figure 1.

The axiom ( ax ) is relevant (there is no weakening) and the rules (app) and (es) are multiplicative. Note that the argument of a bang is typed $\#(I)$ times by the premises of rule (bg). A particular case is when $I=\emptyset$ : the subterm $t$ occurring in the typed term $!t$ turns out not to be typed (we often say that $t$ is untyped in this case).

A (type) derivation is a tree obtained by applying the (inductive) typing rules of system $\mathcal{U}$. The notation $\triangleright \mathcal{U} \Gamma \vdash t: \sigma$ means there is a derivation of

$$
\begin{aligned}
& \frac{\Gamma \vdash[\sigma] \vdash x: \sigma}{x}(\mathrm{ax}) \quad \frac{\Gamma \vdash t: \sigma \quad \Delta \vdash u: \Gamma(x)}{(\Gamma \Downarrow x)+\Delta \vdash t[x \backslash u]: \sigma}(\mathrm{es}) \quad \frac{\Gamma \vdash t: \tau}{\Gamma \Downarrow x \vdash \lambda x . t: \Gamma(x) \rightarrow \tau}(\mathrm{abs}) \\
& \frac{\Gamma \vdash t: \mathrm{M} \rightarrow \tau \quad \Delta \vdash u: \mathrm{M}}{\Gamma+\Delta \vdash t u: \tau}(\mathrm{app}) \quad \frac{\left(\Gamma_{i} \vdash t: \sigma_{i}\right)_{i \in I}}{+_{i \in I} \Gamma_{i} \vdash!t:\left[\sigma_{i}\right]_{i \in I}}(\mathrm{bg}) \quad \frac{\Gamma \vdash t:[\sigma]}{\Gamma \vdash \operatorname{der} t: \sigma}(\mathrm{dr})
\end{aligned}
$$

Figure 1: System $\mathcal{U}$ for the $\lambda$ !-calculus.
the judgement $\Gamma \vdash t: \sigma$ in system $\mathcal{U}$. The term $t$ is typable in system $\mathcal{U}$, or $\mathcal{U}$-typable, iff there are $\Gamma$ and $\sigma$ such that $\triangleright \mathcal{U}^{\Gamma} \vdash t: \sigma$. We use the capital Greek letters $\Phi, \Psi, \ldots$ to name type derivations, by writing for example $\Phi \triangleright \mathcal{U} \Gamma \vdash t: \sigma$. The size of the derivation $\Phi$, denoted by $\mathrm{sz}(\Phi)$, is defined as the number of rules in the type derivation $\Phi$ except rule (bg), which does not count. Note in particular that, given a derivation $\Phi_{t}$ for a term $t$, we always have sz $\left(\Phi_{t}\right) \geq|t|_{\mathrm{w}}$, as $\left.\left.\right|_{-}\right|_{w}$ does not count in turn subterms prefixed by a bang constructor.

Example 3.1. The following tree $\Phi_{0}$ is a type derivation for term $t_{0}$ of Example 2.2 .

Note that sz $\left(\Phi_{0}\right)=8 \geq 3=\left|t_{0}\right|_{\mathrm{w}}$, the normal form of $t_{0}$ is I with size 1 , and $t_{0}$ reduces to I in 5 steps. We will see in Theorem 3.9 that the size of a derivation $\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \sigma$ is always an upper bound of the w-size of the w-normal form of $t$ plus the length of the reduction of $t$ to its w-normal form.

The typability of a term may provide additional information about the neutrality/normality of its subterms:

Lemma 3.2. Let $u \in \mathcal{T}$ :

1. If $t \in \mathrm{na}_{\mathrm{w}}$ and $t u$ is $\mathcal{U}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
2. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $u[x \backslash t]$ is $\mathcal{U}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
3. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $\operatorname{der} t$ is $\mathcal{U}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
4. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $u t$ is $\mathcal{U}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
5. If $t \in \mathrm{no}_{\mathrm{w}}$ and $u t$ is $\mathcal{U}$-typable, then $t \in \mathrm{na}_{\mathrm{w}}$.

Proof. Straightforward case analysis using the characterisation of w-normal forms in Proposition 2.11

The quantitative aspect of system $\mathcal{U}$ is materialised in the following weighted subject reduction (WSR) and expansion (WSE) properties. As usual, a substitution lemma must be proved.

Lemma 3.3 (Substitution). If $\Phi_{t} \triangleright \mathcal{U} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash t: \tau$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{U}} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}$, then there exists $\Phi_{t\{x \backslash u\}} \triangleright_{\mathcal{U}} \Gamma+_{i \in I} \Delta_{i} \vdash t\{x \backslash u\}: \tau$ such that sz $\left(\Phi_{t\{x \backslash u\}}\right)=$ $\mathrm{sz}\left(\Phi_{t}\right)+_{i \in I} \mathrm{sz}\left(\Phi_{u}^{i}\right)-|I|$.
Proof. By induction on $\Phi_{t}$. If $\Phi_{t}$ is (ax) and $t=x$, then $t\{x \backslash u\}=u$ and $\Phi_{t}$ is of the form $x:[\sigma] \vdash x: \sigma$, so that $\Gamma=\emptyset$ and $I=\left\{i_{0}\right\}$ and $\tau=\sigma$. We let $\Phi_{t\{x \backslash u\}}=\Phi_{u}^{i_{0}}$. We conclude since $\mathrm{sz}\left(\Phi_{t}\right)=1$ and $|I|=1$. If $\Phi_{t}$ is (ax) and $t=y \neq x$, then $t\{x \backslash u\}=y, \Gamma=y:[\tau]$, and $I=\emptyset$. We let $\Phi_{t\{x \backslash u\}}=\Phi_{y}$. We conclude since $|I|=0$.

If $\Phi_{t}$ ends with (app), then $t=t_{1} t_{2}, \Gamma=\Gamma_{1}+\Gamma_{2}$ and there exist a type $M$ and two derivations $\Phi_{t_{1}} \triangleright \mathcal{U} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash t_{1}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}} \triangleright \mathcal{U} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash t_{2}: \mathrm{M}$ such that $I=I_{1} \uplus I_{2}$. Using the i.h. on $\Phi_{t_{i}}$ and $\left(\Phi_{u}^{j}\right)_{j \in I_{i}}$, for $i=1,2$, we get two derivations $\Phi_{t_{1}\{x \backslash u\}} \triangleright \mathcal{U} \Gamma_{1}+{ }_{i \in I_{1}} \Delta_{i} \vdash t_{1}\{x \backslash u\}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{U}}$ $\Gamma_{2}+_{i \in I_{2}} \Delta_{i} \vdash t_{2}\{x \backslash u\}:$ M such that sz $\left(\Phi_{t_{i}\{x \backslash u\}}\right)=\mathbf{s z}\left(\Phi_{t_{i}}\right)+_{j \in I_{i}} \mathbf{s z}\left(\Phi_{u}^{j}\right)-\left|I_{i}\right|$, for $i=1,2$. By observing that $t\{x \backslash u\}=t_{1}\{x \backslash u\} t_{2}\{x \backslash u\}$ and by using (app), we get a derivation $\Phi_{t\{x \backslash u\}} \triangleright \mathcal{U} \Gamma+_{i \in I} \Delta_{i} \vdash t\{x \backslash u\}: \tau$ such that $\mathrm{sz}\left(\Phi_{t\{x \backslash u\}}\right)=$ $\left(\mathbf{s z}\left(\Phi_{t_{1}}\right)+_{j \in I_{1}} \mathbf{s z}\left(\Phi_{u}^{j}\right)-\left|I_{1}\right|\right)+\left(\mathbf{s z}\left(\Phi_{t_{2}}\right)+_{j \in I_{2}} \mathbf{s z}\left(\Phi_{u}^{j}\right)-\left|I_{2}\right|\right)+1=\left(\mathbf{s z}\left(\Phi_{t_{1}}\right)+\right.$ $\left.\left.\mathrm{sz}\left(\Phi_{t_{2}}\right)+1\right)+_{i \in I} \mathrm{sz}\left(\Phi_{u}^{i}\right)-\left(\left|I_{1}\right|\right)+\left|I_{2}\right|\right)=\mathrm{sz}\left(\Phi_{t}\right)+_{i \in I} \mathrm{sz}\left(\Phi_{u}^{i}\right)-|I|$.

All the other cases proceed similarly by the i.h.
Lemma 3.4 (Weighted Subject Reduction). Let $\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau$. If $t \rightarrow_{\mathrm{w}} t^{\prime}$, then there is $\Phi^{\prime} \triangleright_{\mathcal{U}} \Gamma \vdash t^{\prime}: \tau$ such that $\mathrm{sz}(\Phi)>\operatorname{sz}\left(\Phi^{\prime}\right)$.
Proof. By induction on $t \rightarrow_{\mathrm{w}} t^{\prime}$.

- For the base cases we have to consider three rules:
- Rule dB. Then $t=\mathrm{L}\langle\lambda x . s\rangle u$ and $t^{\prime}=\mathrm{L}\langle s[x \backslash u]\rangle$. We proceed by induction on L .
$* \mathrm{~L}=\square$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ s.t. $\Gamma^{\prime}(x)=\mathrm{M}$ and

$$
\frac{\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash s: \tau}{\Gamma^{\prime} \backslash x \vdash \lambda x . s: \mathrm{M} \rightarrow \tau}(\mathrm{abs}) \quad \Phi_{u} \triangleright_{\mathcal{U}} \Delta \vdash u: \mathrm{M}}{\Phi \triangleright_{\mathcal{U}} \Gamma \vdash(\lambda x . s) u: \tau}(\mathrm{app})
$$

and we conclude with

$$
\frac{\Phi_{s} \triangleright \mathcal{U} \Gamma^{\prime} \vdash s: \tau \quad \Phi_{u} \triangleright \mathcal{U} \Delta \vdash u: \mathrm{M}}{\Phi^{\prime} \triangleright \mathcal{U} \Gamma \vdash s[x \backslash u]: \tau}(\mathrm{es})
$$

Note that sz $(\Phi)=\mathbf{s z}\left(\Phi^{\prime}\right)+1$.

$$
\begin{gathered}
* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r] . \text { Then } \Gamma=\left(\left(\Gamma^{\prime} \backslash y\right)+\Delta^{\prime}\right)+\Delta \text { s.t. }\left(\left(\Gamma^{\prime} \backslash y\right)+\right. \\
\left.\Delta^{\prime}\right)(x)=\mathrm{M} \text { and } \Gamma^{\prime}(y)=\mathrm{M}^{\prime} \text { and } \\
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau \quad \Phi_{r} \triangleright_{\mathcal{U}} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{\frac{\left(\Gamma^{\prime} \backslash y\right)+\Delta^{\prime} \vdash \mathrm{L}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau}{}(\mathrm{es}) \quad \Phi_{u} \triangleright \mathcal{U} \Delta \vdash u: \mathrm{M}}(\mathrm{ap} \\
\hline \triangleright \mathcal{U} \Gamma \vdash \mathrm{L}\langle\lambda x . s\rangle u: \tau
\end{gathered}
$$

Thus, we build

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau \quad \Phi_{u} \triangleright_{\mathcal{U}} \Delta \vdash u: \mathrm{M}}{\Psi \triangleright_{\mathcal{U}} \Gamma^{\prime}+\Delta \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle u: \tau}(\mathrm{app})
$$

and by $i . h$. there exists $\Psi^{\prime} \triangleright \mathcal{U} \Gamma^{\prime}+\Delta \vdash \mathrm{L}^{\prime}\langle s[x \backslash u]\rangle: \tau$ such that sz $(\Psi)>s z\left(\Psi^{\prime}\right)$. Then, we conclude with

$$
\frac{\Psi^{\prime} \triangleright_{\mathcal{U}} \Gamma^{\prime}+\Delta \vdash \mathrm{L}^{\prime}\langle s[x \backslash u]\rangle: \tau \quad \Phi_{r} \triangleright \mathcal{U} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{\Phi^{\prime} \triangleright \mathcal{U}\left(\left(\Gamma^{\prime} \backslash y\right)+\Delta^{\prime}\right)+\Delta \vdash \mathrm{L}\langle s[x \backslash u]\rangle: \tau}(\mathrm{es})
$$

since we may assume that $y \notin \operatorname{supp}(\Delta)$. Notice that $\operatorname{sz}(\Phi)=$ $\mathrm{sz}(\Psi)+\mathbf{s z}\left(\Phi_{r}\right)+1>\mathbf{s z}\left(\Psi^{\prime}\right)+\mathbf{s z}\left(\Phi_{r}\right)+1=\mathbf{s z}\left(\Phi^{\prime}\right)$.

- Rule s!. Then $t=s[x \backslash \mathrm{~L}\langle!u\rangle]$ and $t^{\prime}=\mathrm{L}\langle s\{x \backslash u\}\rangle$. We proceed by induction on L .
$* \mathrm{~L}=\square$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+_{i \in I} \Delta_{i}$ s.t. $\Gamma^{\prime}(x)=[\sigma]_{i \in I}$ and

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash s: \tau \quad \frac{\left(\Phi_{u}^{i} \triangleright \mathcal{U} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}}{+_{i \in I} \Delta_{i} \vdash!u:[\sigma]_{i \in I}}(\mathrm{bg})}{\Phi \triangleright \mathcal{U} \Gamma \vdash s[x \backslash!u]: \tau}(\mathrm{es})
$$

Thus, we conclude directly by Lemma 3.3 with $\Phi_{s}$ and $\left(\Phi_{u}^{i}\right)_{i \in I}$. Notice that $s z(\Phi)=1+s z\left(\Phi_{s}\right)+_{i \in I} s z\left(\Phi_{u}^{i}\right)$, while $s z\left(\Phi^{\prime}\right)=$ $\mathrm{sz}\left(\Phi_{s}\right)+_{i \in I} \mathrm{sz}\left(\Phi_{u}^{i}\right)-|I|$.
$* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r]$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+(\Delta \backslash y)+\Delta^{\prime}$ with $\Gamma^{\prime}(x)=\mathrm{M}$, $\Delta(y)=\mathrm{M}^{\prime}$ and

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash s: \tau \quad \frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Delta \vdash \mathrm{L}^{\prime}\langle!u\rangle: \mathrm{M} \quad \Phi_{r} \triangleright_{\mathcal{U}} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{(\Delta \backslash \backslash y)+\Delta^{\prime} \vdash \mathrm{L}^{\prime}[y \backslash r]\langle!u\rangle: \mathrm{M}}(\mathrm{es})}{\Phi \triangleright_{\mathcal{U}} \Gamma \vdash s[x \backslash \mathrm{~L}\langle!u\rangle]: \tau}
$$

Thus, we build

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash s: \tau \quad \Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Delta \vdash \mathrm{L}^{\prime}\langle!u\rangle: \mathrm{M}}{\Psi \triangleright_{\mathcal{U}}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash s\left[x \backslash \mathrm{~L}^{\prime}\langle!u\rangle\right]: \tau}(\mathrm{es})
$$

and by the $i . h$. there exists $\Psi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\{x \backslash u\}\rangle: \tau$ such that $\mathbf{s z}(\Psi)>\mathbf{s z}\left(\Psi^{\prime}\right)$. Then, we conclude with

$$
\frac{\Psi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\{x \backslash u\}\rangle: \tau \quad \Phi_{r} \triangleright \mathcal{U} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{\Phi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash x\right)+(\Delta \mathbb{\mathrm { L }})}
$$

since we may assume that $y \notin \operatorname{supp}\left(\Gamma^{\prime}\right)$. Notice that sz $(\Phi)=$ $\mathbf{s z}(\Psi)+\mathbf{s z}\left(\Phi_{r}\right)+1>\mathbf{s z}\left(\Psi^{\prime}\right)+\mathbf{s z}\left(\Phi_{r}\right)+1=\mathbf{s z}\left(\Phi^{\prime}\right)$.

- Rule d!. Then $t=\operatorname{der} \mathrm{L}\langle!s\rangle$ and $t^{\prime}=\mathrm{L}\langle s\rangle$. We proceed by induction on L.
* $\mathrm{L}=$This case is immediate since

$$
\frac{\frac{\Phi^{\prime} \triangleright \mathcal{U} \Gamma \vdash s: \tau}{\Gamma \vdash!s:[\tau]}}{\Phi \triangleright \mathrm{bg})}(\mathrm{dr})
$$

$* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r]$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ with $\Gamma^{\prime}(x)=\mathrm{M}$ and

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle!s\rangle:[\tau] \quad \Phi_{r} \triangleright_{\mathcal{U}} \Delta \vdash r: \mathrm{M}}{\frac{\Gamma \vdash \mathrm{~L}\langle!s\rangle:[\tau]}{\Phi \triangleright \mathcal{U} \Gamma \vdash \operatorname{der} \mathrm{L}\langle!s\rangle: \tau}(\mathrm{dr})}
$$

Thus, we build

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle!s\rangle:[\tau]}{\Psi \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \operatorname{der} \mathrm{L}^{\prime}\langle!s\rangle: \tau}(\mathrm{dr})
$$

and by the $i . h$. there exists $\Psi^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle s\rangle: \tau$. Hence we conclude

$$
\frac{\Psi^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle s\rangle: \tau \quad \Phi_{r} \triangleright \mathcal{U} \Delta \vdash r: \mathrm{M}}{\Phi^{\prime} \triangleright \mathcal{U} \Gamma \vdash \mathrm{L}\langle s\rangle: \tau}(\mathrm{es})
$$

Notice that $\mathbf{s z}(\Phi)=\mathbf{s z}(\Psi)+\mathbf{s z}\left(\Phi_{r}\right)+1>\mathbf{s z}\left(\Psi^{\prime}\right)+\mathbf{s z}\left(\Phi_{r}\right)+1=$ sz $\left(\Phi^{\prime}\right)$.

- All the inductive cases for $t \rightarrow_{\mathrm{w}} t^{\prime}$ are straightforward by the $i . h$.

In order to prove subject expansion, an anti-substitution lemma is needed:
Lemma 3.5 (Anti-Substitution). If $\Phi_{t\{x \backslash u\}} \triangleright \mathcal{U} \Gamma^{\prime} \vdash t\{x \backslash u\}: \tau$, then there exists $\Phi_{t} \triangleright \mathcal{U} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash t: \tau$ and $\left(\Phi_{u}^{i} \triangleright \mathcal{U} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma^{\prime}=\Gamma+_{i \in I}$ $\Delta_{i}$ and $\mathbf{s z}\left(\Phi_{t\{x \backslash u\}}\right)=\mathbf{s z}\left(\Phi_{t}\right)+_{i \in I} \mathbf{s z}\left(\Phi_{u}^{i}\right)-|I|$.

Proof. By induction on $t$.
Lemma 3.6 (Weighted Subject Expansion). Let $\Phi^{\prime} \triangleright \mathcal{U} \Gamma \vdash t^{\prime}: \tau$. If $t \rightarrow_{\mathrm{w}} t^{\prime}$, then there is $\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau$ such that $\mathbf{s z}(\Phi)>\mathbf{s z}\left(\Phi^{\prime}\right)$.

Proof. By induction on $t \rightarrow_{\mathrm{w}} t^{\prime}$.

- For the base cases we have to consider three rules:
- Rule dB. Then, $t=\mathrm{L}\langle\lambda x . s\rangle u$ and $t^{\prime}=\mathrm{L}\langle s[x \backslash u]\rangle$. We proceed by induction on L .
$* \mathrm{~L}=\square$. Then, $\Gamma=\left(\Gamma^{\prime} \ \backslash x\right)+\Delta$ such that

$$
\frac{\Phi_{s}^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash s: \tau \quad \Phi_{u}^{\prime} \triangleright \mathcal{U} \Delta \vdash u: \Gamma^{\prime}(x)}{\Phi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash s[x \backslash u]: \tau}(\mathrm{es})
$$

then, we construct

$$
\frac{\frac{\Phi_{s}^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash s: \tau}{\Gamma^{\prime} \backslash x \vdash \lambda x . s: \Gamma^{\prime}(x) \rightarrow \tau}(\mathrm{abs})}{\Phi \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash(\lambda x . s) u: \tau} \Phi_{u}^{\prime} \triangleright \mathcal{U} \Delta \vdash u: \Gamma^{\prime}(x)(\mathrm{app})
$$

Note that $\mathbf{s z}(\Phi)=\mathbf{s z}\left(\Phi^{\prime}\right)+1$.
$* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r]$. Then, $\Gamma=\left(\Gamma^{\prime} \backslash y\right)+\Delta$ such that

$$
\frac{\Psi^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle s[x \backslash u]\rangle: \tau \quad \Phi_{r}^{\prime} \triangleright \mathcal{U} \Delta \vdash r: \Gamma^{\prime}(y)}{\Phi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s[x \backslash u]\rangle[y \backslash r]: \tau}(\mathrm{es})
$$

By i.h. on $\Psi^{\prime}$ we have a derivation $\Psi \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle u: \tau$ with $\mathbf{s z}(\Psi)>\mathbf{s z}\left(\Psi^{\prime}\right)$. Moreover, by rule (app), $\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}$ and

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma_{1}^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau \quad \Phi_{u} \triangleright_{\mathcal{U}} \Gamma_{2}^{\prime} \vdash u: \mathrm{M}}{\Psi \triangleright \mathcal{U} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle u: \tau}(\mathrm{app})
$$

Moreover, by hypothesis of rule $\mathrm{dB}, y \notin \mathrm{fv}(u)$. Thus, in particular, $y \notin \operatorname{supp}\left(\Gamma_{2}^{\prime}\right)$ and $\Gamma^{\prime}(y)=\Gamma_{1}^{\prime}(y)$. Then, we construct
$\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \mathcal{U} \Gamma_{1}^{\prime} \vdash \mathrm{L}^{\prime}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau \quad \Phi_{r}^{\prime} \triangleright \mathcal{U} \Delta \vdash r: \Gamma_{1}^{\prime}(y)}{\frac{\left(\Gamma_{1}^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}\langle\lambda x . s\rangle: \mathrm{M} \rightarrow \tau}{\Phi \triangleright_{\mathcal{U}}\left(\Gamma^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}\langle\lambda x . s\rangle u: \tau} \quad \Phi_{u} \triangleright_{\mathcal{U}} \Gamma_{2}^{\prime} \vdash u: \mathrm{M}}(\mathrm{app})$
and conclude with

$$
\begin{aligned}
\mathrm{sz}(\Phi) & =\mathrm{sz}\left(\Phi_{\mathrm{L}^{\prime}}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+\mathrm{sz}\left(\Phi_{u}\right)+2 \\
& =\mathrm{sz}(\Psi)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+2 \\
& >\mathrm{sz}\left(\Psi^{\prime}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+1 \\
& =\mathrm{sz}\left(\Phi^{\prime}\right)
\end{aligned}
$$

- Rule s!. Then, $t=s[x \backslash \mathrm{~L}\langle!u\rangle]$ and $t^{\prime}=\mathrm{L}\langle s\{x \backslash u\}\rangle$. We proceed by induction on L .
* $\mathrm{L}=\square$. By Lemma 3.5 with $\Phi^{\prime}$, there exist $\Phi_{s} \triangleright \mathcal{U} \Gamma^{\prime} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash s: \tau$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{U}} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma=\Gamma^{\prime}+_{i \in I} \Delta_{i}$ and sz $\left(\Phi^{\prime}\right)=$ sz $\left(\Phi_{s}\right)+_{i \in I} \mathbf{s z}\left(\Phi_{u}^{i}\right)-|I|$. Then, we construct

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma^{\prime} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash s: \tau}{\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau} \frac{\left(\Phi_{u}^{i} \triangleright_{\mathcal{U}} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}}{+_{i \in I} \Delta_{i} \vdash!u:\left[\sigma_{i}\right]_{i \in I}}(\mathrm{bg})
$$

and conclude since sz $(\Phi)=\mathbf{s z}\left(\Phi_{s}\right)+_{i \in I} \mathbf{s z}\left(\Phi_{u}^{i}\right)+1>s z\left(\Phi^{\prime}\right)$.
$* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r]$. Then, $\Gamma=\left(\Gamma^{\prime} \backslash y\right)+\Delta$ such that

$$
\frac{\Psi^{\prime} \triangleright \mathcal{U} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle s\{x \backslash u\}\rangle: \tau \quad \Phi_{r}^{\prime} \triangleright \mathcal{U} \Delta \vdash r: \Gamma^{\prime}(y)}{\Phi^{\prime} \triangleright_{\mathcal{U}}\left(\Gamma^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\{x \backslash u\}\rangle[y \backslash r]: \tau}
$$

By $i . h$. on $\Psi^{\prime}$ we have a derivation $\Psi \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash s\left[x \backslash \mathrm{~L}^{\prime}\langle!u\rangle\right]: \tau$ with sz $(\Psi)>s z\left(\Psi^{\prime}\right)$. Moreover, by rule (es), $\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}$ and

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma_{1}^{\prime} \vdash s: \tau \quad \Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma_{2}^{\prime} \vdash \mathrm{L}^{\prime}\langle!u\rangle: \Gamma_{1}^{\prime}(x)}{\Psi \triangleright \mathcal{U} \Gamma^{\prime} \vdash s\left[x \backslash \mathrm{~L}^{\prime}\langle!u\rangle\right]: \tau}(\mathrm{es})
$$

Moreover, by hypothesis of rule s!, $y \notin \mathrm{fv}(s)$. Thus, in particular, $y \notin \operatorname{supp}\left(\Gamma_{1}^{\prime}\right)$ and $\Gamma^{\prime}(y)=\Gamma_{2}^{\prime}(y)$. Then, we construct

$$
\frac{\Phi_{s} \triangleright_{\mathcal{U}} \Gamma_{1}^{\prime} \vdash s: \tau \quad \frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma_{2}^{\prime} \vdash \mathrm{L}^{\prime}\langle!u\rangle: \Gamma_{1}^{\prime}(x) \quad \Phi_{r}^{\prime} \triangleright \mathcal{U} \Delta \vdash r: \Gamma_{2}^{\prime}(y)}{\left(\Gamma_{2}^{\prime} \backslash \backslash y\right)+\Delta \vdash \mathrm{L}\langle!u\rangle: \Gamma_{1}^{\prime}(x)}(\mathrm{es})}{\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau}
$$

and conclude with

$$
\begin{aligned}
\mathrm{sz}(\Phi) & =\mathrm{sz}\left(\Phi_{s}\right)+\mathrm{sz}\left(\Phi_{\mathrm{L}^{\prime}}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+2 \\
& =\mathrm{sz}(\Psi)+\mathbf{s z}\left(\Phi_{r}^{\prime}\right)+2 \\
& >\mathrm{sz}\left(\Psi^{\prime}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+1 \\
& =\mathrm{sz}\left(\Phi^{\prime}\right)
\end{aligned}
$$

- Rule d!. Then, $t=\operatorname{der}(\mathrm{L}\langle!s\rangle)$ and $t^{\prime}=\mathrm{L}\langle s\rangle$. We proceed by induction on L .
* $\mathrm{L}=\square$. We have a derivation $\Phi^{\prime} \triangleright_{\mathcal{U}} \Gamma \vdash s: \tau$ and we construct

$$
\frac{\frac{\Phi^{\prime} \triangleright \mathcal{U} \Gamma \vdash s: \tau}{\Gamma \vdash!s:[\tau]}(\mathrm{bg})}{\Phi \triangleright \mathcal{U} \Gamma \vdash \operatorname{der}(!s): \tau}(\mathrm{dr})
$$

to conclude since $s z(\Phi)=s z\left(\Phi^{\prime}\right)+1$.
$* \mathrm{~L}=\mathrm{L}^{\prime}[y \backslash r]$. Then, $\Gamma=\left(\Gamma^{\prime} \backslash y\right)+\Delta$ such that

$$
\frac{\Psi^{\prime} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle s\rangle: \tau \quad \Phi_{r}^{\prime} \triangleright_{\mathcal{U}} \Delta \vdash r: \Gamma^{\prime}(y)}{\Phi^{\prime} \triangleright \mathcal{U}\left(\Gamma^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\rangle[y \backslash r]: \tau}(\mathrm{es})
$$

By i.h. on $\Psi^{\prime}$ we have a derivation $\Psi \triangleright \mathcal{U} \Gamma^{\prime} \vdash \operatorname{der}\left(\mathrm{L}^{\prime}\langle!s\rangle\right): \tau$ with $\mathrm{sz}(\Psi)>\mathrm{sz}\left(\Psi^{\prime}\right)$. Moreover, by rule (dr),

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \mathcal{U}^{\prime} \vdash \mathrm{L}^{\prime}\langle!s\rangle:[\tau]}{\Psi \triangleright \mathcal{U} \Gamma^{\prime} \vdash \operatorname{der}\left(\mathrm{L}^{\prime}\langle!s\rangle\right): \tau}(\mathrm{dr})
$$

Then, we construct

$$
\frac{\Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{U}} \Gamma^{\prime} \vdash \mathrm{L}^{\prime}\langle!s\rangle:[\tau] \quad \Phi_{r}^{\prime} \triangleright \mathcal{U} \Delta \vdash r: \Gamma^{\prime}(y)}{\frac{\left(\Gamma^{\prime} \backslash y\right)+\Delta \vdash \mathrm{L}\langle!s\rangle:[\tau]}{\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau}(\mathrm{dr})}
$$

and conclude with

$$
\begin{aligned}
\mathrm{sz}(\Phi) & =\mathrm{sz}\left(\Phi_{\mathrm{L}^{\prime}}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+2 \\
& =\mathrm{sz}(\Psi)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+2 \\
& >\mathrm{sz}\left(\Psi^{\prime}\right)+\mathrm{sz}\left(\Phi_{r}^{\prime}\right)+1 \\
& =\mathrm{sz}\left(\Phi^{\prime}\right)
\end{aligned}
$$

- All the inductive cases for $t \rightarrow_{\mathrm{w}} t^{\prime}$ are straightforward by the $i . h$.

Erasing steps like $y[x \backslash!z] \rightarrow_{\mathrm{s}!} y$ may seem problematic for subject reduction and expansion, but they are not: the variable $x$ is necessarily assigned a type [] in the corresponding typing context, and the term $!z$ is then necessarily typed with [], so there is no loss of information since the contexts allowing to type the redex and the reduced term are the same.

Typable terms are necessarily weak clash-free:
Lemma 3.7. If $\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \sigma$, then $t$ is wcf.
Proof. Assume towards a contradiction that $t$ is not wcf, i.e. there exists a weak context W and a clash $c$ such that $t \rightarrow{ }_{\mathrm{w}} \mathrm{W}\langle c\rangle$. Then, Lemma 3.4 gives $\Phi^{\prime} \triangleright_{\mathcal{U}} \Gamma \vdash \mathrm{W}\langle c\rangle: \sigma$. If we show that a term of the form $\mathrm{W}\langle c\rangle$ cannot be typed in system $\mathcal{U}$, we are done. This follows by straightforward induction on W . The base case is when $\mathrm{W}=\square$. For every possible $c$, it is immediate to see that there is a mismatch between its syntactical form and the typing rules of system $\mathcal{U}$. For instance, if $c=\mathrm{L}\langle!t\rangle u$, then $\mathrm{L}\langle!t\rangle$ should have a functional type by rule (app) but it can only be assigned a multiset type by rules (es) and (bg). As for the inductive case, an easy inspection of the typing rules shows for all terms $t$ and weak contexts $\mathrm{W}, t$ must be typed in order to type $\mathrm{W}\langle t\rangle$.

However, normal terms are not necessarily clash-free, but the type system captures weak clash-freeness of normal terms. Said differently, when restricted to $\mathrm{no}_{\mathrm{w}}$, typability exactly corresponds to weak clash-freeness.

Theorem 3.8. Let $t \in \mathcal{T}$. Then, $t \in \mathrm{no}_{\text {wcf }}$ iff $t \in \mathrm{no}_{\mathrm{w}}$ and $t$ is $\mathcal{U}$-typable.
Proof. By simultaneous induction on the following claims:

1. $t \in \mathrm{ne}_{\mathrm{wcf}}$ iff $t \in \mathrm{ne}_{\mathrm{w}}$ and for every $\tau$ there exists $\Gamma$ such that $\triangleright \mathcal{U}^{\Gamma} \vdash t: \tau$.
2. $t \in \mathrm{na}_{\mathrm{wcf}}$ iff $t \in \mathrm{na}_{\mathrm{w}}$ and there exist $\Gamma$ and M such that $\triangleright_{\mathcal{U}} \Gamma \vdash t: \mathrm{M}$.
3. $t \in \mathrm{nb}_{\mathrm{wcf}}$ iff $t \in \mathrm{nb}_{\mathrm{w}}$ and there exist $\Gamma$ and $\tau$ such that $\triangleright \mathcal{U} \Gamma \vdash t: \tau$.
4. $t \in \mathrm{no}_{\mathrm{wcf}}$ iff $t \in \mathrm{no}_{\mathrm{w}}$ and there exist $\Gamma$ and $\tau$ such that $\triangleright \mathcal{U} \Gamma \vdash t: \tau$.

We first show the left-to-right implications by only analysing the key cases.
If $t=x \in \mathrm{ne}_{\mathrm{wcf}}$, then $t \in \mathrm{ne}_{\mathrm{w}}$ and for every type $\tau$ we conclude by (ax).
If $t=s u \in \mathrm{ne}_{\mathrm{wcf}}$, by definition $s \in \mathrm{ne}_{\mathrm{wcf}}$ and $u \in \mathrm{na}_{\mathrm{wcf}}$. Let $\tau$ be any type. By i.h. (21) $u \in \mathrm{na}_{\mathrm{w}}$ and $\triangleright \mathcal{U} \Delta \vdash u:$ M. Then, by i.h. (1) we get $s \in \mathrm{ne}_{\mathrm{w}}$ and $\triangleright \mathcal{U} \Gamma \vdash s: \mathrm{M} \rightarrow \tau$. Moreover, $s \in \mathrm{ne}_{\mathrm{w}}$ and $u \in \mathrm{na}_{\mathrm{w}}$ imply $s \in \mathrm{na}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$ resp., hence $t \in \mathrm{ne}_{\mathrm{w}}$. Thus, we conclude by (app), $\triangleright_{\mathcal{U}} \Gamma+\Delta \vdash s u: \tau$.

If $t=\lambda x . s \in \mathrm{nb}_{\mathrm{wcf}}$, then $s \in \mathrm{no}_{\mathrm{wcf}}$ by definition. By i.h. (4) $s \in \mathrm{no}_{\mathrm{w}}$ and $\triangleright_{\mathcal{U}} \Gamma \vdash s: \tau$ for some type $\tau$. Then, $t \in \mathrm{nb}_{\mathrm{w}}$ and we conclude by (abs), $\triangleright \mathcal{U} \Gamma \backslash x \vdash \lambda x . s: \Gamma(x) \rightarrow \tau$.

If $t=!s \in \mathrm{na}_{\mathrm{wcf}}$, then $t \in \mathrm{na}_{\mathrm{w}}$ by definition and we conclude by (bg), $\triangleright_{\mathcal{U}} \vdash!s:[]$.

If $t=\operatorname{der} s \in \mathrm{ne}_{\mathrm{wcf}}$, then $s \in \mathrm{ne}_{\mathrm{wcf}}$ by definition. Let $\tau$ be any type, by i.h. (11) we get $s \in \mathrm{ne}_{\mathrm{w}}$ and $\triangleright_{\mathcal{U}} \Gamma \vdash s:[\tau]$. Moreover, $s \in \mathrm{ne}_{\mathrm{w}}$ implies $s \in \mathrm{nb}_{\mathrm{W}}$ and hence $t \in \mathrm{ne}_{\mathrm{w}}$. Thus, we conclude by ( dr ), $\triangleright_{\mathcal{U}} \Gamma \vdash \operatorname{der} s: \tau$.

If $t=s[x \backslash u]$, then $u \in \mathrm{ne}_{\text {wcf }}$ and there are three possible cases: $s \in \mathrm{ne}_{\mathrm{wcf}}$, $s \in \mathrm{na}_{\mathrm{wcf}}$ or $s \in \mathrm{nb}_{\mathrm{wcf}}$. In either case, by the proper $i . h$. we get $\triangleright_{\mathcal{U}} \Gamma \vdash s: \tau$ (resp. M) and conclude by (es), given that i.h. (11) on $u$ implies $u \in \mathrm{ne}_{\mathrm{w}}$ and $\triangleright \mathcal{U} \Delta \vdash u: \Gamma(x)$. Note that $u \in \mathrm{ne}_{\mathrm{w}}$ in turn implies $u \in \mathrm{nb}_{\mathrm{w}}$, thus $t$ remains in the same set as $s\left(\mathrm{ne}_{\mathrm{w}}, \mathrm{na}_{\mathrm{w}}\right.$ or $\mathrm{nb}_{\mathrm{w}}$ resp.).

The right-to-left implications uses Lemma 3.2.
Typability can be shown to (qualitatively and quantitatively) characterise normalisation. The type system $\mathcal{U}$ is sound (all the typable terms are normalising) and complete (all the normalising terms are typable).
Theorem 3.9 (Soundness and Completeness for System $\mathcal{U}$ ). The term $t$ is $\mathcal{U}$ typable iff $t$ w-normalises to a term $p \in \mathrm{no}_{\mathrm{wcf}}$. Moreover, if $\Phi \triangleright_{\mathcal{U}} \Gamma \vdash t: \tau$, then $t \rightarrow{ }_{\mathrm{w}}^{(b, e)} p$ and $\mathbf{s z}(\Phi) \geq b+e+|p|_{\mathrm{w}}$.
Proof. The soundness proof is straightforward by Lemma 3.4 and Theorem 3.8 Observe that the argument is simply combinatorial, no reducibility argument is needed. For the completeness proof, we reason by induction on the length of the w-normalising sequence. For the base case, we use Theorem 3.8 which states that $p \in \mathrm{no}_{\text {wcf }}$ implies $p$ is $\mathcal{U}$-typable. For the inductive case we use Lemma 3.6 The moreover statement holds by Lemma 3.4 and 3.6 and the fact that the size of the type derivation of $p$ is greater than or equal to $|p|_{w}$.

The previous theorem can be illustrated by the term $t_{0}=\operatorname{der}(!\mathrm{K})(!\mathrm{I})(!\Omega)$ defined in Example 2.2 which normalises in 5 steps to a normal form of w-size 1 , the sum of the two being bounded by the size 11 of its type derivation $\Phi_{0}$ given in Example 3.1,

## 4. Capturing Call-by-Name and Call-by-Value

This section explores the CBN/CBV embeddings into the $\lambda$ !-calculus. For CBN, we slightly adapt Girard's translation into LL [30], which preserves normal
forms and is sound and complete with respect to the standard (quantitative) type system [29]. For CBV, however, we reformulate both the translation and the type system, so that preservation of normal forms and completeness are restored. In both cases, we specify the operational semantics of CBN and CBV by means of a very simple notion of explicit substitution, see for example 6].

Terms $\left(\mathcal{T}_{\lambda}\right)$, values and contexts are defined as follows:

| (Terms) | $t, u$ | $::=$ | $v\|t u\| t[x \backslash u]$ |
| ---: | ---: | :--- | :--- |
| (Values) | $v$ | $::=$ | $x \in \mathcal{X} \mid \lambda x . t$ |
| (List Contexts) | L | $::=$ | $\square \mid \mathrm{L}[x \backslash t]$ |
| (Call-by-Name Contexts) | N | $::=$ | $\square\|\mathrm{N} t\| \lambda x . \mathrm{N} \mid \mathrm{N}[x \backslash u]$ |
| (Call-by-Value Contexts) | V | $::=$ | $\square\|\mathrm{V} t\| t \mathrm{~V}\|\mathrm{~V}[x \backslash u]\| t[x \backslash \mathrm{~V}]$ |

As in Sec. 2 we use the predicate abs $(t)$ iff $t=\mathrm{L}\left\langle\lambda x . t^{\prime}\right\rangle$. We also use the predicates $\operatorname{app}(t)$ iff $t=\mathrm{L}\left\langle t^{\prime} t^{\prime \prime}\right\rangle$ and $\operatorname{var}(t)$ iff $t=\mathrm{L}\langle x\rangle$.

The Call-by-Name reduction relation $\rightarrow_{\mathrm{n}}$ is defined as the closure of the rules dB and s presented below under contexts N , while the Call-by-Value reduction relation $\rightarrow_{\mathrm{v}}$ is defined as the closure of the rules dB and Sv below under contexts V . Equivalently, $\rightarrow_{\mathrm{n}} \stackrel{\text { def }}{=} \mathrm{N}\left(\mapsto_{\mathrm{dB}} \cup \mapsto_{\mathrm{s}}\right)$ and $\rightarrow_{\mathrm{v}} \stackrel{\text { def }}{=} \mathrm{V}\left(\mapsto_{\mathrm{dB}} \cup \mapsto_{\mathrm{sv}}\right)$ and

$$
\begin{array}{rll}
\mathrm{L}\langle\lambda x . t\rangle u & \mapsto_{\mathrm{dB}} & \mathrm{~L}\langle t[x \backslash u]\rangle \\
t[x \backslash u] & \mapsto_{\mathrm{s}} & t\{x \backslash u\} \\
t[x \backslash \mathrm{~L}\langle v\rangle] & \mapsto_{\mathrm{sv}} & \mathrm{~L}\langle t\{x \backslash v\}\rangle
\end{array}
$$

We write $t \not \nrightarrow \mathrm{n}$ (resp. $t \not \nrightarrow \mathrm{v}$ ), and call $t$ an n -normal form (resp. v-normal form), if $t$ cannot be reduced by means of $\rightarrow_{\mathrm{n}}\left(\right.$ resp. $\left.\rightarrow_{\mathrm{v}}\right)$.

Observe that we use CBN and CBV formulations based on distinguished multiplicative ( $c f . \mathrm{dB}$ ) and exponential ( $c f . \mathrm{s}$ and sv) rules, inheriting the nature of cut elimination rules in LL. Moreover, CBN is to be understood as head CBN reduction [9], i.e. reduction does not take place in arguments of applications (and in the present case in arguments of substitutions as well) while CBV corresponds to open CBV reduction [6, 2], i.e. reduction does not take place inside abstractions.

Theorem 4.1. The reduction relations $\rightarrow_{\mathrm{n}}$ and $\rightarrow_{\mathrm{v}}$ are both confluent.
Proof. The proofs of these properties are folklore. They use an abstract result in rewriting theory stating that the union of two reduction relations is confluent if both are confluent and commute with each other [47]. More precisely, two arbitrary reduction relations $\rightarrow_{1}$ and $\rightarrow_{2}$ commute iff for all $t_{0}, t_{1}, t_{2}$ such that $t_{0} \rightarrow_{1} t_{1}$ and $t_{0} \rightarrow_{2} t_{2}$, there exists $t_{3}$ such that $t_{1} \rightarrow_{2} t_{3}$ and $t_{2} \rightarrow_{1} t_{3}$. The reasoning in our case is as follows:

- Each rule $\mathrm{a} \in\{\mathrm{dB}, \mathrm{s}, \mathrm{sv}\}$ induces a complete reduction relation $\rightarrow_{\mathrm{a}}$, i.e. confluent and terminating.
- The relation $\rightarrow_{\mathrm{dB}}$ commutes with $\rightarrow_{\mathrm{a}}$, for $\mathrm{a} \in\{\mathrm{s}, \mathrm{sv}\}$.

The two previous points are straightforward.

Embeddings. The CBN and CBV embeddings into the $\lambda$ !-calculus, written ${ }^{\text {cbn }}$ and _ ${ }^{\text {cbv }}$ resp., are inductively defined as:

$$
\begin{array}{rllll}
x^{\mathrm{cbn}} & \stackrel{\text { def }}{=} x & x^{\mathrm{cbv}} & \stackrel{\text { def }}{=}!x \\
(\lambda x . t)^{\mathrm{cbn}} & \stackrel{\text { def }}{=} \lambda x . t^{\mathrm{cbn}} & (\lambda x . t)^{\mathrm{cbv}} & \stackrel{\text { def }}{=}!\lambda x . t^{\mathrm{cbv}} & \\
(t u)^{\mathrm{cbn}} & \stackrel{\text { def }}{=} t^{\mathrm{cbn}}!u^{\mathrm{cbn}} & (t u)^{\mathrm{cbv}} & \stackrel{\text { def }}{=}\left\{\begin{array}{l}
\mathrm{L}\langle s\rangle u^{\mathrm{cbv}} \\
\operatorname{der}\left(t^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}
\end{array}\right. & \text { if } t^{\mathrm{cbv}}=\mathrm{L}\langle!s\rangle \\
(t[x \backslash u])^{\mathrm{cbn}} & \stackrel{\text { def }}{=} t^{\mathrm{cbn}}\left[x \backslash!u^{\mathrm{cbn}]}\right. & (t[x \backslash u])^{\mathrm{cbv}} & \stackrel{\text { def }}{=} t^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}]}\right. &
\end{array}
$$

Both translations extend to list contexts L as expected. Observe that the terms of the $\lambda$ !-calculus that are in the image of the embeddings never contain two consecutive ! constructors. The CBN embedding extends Girard's translation to explicit substitutions, while the CBV one is different. Indeed, the translation of an application $t u$ is usually defined as der $\left(t^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}$ (see for example [27]). This definition does not preserve normal forms, i.e. $x y$ is a v-normal form but its translated version $\operatorname{der}(!x)!y$ is not a w-normal form. We restore this fundamental property by using the notion of superdevelopment [35, 48], so that d!-reductions created by the translation are executed on the fly.

Example 4.2. Special $\lambda$-terms are $\mathrm{I}_{\lambda}=\lambda z . z, \mathrm{~K}_{\lambda}=\lambda x . \lambda y . x, \Delta_{\lambda}=\lambda x . x x$, and $\Omega_{\lambda}=\Delta_{\lambda} \Delta_{\lambda}$. Observe that $\mathrm{I}_{\lambda}{ }^{\mathrm{cbn}}=\mathrm{I}_{\lambda}=\mathrm{I}$ and $\mathrm{K}_{\lambda}{ }^{\mathrm{cbn}}=\mathrm{K}_{\lambda}=\mathrm{K}$. In this example, we compute $t^{\mathrm{cbn}}$ and $t^{\mathrm{cbv}}$, where $t=\mathrm{K}_{\lambda} \mathrm{I}_{\lambda} \Omega_{\lambda}$.

$$
\begin{aligned}
t^{\mathrm{cbn}} & =\left(\mathrm{K}_{\lambda} \mathrm{I}_{\lambda}\right)^{\mathrm{cbn}}!\Omega_{\lambda}^{\mathrm{cbn}} \\
& =\mathrm{K}_{\lambda}^{\mathrm{cbn}}!\mathrm{I}_{\lambda}^{\mathrm{cbn}}!\Omega_{\lambda}^{\mathrm{cbn}} \\
& =\mathrm{K}!\mathrm{I}!\left(\Delta_{\lambda}^{\mathrm{cbn}}!\Delta_{\lambda}^{\mathrm{cbn}}\right) \\
& =\mathrm{K}!\mathrm{I}!((\lambda x \cdot x!x)!(\lambda x \cdot x!x)) \\
& =\mathrm{K}!\mathrm{I}!(\Delta!\Delta) \\
& =\mathrm{K}!\mathrm{I}!\Omega
\end{aligned}
$$

The $\lambda$ !-term $t^{\mathrm{cbn}}$ is the same as the one obtained by performing a d!-step starting from the term $t_{0}$ of Example 2.2. The CBV embedding is a bit more involved, due to the superdevelopments.

$$
\begin{aligned}
t^{\mathrm{cbv}} & =\operatorname{der}\left(\mathrm{K}_{\lambda} \mathrm{I}_{\lambda}\right)^{\mathrm{cbv}} \Omega_{\lambda}^{\mathrm{cbv}} \\
& =\operatorname{der}((\lambda x .!\lambda y .!x)(!\lambda x .!x)) \Omega_{\lambda}^{\mathrm{cbv}} \\
& =\operatorname{der}((\lambda x .!\lambda y .!x)(!\lambda x .!x))((\lambda x . x!x)!(\lambda x . x!x)) \\
& =\operatorname{der}((\lambda x .!\lambda y!!x)(!\lambda x .!x)) \Omega
\end{aligned}
$$

Notice that in this particular case we have $\Omega_{\lambda}{ }^{\mathrm{cbn}}=\Omega_{\lambda}{ }^{\mathrm{cbv}}=\Omega$.
The example above will be continued after the proof of the fact that the embeddings preserve the CBN and CBV reductions.

We first show (Lemma 4.3) that the set of n-normal forms and v-normal forms can be respectively characterised by the following grammars:

```
(CBN Neutral) \(\mathrm{ne}_{\mathrm{n}} \quad::=\quad x \in \mathcal{X} \mid \mathrm{ne}_{\mathrm{n}} t\)
(CBN Normal) \(\mathrm{no}_{\mathrm{n}} \quad::=\lambda x . \mathrm{no}_{\mathrm{n}} \mid \mathrm{ne}_{\mathrm{n}}\)
    (Variable) \(\quad \mathrm{vr}_{\mathrm{v}} \quad::=\quad x \in \mathcal{X} \mid \mathrm{vr}_{\mathrm{v}}\left[x \backslash n \mathrm{ne}_{\mathrm{v}}\right]\)
(CBV Neutral) ne \(\mathrm{v}_{\mathrm{v}} \quad::=\quad \mathrm{vr}_{\mathrm{v}} \mathrm{no}_{\mathrm{v}}\left|\mathrm{ne}_{\mathrm{v}} \mathrm{no}_{\mathrm{v}}\right| \mathrm{ne}_{\mathrm{v}}\left[x \backslash n \mathrm{ne}_{\mathrm{v}}\right]\)
(CBV Normal) \(\mathrm{no}_{\mathrm{v}} \quad::=\quad \lambda x . t\left|\mathrm{vr}_{\mathrm{v}}\right| \mathrm{ne}_{\mathrm{v}} \mid \mathrm{no}_{\mathrm{v}}\left[x \backslash \mathrm{ne}_{\mathrm{v}}\right]\)
```

Note that the grammar for neutral and normal forms for CBN does not use the ES operator．The reason is that the s－rule executing ES can always be fired in a CBN context，so that ES only appear in the right－hand sides of applications of neutral terms．Note that variables are left out of the definition of neutral terms for the CBV case，since they are now considered values．Moreover，in this way both CBN and CBV neutral terms translate to neutral terms of the $\lambda$ ！－calculus．

Lemma 4．3．Let $t \in \mathcal{T}_{\lambda}$ ．
1．$t \in \mathrm{no}_{\mathrm{n}}$ iff $t \not \not 一 力_{\mathrm{n}}$ ．
2．$t \in \mathrm{no}_{\mathrm{v}}$ iff $t \not \nrightarrow \mathrm{r}_{\mathrm{v}}$ ．
Proof．1．We prove simultaneously the following statements：
（a）$t \in \mathrm{ne}_{\mathrm{n}}$ iff $t \not \nrightarrow \mathrm{n}_{\mathrm{n}}$ and $\neg \mathrm{abs}(t)$ ．
（b）$t \in \mathrm{no}_{\mathrm{n}}$ iff $t \not \nrightarrow \mathrm{n}_{\mathrm{n}}$ ．
（a）The left－to－right implication is straightforward．For the right－to－left implication we reason by induction on $t$ ．Suppose $t \nrightarrow 力_{\mathrm{n}}$ and $\neg$ abs $(t)$ ． If $t$ is a variable，then $t \in \mathrm{ne}_{\mathrm{n}}$ ．If $t$ is a substitution，then rule s is applicable，so this case does not apply．If $t=u u^{\prime}$ ，then $\neg$ abs $(u)$ （otherwise dB would be applicable）and $u{\not ر_{\mathrm{n}}}$（otherwise $t$ would be n－reducible）．The i．h．（1a）gives $u \in \mathrm{ne}_{\mathrm{n}}$ and thus we conclude $t=u u^{\prime} \in \mathrm{ne}_{\mathrm{n}}$ ．
（b）The left－to－right implication is straightforward．For the right－to－left implication we reason by induction on $t$ ．If $t=\lambda x . u$ ，then $u \not 力_{\mathrm{n}}$ ， and the i．h．（1b）gives $u \in \mathrm{no}_{\mathrm{n}}$ ，which implies in turn $\lambda x . u \in \mathrm{no}_{\mathrm{n}}$ ． Otherwise，we apply the previous case．

2．We prove simultaneously the following statements：
（a）$t \in \mathrm{vr}_{\mathrm{v}}$ iff $t \not \nrightarrow \mathrm{v}$ and $\neg \operatorname{abs}(t)$ and $\neg \operatorname{app}(t)$ ．
（b）$t \in \mathrm{ne}_{\mathrm{v}}$ iff $t \not \not_{\mathrm{v}}$ and $\neg \operatorname{abs}(t)$ and $\neg \operatorname{var}(t)$ ．
（c）$t \in \mathrm{no}_{\mathrm{v}}$ iff $t \not \nrightarrow \mathrm{v}$ ．
（a）The left－to－right implication is straightforward．For the right－to－left implication we reason by induction on $t$ ．Suppose $t \nrightarrow_{\mathrm{v}}$ and $\neg$ abs $(t)$ and $\neg \operatorname{app}(t)$ ．Then necessarily $\operatorname{var}(t)$ ．If $t=u\left[x \backslash u^{\prime}\right]$ ，then $u, u^{\prime} \nrightarrow_{\mathrm{v}}$ （otherwise $t$ would be v－reducible），$\neg \operatorname{abs}\left(u^{\prime}\right)$ and $\neg \operatorname{var}\left(u^{\prime}\right)$（otherwise
$t$ would be sv－reducible），$\neg \operatorname{abs}(u)$ and $\neg \operatorname{app}(u)$ ．The i．h．（2a）gives $u \in \mathrm{vr}_{\mathrm{v}}$ and the $i . h$ ．（2b）gives $u^{\prime} \in \mathrm{ne}_{\mathrm{v}}$ ，so that we conclude $t \in \mathrm{vr}_{\mathrm{v}}$ ． If $t=x$ we trivially conclude $t \in \mathrm{vr}_{\mathrm{v}}$ ．
（b）The left－to－right implication is straightforward．For the right－to－left implication we reason by induction on $t$ ．Suppose $t \nrightarrow_{\mathrm{v}}$ and $\neg \mathrm{abs}(t)$ and $\neg \operatorname{var}(t)$ ．Then necessarily $\operatorname{app}(t)$ ．If $t=u\left[x \backslash u^{\prime}\right]$ ，then $u, u^{\prime} \nrightarrow_{\mathrm{v}}$ （otherwise $t$ would be v－reducible），$\neg \operatorname{abs}\left(u^{\prime}\right)$ and $\neg \operatorname{var}\left(u^{\prime}\right)$（otherwise $t$ would be sv－reducible），$\neg \operatorname{abs}(u)$ and $\neg \operatorname{var}(u)$ ．The i．h．（2b）gives $u \in \mathrm{ne}_{\mathrm{v}}$ and $u^{\prime} \in \mathrm{ne}_{\mathrm{v}}$ ，so that we conclude $t \in \mathrm{ne}_{\mathrm{v}}$ ．If $t=u u^{\prime}$ ， then $u, u^{\prime}{\nrightarrow 力_{\mathrm{v}}}$（otherwise $t$ would be v－reducible），$\neg \mathrm{abs}(u)$（otherwise $t$ would be dB－reducible）．The $i . h$ ．（2C）gives $u^{\prime} \in \mathrm{no}_{\mathrm{v}}$ ．Moreover，if $\neg \operatorname{app}(u)$（resp．$\neg \operatorname{var}(u)$ ），then the i．h．（2a）gives $u \in \mathrm{vr}_{\mathrm{v}}$（resp．the i．h．（2b）gives $u \in \mathrm{ne}_{\mathrm{v}}$ ）In both cases we conclude $t=u u^{\prime} \in \mathrm{ne}_{\mathrm{v}}$ ． Since $\neg \operatorname{app}(u)$ or $\neg \operatorname{var}(u))$ holds，we are done．
（c）The left－to－right implication is straightforward．For the right－to－left implication we reason by induction on $t$ ．Suppose $t \nrightarrow_{\mathrm{v}}$ ．If $t=$ $\lambda x . u$ ，then $t \in \mathrm{no}_{\mathrm{v}}$ is straightforward．If $t=u\left[x \backslash u^{\prime}\right]$ ，then $u, u^{\prime} \nrightarrow 力_{\mathrm{v}}$ （otherwise $t$ would be v－reducible），$\neg$ abs $\left(u^{\prime}\right)$ and $\neg \operatorname{var}\left(u^{\prime}\right)$（otherwise $t$ would be sv－reducible）．The i．h．（2C）and（2b）give $u \in \mathrm{no}_{\mathrm{v}}$ and $u^{\prime} \in \mathrm{ne}_{\mathrm{v}}$ ，so that we conclude $t \in \mathrm{no}_{\mathrm{v}}$ ．If $t=u u^{\prime}$ ，then $u \not \nrightarrow \mathrm{v}_{\mathrm{v}}$ and $u^{\prime} \not \nrightarrow \mathrm{v}_{\mathrm{v}}$ ．The $i . h$ ．（2C）on $u^{\prime}$ gives $u^{\prime} \in \mathrm{no}_{\mathrm{v}}$ ．Moreover，$\neg \operatorname{abs}(u)$ ， otherwise $t$ would be v－reducible．Since either $\neg \operatorname{var}(u)$ or $\neg \operatorname{app}(u)$ must hold，then the $i . h$ ．（2a）or（2b）gives $u \in \mathrm{vr}_{\mathrm{v}}$ or $u \in \mathrm{ne}_{\mathrm{v}}$ ．Thus $u u^{\prime} \in \mathrm{ne}_{\mathrm{v}} \subseteq \mathrm{no}_{\mathrm{v}}$ as required．

The following lemma shows that the CBN and CBV embeddings into the $\lambda!$－calculus preserve the property of being in normal form．

Lemma 4．4．Let $t \in \mathcal{T}_{\lambda}$ ．
1．If $t \not \nrightarrow 力_{\mathrm{n}}$ ，then $t^{\mathrm{cbn}} \not \not_{\mathrm{w}}$ ．
2．If $t \not \nrightarrow ⿱ ⿱ 卄 一 八$ ，then $t^{\mathrm{cbv}}{\nrightarrow 力_{\mathrm{w}}}$ ．
Proof．
1．By Lemma 4.3 and Proposition 2.11 it is sufficient to prove that $t \in \mathrm{no}_{\mathrm{n}}$ implies $t^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$ ．This is done by simultaneously showing the following points：
（a）If $t \in \mathrm{ne}_{\mathrm{n}}$ ，then $t^{\mathrm{cbn}} \in \mathrm{ne}_{\mathrm{w}}$ ．
（b）If $t \in \mathrm{no}_{\mathrm{n}}$ ，then $t^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$ ．
－If $t=x \in \mathrm{ne}_{\mathrm{n}}$ ，then $x^{\mathrm{cbn}}=x$ and thus $x \in \mathrm{ne}_{\mathrm{w}}$ holds．

- If $t=s u \in \mathrm{ne}_{\mathrm{n}}$ comes from $s \in \mathrm{ne}_{\mathrm{n}}$, then the i.h. (1a) gives $s^{\mathrm{cbn}} \in \mathrm{ne}_{\mathrm{w}}$ and hence $s^{\mathrm{cbn}} \in \mathrm{na}_{\mathrm{w}}$. Moreover ! $u^{\mathrm{cbn}} \in \mathrm{na}_{\mathrm{w}}$ also holds, which implies $!u^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$. We can then conclude $s^{\mathrm{cbn}}!u^{\mathrm{cbn}} \in \mathrm{ne} \mathrm{e}_{\mathrm{w}}$, i.e. $(s u)^{\mathrm{cbn}} \in \mathrm{ne}_{\mathrm{w}}$.
- If $t \in \mathrm{no}_{\mathrm{n}}$ comes from $t \in \mathrm{ne}_{\mathrm{n}}$, then $t^{\mathrm{cbn}} \in \mathrm{ne}_{\mathrm{w}}$ holds by the previous point, which gives $t^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$.
- If $t=\lambda x . u \in \mathrm{no}_{\mathrm{n}}$ comes from $u \in \mathrm{no}_{\mathrm{n}}$, then the i.h. (1b) gives $u^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$, which implies $\lambda x . u^{\mathrm{cbn}} \in \mathrm{nb}_{\mathrm{w}}$, thus giving $\lambda x . u^{\mathrm{cbn}} \in \mathrm{no}_{\mathrm{w}}$. We conclude since $(\lambda x . u)^{\mathrm{cbn}}=\lambda x . u^{\mathrm{cbn}}$.

2. By Lemma 4.3 and Proposition 2.11 it is sufficient to prove that $t \in \mathrm{no}_{\mathrm{v}}$ implies $t^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$. This is done by simultaneously showing the following points:
(a) If $t \in \mathrm{vr}_{\mathrm{v}}$, then $t^{\mathrm{cbv}} \in \mathrm{na}_{\mathrm{w}}$.
(b) If $t \in \mathrm{ne}_{\mathrm{v}}$, then $t^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$.
(c) If $t \in \mathrm{no}_{\mathrm{v}}$, then $t^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$.

- If $t=x \in \mathrm{vr}_{\mathrm{v}}$, then $x^{\mathrm{cbv}}=!x$ and $!x \in \mathrm{na}_{\mathrm{w}}$.
- If $t=s[x \backslash u] \in \mathrm{vr}_{\mathrm{v}}$ comes from $s \in \mathrm{vr}_{\mathrm{v}}$ and $u \in \mathrm{ne}_{\mathrm{v}}$, then the i.h. (2a) gives $s^{\mathrm{cbv}} \in \mathrm{na}_{\mathrm{w}}$ and $u^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$, which in turn implies $u^{\mathrm{cbv}} \in \mathrm{nb}_{\mathrm{w}}$. Hence, it allows us to conclude $s^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right] \in \mathrm{na}_{\mathrm{w}}$.
- If $t=s u \in \mathrm{ne}_{\mathrm{v}}$ comes from $s \in \mathrm{vr}_{\mathrm{v}}$ and $u \in \mathrm{no}_{\mathrm{v}}$. It is immediate to see that $s^{\mathrm{cbv}}=\mathrm{L}\langle!x\rangle$ for some variable $x$. Thus, $t^{\mathrm{cbv}}=\mathrm{L}\langle x\rangle u^{\mathrm{cbv}}$. Moreover, by i.h. (2a) $s^{\mathrm{cbv}} \in \mathrm{na}_{\mathrm{w}}$ holds, which implies $r \in \mathrm{nb}_{\mathrm{w}}$ for all $[y \backslash r]$ in L. Thus, $\mathrm{L}\langle x\rangle \in \mathrm{ne}_{\mathrm{w}}$ holds. This implies $\mathrm{L}\langle x\rangle \in \mathrm{na}_{\mathrm{w}}$, by definition. Also, the $i . h$. (2c) on $u \in$ no $_{v}$ gives $u^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$. Hence, we conclude $t^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$.
- If $t=s u \in \mathrm{ne}_{\mathrm{v}}$ comes from $s \in \mathrm{ne}_{\mathrm{v}}$ and $u \in \mathrm{no}_{\mathrm{v}}$. By i.h. (2b) $s^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$ holds. In particular, it implies $s^{\mathrm{cbv}} \in \mathrm{nb}_{\mathrm{W}}$ (i.e. $\neg \operatorname{bang}\left(s^{\mathrm{cbv}}\right)$ ). Hence, $t^{\mathrm{cbv}}=\operatorname{der}\left(s^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}$ and $\operatorname{der}\left(s^{\mathrm{cbv}}\right) \in \mathrm{ne}_{\mathrm{w}}$. Moreover, the latter implies $\operatorname{der}\left(s^{\mathrm{cbv}}\right) \in \mathrm{na}_{\mathrm{w}}$, by definition. Also, the $i . h$. (2c) on $u \in \mathrm{no}_{\mathrm{v}}$ gives $u^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$. Thus, we conclude $t^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$.
- If $t=s[x \backslash u] \in \mathrm{ne}_{\mathrm{v}}$ comes from $s \in \mathrm{ne}_{\mathrm{v}}$ and $u \in \mathrm{ne}_{\mathrm{v}}$, then the $i . h$. (2b) gives $s^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$ and $u^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$, which in turn implies $u^{\mathrm{cbv}} \in \mathrm{nb}_{\mathrm{w}}$. Hence, it allows us to conclude $s^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right] \in \mathrm{ne}_{\mathrm{w}}$.
- If $t=\lambda x . s \in \mathrm{no}_{\mathrm{v}}$, then $t^{\mathrm{cbv}}=!\lambda x . s^{\mathrm{cbv}}$. Thus, $t^{\mathrm{cbv}} \in \mathrm{na}_{\mathrm{w}}$ holds and we conclude $t^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$.
- If $t \in \mathrm{no}_{\mathrm{v}}$ comes from $t \in \mathrm{vr}_{\mathrm{v}}$. Then, item (2a) gives $t^{\mathrm{cbv}} \in \mathrm{na}_{\mathrm{w}}$ which implies $t^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$ by definition.
- If $t \in \mathrm{no}_{\mathrm{v}}$ comes from $t \in \mathrm{ne}_{\mathrm{v}}$. Then, item (2b) gives $t^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$ which implies $t^{\text {cbv }} \in \mathrm{no}_{\mathrm{W}}$.
- If $t=s[x \backslash u] \in \mathrm{no}_{\mathrm{v}}$ comes from $s \in \mathrm{no}_{\mathrm{v}}$ and $u \in \mathrm{ne}_{\mathrm{v}}$, then the i.h. (2c) and (2b) gives $s^{\mathrm{cbv}} \in \mathrm{no}_{\mathrm{w}}$ and $u^{\mathrm{cbv}} \in \mathrm{ne}_{\mathrm{w}}$, which in turn implies $u^{\mathrm{cbv}} \in \mathrm{nb}_{\mathrm{w}}$. Hence, it allows us to conclude $s^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right] \in \mathrm{no}_{\mathrm{w}}$.

Simulation of CBN and CBV reductions in the $\lambda!$-calculus can be shown by induction on the reduction relations. We start with a technical lemma showing that the substitution is compatible with the CBN translation, whereas in the case of the CBV translation the argument of the substitution must be adjusted.

Lemma 4.5. Let $t, u, v \in \mathcal{T}_{\lambda}$ with $v$ a value.

1. $t\{x \backslash u\}^{\mathrm{cbn}}=t^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}$.
2. $t\{x \backslash v\}^{\mathrm{cbv}}=t^{\mathrm{cbv}}\{x \backslash u\}$, where $v^{\mathrm{cbv}}=!u$.

Proof.

1. By induction on $t$.

- $t=x$. Then, $x\{x \backslash u\}^{\mathrm{cbn}}=u^{\mathrm{cbn}}=x\left\{x \backslash u^{\mathrm{cbn}}\right\}=x^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}$.
- $t=y$. Then, $y\{x \backslash u\}^{\mathrm{cbn}}=y=y\left\{x \backslash u^{\mathrm{cbn}}\right\}=y^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}$.
- $t=t_{0} t_{1}$. Then,

$$
\begin{aligned}
\left(t_{0} t_{1}\right)\{x \backslash u\}^{\mathrm{cbn}} & = & & \left(t_{0}\{x \backslash u\} t_{1}\{x \backslash u\}\right)^{\mathrm{cbn}} \\
& = & & t_{0}\{x \backslash u\}^{\mathrm{cbn}}!\left(t_{1}\{x \backslash u\}^{\mathrm{cbn}}\right) \\
& =i . h . & & t_{0}^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}!t_{1}^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\} \\
& = & & \left(t_{0}^{\mathrm{cbn}}!t_{1}^{\mathrm{cbn}}\right)\left\{x \backslash u^{\mathrm{cbn}}\right\} \\
& = & & \left(t_{0} t_{1}\right)^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}
\end{aligned}
$$

- $t=\lambda y . t_{0}$. Straightforward by the $i . h$.
- $t=t_{0}\left[y \backslash t_{1}\right]$. Similar to the previous case.

2. By induction on $t$.

- $t=x$. Then, $x\{x \backslash v\}^{\mathrm{cbv}}=v^{\mathrm{cbv}}=!u=(!x)\{x \backslash u\}=x^{\mathrm{cbv}}\{x \backslash u\}$.
- $t=y$. Then, $y\{x \backslash v\}^{\mathrm{cbv}}=y^{\mathrm{cbv}}=y^{\mathrm{cbv}}\{x \backslash u\}$.
- $t=t_{0} t_{1}$. There are two cases:
- If $t_{0}{ }^{\mathrm{cbv}}=\mathrm{L}\langle!s\rangle$, then the $i . h$. gives $t_{0}\{x \backslash v\}^{\mathrm{cbv}}=t_{0}{ }^{\mathrm{cbv}}\{x \backslash u\}=$ $\mathrm{L}\langle!s\rangle\{x \backslash u\}=\mathrm{L}\{x \backslash u\}\langle!s\{x \backslash u\}\rangle$. Then,

$$
\begin{array}{rlrl}
\left(t_{0} t_{1}\right)\{x \backslash v\}^{\mathrm{cbv}} & = & & \left(t_{0}\{x \backslash v\} t_{1}\{x \backslash v\}\right)^{\mathrm{cbv}} \\
& = & \mathrm{L}\{x \backslash u\}\langle s\{x \backslash u\}\rangle\left(t_{1}\{x \backslash v\}\right)^{\mathrm{cbv}} \\
& =\text { i.h. } & \mathrm{L}\{x \backslash u\}\langle s\{x \backslash u\}\rangle\left(t_{1}^{\mathrm{cbv}}\{x \backslash u\}\right) \\
& = & \mathrm{L}\langle s\rangle t_{1}^{\mathrm{cbv}}\{x \backslash u\} \\
& = & \left(t_{0} t_{1}\right)^{\mathrm{cbv}}\{x \backslash u\}
\end{array}
$$

- Otherwise, $\left(t_{0} t_{1}\right)\{x \backslash v\}^{\mathrm{cbv}}=\left(t_{0}\{x \backslash v\} t_{1}\{x \backslash v\}\right)^{\mathrm{cbv}}$. The i.h. gives $t_{0}\{x \backslash v\}^{\mathrm{cbv}}=t_{0}{ }^{\mathrm{cbv}}\{x \backslash u\}$ and $t_{1}\{x \backslash v\}^{\mathrm{cbv}}=t_{1}{ }^{\mathrm{cbv}}\{x \backslash u\}$. Moreover, $t_{0}{ }^{\mathrm{cbv}} \neq \mathrm{L}\langle!s\rangle$ implies the same for $t_{0}{ }^{\mathrm{cbv}}\{x \backslash u\}$, simply because $u$ is a variable or an abstraction. Then,

$$
\begin{aligned}
\left(t_{0}\{x \backslash v\} t_{1}\{x \backslash v\}\right)^{\mathrm{cbv}} & =\operatorname{der}\left(t_{0}{ }^{\mathrm{cbv}}\{x \backslash u\}\right) t_{1} \mathrm{cbv}\{x \backslash u\} \\
& =\left(\operatorname{der}\left(t_{0} \mathrm{cbv}\right) t_{1} \mathrm{cbv}\right)\{x \backslash u\} \\
& =\left(t_{0} t_{1}\right)^{\mathrm{cbv}}\{x \backslash u\}
\end{aligned}
$$

- $t=\lambda y . t_{0}$. Then,

$$
\begin{aligned}
\left(\lambda y \cdot t_{0}\right)\{x \backslash v\}^{\mathrm{cbv}} & = & & \left(\lambda y \cdot t_{0}\{x \backslash v\}\right)^{\mathrm{cbv}} \\
& = & & !\lambda y \cdot\left(t_{0}\{x \backslash v\}\right)^{\mathrm{cbv}} \\
& =\text { i.h. } & & !\lambda y \cdot\left(t_{0} \mathrm{cbv}\{x \backslash u\}\right) \\
& = & & !\left(\lambda y \cdot t_{0} \mathrm{cbv}\right)\{x \backslash u\} \\
& = & & \left(\lambda y \cdot t_{0}\right)^{\mathrm{cbv}}\{x \backslash u\}
\end{aligned}
$$

- $t=t_{0}\left[y \backslash t_{1}\right]$. This case is straightforward by the $i . h$.

The following lemma shows that CBN and CBV reductions are simulated via the embeddings. One CBN reduction step is simulated by exactly one step of weak reduction, whereas a single CBV reduction step may give rise to several steps of weak reduction An example is given after the proof of the lemma.

Lemma 4.6. Let $t, s \in \mathcal{T}_{\lambda}$.

1. If $t \rightarrow_{\mathrm{n}} s$, then $t^{\mathrm{cbn}} \rightarrow_{\mathrm{w}} s^{\mathrm{cbn}}$.
2. If $t \rightarrow_{\mathrm{v}} s$, then $t^{\mathrm{cbv}} \rightarrow_{\mathrm{w}}^{+} s^{\mathrm{cbv}}$.

Moreover, exponential steps in CBN/CBV are always simulated by exponential steps in $\lambda$ !, while multiplicative steps in CBN/CBV are simulated by at least one multiplicative step in $\lambda$ !.

Proof. Both proofs are by induction on the reduction relations.

1. Case $t \rightarrow_{\mathrm{n}} s$.

- $t=\mathrm{L}\langle\lambda x . r\rangle u \mapsto_{\mathrm{dB}} \mathrm{L}\langle r[x \backslash u]\rangle=s$. Then,

$$
t^{\mathrm{cbn}}=\mathrm{L}^{\mathrm{cbn}}\left\langle\lambda x . r^{\mathrm{cbn}}\right\rangle!u^{\mathrm{cbn}} \rightarrow_{\mathrm{dB}} \mathrm{~L}^{\mathrm{cbn}}\left\langle r^{\mathrm{cbn}}\left[x \backslash!u^{\mathrm{cbn}}\right]\right\rangle=s^{\mathrm{cbn}}
$$

- $t=r[x \backslash u] \mapsto_{\mathrm{s}} r\{x \backslash u\}=s$. Then,

$$
t^{\mathrm{cbn}}=r^{\mathrm{cbn}}\left[x \backslash!u^{\mathrm{cbn}}\right] \rightarrow_{\mathrm{s}!} r^{\mathrm{cbn}}\left\{x \backslash u^{\mathrm{cbn}}\right\}={ }_{L} 4.5 s^{\mathrm{cbn}}
$$

- $t=r u \rightarrow_{\mathrm{n}} r^{\prime} u=s$, or $t=\lambda x . r \rightarrow_{\mathrm{n}} \lambda x . r^{\prime}=s$, or $t=r[x \backslash u] \rightarrow_{\mathrm{n}}$ $r^{\prime}[x \backslash u]=s$, where $r \rightarrow_{\mathrm{n}} r^{\prime}$. Then, we easily conclude by the i.h.

2. Case $t \rightarrow_{\mathrm{v}} s$.

- $t=\mathrm{L}\langle\lambda x . r\rangle u \mapsto_{\mathrm{dB}} \mathrm{L}\langle r[x \backslash u]\rangle=s$. Then,

$$
t^{\mathrm{cbv}}=\mathrm{L}^{\mathrm{cbv}}\left\langle\lambda x \cdot r^{\mathrm{cbv}}\right\rangle u^{\mathrm{cbv}} \rightarrow_{\mathrm{dB}} \mathrm{~L}^{\mathrm{cbv}}\left\langle r^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right]\right\rangle=s^{\mathrm{cbv}}
$$

- $t=r[x \backslash \mathrm{~L}\langle v\rangle] \mapsto_{\mathrm{sv}} \mathrm{L}\langle r\{x \backslash v\}\rangle=s$. Then,

$$
t^{\mathrm{cbv}}=r^{\mathrm{cbv}}\left[x \backslash \mathrm{~L}^{\mathrm{cbv}}\left\langle v^{\mathrm{cbv}}\right\rangle\right] \rightarrow_{\mathrm{s}!} \mathrm{L}^{\mathrm{cbv}}\left\langle r^{\mathrm{cbv}}\{x \backslash u\}\right\rangle={ }_{L} 4.5 s^{\mathrm{cbv}}
$$

where $v^{\mathrm{cbv}}=!u$.

- $t=r_{0} u \rightarrow_{\mathrm{v}} r_{1} u=s$ comes from $r_{0} \rightarrow_{\mathrm{v}} r_{1}$. The i.h. gives $r_{0}{ }^{\mathrm{cbv}} \rightarrow_{\mathrm{w}}^{+}$ $r_{1}{ }^{\mathrm{cbv}}$. There are two cases:
- If $r_{0}{ }^{\mathrm{cbv}}=\mathrm{L}\langle!r\rangle$, then $t^{\mathrm{cbv}}=\mathrm{L}\langle r\rangle u^{\mathrm{cbv}}$. The $i . h$. implies in particular that $r_{1}{ }^{\mathrm{cbv}}=\mathrm{L}^{\prime}\left\langle!r^{\prime}\right\rangle$, which implies in turn $\mathrm{L}\langle r\rangle \rightarrow_{\mathrm{w}}^{+} \mathrm{L}^{\prime}\left\langle r^{\prime}\right\rangle$. We then have $t^{\mathrm{cbv}}=\mathrm{L}\langle r\rangle u^{\mathrm{cbv}} \rightarrow_{\mathrm{w}}^{+} \mathrm{L}^{\prime}\left\langle r^{\prime}\right\rangle u^{\mathrm{cbv}}=s^{\mathrm{cbv}}$.
- Otherwise, $t^{\mathrm{cbv}}=\operatorname{der}\left(r_{0}{ }^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}$. We have again two cases.
* If $r_{1}{ }^{\mathrm{cbv}}=\mathrm{L}\langle!r\rangle$, then
$t^{\mathrm{cbv}}=\operatorname{der}\left(r_{0}{ }^{\mathrm{cbv}}\right) u^{\mathrm{cbv}} \rightarrow_{\mathrm{w}}^{+} \operatorname{der}\left(r_{1}{ }^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}=\operatorname{der}(\mathrm{L}\langle!r\rangle) u^{\mathrm{cbv}} \rightarrow_{\mathrm{d}!} \mathrm{L}\langle r\rangle u^{\mathrm{cbv}}=s^{\mathrm{cbv}}$
* Otherwise, the i.h. allows us to conclude

$$
t^{\mathrm{cbv}}=\operatorname{der}\left(r_{0}{ }^{\mathrm{cbv}}\right) u^{\mathrm{cbv}} \rightarrow_{\mathrm{w}}^{+} \operatorname{der}\left(r_{1}{ }^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}=s^{\mathrm{cbv}}
$$

- $t=r u \rightarrow_{\mathrm{v}} r u^{\prime}=s$ comes from $u \rightarrow_{\mathrm{v}} u^{\prime}$. Then, there are two cases according to $r^{\mathrm{cbv}}$ but the proof is easy using the i.h.
- $t=r[x \backslash u] \rightarrow_{\mathrm{v}} r^{\prime}[x \backslash u]=s$ or $t=u[x \backslash r] \rightarrow_{\mathrm{v}} u\left[x \backslash r^{\prime}\right]=s$, where $r \rightarrow_{\mathrm{v}} r^{\prime}$. Then, we easily conclude by the $i . h$.

As mentioned above, the CBV case may require several reduction steps between $t^{\mathrm{cbv}}$ and $s^{\mathrm{cbv}}$. For instance, if $t=\mathrm{I} y z \rightarrow_{\mathrm{v}} w[w \backslash y] z=s$, then $t^{\mathrm{cbv}}=\operatorname{der}((\lambda w!!w)!y)!z \rightarrow_{\mathrm{w}} \operatorname{der}(!w[w \backslash!y])!z \rightarrow_{\mathrm{w}} w[w \backslash!y]!z=s^{\mathrm{cbv}}$.

One may as well wonder if the converse of Lemma 4.6 also holds. Indeed, the property holds for the CBN case, i.e. $t^{\mathrm{cbn}} \rightarrow_{\mathrm{w}} s^{\mathrm{cbn}}$ implies $t \rightarrow_{\mathrm{n}} s$. However, the following example [8] shows that the property does not hold for CBV. Let $t=(\lambda x .(\lambda y . y) z) z$ and $s=(\lambda x . y[y \backslash z]) z$. Then $t^{\mathrm{cbv}}=(\lambda x .(\lambda y .!y)!z)!z \rightarrow_{\mathrm{w}}$ $(\lambda x .(!y)[y \backslash!z])!z=s^{\mathrm{cbv}}$, but $t$ does not reduce to $s$ in CBV.

Example 4.7. Let us consider again the term $t=\mathrm{K}_{\lambda} \mathrm{I}_{\lambda} \Omega_{\lambda}$ of the Example 4.2 We have seen that $t^{\mathrm{cbn}}=\mathrm{K}!\mathrm{I}!\Omega$ and $t^{\mathrm{cbv}}=\operatorname{der}((\lambda x .!\lambda y .!x)(!\lambda x .!x)) \Omega$. The CBN reduction of $t$ is the following:

$$
\mathrm{K}_{\lambda} \mathrm{I}_{\lambda} \Omega_{\lambda} \rightarrow_{\mathrm{dB}}\left(\lambda y \cdot x\left[x \backslash \mathrm{I}_{\lambda}\right]\right) \Omega_{\lambda} \rightarrow_{\mathrm{s}}\left(\lambda y \cdot \mathrm{I}_{\lambda}\right) \Omega_{\lambda} \rightarrow_{\mathrm{dB}} \mathrm{I}_{\lambda}\left[y \backslash \Omega_{\lambda}\right] \rightarrow_{\mathrm{s}} \mathrm{I}_{\lambda}
$$

## System $\mathcal{N}$ for Call-by-Name

$$
\begin{array}{cc}
\frac{\Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash t: \tau \quad\left(\Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}}{\Gamma:[\sigma] \vdash x: \sigma}\left(\mathrm{ax}_{\mathrm{n}}\right) & \frac{\left.\Gamma \mathrm{es}_{\mathrm{n}}\right)}{} \\
\frac{\Gamma \vdash t: \tau}{\Gamma \Downarrow x \vdash t[x \backslash u]: \tau} \\
& \frac{\Gamma \vdash t:\left[\sigma_{i}\right]_{i \in I} \rightarrow \tau \quad\left(\Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}}{\Gamma+{ }_{i \in I} \Delta_{i} \vdash t u: \tau}\left(\mathrm{app}_{\mathrm{n}}\right)
\end{array}
$$

## System $\mathcal{V}$ for Call-by-Value

$$
\left.\begin{array}{c}
\frac{\Gamma \vdash t: \sigma}{x: \mathrm{M} \vdash x: \mathrm{M}}\left(\mathrm{ax}_{\mathrm{v}}\right)
\end{array} \frac{\Delta \vdash u: \Gamma(x)}{(\Gamma \Downarrow x)+\Delta \vdash t[x \backslash u]: \sigma}\left(\mathrm{es}_{\mathrm{v}}\right), ~\left(\Gamma_{i} \vdash t: \tau_{i}\right)_{i \in I} \quad \frac{\Gamma \vdash t:[\mathrm{M} \rightarrow \tau] \quad \Delta \vdash u: \mathrm{M}}{\Gamma+\Delta \vdash t u: \tau}\left(\mathrm{app}_{\mathrm{v}}\right)\right)
$$

Figure 2: Typing schemes for CBN/CBV.
The corresponding reduction of $t^{\mathrm{cbn}}$ in the $\lambda$ !-calculus is:

$$
\mathrm{K}!\mathrm{I}!\Omega \rightarrow_{\mathrm{dB}}(\lambda y \cdot x[x \backslash!\mathrm{I}])!\Omega \rightarrow_{\mathrm{s}!}(\lambda y . \mathrm{I})!\Omega \rightarrow_{\mathrm{dB}} \mathrm{I}[y \backslash \Omega] \rightarrow_{\mathrm{s}!} \mathrm{I}
$$

The CBV reduction of $t$ is the following:

$$
\begin{gathered}
\mathrm{K}_{\lambda} \mathrm{I}_{\lambda} \Omega_{\lambda} \rightarrow_{\mathrm{dB}}\left(\lambda y \cdot x\left[x \backslash \mathrm{I}_{\lambda}\right]\right) \Omega_{\lambda} \rightarrow_{\mathrm{sv}}\left(\lambda y \cdot \mathrm{I}_{\lambda}\right) \Omega_{\lambda} \rightarrow_{\mathrm{dB}} \\
\rightarrow_{\mathrm{dB}} \mathrm{I}_{\lambda}\left[y \backslash \Omega_{\lambda}\right] \rightarrow_{\mathrm{dB}} \mathrm{I}_{\lambda}\left[y \backslash(x x)\left[x \backslash \Delta_{\lambda}\right]\right]
\end{gathered}
$$

Since $\mathrm{I}_{\lambda}\left[y \backslash(x x)\left[x \backslash \Delta_{\lambda}\right]\right] \rightarrow_{\text {sv }} \mathrm{I}_{\lambda}\left[y \backslash \Omega_{\lambda}\right]$, the reduction enters a loop. Accordingly, the reduction of $t^{\mathrm{cbv}}$ is:

$$
\begin{gathered}
\operatorname{der}((\lambda x .!\lambda y .!x)(!\lambda x .!x)) \Omega \rightarrow_{\mathrm{dB}} \operatorname{der}((!\lambda y .!x)[x \backslash!\lambda x .!x]) \Omega \rightarrow_{\mathrm{d}!} \\
\rightarrow_{\mathrm{d}!}(\lambda y .!x)[x \backslash!\lambda x .!x] \Omega \rightarrow_{\mathrm{dB}}(!x)[x \backslash!\lambda x .!x][y \backslash \Omega] \rightarrow_{\mathrm{s}!}(!\lambda x .!x)[y \backslash \Omega] \rightarrow_{\mathrm{dB}} \\
\rightarrow_{\mathrm{dB}}(!\lambda x .!x)[y \backslash(x!x)[x \backslash!\Delta]]
\end{gathered}
$$

Since $(!\lambda x .!x)[y \backslash(x!x)[x \backslash!\Delta]] \rightarrow_{s!}(!\lambda x!!x)[y \backslash \Omega]$, the reduction enters a loop.
Non-Idempotent Types for Call-by-Name and Call-by-Value. For CBN we use the non-idempotent type system defined in [37] for explicit substitutions, that we present in Figure 2 (top), and which is an extension of that in 29]. For CBV, we slightly reformulate the non-idempotent system in 33], tha we present in Figure 2 (bottom), in order to recover completeness of the (typed) CBV translation. In both cases, the types of our systems are the same as the ones in system $\mathcal{U}$.

We write $\triangleright_{\mathcal{N}} \Gamma \vdash t: \sigma\left(\right.$ resp. $\left.\triangleright_{\mathcal{V}} \Gamma \vdash t: \sigma\right)$ if there exists a type derivation in system $\mathcal{N}$ (resp. $\mathcal{V})$. We use names for type derivations as in Sec.3. A key point in rule $\left(\mathrm{app}_{\mathrm{v}}\right)$ is that left hand sides of applications are typed with multisets
of the form $[\mathrm{M} \rightarrow \tau]$, where $\tau$ is any type, potentially a base one, while [33] necessarily requires a multiset of the form $\left[\mathrm{M} \rightarrow \mathrm{M}^{\prime}\right]$, a subtle difference which breaks completeness. System $\mathcal{N}$ (resp. $\mathcal{V}$ ) can be understood as a relational model of the Call-by-Name (resp. Call-by-Value) calculus, in the sense that typing is stable by reduction and expansion (see Sec. 4.1 and 4.2).

The CBV translation is not complete for the system in [33], i.e. there exists a $\lambda$-term $t$ such that $\Gamma \vdash t^{\text {cbv }}: \sigma$ is derivable in $\mathcal{U}$ but $\Gamma \vdash t: \sigma$ is not derivable in their system (see [33] Proposition 15). In this article, we recover the completeness of the translations. More precisely, our two embeddings are sound and complete w.r.t. system $\mathcal{U}$ :

Theorem 4.8 (Soundness/Completeness of the Embeddings). Let $t \in \mathcal{T}_{\lambda}$.

1. $\triangleright_{\mathcal{N}} \Gamma \vdash t: \sigma$ iff $\triangleright_{\mathcal{U}} \Gamma \vdash t^{\mathrm{cbn}}: \sigma$.
2. $\triangleright_{\mathcal{V}} \Gamma \vdash t: \sigma$ iff $\triangleright_{\mathcal{U}} \Gamma \vdash t^{\mathrm{cbv}}: \sigma$.

Proof. 1. $\Rightarrow$ ) By induction on $t$.

- $t=x$. Then, $\Gamma=x:[\sigma]$ and we conclude by rule (ax), since $t^{\mathrm{cbn}}=x$.
- $t=s u$. Then, $\Gamma=\Gamma^{\prime}+_{i \in I} \Delta_{i}, \Gamma^{\prime} \vdash s:\left[\tau_{i}\right]_{i \in I} \rightarrow \sigma$ and $\left(\Delta_{i} \vdash u: \tau_{i}\right)_{i \in I}$. By i.h. $\Gamma^{\prime} \vdash s^{\mathrm{cbn}}:\left[\tau_{i}\right]_{i \in I} \rightarrow \sigma$ and $\left(\Delta_{i} \vdash u^{\mathrm{cbn}}: \tau_{i}\right)_{i \in I}$. Moreover, by definition $t^{\mathrm{cbn}}=s^{\mathrm{cbn}}!u^{\mathrm{cbn}}$. Thus, we conclude by rules ( bg ) and (app)

$$
\frac{\Gamma^{\prime} \vdash s^{\mathrm{cbn}}:\left[\tau_{i}\right]_{i \in I} \rightarrow \sigma}{\Gamma^{\prime}+_{i \in I} \Delta_{i} \vdash s^{\mathrm{cbn}}!u^{\mathrm{cbn}}: \sigma} \frac{\left(\Delta_{i} \vdash u^{\mathrm{cbn}}: \tau_{i}\right)_{i \in I}}{+_{i \in I} \Delta_{i} \vdash!u^{\mathrm{cbn}}:\left[\tau_{i}\right]_{i \in I}}
$$

- $t=\lambda x$.s. Then, $\sigma=\Gamma^{\prime}(x) \rightarrow \tau, \Gamma=\Gamma^{\prime} \Downarrow x$ and $\Gamma^{\prime} \vdash s: \tau$. By i.h. $\Gamma^{\prime} \vdash s^{\mathrm{cbn}}: \tau$. Moreover, by definition $t^{\mathrm{cbn}}=\lambda x . s^{\mathrm{cbn}}$, hence we conclude by rule (abs).
- $t=s[x \backslash u]$. Then, $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+_{i \in I} \Delta_{i}, \Gamma^{\prime} \vdash s: \sigma$ and $\left(\Delta_{i} \vdash u: \tau_{i}\right)_{i \in I}$ with $\Gamma^{\prime}(x)=\left[\tau_{i}\right]_{i \in I}$. By i.h. $\Gamma^{\prime} \vdash s^{\mathrm{cbn}}: \sigma$ and $\left(\Delta_{i} \vdash u^{\mathrm{cbn}}: \tau_{i}\right)_{i \in I}$ hold. Finally, since $t^{\mathrm{cbn}}=s^{\mathrm{cbn}}\left[x \backslash!u^{\mathrm{cbn}}\right]$, we conclude by rules (bg) and (es).
$\Leftarrow) \mathrm{By}$ induction on $t$.
- $t=x$. Then, $x^{\mathrm{cbn}}=x, \Gamma=x:[\sigma]$ and the result is immediate using rule $\left(\mathrm{ax}_{\mathrm{n}}\right)$.
- $t=s u$. Then, $t^{\mathrm{cbn}}=s^{\mathrm{cbn}!} u^{\mathrm{cbn}}, \Gamma=\Gamma^{\prime}+\Delta, \Gamma^{\prime} \vdash s^{\mathrm{cbn}}: \mathrm{M} \rightarrow \sigma$ and $\Delta \vdash!u^{\mathrm{cbn}}: \mathrm{M}$. Moreover, by rule $(\mathrm{bg}), \Delta=+_{i \in I} \Delta_{i}, \mathrm{M}=$ $\left[\tau_{i}\right]_{i \in I}$ and $\left(\Delta_{i} \vdash u^{\mathrm{cbn}}: \tau_{i}\right)_{i \in I}$. The i.h. gives $\Gamma^{\prime} \vdash s:\left[\tau_{i}\right]_{i \in I} \rightarrow \sigma$ and $\left(\Delta_{i} \vdash u: \tau_{i}\right)_{i \in I}$. Finally, we conclude by rule $\left(\mathrm{app}_{\mathrm{n}}\right)$.
- $t=\lambda x$.s. Then $t^{\mathrm{cbn}}=\lambda x . s^{\mathrm{cbn}}, \sigma=\Gamma^{\prime}(x) \rightarrow \tau, \Gamma=\Gamma^{\prime} \mathbb{} \backslash x$ and $\Gamma^{\prime} \vdash s^{\mathrm{cbn}}: \tau$. The result follow immediately from the $i . h$. using rule $\left(\mathrm{abs}_{\mathrm{n}}\right)$.
- $t=s[x \backslash u]$. Then, $t^{\mathrm{cbn}}=s^{\mathrm{cbn}}\left[x \backslash!u^{\mathrm{cbn}}\right], \Gamma=\left(\Gamma^{\prime} \| x\right)+\Delta, \Gamma^{\prime} \vdash$ $s^{\mathrm{cbn}}: \sigma$ and $\Delta \vdash!u^{\mathrm{cbn}}: \Gamma^{\prime}(x)$. Moreover, by rule (bg), $\Delta=+_{i \in I} \Delta_{i}$, $\Gamma^{\prime}(x)=\left[\tau_{i}\right]_{i \in I}$ and $\left(\Delta_{i} \vdash u^{\mathrm{cbn}}: \tau_{i}\right)_{i \in I}$. The i.h. gives $\Gamma^{\prime} \vdash s: \sigma$ and $\left(\Delta_{i} \vdash u: \tau_{i}\right)_{i \in I}$. Finally, we conclude by rule $\left(\mathrm{es}_{\mathrm{n}}\right)$.

2 . $\Rightarrow$ ) By induction on $t$.

- $t=x$. Then, $\sigma=\mathrm{M}$ and $\Gamma=x: \mathrm{M}$ where, by definition, $\mathrm{M}=$ $\left[\tau_{i}\right]_{i \in I}$. Moreover, by definition of the embedding, $t^{\mathrm{cbv}}=!x$. Thus, we conclude by rules ( ax ) and ( bg )

$$
\frac{\left(\overline{x:\left[\tau_{i}\right] \vdash x: \tau_{i}}\right)_{i \in I}}{x:\left[\tau_{i}\right]_{i \in I} \vdash!x:\left[\tau_{i}\right]_{i \in I}}
$$

- $t=s u$. Then, $\Gamma=\Gamma^{\prime}+\Delta, \Gamma^{\prime} \vdash s:[\mathrm{M} \rightarrow \sigma]$ and $\Delta \vdash u: \mathrm{M}$. By i.h. $\Gamma^{\prime} \vdash s^{\mathrm{cbv}}:[\mathrm{M} \rightarrow \sigma]$ and $\Delta \vdash u^{\mathrm{cbv}}: \mathrm{M}$. There are two cases:
- If $s^{\mathrm{cbv}}=\mathrm{L}\langle!r\rangle$, then $t^{\mathrm{cbv}}=\mathrm{L}\langle r\rangle u^{\mathrm{cbv}}$. From $\Gamma^{\prime} \vdash \mathrm{L}\langle!r\rangle:[\mathrm{M} \rightarrow \sigma]$ it is immediate to see, by induction on L using (es), that there exists $\Gamma^{\prime \prime}$ s.t. $\Gamma^{\prime \prime} \vdash!r:[\mathrm{M} \rightarrow \sigma]$. Then, $\Gamma^{\prime \prime} \vdash r: \mathrm{M} \rightarrow \sigma$ holds from rule (bg). Moreover, by straighforward induction on $L$ once again, we get $\Gamma^{\prime} \vdash \mathrm{L}\langle r\rangle: \mathrm{M} \rightarrow \sigma$. Finally, we conclude using (app) by using also $\Delta \vdash u^{\mathrm{cbv}}: \mathrm{M}$.
- Otherwise, $t^{\mathrm{cbv}}=\operatorname{der}\left(s^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}$. Then, we resort to rules (dr) and (app) to conclude

$$
\frac{\frac{\Gamma^{\prime} \vdash s^{\mathrm{cbv}}:[\mathrm{M} \rightarrow \sigma]}{\Gamma^{\prime} \vdash \operatorname{der} s^{\mathrm{cbv}}: \mathrm{M} \rightarrow \sigma} \quad \Delta \vdash u^{\mathrm{cbv}}: \mathrm{M}}{\Gamma^{\prime}+\Delta \vdash \operatorname{der}\left(s^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}: \sigma}
$$

- $t=\lambda x$.s. Then $\sigma=\left[\Gamma_{i}(x) \rightarrow \tau_{i}\right]_{i \in I}, \Gamma=+_{i \in I} \Gamma_{i} \backslash x$ and $\left(\Gamma_{i} \vdash s: \tau_{i}\right)_{i \in I}$. By i.h. $\left(\Gamma_{i} \vdash s^{\mathrm{cbv}}: \tau_{i}\right)_{i \in I}$. Thus, we conclude with rules (abs) and (bg) since, by definition, $t^{\mathrm{cbv}}=!\lambda x . s^{\mathrm{cbv}}$

$$
\frac{\left(\frac{\Gamma_{i} \vdash s^{\mathrm{cbv}}: \tau_{i}}{\Gamma_{i} \backslash x \vdash \lambda x . s^{\mathrm{cbv}}: \Gamma_{i}(x) \rightarrow \tau_{i}}\right)_{i \in I}}{+_{i \in I} \Gamma_{i} \backslash x \vdash!\lambda x . s^{\mathrm{cbv}}:\left[\Gamma_{i}(x) \rightarrow \tau_{i}\right]_{i \in I}}
$$

- $t=s[x \backslash u]$. Then, $\Gamma=\left(\Gamma^{\prime} \ x x\right)+\Delta, \Gamma^{\prime} \vdash s: \sigma$ and $\Delta \vdash u: \Gamma^{\prime}(x)$. By i.h. $\Gamma^{\prime} \vdash s^{\mathrm{cbv}}: \sigma$ and $\Delta \vdash u^{\mathrm{cbv}}: \Gamma^{\prime}(x)$ hold. Hence, since $t^{\mathrm{cbv}}=$ $s^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right]$, we conclude by rule (es).
$\Leftarrow)$ By induction on $t$.
- $t=x$. Then $x^{\mathrm{cbv}}=!x, \sigma=\left[\tau_{i}\right]_{i \in I}$ and $\Gamma=x:\left[\tau_{i}\right]_{i \in I}$. The result is immediate using rule ( $\mathrm{ax}_{\mathrm{v}}$ ).
- $t=s u$. There are two cases to consider:
- If $s^{\mathrm{cbv}}=\mathrm{L}\langle!r\rangle$, then $t^{\mathrm{cbv}}=\mathrm{L}\langle r\rangle u^{\mathrm{cbv}}$. Then, $\Gamma=\Gamma^{\prime}+\Delta$, $\Gamma^{\prime} \vdash \mathrm{L}\langle r\rangle: \mathrm{M} \rightarrow \sigma$ and $\Delta \vdash u^{\mathrm{cbv}}: \mathrm{M}$. Moreover, by induction on L using (es), it is immediate to see that there exists $\Gamma^{\prime \prime}$ s.t. $\Gamma^{\prime \prime} \vdash r: \mathrm{M} \rightarrow \sigma$. By rule (bg) we get $\Gamma^{\prime \prime} \vdash!r:[\mathrm{M} \rightarrow \sigma]$ and, by straightforward induction on L once again, $\Gamma^{\prime} \vdash \mathrm{L}\langle!r\rangle:[\mathrm{M} \rightarrow \sigma]$. Recall that the induction is on $t=s u$ and $s^{\mathrm{cbv}}=\mathrm{L}\langle!r\rangle$, hence by the $i . h . \Gamma^{\prime} \vdash s:[\mathrm{M} \rightarrow \sigma]$ and $\Delta \vdash u: \mathrm{M}$. Thus, we conclude by rule $\left(\mathrm{app}_{\mathrm{v}}\right)$.
- Otherwise, $t^{\mathrm{cbv}}=\operatorname{der}\left(s^{\mathrm{cbv}}\right) u^{\mathrm{cbv}}$. We then have $\Gamma=\Gamma^{\prime}+\Delta$, $\Gamma^{\prime} \vdash \operatorname{der}\left(s^{\mathrm{cbv}}\right): \mathrm{M} \rightarrow \sigma$ and $\Delta \vdash u^{\mathrm{cbv}}: \mathrm{M}$. Moreover, from rule (dr), we have $\Gamma^{\prime} \vdash s^{\mathrm{cbv}}:[\mathrm{M} \rightarrow \sigma]$. Then, by i.h. $\Gamma^{\prime} \vdash s:[\mathrm{M} \rightarrow \sigma]$ and $\Delta \vdash u:$ M. Finally, we conclude by rule $\left(\mathrm{app}_{\mathrm{v}}\right)$.
- $t=\lambda x . s$. Then, $t^{\mathrm{cbv}}=!\lambda x . s^{\mathrm{cbv}}, \sigma=\left[\sigma_{i}\right]_{i \in I}, \Gamma=+_{i \in I} \Gamma_{i}$ and $\left(\Gamma_{i} \vdash \lambda x . s^{\mathrm{cbv}}: \sigma_{i}\right)_{i \in I}$. Moreover, by rule (abs), for every $i \in I$ we have $\sigma_{i}=\Gamma_{i}^{\prime}(x) \rightarrow \tau_{i}, \Gamma_{i}=\Gamma_{i}^{\prime} \backslash x$ and $\Gamma_{i}^{\prime} \vdash s^{\mathrm{cbv}}: \tau_{i}$. By i.h. $\left(\Gamma_{i}^{\prime} \vdash s: \tau_{i}\right)_{i \in I}$ and we conclude by rule $\left(\mathrm{abs}_{\mathrm{v}}\right)$.
- $t=s[x \backslash u]$. Then, $t^{\mathrm{cbv}}=s^{\mathrm{cbv}}\left[x \backslash u^{\mathrm{cbv}}\right], \Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta, \Gamma^{\prime} \vdash$ $s^{\mathrm{cbv}}: \sigma$ and $\Delta \vdash u^{\mathrm{cbv}}: \Gamma^{\prime}(x)$. The result follow immediately from the $i . h$. using rule $\left(\mathrm{es}_{\mathrm{v}}\right)$.

To illustrate our previous theorem, we take the term which is the counterexample to completeness in [33] Proposition 15. Indeed, we show how $t=\lambda x . x x$, and $t^{\mathrm{cbv}}=!\lambda x . x!x$ can now be typed with the same type context and type in the systems $\mathcal{V}$ and $\mathcal{U}$ respectively. Let $\sigma=[] \rightarrow[]$.

$$
\begin{gathered}
\overline{x:[[\sigma] \rightarrow \sigma] \vdash x:[[\sigma] \rightarrow \sigma]}\left(\mathrm{ax}_{\mathrm{v}}\right) \quad \overline{x:[\sigma] \vdash x:[\sigma]}\left(\mathrm{ax}_{\mathrm{v}}\right) \\
\frac{x:[[\sigma] \rightarrow \sigma, \sigma] \vdash x x: \sigma}{\vdash \lambda x \cdot x x:[[[\sigma] \rightarrow \sigma, \sigma] \rightarrow \sigma]}\left(\mathrm{app}_{\mathrm{v}}\right) \\
\frac{\overline{x:[[\sigma] \rightarrow \sigma] \vdash x:[\sigma] \rightarrow \sigma}(\mathrm{ax}) \quad \frac{x:[\sigma] \vdash x: \sigma}{x:[\sigma] \vdash!x:[\sigma]}}{}(\mathrm{ax}) \\
\frac{x:[[\sigma] \rightarrow \sigma, \sigma] \vdash x!x: \sigma}{\vdash}(\mathrm{app}) \\
\frac{\vdash \lambda x \cdot x!x:[[\sigma] \rightarrow \sigma, \sigma] \rightarrow \sigma}{\vdash!\lambda x \cdot x!x:[[[\sigma] \rightarrow \sigma, \sigma] \rightarrow \sigma]}(\mathrm{abs})
\end{gathered}
$$

The type system $\mathcal{N}$ (resp. $\mathcal{V}$ ) characterises n-normalisation (resp. v-normalisation). More precisely:

Theorem 4.9 (Characterisation of CBN/CBV Normalisation). Let $t \in \mathcal{T}_{\lambda}$.

- $t$ is $\mathcal{N}$-typable iff $t$ is n -normalising.
- $t$ is $\mathcal{V}$-typable iff $t$ is v-normalising.

Proof. Sec. 4.1 and 4.2 below show the two statements separately as Thm. 4.15 and Thm. 4.24 respectively.

### 4.1. Call-by-Name

The following lemmas aim to prove that $\mathcal{N}$ is indeed a model for the Call-byName reduction strategy presented at the beginning of Sec. 4. Similar results have been already presented in the literature for different formulations of call-byname languages, with or without explicit substitutions, some examples are [14, 37]. In this subsection we measure $\mathcal{N}$-derivations by using a function $\mathrm{sz}_{\mathrm{n}}\left({ }^{( }\right)$ which counts 1 for all typing rules. We follow here the same pattern used in Sec. 3 for system $\mathcal{U}$ : a substitution (resp. anti-substitution) lemma is established and used in the proof of subject reduction (resp. expansion). In the following subsection the same methodology is used for $\mathcal{V}$.

Lemma 4.10 (Substitution). Let $\Phi_{t} \triangleright_{\mathcal{N}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash t: \tau$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{N}} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}$. Then, there exists $\Phi_{t\{x \backslash u\}} \triangleright_{\mathcal{N}} \Gamma+_{i \in I} \Delta_{i} \vdash t\{x \backslash u\}: \tau$ such that $\mathrm{sz}_{\mathrm{n}}\left(\Phi_{t\{x \backslash u\}}\right)=$ $\mathbf{s} \mathbf{z}_{\mathrm{n}}\left(\Phi_{t}\right)+_{i \in I} \mathrm{sz}_{\mathrm{n}}\left(\Phi_{u}^{i}\right)-|I|$.

Proof. By induction on $\Phi_{t}$, reasoning exactly as in Lemma 3.3,
Lemma 4.11 (Weighted Subject Reduction). Let $\Phi \triangleright_{\mathcal{N}} \Gamma \vdash t: \tau$ and $t \rightarrow_{\mathrm{n}} t^{\prime}$. Then, there exists $\Phi^{\prime} \triangleright_{\mathcal{N}} \Gamma \vdash t^{\prime}: \tau$ such that $\mathrm{sz}_{\mathrm{n}}(\Phi)>\mathrm{sz}_{\mathrm{n}}\left(\Phi^{\prime}\right)$.

Proof. By induction on $t \rightarrow_{\mathrm{n}} t^{\prime}$, where, in particular, Lemma 4.10 is used in the base case $t[x \backslash u]_{\mathrm{s}} t\{x \backslash u\}$.

Lemma 4.12 (Anti-Substitution). If $\Phi_{t\{x \backslash u\}} \triangleright_{\mathcal{N}} \Gamma^{\prime} \vdash t\{x \backslash u\}: \tau$, then there exist $\Phi_{t} \triangleright_{\mathcal{N}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash t: \tau$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{N}} \Delta_{i} \vdash u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma^{\prime}=\Gamma+_{i \in I}$ $\Delta_{i}$ and $\mathbf{s z}_{\mathrm{n}}\left(\Phi_{t\{x \backslash u\}}\right)=\mathbf{s z _ { \mathrm { n } }}\left(\Phi_{t}\right)+_{i \in I} \mathrm{sz}_{\mathrm{n}}\left(\Phi_{u}^{i}\right)-|I|$.

Proof. By induction on $t$, reasoning exactly as in Lemma 3.5.
Lemma 4.13 (Weighted Subject Expansion). Let $\Phi^{\prime} \triangleright_{\mathcal{N}} \Gamma \vdash t^{\prime}: \tau$ and $t \rightarrow_{\mathrm{n}} t^{\prime}$. Then, there exists $\Phi \triangleright_{\mathcal{N}} \Gamma \vdash t: \tau$ such that $\mathrm{sz}_{\mathrm{n}}(\Phi)>\mathrm{sz}_{\mathrm{n}}\left(\Phi^{\prime}\right)$.

Proof. By induction on $t \rightarrow_{\mathrm{n}} t^{\prime}$, where, in particular, Lemma 4.12 is used in the base case $t[x \backslash u] \mapsto_{\mathrm{s}} t\{x \backslash u\}$.

The following lemma, showing that n -normal forms are typable, is used in the proof of Theorem 4.15 as the base case for establishing that a n-normalising term is typable.

Lemma 4.14. Let $t \not \nrightarrow n_{\mathrm{n}}$. Then, $t$ is $\mathcal{N}$-typable.

Proof. By straightforward induction on n-normal forms, using Lemma 4.3,
We now relate $\mathcal{N}$-typability with n-normalisation. We also put in evidence the quantitative aspect of system $\mathcal{N}$, so that we introduce the n -size of terms as: $|x|_{\mathrm{n}} \stackrel{\text { def }}{=} 0,|\lambda x . t|_{\mathrm{n}} \xlongequal{\text { def }} 1+|t|_{\mathrm{n}},|t u|_{\mathrm{n}} \stackrel{\text { def }}{=} 1+|t|_{\mathrm{n}}$, and $|t[x \backslash u]|_{\mathrm{n}} \xlongequal{\text { def }} 1+|t|_{\mathrm{n}}$. Note in particular that given a derivation $\Phi_{t}$ for a term $t$ we always have $\mathbf{s z}_{\mathrm{n}}\left(\Phi_{t}\right) \geq|t|_{\mathrm{n}}$.

Theorem 4.15 (Soundness and Completeness for System $\mathcal{N}$ ). Let $t \in \mathcal{T}_{\lambda}$. Then, $t$ is $\mathcal{N}$-typable iff $t$ is n-normalising. Moreover, if $\Phi \triangleright_{\mathcal{N}} \Gamma \vdash t: \tau$, then there exists $p \in \mathrm{no}_{\mathrm{n}}$ such that $t \rightarrow_{\mathrm{n}}^{(b, e)} p$ and $\mathrm{sz}_{\mathrm{n}}(\Phi) \geq b+e+|p|_{\mathrm{n}}$.

Proof. The $\Rightarrow$ direction holds by WSR (Lemma 4.11), while the $\Leftarrow$ direction follows from Lemma 4.14 and WSE (Lemma 4.13). The moreover statement holds by Lemma 4.11 and the fact that the size of the type derivation of $p$ is greater than or equal to $|p|_{\mathrm{n}}$.

### 4.2. Call-by-Value

The following lemmas aim to prove that $\mathcal{V}$ is indeed a model for the Call-byValue reduction strategy. In this subsection we measure $\mathcal{V}$-derivations by using a function $\mathbf{s z}_{\mathrm{v}}(\mathrm{-})$. Formally,

Definition 4.16. Size of derivations is defined by induction as follows:

- $\mathrm{sZ}_{\mathrm{v}}\left(\frac{}{x: \mathrm{M} \vdash x: \mathrm{M}}\left(\mathrm{ax}_{\mathrm{v}}\right)\right)=|\mathrm{M}|$,
- $\mathbf{s z}_{\mathrm{v}}\left(\frac{\Pi_{1} \quad \Pi_{2}}{\Gamma \vdash t[x \backslash u]: \sigma}\left(\mathrm{es}_{\mathrm{v}}\right)\right)=1+\mathbf{s z _ { \mathrm { v } }}\left(\Pi_{1}\right)+\mathbf{s z _ { \mathrm { v } }}\left(\Pi_{2}\right)$,
- $\mathbf{s z}_{\mathrm{v}}\left(\frac{\Pi_{1} \Pi_{2}}{\Gamma \vdash t u: \tau}\left(\operatorname{app}_{\mathrm{v}}\right)\right)=1+\mathbf{s z}_{\mathrm{v}}\left(\Pi_{1}\right)+\mathbf{s z _ { \mathrm { v } }}\left(\Pi_{2}\right)$,
- $\mathbf{s z}_{\mathrm{v}}\left(\frac{\left(\Pi_{i}\right)_{i \in I}}{+_{i \in I} \Gamma_{i} \backslash x \vdash \lambda x . t:\left[\Gamma_{i}(x) \rightarrow \tau_{i}\right]_{i \in I}}\left(\mathrm{abs}_{\mathrm{v}}\right)\right)=|I|+_{i \in I} \mathbf{s z}_{\mathrm{v}}\left(\Pi_{i}\right)$.

Thus, $\mathrm{sz}_{\mathrm{v}}(\Pi)$ counts 1 for rule ( $\mathrm{app}_{\mathrm{v}}$ ), while rule ( $\mathrm{es}_{\mathrm{v}}$ ) does not count, the axiom $\left(\mathrm{ax}_{\mathrm{v}}\right) x: \mathrm{M} \vdash x: \mathrm{M}$ counts $|\mathrm{M}|$, and $\left(\mathrm{abs}_{\mathrm{v}}\right)$ contributes with its number of premises.

In order to prove a substitution lemma for CBV, we need an additional lemma allowing the decomposition of the multiset types of values.

Lemma 4.17 (Split type for value). Let $\Phi_{v} \triangleright \mathcal{V} \Gamma \vdash v: \mathrm{M}$ such that $\mathrm{M}=+_{i \in I} \mathrm{M}_{i}$. Then, there exist $\left(\Phi_{v}^{i} \triangleright \mathcal{V} \Gamma_{i} \vdash v: \mathrm{M}_{i}\right)_{i \in I}$ such that $\Gamma=+_{i \in I} \Gamma_{i}$ and $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{v}\right)=$ $+_{i \in I} \mathbf{S Z}_{\mathrm{v}}\left(\Phi_{v}^{i}\right)$.

Proof. By case analysis on the shape of $v$.

- $v=x$. By $\left(\mathrm{ax}_{\mathrm{v}}\right), \Phi_{v} \triangleright \mathcal{V} x: \mathrm{M} \vdash x: \mathrm{M}$. From $\mathrm{M}=+_{i \in I} \mathrm{M}_{i}$ we get $\left(\Phi_{v}^{i} \triangleright_{\mathcal{V}} x: \mathrm{M}_{i} \vdash x: \mathrm{M}_{i}\right)_{i \in I}$ by $\left(\mathrm{ax}_{\mathrm{v}}\right)$ several times. We then conclude, since $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{v}\right)=|\mathrm{M}|=+_{i \in I}\left|\mathrm{M}_{i}\right|=+_{i \in I} \mathbf{S Z}_{\mathrm{v}}\left(\Phi_{v}^{i}\right)$.
- $v=\lambda x$.s. By $\left(\mathrm{abs}_{\mathrm{v}}\right)$, we have $\Gamma=+_{j \in J} \Gamma_{j} \backslash x, \mathrm{M}=\left[\Gamma_{j}(x) \rightarrow \tau_{j}\right]_{j \in J}$ and $\left(\Phi_{s}^{j} \triangleright \mathcal{V} \Gamma_{j} \vdash s: \tau_{j}\right)_{j \in J}$ for some $J$. Since $\mathrm{M}=+_{i \in I} \mathrm{M}_{i}$, we have also $J=+_{i \in I} J_{i}$, with $M_{i}=\left[\Gamma_{j}(x) \rightarrow \tau_{j}\right]_{j \in J_{i}}$ for each $i \in I$. We construct the type derivations $\left(\Phi_{v}^{i} \triangleright \mathcal{V}+{ }_{j \in J_{i}} \Gamma_{j} \backslash x \vdash \lambda x . s: \mathrm{M}_{i}\right)_{i \in I}$, using for each $i \in I$ the rule ( $\mathrm{abs}_{\mathrm{v}}$ ) with premises $\left(\Phi_{s}^{j} \triangleright \mathcal{V} \Gamma_{j} \vdash s: \tau_{j}\right)_{j \in J_{i}}$. We have that $\Gamma=+_{j \in J} \Gamma_{j} \backslash x=+_{i \in I}\left(+_{j \in J_{i}} \Gamma_{j} \backslash x\right)$. We conclude since $\mathbf{s z _ { \mathbf { v } }}\left(\Phi_{v}\right)=|J|+{ }_{j \in J} \mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{s}^{j}\right)=+_{i \in I}\left(\left|J_{i}\right|+{ }_{j \in J_{i}} \mathbf{s z}_{\mathrm{v}}\left(\Phi_{v}^{j}\right)\right)=+_{i \in I} \mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}^{i}\right)$.

Lemma 4.18 (Substitution). Let $\Phi_{t} \triangleright \mathcal{\nu} \Gamma \vdash t: \tau$ and $\Phi_{v} \triangleright \mathcal{V} \Delta \vdash v: \Gamma(x)$. Then, there exists $\Phi_{t\{x \mid v\}} \triangleright \mathcal{V} \Gamma \geqq x+\Delta \vdash t\{x \backslash v\}: \tau$ such that $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t\{x \backslash v\}}\right)=\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t}\right)+$ $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{v}\right)-|\Gamma(x)|$.
Proof. By induction on $\Phi_{t}$. If $\Phi_{t}$ is ( $\mathrm{ax}_{\mathrm{v}}$ ) and $t=x$, then $t\{x \backslash v\}=v$ and $\Phi_{t}$ is of the form $x: \Gamma(x) \vdash x: \Gamma(x)$. We let $\Phi_{t\{x \backslash v\}}=\Phi_{v}$. We conclude since $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t}\right)=|\Gamma(x)|$. If $\Phi_{t}$ is $\left(\mathrm{ax}_{\mathrm{v}}\right)$ and $t=y \neq x$, then $t\{x \backslash v\}=y$, $\Gamma(x)=[]$ (so that $|\Gamma(x)|=0$ ) and $\Phi_{y}$ is of the form $y: \mathrm{M} \vdash y: \mathrm{M}$. We let $\Phi_{t\{x \backslash v\}}=\Phi_{y}$. Moreover, $\Phi_{v} \triangleright \mathcal{V} \Delta \vdash v:[]$ implies $\mathrm{sz}_{\mathrm{v}}\left(\Phi_{v}\right)=0$. Then, we immediately conclude.

If $\Phi_{t}$ ends with $\left(\mathrm{app}_{\mathrm{v}}\right)$, then $t=t_{1} t_{2}, \Gamma=\Gamma_{1}+\Gamma_{2}$ and there exist a type M and two derivations $\Phi_{t_{1}} \triangleright \mathcal{V} \Gamma_{1} \vdash t_{1}:[\mathrm{M} \rightarrow \tau]$ and $\Phi_{t_{2}} \triangleright \mathcal{V} \Gamma_{2} \vdash t_{2}: M$. Since $\Phi_{v} \triangleright \mathcal{V} \Delta \vdash v: \Gamma(x)$ and $\Gamma(x)=\Gamma_{1}(x)+\Gamma_{2}(x)$, Lemma 4.17 ensures that there exist two derivations $\Phi_{v}^{1} \triangleright \mathcal{\nu} \Delta_{1} \vdash v: \Gamma_{1}(x)$ and $\Phi_{v}^{2} \triangleright \mathcal{V}^{\Delta_{2}} \vdash v: \Gamma_{2}(x)$ such that $\Delta=\Delta_{1}+\Delta_{2}$ and $\mathbf{s z}\left(\Phi_{v}\right)=\mathbf{s z}\left(\Phi_{v}^{1}\right)+\mathbf{s z}\left(\Phi_{v}^{2}\right)$. Using the i.h. on $\Phi_{t_{i}}, \Phi_{v}^{i}$, $i=1,2$, we get two derivations $\Phi_{t_{1}\{x \backslash v\}} \triangleright \mathcal{V} \Gamma_{1} \backslash x+\Delta_{1} \vdash t_{1}\{x \backslash v\}:[\mathrm{M} \rightarrow \tau]$ and $\Phi_{t_{2}\{x \backslash v\}} \triangleright{ }_{\nu} \Gamma_{2} \backslash x+\Delta_{2} \vdash t_{2}\{x \backslash v\}: \mathrm{M}$ such that $\mathbf{s z}\left(\Phi_{t_{i}\{x \backslash v\}}\right)=\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t_{i}}\right)+$ $\mathbf{s z}_{\mathbf{v}}\left(\Phi_{v}^{i}\right)-\left|\Gamma_{i}(x)\right|$, for $i=1,2$. By observing that $t\{x \backslash v\}=t_{1}\{x \backslash v\} t_{2}\{x \backslash v\}$ and by using ( $\mathrm{app}_{\mathrm{v}}$ ), we get a derivation $\Phi_{t\{x \backslash v\}} \triangleright \nu \Gamma \backslash x+\Delta \vdash t\{x \backslash v\}: \tau$ such that $\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t\{x \backslash v\}}\right)=\left(\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t_{1}}\right)+\mathbf{s z}_{\mathrm{v}}\left(\Phi_{v}^{1}\right)-\left|\Gamma_{1}(x)\right|\right)+\left(\mathbf{s z}_{\mathrm{v}}\left(\Phi_{t_{2}}\right)+\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi_{v}^{2}\right)-\right.$ $\left.\left|\Gamma_{2}(x)\right|\right)+1=\left(\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{t_{1}}\right)+\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{t_{2}}\right)+1\right)+\left(\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}^{1}\right)+\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}^{2}\right)\right)-\left(\left|\Gamma_{1}(x)\right|\right)+$ $\left.\left|\Gamma_{2}(x)\right|\right)=\mathbf{s z}_{\mathbf{v}}\left(\Phi_{t}\right)+\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}\right)-|\Gamma(x)|$.

All the other cases are similar to the previous one, and follow easily from the i.h. and Lemma 4.17

Lemma 4.19 (Weighted Subject Reduction). Let $\Phi \triangleright_{\mathcal{V}} \Gamma \vdash t: \tau$ and $t \rightarrow_{\mathrm{v}} t^{\prime}$.


Proof. By induction on $t \rightarrow_{\mathrm{v}} t^{\prime}$. We only show the base case $t \rightarrow_{\mathrm{sv}} t^{\prime}$ as the case $t \rightarrow_{\mathrm{dB}} t^{\prime}$ is very similar to the one in Lemma 3.4 The inductive cases are straightforward.

Let $t=s[x \backslash \mathrm{~L}\langle v\rangle]$ and $t^{\prime}=\mathrm{L}\langle s\{x \backslash v\}\rangle$. We proceed by induction on L .

- $\mathrm{L}=\square$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ s.t. $\Gamma^{\prime}(x)=\mathrm{M}$ and $\Phi$ has the following form

$$
\frac{\Phi_{s} \triangleright_{\mathcal{V}} \Gamma^{\prime} \vdash s: \tau \quad \Phi_{v} \triangleright_{\mathcal{V}} \Delta \vdash v: \mathrm{M}}{\Gamma \vdash s[x \backslash v]: \tau}\left(\mathrm{es}_{\mathrm{v}}\right)
$$

Thus, we conclude directly by Lemma 4.18 with $\Phi_{s}$ and $\Phi_{v}$. Notice that $\mathbf{s z _ { \mathrm { v } }}(\Phi)=1+\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi_{s}\right)+\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi_{v}\right)$, while $\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi^{\prime}\right)=\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{s}\right)+\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}\right)-|\mathrm{M}|$.

- $\mathrm{L}=\mathrm{L}^{\prime}[y \backslash r]$. Then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+(\Delta \backslash y)+\Delta^{\prime}$ with $\Gamma^{\prime}(x)=\mathrm{M}, \Delta(y)=\mathrm{M}^{\prime}$ and

$$
\frac{\Phi_{s} \triangleright_{\mathcal{V}} \Gamma^{\prime} \vdash s: \tau}{\Gamma \vdash s[x \backslash \mathrm{~L}\langle v\rangle]: \tau} \frac{\Phi_{\mathrm{L}^{\prime}} \triangleright \mathcal{V} \Delta \vdash \mathrm{L}^{\prime}\langle v\rangle: \mathrm{M} \quad \Phi_{r} \triangleright \mathcal{V} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{(\Delta \backslash y y)+\Delta^{\prime} \vdash \mathrm{L}^{\prime}[y \backslash r]\langle v\rangle: \mathrm{M}}\left(\mathrm{es}_{\mathrm{v}}\right)
$$

Thus, we build

$$
\frac{\Phi_{s} \triangleright_{\mathcal{V}} \Gamma^{\prime} \vdash s: \tau \quad \Phi_{\mathrm{L}^{\prime}} \triangleright_{\mathcal{V}} \Delta \vdash \mathrm{L}^{\prime}\langle v\rangle: \mathrm{M}}{\Psi \triangleright_{\mathcal{V}}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash s\left[x \backslash \mathrm{~L}^{\prime}\langle v\rangle\right]: \tau}\left(\mathrm{es}_{\mathrm{v}}\right)
$$

and by the $i . h$. there exists $\Psi^{\prime} \triangleright \mathcal{V}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\{x \backslash v\}\rangle: \tau$ such that $\mathrm{sz}_{\mathrm{v}}(\Psi)>\mathrm{sz}_{\mathrm{v}}\left(\Psi^{\prime}\right)$. Then, we conclude with

$$
\frac{\Psi^{\prime} \triangleright_{\mathcal{V}}\left(\Gamma^{\prime} \backslash x\right)+\Delta \vdash \mathrm{L}^{\prime}\langle s\{x \backslash v\}\rangle: \tau \quad \Phi_{r} \triangleright \mathcal{V} \Delta^{\prime} \vdash r: \mathrm{M}^{\prime}}{\Phi^{\prime} \triangleright_{\mathcal{V}}\left(\Gamma^{\prime} \backslash x\right)+\left(\Delta \mathbb{}\left(\mathrm{es}_{\mathrm{v}}\right)\right.}
$$

since we may assume that $y \notin \operatorname{supp}\left(\Gamma^{\prime}\right)$. Notice that $\mathbf{s z}_{\mathrm{v}}(\Phi)=\mathbf{s \mathbf { z } _ { \mathrm { v } }}(\Psi)+$ $\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{r}\right)+1>\mathbf{s z}_{\mathrm{v}}\left(\Psi^{\prime}\right)+\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi_{r}\right)+1=\mathbf{s \mathbf { z } _ { \mathrm { v } }}\left(\Phi^{\prime}\right)$.

The ability of merging different type derivations of a given value is necessary for proving an anti-substitution lemma for CBV.

Lemma 4.20 (Merge type for value). Let $\left(\Phi_{v}^{i} \triangleright \mathcal{V} \Gamma_{i} \vdash v: \mathrm{M}_{i}\right)_{i \in I}$. Then, there exists $\Phi_{v} \triangleright \mathcal{\nu} \Gamma \vdash v: \mathrm{M}$ such that $\Gamma=+_{i \in I} \Gamma_{i}, \mathrm{M}=+_{i \in I} \mathrm{M}_{i}$ and $\mathrm{sz}_{\mathrm{v}}\left(\Phi_{v}\right)=$ $+_{i \in I} \mathbf{s} \mathbf{Z}_{\mathrm{v}}\left(\Phi_{v}^{i}\right)$.

Proof. The proof is straightforward by case analysis on $v$.
Lemma 4.21 (Anti-Substitution). Let $\Phi_{t\{x \backslash v\}} \triangleright_{\mathcal{V}} \Gamma \vdash t\{x \backslash v\}: \tau$. Then, there exist $\Phi_{t} \triangleright_{\mathcal{V}} \Gamma^{\prime} \vdash t: \tau$ and $\Phi_{v} \triangleright_{\mathcal{V}} \Delta \vdash v: \Gamma^{\prime}(x)$ such that $\Gamma=\Gamma^{\prime} \ x+\Delta$ and $\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{t\{x \backslash v\}}\right)=\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{t}\right)+\mathbf{s} \mathbf{z}_{\mathrm{v}}\left(\Phi_{v}\right)-\left|\Gamma^{\prime}(x)\right|$.

Proof. By induction on $t$. If $t=x$, then $t\{x \backslash v\}=v$ and it is necessarily the case $\tau=\mathrm{M}$. We let $\Phi_{v}=\Phi_{t\{x \backslash v\}}$ and $\Phi_{t} \triangleright \mathcal{V} x: \mathrm{M} \vdash x: \mathrm{M}$ by $\left(\mathrm{ax}_{\mathrm{v}}\right)$. We conclude since $\Gamma^{\prime}(x)=\mathrm{M}$ and $\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{t}\right)=|\mathrm{M}|$ by definition. If $t=y \neq x$, then $t\{x \backslash v\}=y$ and we let $\Phi_{t}=\Phi_{t\{x \backslash v\}}$. Therefore, by $\left(\mathrm{ax}_{\mathrm{v}}\right), \Gamma^{\prime}(x)=[]$. Moreover, by rules $\left(\mathrm{ax}_{\mathrm{v}}\right)$ and $\left(\mathrm{abs}_{\mathrm{v}}\right), \Phi_{v} \triangleright \mathcal{V} \vdash v:[]$. Thus, $\mathrm{sz}_{\mathrm{v}}\left(\Phi_{v}\right)=0=\left|\Gamma^{\prime}(x)\right|$ and we conclude.

If $t=t_{1} t_{2}$, then $t\{x \backslash v\}=t_{1}\{x \backslash v\} t_{2}\{x \backslash v\}$ and there exist a type M and two derivations $\Phi_{t_{1}\{x \backslash v\}} \triangleright_{\mathcal{V}} \Gamma_{1} \backslash x+\Delta_{1} \vdash t_{1}\{x \backslash v\}:[\mathrm{M} \rightarrow \tau]$ and $\Phi_{t_{2}\{x \backslash v\}} \triangleright_{\mathcal{V}}$ $\Gamma_{2} \backslash x+\Delta_{2} \vdash t_{2}\{x \backslash v\}:$ M such that $\Gamma=\Gamma_{1}+\Gamma_{2}$ and sz $\left(\Phi_{t\{x \backslash v\}}\right)=\mathbf{s z}\left(\Phi_{t_{1}\{x \backslash v\}}\right)+$ $\mathrm{sz}\left(\Phi_{t_{2}\{x \backslash v\}}\right)+1$. Using the i.h. on $t_{1}$ and $t_{2}$ we get four derivations:

1. $\Phi_{t_{1}} \triangleright \mathcal{V} \Gamma_{1}^{\prime} \vdash t_{1}:[\mathrm{M} \rightarrow \tau]$
2. $\Phi_{v}^{1} \triangleright_{\mathcal{V}} \Delta_{1} \vdash v: \Gamma_{1}^{\prime}(x)$
3. $\Phi_{t_{2}} \triangleright \mathcal{V} \Gamma_{2}^{\prime} \vdash t_{2}: \mathrm{M}$
4. $\Phi_{v}^{2} \triangleright \mathcal{V} \Delta_{2} \vdash v: \Gamma_{2}^{\prime}(x)$
such that $\Gamma_{i}=\Gamma_{i}^{\prime} \backslash x+\Delta_{i}$ and $\mathrm{sz}\left(\Phi_{t_{i}\{x \backslash v\}}\right)=\mathrm{sz}\left(\Phi_{t_{i}}\right)+\mathrm{sz}\left(\Phi_{v}^{i}\right)-\left|\Gamma_{i}^{\prime}(x)\right|$ for $i=1,2$. Lemma 4.20 applied to (21) and (44) above provides a derivation $\Phi_{v} \triangleright_{\mathcal{V}} \Delta \vdash v: \Gamma^{\prime}(x)$ where $\Delta=\Delta_{1}+\Delta_{2}$ and $\Gamma^{\prime}(x)=\Gamma_{1}^{\prime}(x)+\Gamma_{2}^{\prime}(x)$. Using (11) and (3) above we get $\Phi_{t} \triangleright_{\mathcal{V}} \Gamma^{\prime} \vdash t: \tau$ where $\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}$ and $\mathrm{sz}\left(\Phi_{t}\right)=$ $\mathbf{s z}\left(\Phi_{t_{1}}\right)+\mathbf{s z}\left(\Phi_{t_{2}}\right)+1$. We have that $\Gamma=\Gamma^{\prime} \ \backslash x+\Delta$ and $\mathbf{s z}_{\mathbf{v}}\left(\Phi_{t\{x \backslash v\}}\right)=$ $\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{t}\right)+\mathbf{s z _ { \mathrm { v } }}\left(\Phi_{v}\right)-\left|\Gamma^{\prime}(x)\right|$, and we conclude.

All the other cases are similar to the previous one, and follow easily from the i.h. and Lemma 4.20

Lemma 4.22 (Weighted Subject Expansion). Let $\Phi^{\prime} \triangleright_{\mathcal{V}} \Gamma \vdash t^{\prime}: \tau$ and $t \rightarrow_{\mathrm{v}} t^{\prime}$. Then, there exists $\Phi \triangleright \mathcal{V} \Gamma \vdash t: \tau$ such that $\mathbf{s z}_{\mathrm{v}}(\Phi)>\mathbf{s z}_{\mathrm{v}}\left(\Phi^{\prime}\right)$.

Proof. By induction on $t \rightarrow_{\mathrm{v}} t^{\prime}$ where, in particular, Lemma 4.21 is used in the base case $t[x \backslash v] \mapsto_{\mathrm{sv}} t\{x \backslash v\}$.

As in the case of CBN, a lemma establishing that normal forms are typable is needed.

Lemma 4.23. Let $t \not \not_{\mathrm{v}}$. Then, $t$ is $\mathcal{V}$-typable.
Proof. By induction on $t$, we prove simultaneously the following statements:

1. If $t \in \mathrm{vr}_{\mathrm{v}}$, then for every multiset type M there exists $\Gamma$ such that $\triangleright \mathcal{V}$ $\Gamma \vdash t: \mathrm{M}$.
2. If $t \in \mathrm{ne}_{\mathrm{v}}$, then for every type $\tau$ there exists $\Gamma$ such that $\triangleright_{\mathcal{V}} \Gamma \vdash t: \tau$.

We only analyse the key cases. If $t=x \in \mathrm{vr}_{\mathrm{v}}$ we conclude by ( $\mathrm{ax}_{\mathrm{v}}$ ). If $t=s u \in$ $\mathrm{ne}_{\mathrm{v}}$, then $u \in \mathrm{no}_{\mathrm{v}}$ and there are two possible cases: $s \in \mathrm{Vr}_{\mathrm{v}}$ or $s \in \mathrm{ne}_{\mathrm{v}}$. Let $\tau$ by any type. By i.h. (3), $\triangleright_{\mathcal{V}} \Delta \vdash u:$ M. Moreover, by i.h. (1) or (2) resp., we have $\triangleright_{\mathcal{V}} \Gamma \vdash s:[\mathrm{M} \rightarrow \tau]$ and we conclude by $\left(\mathrm{app}_{\mathrm{v}}\right)$. If $t=\lambda x . s \in \mathrm{no}_{\mathrm{v}}$ we conclude by $\left(\mathrm{abs}_{\mathrm{v}}\right)$ with $\triangleright_{\mathcal{V}} \Gamma \vdash t:$ []. If $t=s[x \backslash u]$, there are three possible cases: $s \in \mathrm{vr}_{\mathrm{v}}$, $s \in \mathrm{ne}_{\mathrm{v}}$ or $s \in \mathrm{no}_{\mathrm{v}}$. In either case, by the proper $i . h$. we get $\triangleright_{\mathcal{V}} \Gamma \vdash s: \mathrm{M}$ (resp. $\tau$ ) and conclude by $\left(\mathrm{es}_{\mathrm{v}}\right)$, given that $u \in \mathrm{ne}_{\mathrm{v}}$ and $\triangleright \mathcal{V} \Delta \vdash u: \Gamma(x)$ by i.h. (2).

We now relate $\mathcal{V}$-typability with v-normalisation. We also put in evidence the quantitative aspect of system $\mathcal{V}$, so that we introduce the v -size of terms as: $|x|_{\mathrm{v}} \stackrel{\text { def }}{=} 0,|\lambda x . t|_{\mathrm{v}} \stackrel{\text { def }}{=} 0,|t u|_{\mathrm{v}} \stackrel{\text { def }}{=} 1+|t|_{\mathrm{v}}+|u|_{\mathrm{v}}$, and $|t[x \backslash u]|_{\mathrm{v}} \stackrel{\text { def }}{=} 1+|t|_{\mathrm{v}}+|u|_{\mathrm{v}}$. Note in particular that given a derivation $\Phi_{t}$ for a term $t$ we always have $s z_{\mathrm{v}}\left(\Phi_{t}\right) \geq|t|_{\mathrm{v}}$.

Theorem 4.24 (Soundness and Completeness for System $\mathcal{V}$ ). Let $t \in \mathcal{T}_{\lambda}$. Then, $t$ is $\mathcal{V}$-typable iff $t$ is v-normalising. Moreover, if $\Phi \triangleright \mathcal{V} \Gamma \vdash t: \tau$, then there exists $p \in \mathrm{no}_{\mathrm{v}}$ such that $t \rightarrow{ }_{\mathrm{v}}^{(b, e)} p$ and $\mathbf{s z}_{\mathrm{v}}(\Phi) \geq b+e+|p|_{\mathrm{v}}$.

Proof. The $\Rightarrow$ direction holds by WSR (Lemma 4.19), while the $\Leftarrow$ direction follows from Lemma 4.23 and WSE (Lemma 4.22). The moreover statement holds by Lemma 4.19 and the fact that the size of the type derivation of $p$ is greater than or equal to $|p|_{\mathrm{v}}$.

## 5. A Tight Type System Giving Exact Bounds

In order to count exactly the length of w-reduction sequences to normal forms, we first fix a deterministic strategy for the $\lambda$ !-calculus, called dw, which computes the same w-normal forms. We then define the tight type system $\mathcal{E}$, being able to count exactly the length of dw-reduction sequences. Theorem 2.8 stating that any two different reduction paths to normal form have the same length, guarantees that system $\mathcal{E}$ is able to count exactly the length of any w-reduction sequence to w-normal form.

A Deterministic Strategy for the $\lambda$ !-Calculus. The reduction relation $\rightarrow_{d w}$ defined below is a deterministic version of $\rightarrow_{\mathrm{w}}$ and is used, as explained, as a technical tool of our development.

$$
\begin{aligned}
& \overline{\mathrm{L}\langle\lambda x . t\rangle u \rightarrow_{\mathrm{dw}} \mathrm{~L}\langle t[x \backslash u]\rangle} \quad \overline{t[x \backslash \mathrm{~L}\langle!u\rangle] \rightarrow_{\mathrm{dw}} \mathrm{~L}\langle t\{x \backslash u\}\rangle} \quad \overline{\operatorname{der}(\mathrm{L}\langle!t\rangle) \rightarrow_{\mathrm{dw}} \mathrm{~L}\langle t\rangle} \\
& \frac{t \rightarrow_{\mathrm{dw}} u}{\lambda x . t \rightarrow_{\mathrm{dw}} \lambda x . u} \quad \frac{t \rightarrow_{\mathrm{dw}} u \quad \neg \operatorname{bang}(t)}{r[x \backslash t] \rightarrow_{\mathrm{dw}} r[x \backslash u]} \quad \frac{t \rightarrow_{\mathrm{dw}} u \quad \neg \operatorname{bang}(t)}{\operatorname{der} t \rightarrow_{\mathrm{dw}} \operatorname{der} u} \\
& \frac{t \rightarrow_{\mathrm{dw}} u \quad \neg^{\operatorname{abs}(t)}}{t r \rightarrow_{\mathrm{dw}} u r} \quad \frac{t \rightarrow_{\mathrm{dw}} u \quad r \in \mathrm{na}_{\mathrm{w}}}{r t \rightarrow_{\mathrm{dw}} r u} \quad \frac{t \rightarrow_{\mathrm{dw}} u \quad r \in \mathrm{nb}_{\mathrm{w}}}{t[x \backslash r] \rightarrow_{\mathrm{dw}} u[x \backslash r]}
\end{aligned}
$$

The rules in the first line correspond to the base cases. The first of these rules is the multiplicative case, while the other two are the exponential cases. The six remaining rules specify the closure by weak contexts.

Normal forms of $\rightarrow_{\mathrm{w}}$ and $\rightarrow_{\mathrm{dw}}$ are the same, both characterised by the set $\mathrm{no}_{\mathrm{w}}$.
Proposition 5.1. Let $t \in \mathcal{T}$. Then, (1) $t \not \nrightarrow \mathrm{w}_{\mathrm{w}}$ iff (2) $t \not{\nrightarrow \overbrace{\mathrm{dw}}}$ iff (3) $t \in \mathrm{no}_{\mathrm{w}}$.
Proof. Notice that (11) $\Longrightarrow$ (2) follows from $\rightarrow_{\mathrm{dw}} \subset \rightarrow_{\mathrm{w}}$. Moreover, (1) iff (3) holds by Proposition 2.11. The proof of (2) $\Longrightarrow$ (3) follows from a straightforward adaptation of the proof of Proposition 2.11

The Type System $\mathcal{E}$. We now extend the type system $\mathcal{U}$ to a tight one, called $\mathcal{E}$, being able to provide exact bounds for dw-normalising sequences and size of normal forms. The technique is based on [1], which defines type systems to count reduction lengths for different strategies in the $\lambda$-calculus. The notion of tight derivation turns out to be a particular implementation of minimal derivation, pioneered by de Carvalho in [20], where exact bounds for CBN abstract machines are inferred from minimal type derivations.

We define the following sets of types:

$$
\begin{array}{rrll}
\text { (Tight Constants) } & \mathrm{tt} & ::= & \mathrm{a}|\mathrm{~b}| \mathrm{n} \\
\text { (Types) } & \sigma, \tau & ::= & \mathrm{tt}|\mathrm{M}| \mathrm{M} \rightarrow \sigma \\
\text { (Multiset Types) } & \mathrm{M} & ::= & {\left[\sigma_{i}\right]_{i \in I} \text { where } I \text { is a finite set }}
\end{array}
$$

In contrast with $\mathcal{U}$ where an infinite countable set of variable is used, following the standard presentations in the literature, system $\mathcal{E}$ relies only in the use of a few type constants with a specific semantics. Inspired by [1], which only uses two constant types a and $n$ for abstractions and neutral terms respectively, we now use three tight constants. Indeed, the constant a (resp. b) types terms whose normal form has the shape $\mathrm{L}\langle\lambda x . t\rangle$ (resp. $\mathrm{L}\langle!t\rangle$ ), and the constant n types terms whose normal form is in $\mathrm{ne}_{\text {wcf }}$. As a matter of notation, given an arbitrary tight constant $\mathrm{tt}_{0}$ we write $\overline{\mathrm{tt}_{0}}$ to denote a tight constant different from $\mathrm{tt}_{0}$. Thus for instance, $\overline{\mathrm{a}} \in\{\mathrm{b}, \mathrm{n}\}$.

Typing contexts are functions from variables to multiset types, assigning the empty multiset to all but a finite number of variables. Sequents are of the form $\Gamma \vdash^{(b, e, s)} t: \sigma$, where the natural numbers $b, e$ and $s$ provide information on the reduction of $t$ to normal form, and on the size of its normal form. More precisely, $b$ (resp. e) indicates the number of multiplicative (resp. exponential) steps to normal form, while $s$ indicates the w-size of this normal form. Observe that we do not count s! and d! steps separately, because both of them are exponential steps of the same nature. It is also worth noticing that only two counters suffice in the case of the $\lambda$-calculus [1], one to count $\beta$-reduction steps, and another to count the w-size of normal forms. The difficulty in the case of the $\lambda$ !-calculus is to statically discriminate between multiplicative and exponential steps.

A multiset type $\left[\sigma_{i}\right]_{i \in I}$ is tight, written $\operatorname{tight}\left(\left[\sigma_{i}\right]_{i \in I}\right)$, if $\sigma_{i} \in \mathrm{tt}$ for all $i \in I$. A context $\Gamma$ is said to be tight if it assigns tight multisets to all variables. A type derivation $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ is tight if $\Gamma$ is tight and $\sigma \in \mathrm{tt}$.

Typing rules (Figure 3) are split in two groups: the persistent and the consuming ones. A constructor is consuming (resp. persistent) if it is consumed (resp. not consumed) during w-reduction to w-normal form. For instance, in $\operatorname{der}(!K)(!\mathrm{I})(!\Omega)$ the two abstractions of K are consuming, while the abstraction of I is persistent, and all the other constructors are also consuming, except those of $\Omega$ that turns out to be an untyped subterm. This dichotomy between consuming/persistent constructors has been used in 1] for the $\lambda$-calculus, and adapted here for the $\lambda!$-calculus.

Observe that in every typing rule the counters of the conclusion are at least the sums of the corresponding counters of the premises. In some cases, one

## Persistent Typing Rules

$$
\begin{gathered}
\frac{\Gamma \vdash^{(b, e, s)} t: \mathrm{n} \quad \Delta \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} u: \overline{\mathrm{a}}}{\Gamma+\Delta \vdash^{\left(b+b^{\prime}, e+e^{\prime}, s+s^{\prime}+1\right)} t u: \mathrm{n}}\left(\mathrm{ae}_{\mathrm{p}}\right) \quad \frac{\Gamma \vdash^{(b, e, s)} t: \mathrm{tt} \operatorname{tight}(\Gamma(x))}{\Gamma \backslash x \vdash^{(b, e, s+1)} \lambda x . t: \mathrm{a}}\left(\mathrm{ai}_{\mathrm{p}}\right) \\
\frac{\vdash^{(0,0,0)}!t: \mathrm{b}}{}\left(\mathrm{bg}_{\mathrm{p}}\right) \quad \frac{\Gamma \vdash^{(b, e, s)} t: \mathrm{n}}{\Gamma \vdash^{(b, e, s+1)} \operatorname{der} t: \mathrm{n}}\left(\mathrm{dr}_{\mathrm{p}}\right) \\
\frac{\Gamma \vdash^{(b, e, s)} t: \tau \quad \Delta \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} u: \mathrm{n} \operatorname{tight}(\Gamma(x))}{(\Gamma \ x)+\Delta \vdash^{\left(b+b^{\prime}, e+e^{\prime}, s+s^{\prime}\right)} t[x \backslash u]: \tau}\left(\mathrm{es}_{\mathrm{p}}\right)
\end{gathered}
$$

## Consuming Typing Rules

$$
\begin{aligned}
& \overline{x:[\sigma] \vdash^{(0,0,0)} x: \sigma}\left(\mathrm{ax}_{\mathrm{c}}\right) \quad \frac{\Gamma \vdash^{(b, e, s)} t: \mathrm{M} \rightarrow \tau \quad \Delta \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} u: \mathrm{M}}{\Gamma+\Delta \vdash^{\left(b+b^{\prime}+1, e+e^{\prime}, s+s^{\prime}\right)} t u: \tau}\left(\mathrm{ae}_{\mathrm{c} 1}\right) \\
& \frac{\Gamma \vdash^{(b, e, s)} t: \mathrm{M} \rightarrow \tau \quad \Delta \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} u: \mathrm{n} \quad \operatorname{tight}(\mathrm{M})}{\Gamma+\Delta \vdash^{\left(b+b^{\prime}+1, e+e^{\prime}, s+s^{\prime}\right)} t u: \tau}\left(\mathrm{ae}_{\mathrm{c} 2}\right) \\
& \frac{\Gamma \vdash^{(b, e, s)} t: \tau}{\Gamma \backslash x \vdash^{(b, e, s)} \lambda x . t: \Gamma(x) \rightarrow \tau}\left(\mathrm{ai}_{\mathrm{c}}\right) \quad \frac{\left(\Gamma_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} t: \sigma_{i}\right)_{i \in I}}{+_{i \in I} \Gamma_{i} \vdash^{\left(+{ }_{i \in I} b_{i}, 1+{ }_{i \in I} e_{i},+{ }_{i \in I} s_{i}\right)}!t:\left[\sigma_{i}\right]_{i \in I}}\left(\mathrm{bg}_{\mathrm{c}}\right) \\
& \frac{\Gamma \vdash^{(b, e, s)} t:[\sigma]}{\Gamma \vdash^{(b, e, s)} \operatorname{der} t: \sigma}\left(\mathrm{dr}_{\mathrm{c}}\right) \quad \frac{\Gamma \vdash^{(b, e, s)} t: \sigma \quad \Delta \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} u: \Gamma(x)}{(\Gamma \backslash x)+\Delta \vdash^{\left(b+b^{\prime}, e+e^{\prime}, s+s^{\prime}\right)} t[x \backslash u]: \sigma}\left(\mathrm{es}_{\mathrm{c}}\right)
\end{aligned}
$$

Figure 3: System $\mathcal{E}$ for the $\lambda!$-Calculus.
counter may undergo an additional increment, as explained below. The persistent rules are those typing persistent constructors, so that none of them increases the first two counters, but only possibly the third one, which contributes to the size of the normal form. The consuming rules type consuming constructors, so that they may increase one of the first two counters, contributing to the length of the normalisation sequence. More precisely, rules $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$ and $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$ increment the first counter because the (consuming) application will be used to perform a dB -step, while rule $\left(\mathrm{bg}_{\mathrm{c}}\right)$ increments the second counter because the (consuming) bang will be used to perform either a s! or a d!-step. Rule ( $\mathrm{ae}_{\mathrm{c} 2}$ ) is particularly useful to type dB-redexes whose reduction does not create an exponential redex, because the argument of the substitution created by the dB-step does not reduce to a bang.

Example 5.2. The following tight typing can be derived for term $t_{0}$ of Example 2.2.

Note that the only persistent rule used is $\left(\mathrm{ai}_{\mathrm{p}}\right)$ when typing I , thus contributing to count the w-size of the w-normal form of $t_{0}$. Indeed, I is the w-normal form of $t_{0}$.

Soundness. We now study soundness of the type system $\mathcal{E}$, which does not only guarantee that typable terms are normalising -a qualitative property- but also provides quantitative (exact) information for normalising sequences. More precisely, given a tight type derivation $\Phi$ with counters $(b, e, s)$ for a term $t, t$ is w-normalisable in $(b+e)$-steps and its w-normal form has w-size $s$. Therefore, information about a dynamic behaviour of $t$, is extracted from a static typing property of $t$. The soundness proof is mainly based on a subject reduction property (Lemma 5.12), as well as on some auxiliary results.

We start by the following remark, which is proved by inspecting the typing rules:

Remark 5.3. If $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ then:

- abs $(t)$ implies $\sigma=\mathrm{a}$ or $\sigma=\mathrm{M} \rightarrow \tau$.
- bang $(t)$ implies $\sigma=\mathrm{b}$ or $\sigma=\mathrm{M}$.

As in system $\mathcal{U}$, typable terms are weakly clash-free, as stated in the following lemma. This lemma is needed for proving Lemma 5.5 which is one of the two tight spreading lemmas established here. By tight spreading we mean that in a type derivation the tightness of the final context implies (under some additional hypotheses) the tightness of the final type, and hence of the derivation itself.

Lemma 5.4. If $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$, then $t$ is wcf.
Proof. By straightforward induction in $t$.
The following tight spreading lemmas will be used in Lemma 5.6 and Lemma 5.7, which in turn ensure that in tight derivations the counters work as expected for normal forms.

Lemma 5.5 (Tight Spreading for Neutral Terms). Let $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ such that $t \in \mathrm{ne}_{\mathrm{w}}$. If $\Gamma$ is tight, then $\sigma \in \mathrm{tt}$.

Proof. We reason by induction on $t$. Notice that by i.h. for every subterm $u$ of $t$ verifying $u \in \mathrm{ne}_{\mathrm{w}}$, every derivation of $u$ having a tight typing context must also have a tight type subject.

- $t=x$. Then, $\Phi$ ends with rule $\left(\mathrm{ax}_{\mathrm{c}}\right)$ and $\Gamma=x:[\sigma]$ tight implies $\sigma \in \mathrm{tt}$.
- $t=r u$. By definition of $t \in \mathrm{ne}_{\mathrm{w}}, r \in \mathrm{na}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$ hold. Moreover, $\neg \operatorname{bang}(r)$ by Lemma 5.4. Then, by Remark 2.10, it is necessarily the case that $r \in \mathrm{nb}_{\mathrm{w}}$ holds, and hence $r \in \mathrm{ne}_{\mathrm{w}}$ as well. There are three cases for $\Phi$ :

1. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{p}}\right)$, then $\sigma=\mathrm{n}$ and the statement trivially holds.
2. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$, then $\Gamma=\Gamma^{\prime}+\Delta, b=b^{\prime}+b^{\prime \prime}+1$, $e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{r} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \mathrm{M} \rightarrow \tau$ with $\Gamma^{\prime}$ tight (since $\Gamma$ is tight). Then, $r \in \mathrm{ne}_{\mathrm{w}}$ gives $\mathrm{M} \rightarrow \tau \in \mathrm{tt}$ by $i . h$. This is clearly a contradiction. Hence, this case does not apply.
3. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$, then we reason exactly as in the previous case, so that this case does not apply neither.

- $t=\operatorname{der} u$. By definition of $t \in \mathrm{ne}_{\mathrm{w}}, u \in \mathrm{nb}_{\mathrm{w}}$ holds. Moreover, $\neg \mathrm{abs}(u)$ by Lemma 5.4. Then, by Remark 2.10 it is necessarily the case that $u \in \mathrm{na}_{\mathrm{w}}$ holds, and hence $u \in \mathrm{ne}_{\mathrm{w}}$ as well. Then, there are two cases for $\Phi$ :

1. if $\Phi$ ends with rule $\left(d r_{\mathrm{p}}\right)$, then $\sigma=\mathrm{n}$ and the statement trivially holds.
2. if $\Phi$ ends with rule $\left(\mathrm{dr}_{\mathrm{c}}\right)$, then $\Phi_{u} \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} u:[\sigma]$. Therefore, $u \in \mathrm{ne}_{\mathrm{w}}$ gives $[\sigma] \in \mathrm{tt}$ by $i . h$. This is clearly a contradiction. Hence, this case does not apply.

- $t=r[x \backslash u]$. By definition of $t \in \mathrm{ne}_{\mathrm{w}}, r \in \mathrm{ne}_{\mathrm{w}}$ and $u \in \mathrm{nb}_{\mathrm{w}}$ hold. Moreover, $\neg \operatorname{abs}(u)$ by Lemma 5.4. Then, by Remark 2.10, it is necessarily the case that $u \in \mathrm{na}_{\mathrm{w}}$ holds, and hence $u \in \mathrm{ne}_{\mathrm{w}}$ as well. Then, there are two cases for $\Phi$ :

1. if $\Phi$ ends with rule ( $\mathrm{es}_{\mathrm{c}}$ ), $\Gamma=\Gamma^{\prime} \ x+\Delta, b=b^{\prime}+b^{\prime \prime}, e=e^{\prime}+e^{\prime \prime}$, $s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{u} \triangleright_{\mathcal{E}} \Delta \vdash^{\left(b^{\prime \prime}, e^{\prime \prime}, s^{\prime \prime}\right)} u: \Gamma^{\prime}(x)$ with $\Delta$ tight (since $\Gamma$ is tight). Then, $u \in \mathrm{ne}_{\mathrm{w}}$ gives $\Gamma^{\prime}(x) \in \mathrm{tt}$ by i.h. This leads to a contradiction, since $\Gamma^{\prime}(x)$ is a multiset type by definition. Hence, this case does not apply.
2. if $\Phi$ ends with rule $\left(\mathrm{es}_{\mathrm{p}}\right)$, then $\Gamma=\Gamma^{\prime} \ x+\Delta$, $\operatorname{tight}\left(\Gamma^{\prime}(x)\right), b=b^{\prime}+$ $b^{\prime \prime}, e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{r} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \sigma$. Moreover, $\Gamma$ tight and tight $\left(\Gamma^{\prime}(x)\right)$ give $\Gamma^{\prime}$ tight as well. We conclude by i.h. with $r \in \mathrm{ne}_{\mathrm{w}}$ that $\sigma \in \mathrm{tt}$.

Lemma 5.6 (Tight Spreading for Zero Counters). Let $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ such that $b=e=0$ and $\sigma$ is not an arrow type. If $\Gamma$ is tight, then $\sigma \in \mathrm{tt}$.

Proof. By induction on $\Phi$. Note that the statement trivially holds for the rules $\left(a e_{p}\right),\left(a i_{p}\right),\left(b g_{p}\right),\left(d r_{p}\right)$ in Figure 3 since all of them conclude with $\sigma \in t t$. We proceed by analysing the other rules in Figure 3

- $\left(\mathrm{es}_{\mathrm{p}}\right)$. Then $\Gamma=\Gamma^{\prime} \| x+\Delta$, $\operatorname{tight}\left(\Gamma^{\prime}(x)\right), s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{r} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} r: \sigma$. Moreover, $\Gamma$ tight and tight $\left(\Gamma^{\prime}(x)\right)$ give $\Gamma^{\prime}$ tight as well. We then conclude directly by i.h. with $\Phi_{r}$ that $\sigma \in \mathrm{tt}$.
- $\left(\mathrm{ax}_{\mathrm{c}}\right)$. Then $t=x$ and $\Gamma=x:[\sigma]$ tight which implies $\sigma \in \mathrm{tt}$.
- $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$. This case does not apply since it concludes with $b>0$.
- $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$. This case does not apply since it concludes with $b>0$.
- ( $\mathrm{ai}_{\mathrm{c}}$ ). This case does not apply since it concludes with an arrow type.
- $\left(\mathrm{bg}_{\mathrm{c}}\right)$. This case does not apply since it concludes with $e>0$.
- $\left(\mathrm{dr}_{\mathrm{c}}\right)$. Then $t=\operatorname{der} u$ and $\Gamma \vdash^{(0,0, s)} u:[\sigma]$ is derivable. Then, the i.h. gives $[\sigma] \in \mathrm{tt}$ which is clearly a contradiction. Thus, this case does not apply either.
- ( $\mathrm{es}_{\mathrm{c}}$ ). Then $t=r[x \backslash u], s=s^{\prime}+s^{\prime \prime}$ and $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ tight such that, in particular $\Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \Gamma(x)$ with $\Delta$ tight. The i.h. gives $\Gamma(x) \in \mathrm{tt}$ which leads to a contradiction since it is a multiset type. Hence, this case does not apply.

The following two lemmas are needed to establish the base case of the induction proving soundness.

Lemma 5.7. If $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ is tight, then $b=e=0$ iff $t \in \mathrm{no}_{\mathrm{w}}$.
Proof. $\Rightarrow)$ By induction on $\Phi$.

- $\left(\mathrm{ax}_{\mathrm{c}}\right)$. Then $t=x, \Gamma=[\sigma]$ with $\sigma \in \mathrm{tt}$. By definition $t \in \mathrm{ne}_{\mathrm{w}}$ holds, which implies $t \in \mathrm{no}_{\mathrm{w}}$ as well.
- $\left(\mathrm{ae}_{\mathrm{p}}\right)$. Then $t=r u, \sigma=\mathrm{n}, \Gamma=\Gamma^{\prime}+\Delta, s=s^{\prime}+s^{\prime \prime}+1, \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} r: \mathrm{n}$, $\Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \overline{\mathrm{a}}$. Then, $\Gamma^{\prime}$ and $\Delta$ are both tight, hence the $i . h$. gives $r \in \mathrm{no}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$. There are two cases to consider based on $r \in \mathrm{no}_{\mathrm{w}}$ :

1. if $r \in \mathrm{na}_{\mathrm{w}}$ the result is immediate.
2. if $r \in \mathrm{nb}_{\mathrm{w}}$ then we can assume $r \notin \mathrm{na}_{\mathrm{w}}$ too (since $r \in \mathrm{na} \mathrm{w}_{\mathrm{w}}$ is already considered). Then, by Remark 2.10, abs( $r$ ) holds. This leads to a contradiction with $r$ having type $\mathrm{n}(c f$. Remark 5.3). Hence, this case does not apply.

- $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$. This case does not apply since it concludes with $b>0$.
- $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$. This case does not apply since it concludes with $b>0$.
- ( $\mathrm{ai}_{\mathrm{p}}$ ). Then $t=\lambda x . u, \sigma=\mathrm{a}, \Gamma=\Gamma^{\prime} \backslash x, s=s^{\prime}+1, \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} u$ : tt and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right), \Gamma^{\prime}$ is tight as well. Then, by i.h. $u \in \mathrm{no}_{\mathrm{w}}$ holds, which implies $t \in \mathrm{no}_{\mathrm{w}}$ too.
- ( $\mathrm{ai}_{\mathrm{c}}$ ). Then $\sigma=\mathrm{M} \rightarrow \tau$ which contradicts the hypothesis of $\Phi$ being tight. Hence, this case does not apply.
- $\left(\mathrm{bg}_{\mathrm{p}}\right)$. Then $t=!u$ which implies $t \in \mathrm{na} \mathrm{w}_{\mathrm{w}}$ and hence $t \in \mathrm{no} \mathrm{w}_{\mathrm{w}}$.
- $\left(\mathrm{bg}_{\mathrm{c}}\right)$. This case does not apply since it concludes with $e>0$.
- $\left(\mathrm{dr}_{\mathrm{p}}\right)$. Then $t=\operatorname{der} u, \sigma=\mathrm{n}, s=s^{\prime}+1$ and $\Gamma \vdash^{\left(0,0, s^{\prime}\right)} u: \mathrm{n}$. By i.h. $u \in \mathrm{no}_{\mathrm{w}}$ holds. There are two cases to consider:

1. if $u \in \mathrm{nb}_{\mathrm{w}}$ the result is immediate.
2. if $u \in \mathrm{na}_{\mathrm{w}}$ and $u \notin \mathrm{nb}_{\mathrm{w}}$, then bang $(u)$ holds by Remark 2.10. This leads to a contradiction with $u$ having type n ( $c f$. Remark 5.3). Hence, this case does not apply.

- $\left(\mathrm{dr}_{\mathrm{c}}\right)$. Then $t=\operatorname{der} u$ and $\Phi_{u} \triangleright_{\mathcal{E}} \Gamma \vdash^{(0,0, s)} u:[\sigma]$. Then, Lemma 5.6 on $\Phi_{u}$ give $[\sigma] \in \mathrm{tt}$ which is clearly a contradiction. Thus, this case does not apply.
- $\left(\mathrm{es}_{\mathrm{p}}\right)$. Then $t=r[x \backslash u], \Gamma=\left(\Gamma^{\prime} \ x\right)+\Delta, s=s^{\prime}+s^{\prime \prime}, \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} r: \sigma$, $\Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \mathrm{n}$ and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$, then $\Gamma^{\prime}$ and $\Delta$ are both tight as well. Thus, i.h. gives $r \in \mathrm{no}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$. Moreover, by definition $r \in \mathrm{no}_{\mathrm{w}}$ means $r \in \mathrm{na}_{\mathrm{w}}$ or $r \in \mathrm{nb}_{\mathrm{W}}$. Same for $u \in \mathrm{no}_{\mathrm{w}}$, hence there are two different cases to analyse:

1. if $u \in \mathrm{nb}_{\mathrm{w}}$ the result is immediate.
2. if $u \in \mathrm{na}_{\mathrm{w}}$ and $u \notin \mathrm{nb}_{\mathrm{w}}$, then bang $(u)$ holds by Remark 2.10 This leads to a contradiction with $u$ having type n ( $c f$. Remark 5.3). Hence, this case does not apply.

- $\left(\mathrm{es}_{\mathrm{c}}\right)$. Then $t=r[x \backslash u], s=s^{\prime}+s^{\prime \prime}$ and $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ tight such that, in particular, $\Phi_{u} \triangleright_{\mathcal{E}} \Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \Gamma^{\prime}(x)$ with $\Delta$ tight. By Lemma 5.6 on $\Phi_{u}, \Gamma^{\prime}(x) \in \mathrm{tt}$ which leads to a contradiction, since $\Gamma^{\prime}(x)$ is a multiset type by definition. Hence, this case does not apply.
$\Leftarrow)$ By induction on $t$.
- $t=x$. Then, $\Phi$ ends with rule $\left(\mathrm{ax}_{\mathrm{c}}\right)$ and the statement holds trivially.
- $t=r u$. By definition $t \in \mathrm{no}_{\mathrm{w}}$ gives $r \in \mathrm{na}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$. Thus, $r \in \mathrm{no}_{\mathrm{w}}$ holds too. There are three cases to consider:

1. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{p}}\right)$, then $\sigma=\mathrm{n}, \Gamma=\Gamma^{\prime}+\Delta, b=b^{\prime}+b^{\prime \prime}$, $e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}+1, \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \mathrm{n}, \Delta \vdash^{\left(b^{\prime \prime}, e^{\prime \prime}, s^{\prime \prime}\right)} u: \overline{\mathrm{a}}$. Moreover, $\Gamma^{\prime}$ and $\Delta$ are both tight. Then, the i.h. with $r \in \mathrm{no}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$ gives $b^{\prime}=e^{\prime}=0$ and $b^{\prime \prime}=e^{\prime \prime}=0$. Hence, $b=e=0$.
2. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$, then $\Gamma=\Gamma^{\prime}+\Delta, b=b^{\prime}+b^{\prime \prime}+1$, $e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{r} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \mathrm{M} \rightarrow \tau$. Moreover, by contra-positive of Remark 5.3 with $\Phi_{r}$, it is necessarily the case that $\neg \operatorname{bang}(r)$. Thus, together with $r \in \mathrm{na}_{\mathrm{w}}$ it gives $r \in \mathrm{ne}_{\mathrm{w}}$ ( $c f$. Remark 2.10). Also, $\Gamma$ tight implies $\Gamma^{\prime}$ tight as well. Then, by Lemma 5.5, $\Phi_{r}$ is a tight typing, which leads to a contradiction with $r$ having a functional type. Hence, this case does not apply.
3. if $\Phi$ ends with rule $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$, then $\Gamma=\Gamma^{\prime}+\Delta, b=b^{\prime}+b^{\prime \prime}+1$, $e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}$ and, in particular, $\Phi_{r} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \mathrm{M} \rightarrow \tau$. This case is identical to the previous one.

- $t=\lambda$ x.u. By definition $t \in \mathrm{no}_{\mathrm{w}}$ gives $u \in \mathrm{no}_{\mathrm{w}}$. There are two cases to consider for $\Phi$ :

1. if $\Phi$ ends with rule $\left(\mathrm{ai}_{\mathrm{p}}\right)$, then $\sigma=\mathrm{a}, \Gamma=\Gamma^{\prime} \backslash x, s=s^{\prime}+1, \Phi_{u} \triangleright_{\mathcal{E}}$ $\Gamma^{\prime} \vdash^{\left(b, e, s^{\prime}\right)} u$ : tt and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$, $\Gamma^{\prime}$ is tight as well. Then, the i.h. gives $b=e=0$.
2. if $\Phi$ ends with rule $\left(a i_{c}\right)$, then $\sigma=M \rightarrow \tau$ which contradicts the hypothesis of $\Phi$ being tight. Hence, this case does not apply.

- $t=!u$. There are two cases to consider for $\Phi$ :

1. if $\Phi$ ends with rule $\left(\mathrm{bg}_{\mathrm{p}}\right)$, then $b=e=0$ and the statement holds immediately.
2. if $\Phi$ ends with rule $\left(\mathrm{bg}_{c}\right)$, then $\sigma$ is a multiset type which contradicts the hypothesis of $\Phi$ being tight. Hence, this case does not apply.

- $t=\operatorname{der} u$. By definition $t \in \mathrm{no}_{\mathrm{w}}$ gives $u \in \mathrm{nb}_{\mathrm{w}}$ which implies $u \in \mathrm{no}_{\mathrm{w}}$ as well. There are two cases to consider:

1. if $\Phi$ ends with rule $\left(\mathrm{dr}_{\mathrm{p}}\right)$, then $\sigma=\mathrm{n}, s=s^{\prime}+1$ and $\Gamma \vdash^{\left(b, e, s^{\prime}\right)} u: \mathrm{n}$. Then, the statement follows immediately from the i.h.
2. if $\Phi$ ends with rule $\left(\mathrm{dr}_{\mathrm{c}}\right)$, then $\Phi_{u} \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} u:[\sigma]$. Moreover, by contra-positive of Remark 5.3 with $\Phi_{u}$, it is necessarily the case $\neg$ abs $(u)$. Thus, together with $u \in \mathrm{nb}_{\mathrm{w}}$ it gives $u \in \mathrm{ne}_{\mathrm{w}}$ (cf. Remark 2.10). Then, by Lemma 5.5, $\Phi_{u}$ is a tight typing, which leads to a contradiction with $u$ having a multiset type. Hence, this case does not apply.

- $t=r[x \backslash u]$. By definition $t \in \mathrm{no}_{\mathrm{w}}$ implies $r \in \mathrm{no}_{\mathrm{w}}$ and $u \in \mathrm{nb}_{\mathrm{w}}$, which in turn implies $u \in \mathrm{no}_{\mathrm{w}}$ as well. Then, there are two cases to consider for $\Phi$ :

1. if $\Phi$ ends with rule $\left(\mathrm{es}_{\mathrm{p}}\right)$, then $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta, b=b^{\prime}+b^{\prime \prime}$, $e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+s^{\prime \prime}, \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} r: \sigma, \Delta \vdash^{\left(b^{\prime \prime}, e^{\prime \prime}, s^{\prime \prime}\right)} u: \mathrm{n}$ and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$, then $\Gamma^{\prime}$ and $\Delta$ are both tight as well. Then, the $i . h$. with $r \in \mathrm{no}_{\mathrm{w}}$ and $u \in \mathrm{no}_{\mathrm{w}}$ gives $b^{\prime}=e^{\prime}=0$ and $b^{\prime \prime}=e^{\prime \prime}=0$. Hence, $b=e=0$.
2. if $\Phi$ ends with rule ( $\mathrm{es}_{\mathrm{c}}$ ), then $b=b^{\prime}+b^{\prime \prime}, e=e^{\prime}+e^{\prime \prime}, s=s^{\prime}+$ $s^{\prime \prime}$ and $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ tight such that, in particular, $\Phi_{u} \triangleright_{\mathcal{E}}$ $\Delta \vdash\left(b^{\prime \prime}, e^{\prime \prime}, s^{\prime \prime}\right) u: \Gamma^{\prime}(x)$ with $\Delta$ tight. Moreover, by contra-positive of Remark 5.3 with $\Phi_{u}$, it is necessarily the case $\neg$ abs $(u)$. Thus, together with $u \in \mathrm{nb}_{\mathrm{w}}$ it gives $u \in \mathrm{ne}_{\mathrm{w}}$ ( $c f$. Remark 2.10). Then, by Lemma 5.5, $\Phi_{u}$ is a tight typing, which leads to a contradiction with $u$ having a multiset type. Hence, this case does not apply.

Lemma 5.8. If $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(0,0, s)} t: \sigma$ is tight, then $s=|t|_{w}$.
Proof. By induction on $\Phi$.

- $\left(\mathrm{ax}_{\mathrm{c}}\right)$. Then $t=x$ and $s=0=|t|_{\mathrm{w}}$.
- $\left(\mathrm{ae}_{\mathrm{p}}\right)$. Then $t=r u, \sigma=\mathrm{n}, \Gamma=\Gamma^{\prime}+\Delta, s=s^{\prime}+s^{\prime \prime}+1, \Phi_{r} \triangleright_{\mathcal{E}}$ $\Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} r: \mathrm{n}, \Phi_{u} \triangleright_{\mathcal{E}} \Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u$ : $\overline{\mathrm{a}}$. Then, $\Gamma^{\prime}$ and $\Delta$ are both tight, hence the i.h. gives $s^{\prime}=|r|_{\mathrm{w}}$ and $s^{\prime \prime}=|u|_{\mathrm{w}}$. Hence, $s=|r|_{\mathrm{w}}+|u|_{\mathrm{w}}+1=|t|_{\mathrm{w}}$.
- $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$. This case does not apply since it concludes with $b>0$.
- $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$. This case does not apply since it concludes with $b>0$.
- ( $\mathrm{ai}_{\mathrm{p}}$ ). Then $t=\lambda x \cdot u, \sigma=\mathrm{a}, \Gamma=\Gamma^{\prime} \ x, s=s^{\prime}+1, \Phi_{u} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} u$ : tt and tight $\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and tight $\left(\Gamma^{\prime}(x)\right), \Gamma^{\prime}$ is tight as well. Then, by i.h. $s^{\prime}=|u|_{\mathrm{w}}$, which implies $s=|u|_{\mathrm{w}}+1=|t|_{\mathrm{w}}$.
- ( $\mathrm{ai}_{\mathrm{c}}$ ). Then $\sigma=\mathrm{M} \rightarrow \tau$ which contradicts the hypothesis of $\Phi$ being tight. Hence, this case does not apply.
- $\left(\mathrm{bg}_{\mathrm{p}}\right)$. Then $t=!u$ and $s=0=|t|_{\mathrm{w}}$.
- $\left(\mathrm{bg}_{\mathrm{c}}\right)$. This case does not apply since it concludes with $e>0$.
- $\left(\mathrm{dr}_{\mathrm{p}}\right)$. Then $t=\operatorname{der} u, \sigma=\mathrm{n}, s=s^{\prime}+1$ and $\Phi_{u} \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0, s^{\prime}\right)} u: \mathrm{n}$. By i.h. $s^{\prime}=|u|_{\mathrm{w}}$, which implies $s=|u|_{\mathrm{w}}+1=|t|_{\mathrm{w}}$.
- $\left(\mathrm{dr}_{\mathrm{c}}\right)$. Then $t=\operatorname{der} u$ and $\Phi_{u} \triangleright_{\mathcal{E}} \Gamma \vdash^{(0,0, s)} u:[\sigma]$. Then, Lemma 5.6 on $\Phi_{u}$ gives $[\sigma] \in \mathrm{tt}$ which is clearly a contradiction. Thus, this case does not apply.
- $\left(\mathrm{es}_{\mathrm{p}}\right)$. Then $t=r[x \backslash u], \Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta, s=s^{\prime}+s^{\prime \prime}, \Phi_{r} \triangleright{ }_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(0,0, s^{\prime}\right)} r: \sigma$, $\Phi_{u} \triangleright_{\mathcal{E}} \Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \mathrm{n}$ and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$. Since $\Gamma$ is tight and $\operatorname{tight}\left(\Gamma^{\prime}(x)\right)$, then $\Gamma^{\prime}$ and $\Delta$ are both tight as well. Thus, i.h. gives $s^{\prime}=|r|_{\mathrm{w}}$ and $s^{\prime \prime}=|u|_{\mathrm{w}}$. Then, $s=|r|_{\mathrm{w}}+|u|_{\mathrm{w}}=|t|_{\mathrm{w}}$.
- ( $\mathrm{es}_{\mathrm{c}}$ ). Then $t=r[x \backslash u], s=s^{\prime}+s^{\prime \prime}$ and $\Gamma=\left(\Gamma^{\prime} \backslash x\right)+\Delta$ tight such that, in particular, $\Phi_{u} \triangleright_{\mathcal{E}} \Delta \vdash^{\left(0,0, s^{\prime \prime}\right)} u: \Gamma^{\prime}(x)$ with $\Delta$ tight. By Lemma 5.6 on $\Phi_{u}, \Gamma^{\prime}(x) \in \mathrm{tt}$ which leads to a contradiction, since $\Gamma^{\prime}(x)$ is a multiset type by definition. Hence, this case does not apply.

As well as $\mathcal{U}$-typability, $\mathcal{E}$-typability of a term may provide additional information about the neutrality/normality of its subterms:

Lemma 5.9. Let $u \in \mathcal{T}$ :

1. If $t \in \mathrm{na}_{\mathrm{w}}$ and $t u$ is $\mathcal{E}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
2. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $u[x \backslash t]$ is $\mathcal{E}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
3. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $\operatorname{der} t$ is $\mathcal{E}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
4. If $t \in \mathrm{nb}_{\mathrm{w}}$ and $u t$ is $\mathcal{E}$-typable, then $t \in \mathrm{ne}_{\mathrm{w}}$.
5. If $t \in \mathrm{no}_{\mathrm{w}}$ and $u t$ is $\mathcal{E}$-typable, then $t \in \mathrm{na}_{\mathrm{w}}$.

Proof. Straightforward case analysis using the characterisation in the proof of Proposition 2.11 and resorting to Remark 5.3. Notice that a similar property was shown for $\mathcal{U}$-typability (Lemma 3.2).

As well as the type system $\mathcal{U}$, the type system $\mathcal{E}$ captures clash-freeness of normal terms:

Theorem 5.10. Let $t \in \mathcal{T}$. Then, $t \in \mathrm{no}_{\mathrm{wcf}}$ iff $t \in \mathrm{no}_{\mathrm{w}}$ and $t$ is $\mathcal{E}$-typable.
Proof. A similar property was shown for $\mathcal{U}$-typability (Theorem 3.8). This proof is analogous to that one, but now using Lemma 5.9. Notice that the consuming rules of system $\mathcal{E}$ are essentially the typing rules of system $\mathcal{U}$.

As usual, in order to prove soundness, the key property is subject reduction, stating that every reduction step decreases one of the first two counters of tight derivations by exactly one. We first prove a substitution lemma.

Lemma 5.11 (Substitution). Let us consider $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} t: \tau$ and derivations $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$, then there exists a derivation of the form $\Phi_{t\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma+_{i \in I} \Delta_{i} \vdash^{\left(b+_{i \in I} b_{i}, e+_{i \in I} e_{i}, s+_{i \in I} s_{i}\right)} t\{x \backslash u\}: \tau$.

Proof. Straightforward induction on $\Phi_{t}$. The detailed proof of few chosen cases follow. Suppose that the last rule of $\Phi_{t}$ is:

- $\left(\mathrm{ax}_{\mathrm{c}}\right)$. Then $t=y$ and $\Gamma=y:[\tau]$. If $x \neq y$ then $I=\emptyset$, and the required typing is $\Phi_{t}$. If $x=y$ then $I$ is a singleton $\{*\}$ and the required typing is $\Phi_{u}^{*}$. The counters are as expected since $b=e=s=0$.
- $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$. Then $t=t_{1} t_{2}$ with $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \mathrm{n}$ such that $\Gamma=\Gamma_{1}+\Gamma_{2}, I=I_{1} \uplus I_{2}$, $b=b_{1}+b_{2}+1, e=e_{1}+e_{2}, s=s_{1}+s_{2}$, and M is tight. The $i . h$. provides the typings $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}+{ }_{i \in I_{1}} \Delta_{i} \vdash^{\left(b_{1}++_{i \in I_{1}} b_{i}, e_{1}+{ }_{i \in I_{1}} e_{i}, s_{1}+{ }_{i} \in I_{1} s_{i}\right)} t_{1}\{x \backslash u\}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}+_{i \in I_{2}} \Delta_{i} \vdash^{\left(b_{2}+{ }_{i \in I_{2}} b_{i}, e_{2}+{ }_{i \in I_{2}} e_{i}, s_{2}+{ }_{i \in I_{2}} s_{i}\right)} t_{2}\{x \backslash u\}: \mathrm{n}$. The required typing is obtained by applying the rule $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$ to these, using the fact that $M$ is tight. The counters are as expected.
- ( $\mathrm{es}_{\mathrm{c}}$ ). Then $t=t_{1}\left[y \backslash t_{2}\right]$, and we have $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \tau$ and $\Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \Gamma_{1}(y)$ such that $\Gamma=\left(\Gamma_{1} \backslash y\right)+\Gamma_{2}$, $I=I_{1} \uplus I_{2}, b=b_{1}+b_{2}, e=e_{1}+e_{2}, s=s_{1}+s_{2}$. The $i . h$. provides the typings $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}+_{i \in I_{1}} \Delta_{i} \vdash^{\left(b_{1}+{ }_{i \in I_{1}} b_{i}, e_{1}+_{i \in I_{1}} e_{i}, s_{1}+{ }_{i \in I_{1}} s_{i}\right)} t_{1}\{x \backslash u\}: \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}+_{i \in I_{2}} \Delta_{i} \vdash^{\left(b_{2}+{ }_{i \in I_{2}} b_{i}, e_{2}+{ }_{i \in I_{2}} e_{i}, s_{2}+{ }_{i \in I_{2}} s_{i}\right)} t_{2}\{x \backslash u\}: \Gamma_{1}(y)$ The required typing is obtained by applying the rule (esc $)$ to these. The counters are as expected.

The goal of exact subject reduction is to show that tight derivations are preserved by reduction. To apply the i.h. on a sub-derivation of the original tight type derivation, one would need this sub-derivation to be also tight. However, tightness is a global property not necessarily true for all sub-derivations. A subtle property is then needed, whose precise formulation uses an idea in 1]: the original typed term $t$ is required not to be an abstraction-like term, or tightly typable. This is sufficient to show the desired property. Moreover, subject reduction for the system $\mathcal{E}$ proceeds by induction on the definition of $t \rightarrow_{\mathrm{dw}} t^{\prime}$, and in the three base cases of the recursive definition of $t \rightarrow_{\mathrm{dw}} t^{\prime}$ the list L has arbitrary length. Therefore, for each base case there is a further induction on the length of L. Formally,

Lemma 5.12 (Exact Subject Reduction). Let $\Phi \triangleright_{\mathcal{E}} \Gamma^{\vdash^{(b, e, s)}} t: \sigma$ such that $\Gamma$ is tight, and either $\sigma \in \mathrm{tt}$ or $\neg \operatorname{abs}(t)$. If $t \rightarrow_{\mathrm{dw}} t^{\prime}$, then there exists $\Phi^{\prime} \triangleright_{\mathcal{E}}$ $\Gamma \vdash^{\left(b^{\prime}, e^{\prime}, s\right)} t^{\prime}: \sigma$ such that

- $b^{\prime}=b-1$ and $e^{\prime}=e$ if $t \rightarrow_{\mathrm{dw}} t^{\prime}$ is an m-step.
- $e^{\prime}=e-1$ and $b^{\prime}=b$ if $t \rightarrow_{\mathrm{dw}} t^{\prime}$ is an e-step.

Proof. By induction on $t \rightarrow_{\mathrm{dw}} t^{\prime}$.

- $t=\mathrm{L}\langle\lambda x . u\rangle r \rightarrow_{\mathrm{dw}} \mathrm{L}\langle u[x \backslash r]\rangle=t^{\prime}$. We reason by induction on L .
$-\mathrm{L}=\square$. We first note that $\Phi$ cannot end with rule ( $\mathrm{ae}_{\mathrm{p}}$ ) since $\lambda x . u$ cannot be typed with n , then there are two cases depending on the last rule of $\Phi$.

1. If $\Phi$ has the following form:

$$
\frac{\frac{\Gamma_{u} ; x: \mathrm{M} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma}{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} \lambda x . u: \mathrm{M} \rightarrow \sigma}\left(\mathrm{ai}_{\mathrm{c}}\right) \quad \Gamma_{r} \vdash^{\left(b_{r}, e_{r}, s_{r}\right)} r: \mathrm{M}}{\Gamma_{u}+\Gamma_{r} \vdash^{\left(b_{u}+b_{r}+1, e_{u}+e_{r}, s_{u}+s_{r}\right)}(\lambda x . u) r: \sigma}\left(\mathrm{ae}_{\mathrm{c} 1}\right)
$$

We can then construct the following derivation $\Phi^{\prime}$.

$$
\frac{\Gamma_{u} ; x: \mathrm{M} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma \quad \Gamma_{r} \vdash^{\left(b_{r}, e_{r}, s_{r}\right)} r: \mathrm{M}}{\Gamma_{u}+\Gamma_{r} \vdash^{\left(b_{u}+b_{r}, e_{u}+e_{r}, s_{u}+s_{r}\right)} u[x \backslash r]: \sigma}\left(\mathrm{es}_{c}\right)
$$

The counters verify the expected property.
2. If $\Phi$ has the following form:

$$
\frac{\frac{\Gamma_{u} ; x: \mathrm{M} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma}{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} \lambda x . u: \mathrm{M} \rightarrow \sigma}\left(\mathrm{ai}_{\mathrm{c}}\right)}{\Gamma_{u}+\Gamma_{r} \vdash^{\left(b_{u}+b_{r}+1, e_{u}+e_{r}, s_{u}+s_{r}\right)}(\lambda x . u) r: \sigma} \vdash^{\left(b_{r}, e_{r}, s_{r}\right)} r: \mathrm{n} \quad \operatorname{tight}(\mathrm{M}) \quad\left(\mathrm{ae}_{\mathrm{c} 2}\right)
$$

We can then construct the following derivation $\Phi^{\prime}$.

$$
\frac{\Gamma_{u} ; x: \mathrm{M} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma \quad \Gamma_{r} \vdash^{\left(b_{r}, e_{r}, s_{r}\right)} r: \mathrm{n} \quad \operatorname{tight}(\mathrm{M})}{\Gamma_{u}+\Gamma_{r} \vdash^{\left(b_{u}+b_{r}, e_{u}+e_{r}, s_{u}+s_{r}\right)} u[x \backslash r]: \sigma}\left(\mathrm{es}_{\mathrm{p}}\right)
$$

The counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash s]$. Immediate from the i.h.

- $t=u[x \backslash \mathrm{~L}\langle!r\rangle] \rightarrow_{\mathrm{dw}} \mathrm{L}\langle u\{x \backslash r\}\rangle=t^{\prime}$. We reason by induction on L .
$-\mathrm{L}=\square$. We first note that $\Phi$ cannot end with rule ( $\mathrm{es}_{\mathrm{p}}$ ) since ! $r$ cannot be typed with n , then $\Phi$ has the following form:
$\frac{\Gamma_{u} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma \quad \frac{\left(\Gamma_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} r: \sigma_{i}\right)_{i \in I}}{+_{i \in I} \Gamma_{i} \vdash^{\left(+_{i \in I} b_{i}, 1+{ }_{i \in I} e_{i},+_{i \in I} s_{i}\right)}!r:\left[\sigma_{i}\right]_{i \in I}}\left(\mathrm{bg}_{\mathrm{c}}\right)}{\Gamma_{u}+{ }_{i \in I} \Gamma_{i} \vdash^{\left(b_{u}+{ }_{i \in I} b_{i}, e_{u}+1+{ }_{i \in I} e_{i}, s_{u}+{ }_{i \in I} s_{i}\right)} u[x \backslash!r]: \sigma}\left(\mathrm{es}_{\mathrm{c}}\right)$
By applying Lemma 5.11 to the premises we obtain a derivation

$$
\Phi^{\prime} \triangleright \Gamma_{u}+_{i \in I} \Gamma_{i} \vdash^{\left(b_{u}+{ }_{i \in I} b_{i}, e_{u}+{ }_{i \in I} e_{i}, s_{u}+{ }_{i \in I} s_{i}\right)} u\{x \backslash r\}: \sigma
$$

The counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash s]$. Immediate from the i.h.

- $t=\operatorname{der}(\mathrm{L}\langle!u\rangle) \rightarrow_{\mathrm{dw}} \mathrm{L}\langle u\rangle=t^{\prime}$. We reason by induction on L .
$-L=\square$. Then $\Phi$ has necessarily the following form:

$$
\frac{\frac{\Gamma_{u} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \sigma}{\Gamma_{u} \vdash^{\left(b_{u}, 1+e_{u}, s_{u}\right)}!u:[\sigma]}}{\Gamma_{u} \vdash^{\left(b_{u}, 1+e_{u}, s_{u}\right)} \operatorname{der}!u: \sigma}\left(\mathrm{bg}_{\mathrm{c}}\right)
$$

We conclude with the derivation $\Phi^{\prime}$ given by the premise. The counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash s]$. Immediate from the i.h.

- All the inductive cases for internal reductions are straightforward by $i . h$.

Theorem 5.13 (Soundness for System $\mathcal{E})$. If $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ is tight, then there exists $p$ such that $p \in \mathrm{no}_{\mathrm{wcf}}$ and $t \rightarrow{ }_{\mathrm{w}}^{(b, e)} p$ with $b$ m-steps, $e$ e-steps, and $|p|_{\text {w }}=s$.

Proof. We prove the statement by showing that $t \rightarrow_{d w}^{(b, e)} p$ holds for the deterministic strategy, then we conclude since $\rightarrow_{\mathrm{dw}} \subseteq \rightarrow_{\mathrm{w}}$. Let $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$. We reason by induction on $b+e$.

If $b+e=0$, then $b=e=0$ and Lemma 5.7 gives $t \in \mathrm{no}_{\mathrm{w}}$. Moreover, by Lemma 5.8 and Theorem 5.10 we get both $|t|_{\mathrm{w}}=s$ and $t \in \mathrm{no}_{\text {wcf }}$. Thus, we conclude with $p=t$.

If $b+e>0$, then $t \notin \mathrm{no}_{\mathrm{w}}$ holds by Lemma 5.7 and thus there exists $t^{\prime}$ such that $t \rightarrow{ }_{d w}^{(1,0)} t^{\prime}$ or $t \rightarrow{ }_{\mathrm{dw}}^{(0,1)} t^{\prime}$ by Proposition 5.1. By Lemma 5.12 there is $\Phi^{\prime} \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, s\right)} t^{\prime}: \sigma$ such that $1+b^{\prime}+e^{\prime}=b+e$. By the i.h. there is $p$ such that $p \in \mathrm{no}_{\mathrm{wcf}}$ and $t^{\prime} \rightarrow{ }_{\mathrm{dw}}^{\left(b^{\prime}, e^{\prime}\right)} p$ with $s=|p|_{\mathrm{w}}$. Then $t \rightarrow{ }_{\mathrm{dw}}^{(1,0)} t^{\prime} \rightarrow{ }_{\mathrm{dw}}^{\left(b^{\prime}, e^{\prime}\right)} p$ (resp. $\left.t \rightarrow{ }_{\mathrm{dw}}^{(0,1)} t^{\prime} \ldots\right)$ which means $t \rightarrow{ }_{\mathrm{dw}}^{(b, e)} p$, as expected.

Completeness. We now study completeness of the type system $\mathcal{E}$, which does not only guarantee that normalising terms are typable -a qualitative propertybut also provides a tight type derivation having appropriate counters. More precisely, given a term $t$ which is w-normalisable by means of $b \mathrm{~dB}$-steps and $e$ $\{\mathrm{s}!, \mathrm{d}!\}$-steps, and having a w-normal form of w-size $s$, there is a tight derivation $\Phi$ for $t$ with counters $(b, e, s)$. The completeness proof is mainly based on a subject expansion property (Lemma 5.16), as well as on an auxiliary lemma providing tight derivations with appropriate counters for w-normal weak clashfree terms.
Lemma 5.14. If $t \in \mathrm{no}_{\text {wcf }}$, then there is a tight derivation $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,|t|_{\mathrm{w}}\right)} t: \sigma$.
Proof. We proceed by induction on the derivation of $t \in \mathrm{no}_{\text {wcf }}$ (resp. $t \in \mathrm{ne}_{\mathrm{wcf}}$, $t \in \mathrm{nb}_{\mathrm{wcf}}$ and $t \in \mathrm{na}_{\mathrm{wcf}}$ ). Moreover, we generalise the statement and simultaneously show the following:

1. If $t \in \mathrm{ne}_{\mathrm{wcf}}$, then $\sigma=\mathrm{n}$;
2. If $t \in \mathrm{nb}_{\text {wcf }}$, then $\sigma \in \overline{\mathrm{b}}$;
3. If $t \in \mathrm{na}_{\text {wcf }}$, then $\sigma \in \overline{\mathrm{a}}$.

We analyse the possible cases for $t$.

- Let $x \in \mathrm{ne}_{\mathrm{wcf}}$, so that $|x|_{\mathrm{w}}=0$. Then, take $\triangleright_{\mathcal{E}} x:[\mathrm{n}] \vdash^{(0,0,0)} x: \mathrm{n}$ to conclude.
- Let $\operatorname{der} t \in \mathrm{ne}_{\text {wcf }}$ with $t \in \mathrm{ne}_{\mathrm{wcf}}$. Then, apply the i.h. (1) to obtain $\triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,|t|_{w}\right)} t: \mathrm{n}$ and conclude $\triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,|t|_{w}+1\right)}$ der $t: \mathrm{n}$ with rule $\left(\mathrm{dr}_{\mathrm{p}}\right)$. The size is as expected since $|\operatorname{der} t|_{\mathrm{w}}=|t|_{\mathrm{w}}+1$.
- Let $t u \in \mathrm{ne}_{\mathrm{wcf}}$ with $t \in \mathrm{ne}_{\mathrm{wcf}}$ and $u \in \mathrm{na}_{\mathrm{wcf}}$. Then, apply the $i . h$. (11) and (3) to obtain $\triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,|t|_{w}\right)} t: \mathrm{n}$ and $\triangleright_{\mathcal{E}} \Delta \vdash^{\left(0,0,|u|_{\tilde{v}}\right)} u: \overline{\mathrm{a}}$ resp., and conclude $\triangleright_{\mathcal{E}} \Gamma+\Delta \vdash^{\left(0,0,|t|_{w}+|u|_{w}+1\right)} t u$ : n with rule $\left(\mathrm{ae}_{\mathrm{p}}\right)$. The size is as expected since $|t u|_{\mathrm{w}}=|t|_{\mathrm{w}}+|u|_{\mathrm{w}}+1$.
- Let $t[x \backslash u] \in$ ne $_{\text {wcf }}$ with $t \in \mathrm{ne}_{\text {wcf }}$ and $u \in \mathrm{ne}_{\text {wcf }}$, or $t[x \backslash u] \in \mathrm{na}_{\text {wcf }}$ with $t \in \mathrm{na}_{\text {wcf }}$ and $u \in \mathrm{ne}_{\mathrm{wcf}}$, or $t[x \backslash u] \in \mathrm{nb}_{\text {wcf }}$ with $t \in \mathrm{nb}_{\text {wcf }}$ and $u \in \mathrm{ne}_{\text {wcf }}$. Then, apply the appropriate $i . h$. to obtain $\triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,\left.|t|\right|_{w}\right)} t$ : tt and $\triangleright_{\mathcal{E}}$ $\Delta \vdash^{\left(0,0,|u|_{w}\right)} u: \mathrm{n}$, and conclude $\triangleright_{\mathcal{E}}(\Gamma \backslash x)+\Delta \vdash^{\left(0,0,|t|_{w}+|u|_{\mathrm{w}}\right)} t[x \backslash u]: \mathrm{tt}$ by rule $\left(\mathrm{es}_{\mathrm{p}}\right)$ since $\Gamma(x)$ is tight. The size is as expected since $|t[x \backslash u]|_{\mathrm{w}}=$ $|t|_{\mathrm{w}}+|u|_{\mathrm{w}}$.
- Let $!t \in$ na $_{\text {wcf }}$. Then, conclude $\triangleright_{\mathcal{E}} \vdash^{(0,0,0)}!t: \mathrm{b}$ by $\left(\mathrm{bg}_{\mathrm{p}}\right)$. The size is as expected.
- Let $\lambda x . t \in \mathrm{nb}_{\mathrm{wcf}}$ with $t \in \mathrm{no}_{\mathrm{wcf}}$. Then, apply the appropriate $i . h$. to obtain $\triangleright_{\mathcal{E}} \Gamma \vdash^{\left(0,0,|t|_{\mathfrak{w}}\right)} t$ : tt and conclude $\triangleright_{\mathcal{E}} \Gamma \backslash x \vdash^{\left(0,0,|t|_{w}+1\right)} \lambda x . t$ : a by ( $\mathrm{ai}_{\mathrm{p}}$ ) since $\Gamma(x)$ is tight.

Lemma 5.15 (Anti-Substitution). If $\Phi_{t\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} t\{x \backslash u\}: \tau$, then there exists $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} t: \tau$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma^{\prime}=\Gamma+_{i \in I} \Delta_{i}, b^{\prime}=b+_{i \in I} b_{i}, e^{\prime}=e+_{i \in I} e_{i}$ and $s^{\prime}=s+{ }_{i \in I} s_{i}$.

Proof. By induction on $t$.

- $t=x$. Then, $t\{x \backslash u\}=u$ and we set $I=\{*\}, \sigma_{*}=\tau, \Gamma=\emptyset, \Delta_{*}=\Gamma^{\prime}$, $\Phi_{u}^{*}=\Phi_{t\{x \backslash u\}}$, and $\Phi_{t} \triangleright_{\mathcal{E}} x:[\tau] \vdash^{(0,0,0)} x: \tau$ by rule $\left(\mathrm{ax}_{\mathrm{c}}\right)$.
- $t=y \neq x$. Then, $t\{x \backslash u\}=y$ and we conclude with $I=\emptyset$ (hence, $\left.\left[\sigma_{i}\right]_{i \in I}=[]\right), \Gamma=\Gamma^{\prime}$ and $\Phi_{t}=\Phi_{t\{x \backslash u\}}$.
- $t=t_{1} t_{2}$. Then, $t\{x \backslash u\}=t_{1}\{x \backslash u\} t_{2}\{x \backslash u\}$ and there are three possible cases.

1. $\left(\mathrm{ae}_{\mathrm{p}}\right)$. Then, $\tau=\mathrm{n}, \Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}, b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}, e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}$ and $s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}+1$ with premises $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash^{\left(b_{1}^{\prime}, e_{1}^{\prime}, s_{1}^{\prime}\right)} t_{1}\{x \backslash u\}: \mathrm{n}$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}^{\prime} \vdash\left(b_{2}^{\prime}, e_{2}^{\prime}, s_{2}^{\prime}\right) t_{2}\{x \backslash u\}$ : $\overline{\text { a }}$. By $i . h$. on both, there exist type derivations $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \mathrm{n}, \quad\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{1}}$, $\Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \overline{\mathrm{a}}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{2}}$ (note that $I_{1}$ and $I_{2}$ can be assumed disjoint) such that $\Gamma_{1}^{\prime}=\Gamma_{1}+{ }_{i \in I_{1}}$ $\Delta_{i}, \Gamma_{2}^{\prime}=\Gamma_{2}+_{i \in I_{2}} \Delta_{i}, b_{1}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}, b_{2}^{\prime}=b_{2}+_{i \in I_{2}} b_{i}, e_{1}^{\prime}=e_{1}+{ }_{i \in I_{1}}$ $e_{i}, e_{2}^{\prime}=e_{2}+_{i \in I_{2}} e_{i}, s_{1}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}$ and $s_{2}^{\prime}=s_{2}+_{i \in I_{2}} s_{i}$. Then, by $\left(\mathrm{ae}_{\mathrm{p}}\right)$ we have $\Phi_{t} \triangleright \mathcal{E} \Gamma_{1}+\Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{1} \cup I_{2}} \vdash^{\left(b_{1}+b_{2}, e_{1}+e_{2}, s_{1}+s_{2}+1\right)} t_{1} t_{2}: \mathrm{n}$ and we conclude with $\Gamma=\Gamma_{1}+\Gamma_{2}, I=I_{1} \cup I_{2}, b=b_{1}+b_{2}$, $e=e_{1}+e_{2}$ and $s=s_{1}+s_{2}+1$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}=\Gamma_{1}+_{i \in I_{1}} \Delta_{i}+\Gamma_{2}+_{i \in I_{2}} \Delta_{i}=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}+b_{2}+_{i \in I_{2}} b_{i}=b+_{i \in I} b_{i}$.
$-e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}+e_{2}+_{i \in I_{2}} e_{i}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}+1=s_{1}+_{i \in I_{1}} s_{i}+s_{2}+_{i \in I_{2}} s_{i}+1=s+i \in I s_{i}$.
2. $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$. Then, $\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}, b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}+1, e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}$ and $s^{\prime}=$ $s_{1}^{\prime}+s_{2}^{\prime}$ with premises $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash\left(b_{1}^{\prime}, e_{1}^{\prime}, s_{1}^{\prime}\right) \quad t_{1}\{x \backslash u\}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}^{\prime} \vdash\left(b_{2}^{\prime}, e_{2}^{\prime}, s_{2}^{\prime}\right) t_{2}\{x \backslash u\}: \mathrm{n}$ for some multiset type M such that tight(M). By i.h. there exist $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \mathrm{M} \rightarrow \tau$, $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{1}}, \Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \mathrm{n}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{2}}$ (note that $I_{1}$ and $I_{2}$ can be assumed disjoint) such that $\Gamma_{1}^{\prime}=\Gamma_{1}+_{i \in I_{1}} \Delta_{i}, \Gamma_{2}^{\prime}=\Gamma_{2}+_{i \in I_{2}} \Delta_{i}$, $b_{1}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}, b_{2}^{\prime}=b_{2}+_{i \in I_{2}} b_{i}, e_{1}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}, e_{2}^{\prime}=e_{2}+_{i \in I_{2}} e_{i}$, $s_{1}^{\prime}=s_{1}+{ }_{i \in I_{1}} s_{i}$ and $s_{2}^{\prime}=s_{2}+_{i \in I_{2}} s_{i}$. Then, by $\left(\mathrm{ae}_{\mathrm{c} 2}\right)$ we have the type derivation $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma_{1}+\Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{1} \cup I_{2}} \vdash^{\left(b_{1}+b_{2}+1, e_{1}+e_{2}, s_{1}+s_{2}\right)} t_{1} t_{2}: \tau$ and we conclude with $\Gamma=\Gamma_{1}+\Gamma_{2}, I=I_{1} \cup I_{2}, b=b_{1}+b_{2}+1$, $e=e_{1}+e_{2}$ and $s=s_{1}+s_{2}$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}=\Gamma_{1}+_{i \in I_{1}} \Delta_{i}+\Gamma_{2}+_{i \in I_{2}} \Delta_{i}=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}+1=b_{1}+_{i \in I_{1}} b_{i}+b_{2}+_{i \in I_{2}} b_{i}+1=b+_{i \in I} b_{i}$.
$-e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}+e_{2}+_{i \in I_{2}} e_{i}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}+s_{2}+_{i \in I_{2}} s_{i}=s+{ }_{i \in I} s_{i}$.
3. $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$. Then, $\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}, b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}+1, e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}$ and $s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}$ with premises $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash^{\left(b_{1}^{\prime}, e_{1}^{\prime}, s_{1}^{\prime}\right)} t_{1}\{x \backslash u\}: \mathrm{M} \rightarrow \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}^{\prime} \vdash^{\left(b_{2}^{\prime}, e_{2}^{\prime}, s_{2}^{\prime}\right)} t_{2}\{x \backslash u\}:$ M for some multiset type M. By i.h. there exist derivations $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \mathrm{M} \rightarrow \tau$, $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{1}}, \Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \mathrm{M}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{2}}$ (note that $I_{1}$ and $I_{2}$ can be assumed disjoint) such that $\Gamma_{1}^{\prime}=\Gamma_{1}+_{i \in I_{1}} \Delta_{i}, \Gamma_{2}^{\prime}=\Gamma_{2}+_{i \in I_{2}} \Delta_{i}$, $b_{1}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}, b_{2}^{\prime}=b_{2}+_{i \in I_{2}} b_{i}, e_{1}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}, e_{2}^{\prime}=e_{2}+_{i \in I_{2}} e_{i}, s_{1}^{\prime}=$ $s_{1}+{ }_{i \in I_{1}} s_{i}$ and $s_{2}^{\prime}=s_{2}+_{i \in I_{2}} s_{i}$. Then, by $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$ we obtain the type derivation $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma_{1}+\Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{1} \cup I_{2}} \vdash^{\left(b_{1}+b_{2}+1, e_{1}+e_{2}, s_{1}+s_{2}\right)} t_{1} t_{2}: \tau$
and we conclude with $\Gamma=\Gamma_{1}+\Gamma_{2}, I=I_{1} \cup I_{2}, b=b_{1}+b_{2}+1$, $e=e_{1}+e_{2}$ and $s=s_{1}+s_{2}$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime}+\Gamma_{2}^{\prime}=\Gamma_{1}+_{i \in I_{1}} \Delta_{i}+\Gamma_{2}+_{i \in I_{2}} \Delta_{i}=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}+1=b_{1}+_{i \in I_{1}} b_{i}+b_{2}+_{i \in I_{2}} b_{i}+1=b+_{i \in I} b_{i}$.
$-e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}+e_{2}+_{i \in I_{2}} e_{i}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}+s_{2}+_{i \in I_{2}} s_{i}=s+{ }_{i \in I} s_{i}$.

- $t=\lambda y \cdot t^{\prime}$. By $\alpha$-conversion we assume $y \neq x$ and $y \notin \mathrm{fv}(u)$. Then, $t\{x \backslash u\}=\lambda y \cdot t^{\prime}\{x \backslash u\}$ and there are two possible cases.

1. $\left(\mathrm{ai}_{\mathrm{p}}\right)$. Then, $\tau=\mathrm{a}, \Gamma^{\prime}=\Gamma_{1}^{\prime} \mathbb{y}$ and $s^{\prime}=s_{1}^{\prime}+1$ with premise $\Phi_{t^{\prime}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash\left(b^{\prime}, e^{\prime}, s_{1}^{\prime}\right) t^{\prime}\{x \backslash u\}$ : tt where tight $\left(\Gamma_{1}^{\prime}(y)\right)$ holds. By i.h. there exist $\Phi_{t^{\prime}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b, e, s_{1}\right)} t^{\prime}:$ tt and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma_{1}^{\prime}=\Gamma_{1}+_{i \in I} \Delta_{i}, b^{\prime}=b+_{i \in I} b_{i}, e^{\prime}=e+_{i \in I} e_{i}$ and $s_{1}^{\prime}=$ $s_{1}+{ }_{i \in I} s_{i}$. Moreover, $y \notin \mathrm{fv}(u)$ implies $y \notin \operatorname{supp}\left(\Delta_{i}\right)$ for all $i \in I$ and, hence, $\Gamma_{1}^{\prime}(y)=\Gamma_{1}(y)$. Then, tight $\left(\Gamma_{1}(y)\right)$ holds as well. Finally, by ( $\mathrm{ae}_{\mathrm{p}}$ ) with $y \neq x$ we have $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma_{1} \backslash y ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b, e, s_{1}+1\right)} \lambda y . t^{\prime}: \mathrm{a}$ and we conclude with $\Gamma=\Gamma_{1} \backslash y$ and $s=s_{1}+1$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime} \ y=\left(\Gamma_{1}+_{i \in I} \Delta_{i}\right) \backslash y=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b+_{i \in I} b_{i}$.
$-e^{\prime}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+1=s_{1}+_{i \in I} s_{i}+1=s+{ }_{i \in I} s_{i}$.
2. ( $\mathrm{ai}_{\mathrm{c}}$ ). Then, $\tau=\Gamma_{1}^{\prime}(y) \rightarrow \tau^{\prime}, \Gamma^{\prime}=\Gamma_{1}^{\prime} \backslash y$ with premise $\Phi_{t^{\prime}\{x \backslash u\}} \triangleright_{\mathcal{E}}$ $\Gamma_{1}^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} t^{\prime}\{x \backslash u\}: \tau^{\prime}$. Then, by i.h. there exist type derivations $\Phi_{t^{\prime}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} t^{\prime}: \tau^{\prime}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma_{1}^{\prime}=\Gamma_{1}+_{i \in I} \Delta_{i}, b^{\prime}=b+_{i \in I} b_{i}, e^{\prime}=e+_{i \in I} e_{i}$ and $s^{\prime}=s+_{i \in I} s_{i}$. Moreover, $y \notin \mathrm{fv}(u)$ implies $y \notin \operatorname{supp}\left(\Delta_{i}\right)$ for all $i \in I$ and, hence, $\Gamma_{1}^{\prime}(y)=\Gamma_{1}(y)$. Finally, by $\left(\mathrm{ae}_{\mathrm{c} 1}\right)$ with $y \neq x$ we construct the type derivation $\Phi_{t} \triangleright \mathcal{E} \Gamma_{1} \backslash y ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} \lambda y . t^{\prime}: \Gamma_{1}^{\prime}(y) \rightarrow \tau^{\prime}$ and we conclude with $\Gamma=\Gamma_{1} \backslash y$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime} \mathbb{} \| y=\left(\Gamma_{1}+_{i \in I} \Delta_{i}\right) \ y=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b+{ }_{i \in I} b_{i}$.
$-e^{\prime}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s+{ }_{i \in I} s_{i}$.

- $t=!t^{\prime}$. Then, $t\{x \backslash u\}=!t^{\prime}\{x \backslash u\}$ and there are two possible cases.

1. $\left(\mathrm{bg}_{\mathrm{p}}\right)$. Then, $\tau=\mathrm{b}, \Gamma^{\prime}=\emptyset$ and $b^{\prime}=e^{\prime}=s^{\prime}=0$. We set $I=\emptyset$ (hence, $\left.\left[\sigma_{i}\right]_{i \in I}=[]\right), \Gamma=\emptyset$ and conclude by $\left(\mathrm{bg}_{\mathrm{p}}\right)$ with $\Phi_{t} \triangleright_{\mathcal{E}} \vdash^{(0,0,0)} t: \mathrm{b}$.
2. $\left(\mathrm{bg}_{\mathrm{c}}\right)$. Then, $\tau=\left[\tau_{j}\right]_{j \in J}, \Gamma^{\prime}=+_{j \in J} \Gamma_{j}^{\prime}, b^{\prime}=+_{j \in J} b_{j}^{\prime}, e^{\prime}=1+_{j \in J} e_{j}^{\prime}$, $s^{\prime}=+_{j \in J} s_{j}^{\prime}$ with premise $\left(\Phi_{t^{\prime}\{x \backslash u\}}^{j} \triangleright_{\mathcal{E}} \Gamma_{j}^{\prime} \vdash^{\left(b_{j}^{\prime}, e_{j}^{\prime}, s_{j}^{\prime}\right)} t^{\prime}\{x \backslash u\}: \tau_{j}\right)_{j \in J}$.
By $i . h$. there exist type derivations $\Phi_{t^{\prime}}^{j} \triangleright_{\mathcal{E}} \Gamma_{j} ; x:\left[\sigma_{i}\right]_{i \in I_{j}} \vdash^{\left(b_{j}, e_{j}, s_{j}\right)} t^{\prime}: \tau_{j}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{j}}$ (note that all $I_{j}$ can be assumed
pairwise disjoint) such that $\Gamma_{j}^{\prime}=\Gamma_{j}+_{i \in I_{j}} \Delta_{i}, b_{j}^{\prime}=b_{j}+_{i \in I_{j}} b_{i}$, $e_{j}^{\prime}=e_{j}+_{i \in I_{j}} e_{i}$ and $s_{j}^{\prime}=s_{j}+{ }_{i \in I_{j}} s_{i}$ for each $j \in J$. Then, by $\left(\mathrm{bg}_{c}\right)$ we have $\Phi_{t} \triangleright_{\mathcal{E}}+_{j \in J} \Gamma_{j} ; x:\left[\sigma_{i}\right]_{i \in I_{j}} \vdash\left(+_{j \in J} b_{j}, 1+{ }_{j \in J} e_{j},+_{j \in J} s_{j}\right)!t^{\prime}:\left[\tau_{j}\right]_{j \in J}$ and we conclude with $\Gamma=+_{j \in J} \Gamma_{j}, I=\bigcup_{j \in J} I_{j}, b=+_{j \in J} b_{j}$, $e=1+_{j \in J} e_{j}, s=+_{j \in J} s_{j}$ since:
$-\Gamma^{\prime}=+_{j \in J} \Gamma_{j}^{\prime}=+_{j \in J}\left(\Gamma_{j}+_{i \in I_{j}} \Delta_{i}\right)=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=+_{j \in J} b_{j}^{\prime}=+_{j \in J}\left(b_{j}+_{i \in I_{j}} b_{i}\right)=b+_{i \in I} b_{i}$.
$-e^{\prime}=1+_{j \in J} e_{j}^{\prime}=1+_{j \in J}\left(e_{j}+_{i \in I_{j}} e_{i}\right)=e+_{i \in I} e_{i}$.
$-s^{\prime}=+_{j \in J} s_{j}^{\prime}=+_{j \in J}\left(s_{j}+_{i \in I_{j}} s_{i}\right)=s+_{i \in I} s_{i}$.

- $t=\operatorname{der} t^{\prime}$. Then, $t\{x \backslash u\}=\operatorname{der}\left(t^{\prime}\{x \backslash u\}\right)$ and there are two possible cases.

1. $\left(\mathrm{dr}_{\mathrm{p}}\right)$. Then, $\tau=\mathrm{n}$ and $s^{\prime}=s_{1}^{\prime}+1$ with a type derivation for the premise $\Phi_{t^{\prime}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s_{1}^{\prime}\right)} t^{\prime}\{x \backslash u\}:$ n. By i.h. there exist type derivations $\Phi_{t^{\prime}} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b, e, s_{1}\right)} t^{\prime}: \mathrm{n}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma^{\prime}=\Gamma+_{i \in I} \Delta_{i}, b^{\prime}=b+_{i \in I} b_{i}, e^{\prime}=e+_{i \in I} e_{i}$ and $s_{1}^{\prime}=$ $s_{1}+{ }_{i \in I} s_{i}$. Finally, we conclude by $\left(\mathrm{dr}_{\mathrm{p}}\right)$ with $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b, e, s_{1}+1\right)} \operatorname{der} t^{\prime}: \mathrm{n}$ and $s=s_{1}+1$.
2. $\left(\mathrm{dr}_{\mathrm{c}}\right)$. Then, we have the premise $\Phi_{t^{\prime}\{x \backslash u\}} \triangleright{ }_{\mathcal{E}} \Gamma^{\prime} \vdash^{\left(b^{\prime}, e^{\prime}, s^{\prime}\right)} t^{\prime}\{x \backslash u\}:[\tau]$. By i.h. there exist type derivations $\Phi_{t^{\prime}} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} t^{\prime}:[\tau]$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma^{\prime}=\Gamma+_{i \in I} \Delta_{i}, b^{\prime}=$ $b+_{i \in I} b_{i}, e^{\prime}=e+_{i \in I} e_{i}$ and $s^{\prime}=s+{ }_{i \in I} s_{i}$. Finally, we conclude by $\left(\mathrm{dr}_{\mathrm{c}}\right)$ with $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{(b, e, s)} \operatorname{der} t^{\prime}: \tau$.

- $t=t_{1}\left[y \backslash t_{2}\right]$. By $\alpha$-conversion we assume $y \neq x, y \notin \mathrm{fv}\left(t_{2}\right)$ and $y \notin \mathrm{fv}(u)$. Then, $t\{x \backslash u\}=t_{1}\{x \backslash u\}\left[y \backslash t_{2}\{x \backslash u\}\right]$ and there are two possible cases.

1. $\left(\mathrm{es}_{\mathrm{p}}\right)$. Then, $\Gamma^{\prime}=\Gamma_{1}^{\prime} \ \backslash y+\Gamma_{2}^{\prime}, b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}, e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}$ and $s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}$ with premises $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash^{\left(b_{1}^{\prime}, e_{1}^{\prime}, s_{1}^{\prime}\right)} t_{1}\{x \backslash u\}: \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}^{\prime} \vdash^{\left(b_{2}^{\prime}, e_{2}^{\prime}, s_{2}^{\prime}\right)} t_{2}\{x \backslash u\}: \mathrm{n}$ where tight $\left(\Gamma_{1}^{\prime}(y)\right)$ holds. By i.h. there exist $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \tau, \quad\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{1}}$, $\Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \mathrm{n}$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{2}}$ (note that $I_{1}$ and $I_{2}$ can be assumed disjoint) such that $\Gamma_{1}^{\prime}=\Gamma_{1}+{ }_{i \in I_{1}}$ $\Delta_{i}, \Gamma_{2}^{\prime}=\Gamma_{2}+_{i \in I_{2}} \Delta_{i}, b_{1}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}, b_{2}^{\prime}=b_{2}+_{i \in I_{2}} b_{i}, e_{1}^{\prime}=e_{1}+_{i \in I_{1}}$ $e_{i}, e_{2}^{\prime}=e_{2}+{ }_{i \in I_{2}} e_{i}, s_{1}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}$ and $s_{2}^{\prime}=s_{2}+_{i \in I_{2}} s_{i}$. Moreover, $y \notin \mathrm{fv}(u)$ implies $y \notin \operatorname{supp}\left(\Delta_{i}\right)$ for all $i \in I_{1} \cup I_{2}$ while $y \notin \mathrm{fv}\left(t_{2}\right)$ implies $y \notin \operatorname{supp}\left(\Gamma_{2}\right)$. Hence, $\Gamma_{1}^{\prime}(y)=\Gamma_{1}(y)$ and tight $\left(\Gamma_{1}(y)\right)$ holds as well. Finally, by $\left(\mathrm{es}_{\mathrm{p}}\right)$ with $y \neq x$ we construct the type derivation $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma_{1} \backslash y+\Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{1} \cup I_{2}} \vdash^{\left(b_{1}+b_{2}, e_{1}+e_{2}, s_{1}+s_{2}\right)} t_{1}\left[y \backslash t_{2}\right]: \tau$ and we conclude with $\Gamma=\Gamma_{1} \backslash y+\Gamma_{2}, I=I_{1} \cup I_{2}, b=b_{1}+b_{2}$, $e=e_{1}+e_{2}$ and $s=s_{1}+s_{2}$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime} \ y+\Gamma_{2}^{\prime}=\left(\Gamma_{1}+_{i \in I_{1}} \Delta_{i}\right) \| y+\Gamma_{2}+_{i \in I_{2}} \Delta_{i}$, and this last context is equal to $\Gamma_{1} \backslash y+\Gamma_{2}+_{i \in I} \Delta_{i}=\Gamma+{ }_{i \in I} \Delta_{i}$.
$-b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}+b_{2}+_{i \in I_{2}} b_{i}=b+_{i \in I} b_{i}$.
$-e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}+e_{2}+_{i \in I_{2}} e_{i}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}+s_{2}+_{i \in I_{2}} s_{i}=s+_{i \in I} s_{i}$.
2. $\left(\mathrm{es}_{\mathrm{c}}\right)$. Then, $\Gamma^{\prime}=\Gamma_{1}^{\prime} \backslash y+\Gamma_{2}^{\prime}, b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}, e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}$ and $s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}$ with premises $\Phi_{t_{1}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{1}^{\prime} \vdash\left(b_{1}^{\prime}, e_{1}^{\prime}, s_{1}^{\prime}\right) \quad t_{1}\{x \backslash u\}: \tau$ and $\Phi_{t_{2}\{x \backslash u\}} \triangleright_{\mathcal{E}} \Gamma_{2}^{\prime} \vdash\left(b_{2}^{\prime}, e_{2}^{\prime}, s_{2}^{\prime}\right) t_{2}\{x \backslash u\}: \Gamma_{1}^{\prime}(y)$. Thus, by i.h. there exist type derivations $\Phi_{t_{1}} \triangleright_{\mathcal{E}} \Gamma_{1} ; x:\left[\sigma_{i}\right]_{i \in I_{1}} \vdash^{\left(b_{1}, e_{1}, s_{1}\right)} t_{1}: \tau, \quad\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{1}}$, $\Phi_{t_{2}} \triangleright_{\mathcal{E}} \Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{2}} \vdash^{\left(b_{2}, e_{2}, s_{2}\right)} t_{2}: \Gamma_{1}^{\prime}(y)$ and $\left(\Phi_{u}^{i} \triangleright_{\mathcal{E}} \Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I_{2}}$ (note that $I_{1}$ and $I_{2}$ can be assumed disjoint) such that $\Gamma_{1}^{\prime}=\Gamma_{1}+_{i \in I_{1}}$ $\Delta_{i}, \Gamma_{2}^{\prime}=\Gamma_{2}+_{i \in I_{2}} \Delta_{i}, b_{1}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}, b_{2}^{\prime}=b_{2}+_{i \in I_{2}} b_{i}, e_{1}^{\prime}=e_{1}+_{i \in I_{1}}$ $e_{i}, e_{2}^{\prime}=e_{2}+_{i \in I_{2}} e_{i}, s_{1}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}$ and $s_{2}^{\prime}=s_{2}+_{i \in I_{2}} s_{i}$. Moreover, $y \notin$ $\mathrm{fv}(u)$ implies $y \notin \operatorname{supp}\left(\Delta_{i}\right)$ for all $i \in I_{1} \cup I_{2}$ while $y \notin \mathrm{fv}\left(t_{2}\right)$ implies $y \notin \operatorname{supp}\left(\Gamma_{2}\right)$. Hence, $\Gamma_{1}^{\prime}(y)=\Gamma_{1}(y)$. Finally, by $\left(\mathrm{es}_{\mathrm{c}}\right)$ with $y \neq x$ we have $\Phi_{t} \triangleright_{\mathcal{E}} \Gamma_{1} \backslash y+\Gamma_{2} ; x:\left[\sigma_{i}\right]_{i \in I_{1} \cup I_{2}} \vdash^{\left(b_{1}+b_{2}, e_{1}+e_{2}, s_{1}+s_{2}\right)} t_{1}\left[y \backslash t_{2}\right]: \tau$ and we conclude with $\Gamma=\Gamma_{1} \backslash y+\Gamma_{2}, I=I_{1} \cup I_{2}, b=b_{1}+b_{2}$, $e=e_{1}+e_{2}$ and $s=s_{1}+s_{2}$ since:
$-\Gamma^{\prime}=\Gamma_{1}^{\prime} \ y+\Gamma_{2}^{\prime}=\left(\Gamma_{1}+_{i \in I_{1}} \Delta_{i}\right) \Downarrow y+\Gamma_{2}+_{i \in I_{2}} \Delta_{i}$ and this last context is equal to $\Gamma_{1} \backslash y+\Gamma_{2}+_{i \in I} \Delta_{i}=\Gamma+_{i \in I} \Delta_{i}$.
$-b^{\prime}=b_{1}^{\prime}+b_{2}^{\prime}=b_{1}+_{i \in I_{1}} b_{i}+b_{2}+_{i \in I_{2}} b_{i}=b+_{i \in I} b_{i}$.
$-e^{\prime}=e_{1}^{\prime}+e_{2}^{\prime}=e_{1}+_{i \in I_{1}} e_{i}+e_{2}+_{i \in I_{2}} e_{i}=e+_{i \in I} e_{i}$.
$-s^{\prime}=s_{1}^{\prime}+s_{2}^{\prime}=s_{1}+_{i \in I_{1}} s_{i}+s_{2}+_{i \in I_{2}} s_{i}=s+{ }_{i \in I} s_{i}$.

Lemma 5.16 (Exact Subject Expansion). Let $\Phi^{\prime} \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(b^{\prime}, e^{\prime}, s\right)} t^{\prime}: \sigma$ be a tight derivation. If $t \rightarrow_{\mathrm{dw}} t^{\prime}$, then there exists $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{(b, e, s)} t: \sigma$ such that

- $b^{\prime}=b-1$ and $e^{\prime}=e$ if $t \rightarrow_{\mathrm{dw}} t^{\prime}$ is an m-step.
- $e^{\prime}=e-1$ and $b^{\prime}=b$ if $t \rightarrow_{\mathrm{dw}} t^{\prime}$ is an e-step.

Proof. By induction on $t \rightarrow_{\mathrm{dw}} t^{\prime}$.

- $t=\mathrm{L}\langle\lambda x . s\rangle u \rightarrow_{\mathrm{dw}} \mathrm{L}\langle s[x \backslash u]\rangle=t^{\prime}$. We reason by induction on L.
$-\mathrm{L}=\square$. There are two cases depending on the last rule of $\Phi^{\prime}$.

1. $\left(\mathrm{es}_{\mathrm{p}}\right)$. Then, $\Phi^{\prime}$ has the following form:

$$
\frac{\Gamma^{\prime} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma \quad \Delta \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \mathrm{n} \quad \operatorname{tight}\left(\Gamma^{\prime}(x)\right)}{\Gamma^{\prime} \backslash x+\Delta \vdash^{\left(b_{s}+b_{u}, e_{s}+e_{u}, s_{s}+s_{u}\right)} s[x \backslash u]: \sigma}\left(\mathrm{es}_{\mathrm{p}}\right)
$$

We can then construct the following derivation $\Phi$ :

$$
\frac{\frac{\Gamma^{\prime} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma}{\Gamma^{\prime} \backslash x \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} \lambda x . s: \Gamma^{\prime}(x) \rightarrow \sigma}\left(\mathrm{ai}_{\mathrm{c}}\right) \quad \Delta \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \mathrm{n} \quad \operatorname{tight}\left(\Gamma^{\prime}(x)\right)}{\Gamma^{\prime} \backslash x+\Delta \vdash^{\left(b_{s}+b_{u}+1, e_{s}+e_{u}, s_{s}+s_{u}\right)}(\lambda x . s) u: \sigma}\left(\mathrm{ae}_{\mathrm{c} 2}\right)
$$

The counters verify the expected property.
2. $\left(\mathrm{es}_{\mathrm{c}}\right)$. Then, $\Phi^{\prime}$ has the following form:

$$
\frac{\Gamma^{\prime} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma \quad \Delta \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \Gamma^{\prime}(x)}{\Gamma^{\prime} \backslash x x+\Delta \vdash^{\left(b_{s}+b_{u}, e_{s}+e_{u}, s_{s}+s_{u}\right)} s[x \backslash u]: \sigma}\left(\mathrm{es}_{\mathrm{c}}\right)
$$

We can then construct the following derivation $\Phi$ :

$$
\frac{\frac{\Gamma^{\prime} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma}{\Gamma^{\prime} \backslash x \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} \lambda x . s: \Gamma^{\prime}(x) \rightarrow \sigma}\left(\mathrm{ai}_{\mathrm{c}}\right)}{\Gamma^{\prime} \ x+\Delta \vdash^{\left(b_{s}+b_{u}+1, e_{s}+e_{u}, s_{s}+s_{u}\right)}(\lambda x . s) u: \sigma} \vdash^{\left(b_{u}, e_{u}, s_{u}\right)} u: \Gamma^{\prime}(x){\left(\mathrm{ae}_{\mathrm{c} 1}\right)}^{\text {}}
$$

The counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash r]$. Immediate from the i.h.

- $t=s[x \backslash \mathrm{~L}\langle!u\rangle] \rightarrow_{\mathrm{dw}} \mathrm{L}\langle s\{x \backslash u\}\rangle=t^{\prime}$. We reason by induction on L .
$-\mathrm{L}=\square$. Then $\Phi$ has the form $\Gamma \vdash^{\left(b^{\prime}, e^{\prime}, s\right)} s\{x \backslash u\}: \sigma$. By Lemma 5.15 there exist $\Gamma^{\prime} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma$ and $\left(\Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}$ such that $\Gamma=\Gamma^{\prime}+_{i \in I} \Delta_{i}, b^{\prime}=b_{s}+_{i \in I} b_{i}, e^{\prime}=e_{s}+_{i \in I} e_{i}$ and $s=s_{s}+_{i \in I} s_{i}$. We can then construct the following derivation $\Phi$ :

$$
\frac{\Gamma^{\prime} ; x:\left[\sigma_{i}\right]_{i \in I} \vdash^{\left(b_{s}, e_{s}, s_{s}\right)} s: \sigma \frac{\left(\Delta_{i} \vdash^{\left(b_{i}, e_{i}, s_{i}\right)} u: \sigma_{i}\right)_{i \in I}}{\Gamma^{\prime}+{ }_{i \in I} \Delta_{i} \vdash^{\left(b_{s}+{ }_{i \in I} b_{i}, 1+e_{s}+{ }_{i \in I} e_{i}, s_{s}+{ }_{i \in I} s_{i}\right)} s[x \backslash!u]: \sigma}\left(\mathrm{bg}_{\mathrm{c}}\right)}{+_{i \in I} \vdash^{\left(+{ }_{i \in I} b_{i}, 1+{ }_{i \in I} e_{i},+_{i \in I} s_{i}\right)}!u:\left[\sigma_{i}\right]_{i \in I}}\left(\mathrm{es}_{\mathrm{c}}\right)
$$

The counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash r]$. Immediate from the i.h.

- $t=\operatorname{der}(\mathrm{L}\langle!s\rangle) \rightarrow_{\mathrm{dw}} \mathrm{L}\langle s\rangle=t^{\prime}$. We reason by induction on L .
$-\mathrm{L}=\square$. Then, $t^{\prime}=s$ and from $\Phi^{\prime}$ we construct the following derivation:

$$
\frac{\frac{\Gamma \vdash^{\left(b^{\prime}, e^{\prime}, s\right)} s: \sigma}{\Gamma \vdash^{\left(b^{\prime}, 1+e^{\prime}, s\right)}!s:[\sigma]}}{\Gamma \vdash^{\left(b^{\prime}, 1+e^{\prime}, s\right)} \operatorname{der}!s: \sigma}\left(\mathrm{bg}_{\mathrm{c}}\right)
$$

We conclude since the counters verify the expected property.
$-\mathrm{L}=\mathrm{L}^{\prime}[y \backslash r]$. Immediate from the i.h.

- All the inductive cases for internal reductions are straightforward by i.h.

Theorem 5.17 (Completeness for System $\mathcal{E}$ ). If $t \rightarrow{ }_{\mathrm{w}}^{(b, e)} p$ with $p \in \mathrm{no}_{\text {wcf }}$, then there exists a tight type derivation $\Phi \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(b, e,|p|_{w}\right)} t: \sigma$.

Proof. We prove the statement for $\rightarrow_{{ }_{d w}}$ and then conclude for the general notion of reduction $\rightarrow_{\mathrm{w}}$ by Theorem [2.8, Let $t \rightarrow{ }_{\mathrm{dw}}^{(b, e)} p$. We proceed by induction on $b+e$.

If $b+e=0$, then $b=e=0$ and thus $t=p$, which implies $t \in \mathrm{no}_{\mathrm{wcf}}$. Lemma 5.14 allows us to conclude.

If $b+e>0$, then there exists $t^{\prime}$ such that $t \rightarrow{ }_{\mathrm{dw}}^{(1,0)} t^{\prime} \rightarrow{ }_{\mathrm{dw}}^{(b-1, e)} p$ or $t \rightarrow{ }_{\mathrm{dw}}^{(0,1)}$ $t^{\prime} \rightarrow{ }_{\mathrm{dw}}^{(b, e-1)} p$. By i.h. there exists a tight derivation $\Phi^{\prime} \triangleright_{\mathcal{E}} \Gamma \vdash^{\left(b^{\prime}, e^{\prime},|p|_{w}\right)} t^{\prime}: \sigma$ such that $b^{\prime}+e^{\prime}=b+e-1$. Lemma 5.16 gives a tight derivation $\Phi \triangleright_{\mathcal{E}}$ $\Gamma \vdash^{\left(b^{\prime \prime}, e^{\prime \prime},|p|_{w}\right)} t: \sigma$ such that $b^{\prime \prime}+e^{\prime \prime}=b^{\prime}+e^{\prime}+1$. We then have $b^{\prime \prime}+e^{\prime \prime}=b+e$. The fact that $b^{\prime \prime}=b$ and $e^{\prime \prime}=e$ holds by a simple case analysis.

The main results can be illustrated by the term $t_{0}=\operatorname{der}(!\mathrm{K})(!\mathrm{I})(!\Omega)$ in Sec. 2 which normalises in 2 multiplicative steps and 3 exponential steps to a w-normal form of w-size 1 . A tight derivation for $t_{0}$ with appropriate counters $(2,3,1)$ is given in Example 5.2.

## 6. Conclusion

This paper gives a fresh view of the Bang Calculus, a formalism introduced by T. Ehrhard to study the relation between CBPV and Linear Logic.

Our reduction relation integrates permutative conversions inside the logical original formulation of [26], thus recovering soundness, i.e. avoiding mismatches between terms in normal form that are semantically non-terminating. In contrast to [27], which models permutative conversions as $\sigma$-reduction rules by paying the cost of losing confluence, our at a distance formulation yields a confluent reduction system.

We then define two non-idempotent intersection type systems for our calculus. On the one hand, system $\mathcal{U}$ provides upper bounds for the length of normalising sequences plus the size of normal forms. Moreover, it captures typed CBN and CBV. On the other hand, the quantitative system $\mathcal{U}$ is further refined into system $\mathcal{E}$, being able to provide exact bounds for normalising sequences and size of normal forms, independently. Moreover, our tight system $\mathcal{E}$ is able to discriminate between different kind of steps performed to normalise terms.

Concerning related works, several points should be noticed respect to the closest [33]. First of all, our CBV translation recovers the expected property of preserving the normal forms, as stated in Lemma 4.4 whereas normal forms in CBV do not necessarily translate to normal forms in the Bang calculus in 33]. Moreover the CBV embedding in [33] is not complete w.r.t. their CBV type system, i.e. there exists a $\lambda$-term $t$ such that $\Gamma \vdash t^{\mathrm{cbv}}: \sigma$ is derivable in $\mathcal{U}$ but $\Gamma \vdash t: \sigma$ is not derivable in their CBV system (see [33], Proposition 16). The completeness property in our framework is stated as Theorem 4.8. As a matter of fact in [33] the authors remark that the incompleteness of the CBV embedding is due in particular to the fact that the CBV type system they use, which is the canonical one stemming from the relational model of the CBV $\lambda$-calculus defined in 27], assigns multiset types to all the $\lambda$-terms, and in
particular to all the applications, whereas an application in the target of the CBV translation may well get a non-multiset type in their Bang calculus type system. They propose at the end of the paper an alternative type system, targeting the range of their CBV translation, which is essentialy the one we have adopted here. Nevertheless, they leave (the model theoretic version of) this question of incompleteness for future work.

Several topics deserve future attention. One of them is the study of strong reduction for the $\lambda!$-calculus, which allows us to reduce terms under all the constructors, including !. Another challenging problem is to relate tight typing in CBN/CBV with tight typing in our calculus, thus providing an exact correspondence between (CBN/CBV) reduction steps and $\lambda!$-reduction steps.

## 7. Acknowledgments

We are grateful to Giulio Guerrieri, Giulio Manzonetto and Victor Arrial who provided valuable feedback and insightful discussions during the course of this study. Also, we would like to thank the reviewers for their constructive comments and suggestions which helped to improve the manuscript. This work has been partially supported by LIA INFINIS/IRP SINFIN, the ECOS-Sud program PA17C01, and the ANR COCA HOLA.

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[^1]:    ${ }^{1}$ Indeed, there exist clash-free terms in normal form that are $\sigma$-reducible to normal terms with clashes, see the definition of clash term at the end of Section 2 e.g. $R=$ $\operatorname{der}((\lambda y . \lambda x . z)(\operatorname{der}(y) y)) \equiv_{\sigma} \operatorname{der}(\lambda x .(\lambda y . z)(\operatorname{der}(y) y))$.
    ${ }^{2}$ In particular, the term $R$ is not in normal form in our framework, and it reduces to a clash term in normal form which is filtered by the type system, see Lemma 3.7

[^2]:    ${ }^{3}$ Here is the definition: $\|x\|=1,\|!t\|=\|\operatorname{der} t\|=\|\lambda x \cdot t\|=\|t\|+1,\|t[u]\|=\|t[x \backslash u]\|=$ $\|t\|+\|u\|+1$.

