Automata on Lempel-Ziv Compressed Strings

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Abstract. Using the Lempel-Ziv-78 compression algorithm to compress a string yields a dictionary of substrings, i.e. an edge-labelled tree with an order-compatible enumeration, here called an *LZ*-trie. Queries about strings translate to queries about *LZ*-tries and hence can in principle be answered without decompression. We compare notions of automata accepting *LZ*-tries and consider the relation between acceptable and MSO-definable classes of *LZ*-tries. It turns out that regular properties of strings can be checked efficiently on compressed strings by *LZ*-trie automata.

1 Introduction

We are interested in the *compressed model checking problem*: which properties of strings can be checked given the compressed strings? The challenge is to beat the decompress-and-then-check method. We restrict ourselves to the classical Ziv-Lempel[8] string compression algorithm *LZ*-78. It compresses a string $w \in \Sigma^+$ to a generally much shorter sequence $LZ(w) \in (\mathbb{N} \times \Sigma)^+$, where the numbers point to previous elements of the sequence.

As usual, we view a string w as a colored finite linear order $\mathcal{S}_w = (D, <, U_a)_{a \in \Sigma}$, where $D = \{0, \ldots, |w| - 1\}$ is the set of positions, ordered by $<$ as usual, and $U_a(i)$ means that a occurs at position i of w. Properties of strings are expressed in first-order (FO) or second-order (SO) logic of colored linear orders. In a similar way, we give two representations of *LZ*-78-compressed strings α by relational structures, one by node-labelled *LZ-graphs* $\mathcal{G}_\alpha$ and another one by *LZ-tries* $\mathcal{T}_\alpha$, which are a kind of edge-labelled trees.

A natural approach to the compressed model checking problem is to translate properties of strings to properties of compressed strings.

In fact, monadic second-order (MSO) formulas φ in the language of strings can be translated to dyadic second-order (DSO) formulas φ^{LZ} in the language of LZ-graphs $\mathcal{G}_\alpha$. One can therefore answer queries φ about $\mathcal{S}_w$ by evaluating φ^{LZ} on the smaller structure $\mathcal{G}_{LZ(w)}$. However, since the translation doubles the arity of relation variables, this is not guaranteed to provide an efficient solution.

By Büchi's well-known theorem (cf. [4], Theorem 5.2.3), MSO for strings is equally expressive as regular expressions, or finite automata, are. This raises the question whether MSO for LZ-graphs leads to a notion of LZ-automaton that provides an efficient method for checking a reasonably rich class of properties of compressed strings.

We study this question in the slightly more suitable format of LZ-tries. These come with an enumeration of their nodes and can be viewed as simple acyclic directed graphs. We introduce notions of LZ-*trie automata* by modifying corresponding notions of tree-automata. We show that deterministic LZ-trie-automata are less powerful than non-deterministic ones, and that the latter capture $\exists$-MSO for LZ-tries, where $\exists$-MSO is the set of MSO-formulas of the form $\exists \boldsymbol{X} \psi$ where $\boldsymbol{X}$ are set variables and ψ is an FO-formula. We show that some problems that are difficult for arbitrary graphs or dags become easy for LZ-tries. Finally, we show that MSO-properties of strings can be checked efficiently on LZ-compressed strings by deterministic top-down LZ-automata. Thus, regular expression search in strings can be done on the LZ-compressed strings, without decompression.

2 Lempel-Ziv-78 compression

We fix a finite alphabet Σ and, to avoid structures with empty universe, only consider non-empty strings $w \in \Sigma^+$. The classical Lempel-Ziv-78 compression algorithm has many variations (cf. [2], [3]). It decomposes a string $w \in \Sigma^+$ into a sequence of substrings or *blocks* $B_i \in \Sigma^+$, so that $w = B_0 \cdots B_{m-1}$. The first block B_0 consists of the first letter of w. Suppose for some $n > 0$, we have constructed blocks $B_0, \dots, B_{n-1}$ such that $w = B_0 \cdots B_{n-1} v$ for some $v \in \Sigma^+$. Then B_n is the shortest non-empty prefix of v that is not among $\{B_0, \dots, B_{n-1}\}$, if this exists, otherwise B_n is v.[1] The

[1] Note that if B_n extends one of $B_0, \dots, B_{n-1}$ by exactly one letter.

LZ-compression $LZ(w)$ of w is the sequence $p_0 \cdots p_{m-1}$ of pairs $p_n = (k, a)$ such that $B_n = B_{k-1}a$ (where $B_{-1} := \epsilon$) and $w = B_0 \cdots B_{m-1}$. The decompression is given by $B_k = decode(p_k)$ where $decode((0, a)) = a$ and $decode((n + 1, a)) = decode(p_n)a$.

Example 1. The blocks of $w = abbbaabbabbb$ are $a.b.bb.aa.bba.bbb$, and its compression is $LZ(w) = (0, a)(0, b)(2, b)(1, a)(3, a)(3, b)$.

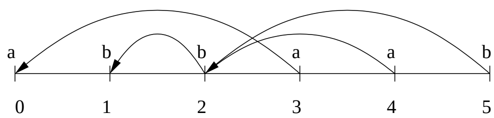
2.1 LZ-graphs

Definition 1. *A compressed string* $\alpha = p_0 \cdots p_{m-1}$ *is represented as a* finite labelled ordered graph

$$\mathcal{G}_\alpha := (D_m, <, U_a, E)_{a\in\Sigma},$$

where $m = |\alpha|$ *is the number of blocks,* $D_m = \{0, \dots, m - 1\}$*,* $<$ *the natural order on* D_m*,* $U_a(i)$ *is true iff the last letter of block* B_i *is* a*. The binary relation* E *describes the reference to previous pairs: if* $p_i = (k, a)$ *for some* $a \in \Sigma$ *and* $k > 0$*, (i.e. the* k*-th block* B_{k-1} *is the longest strict prefix of* B_i*), then there is an edge* $E(i, k - 1)$ *from node* i *to* $k - 1$*.*

Example 1 (Cont.). For $w = a.b.bb.aa.bba.bbb. = B_0B_1B_2B_3B_4B_5$, the graph $\mathcal{G}_{LZ(w)}$ is



Observe that $\mathcal{S}_w$ can be interpreted in $\mathcal{G}_{LZ(w)}$ as a binary relation: a position i in w is mapped to the pair $h(i) = (k, j)$ in $LZ(w)$ iff i lies in block B_k and B_j is the nonempty prefix of B_k ending in i. Note that in $\mathcal{G}_{LZ(w)}$, node j can be reached from k by a path of E-edges.

Example 2. If $w = a.aa.ab.aba.aa$, the positions 5,6,7 occurring in block $B_3 = aba$ are represented by $(3, 0), (3, 2), (3, 3)$, because the nonempty prefixes of B_3 are $a = B_0$, $ab = B_2$, and $aba = B_3$.

Theorem 1 ([1]). *For every MSO-formula* $\varphi(x_1, \dots, x_n, \boldsymbol{X}^{(1)})$ *about strings there is a DSO formula* $\varphi^{LZ}(x_1, y_1, \dots, x_n, y_n, \boldsymbol{X}^{(2)})$ *about LZ-graphs, such that for each* $\mathcal{S}_w$ *and all* $i_1, \dots, i_n \in \mathcal{S}_w$ *and* $\boldsymbol{S} \subseteq \mathcal{S}_w$*,*

$$\mathcal{S}_w \models \varphi[\boldsymbol{i}, \boldsymbol{S}] \iff \mathcal{G}_{LZ(w)} \models \varphi^{LZ}[h(\boldsymbol{i}), h(\boldsymbol{S})].$$

Proof. (Sketch) In DSO, we can define E^*, the reflexive transitive closure of E, and then define φ^{LZ} inductively using

$$\begin{aligned}
(U_a(x_i))^{LZ} &:= U_a(y_i),\\
(x_i \leq x_j)^{LZ} &:= x_i < x_j \vee (x_i = x_j \wedge y_i \leq y_j),\\
(\exists x_{n+1}\varphi)^{LZ} &:= \exists x_{n+1} \exists y_{n+1}\, (E^*(x_{n+1}, y_{n+1}) \wedge \varphi^{LZ}),\\
(\exists X^1 \varphi)^{LZ} &:= \exists X^2 (\forall x \forall y (X^2(x,y) \to E^*(x,y)) \wedge \varphi^{LZ}).
\end{aligned} \tag{1}$$

For the atomic cases, note that in φ^{LZ} a variable x_i stands for a block of w and y_i for a relative position in this block.

A property L of non-empty strings is *definable on strings* (resp. *on compressed strings* or *LZ-graphs*), if for some formula φ of the appropriate language, $L = \{w \mid \mathcal{S}_w \models \varphi\}$ (resp. $L = \{w \mid \mathcal{G}_{LZ(w)} \models \varphi\}$).

Remark 1. There are properties of strings that are FO-definable on strings, but not on LZ-graphs, like $\exists x (U_a(x) \wedge U_b(x+1))$. There are properties of strings that are FO-definable on LZ-graphs, but not even MSO-definable on strings (cf. [1]).

2.2 LZ-tries

While compressing $w = B_0 \cdots B_{n-1} v$, the blocks $B_0, \ldots, B_{n-1}$ found are maintained as a *dictionary* of subwords of w and stored as a tree by sharing common prefixes. The linear order of the blocks in $LZ(w)$ amounts to an enumeration of the nodes of the tree.

Definition 2. *A (finite) Σ-*tree *$(T, \leq, \xleftarrow{a}, 0)_{a \in \Sigma}$ is a (finite) tree $(T, \leq, 0)$ with root 0, where $\{\xleftarrow{a} \subseteq T \times T \mid a \in \Sigma\}$ are pairwise disjoint minimal relations such that $\leq$ is the reflexive transitive closure of their union.*

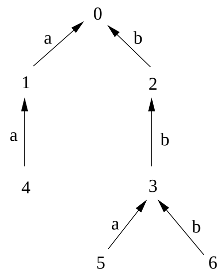
*A Σ-tree is a Σ-*trie *if to each node $n \in T$ and each $a \in \Sigma$ there is at most one $n' \in T$ such that $n \xleftarrow{a} n'$. A (finite)* enumerated Σ-trie

$$\mathcal{T} = (T, \leq, \xleftarrow{a}, 0, Succ)_{a \in \Sigma},$$

*or an LZ-*trie *for short, is a Σ-trie $(T, \leq, \xleftarrow{a}, 0)_{a \in \Sigma}$ with a successor relation[2] $Succ$ on T that is compatible with the partial order $\leq$. We assume that $T = \{0, 1, 2, \ldots, m\}$ and $Succ(i,j)$ iff $i + 1 = j$ in $\mathbb{N}$.*

[2] i.e. a minimal binary relation $Succ$ whose transitive reflexive closure $Succ^*$ is a total ordering of T

Example 1 (Cont.). Enumerating the pairs of $LZ(w)$ by 1,2, etc. in a third component, we obtain a sequence $(0,a,1)(0,b,2)(2,b,3)$ $(1,a,4)(3,a,5)(3,b,6)$ of triples. These represent a tree in which block B_k labels the path from the root 0 to node $k+1$:



A tuple (k,a) of $LZ(w)$ is drawn as an edge $k \xleftarrow{a}$.

We write $\mathcal{T}_\alpha$ for the enumerated trie representing the compressed word α. We always assume that our strings w have a distinguished end symbol; then the final block of $LZ(w)$ is different from the previous ones and the tree of blocks indeed is a trie.

Remark 2. What differs in choosing $\mathcal{G}_\alpha$ or $\mathcal{T}_\alpha$ is the logical language used to talk about *LZ*-compessed strings α. Basically, we have

$$\mathcal{G}_\alpha \models E(i,j) \wedge U_a(i) \iff \mathcal{T}_\alpha \models (j+1) \xleftarrow{a} (i+1).$$

Modulo the additional root node in the trie, $\geq$ in the trie amounts to E^* in the graph, and $\leq$ in the graph to $Succ^*$ in the trie.

Using (1), quantifiers Qx and QX about strings translate to *bounded* quantifiers $Q(x,y) \in E^*$ and $QX \subseteq E^*$. Translating to the language of *LZ*-tries we only quantify over tuples and relations whose tuples lie on *paths* of $\mathcal{T}_{LZ(w)}$. Hence we actually translate to a reasonably nice sublanguage of *path-restricted* DSO over *LZ*-tries.

3 MSO-equivalence for LZ-graphs

For relational structures $\mathcal{A}, \mathcal{B}$ of the same signature, $\mathcal{A} \equiv_r^{MSO} \mathcal{B}$ says that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same MSO-sentences of quantifier rank $\leq r$.

Two facts about MSO for strings imply the existence of finite automata that can check MSO-properties of strings (cf. [4]):

a) for each r, there are only finitely many $\equiv_r^{MSO}$-equivalence classes for word structures $\mathcal{S}_w$, and

b) for compound strings wa, the $\equiv_r^{MSO}$-class of $\mathcal{S}_{wa}$ depends only on the $\equiv_r^{MSO}$-classes of $\mathcal{S}_w$ and $\mathcal{S}_a$.

(The analogous situatation holds for trees over Σ.) *LZ*-compressed words $\alpha = p_0 \cdots p_{m-1}$ are words over the infinite alphabet $\mathbb{N} \times \Sigma$. Can we check MSO-properties of compressed strings by a kind of finite automaton for *LZ*-graphs? Since we deal with a finite relational language, we still have a):

Proposition 1. *For each r, the equivalence relation $\equiv_r^{MSO}$ between LZ-graphs has finite index.*

But what about b)? By extending winning strategies for duplicator in the Ehrenfeucht-Fraisse-game $G_r(\mathcal{G}_\alpha, \mathcal{G}_\beta)$, we can show:

Lemma 1. *(i) For LZ-compressed words $\alpha(0,a), \beta(0,a')$ over Σ,*

$$\mathcal{G}_\alpha \equiv_r^{MSO} \mathcal{G}_\beta \ \wedge a = a' \implies \mathcal{G}_{\alpha(0,a)} \equiv_r^{MSO} \mathcal{G}_{\beta(0,a')}.$$

(ii) For LZ-compressed words $\alpha(k+1,a)$ and $\beta(k'+1,a')$ over Σ,

$$(\mathcal{G}_\alpha, k) \equiv_r^{MSO} (\mathcal{G}_\beta, k') \ \wedge \ a = a' \implies \mathcal{G}_{\alpha(k+1,a)} \equiv_r^{MSO} \mathcal{G}_{\beta(k'+1,a')}.$$

However, in (ii) one has to assume that the elements k, k' pointed to from the new maximal elements share the same properties. Instead, one would need the stronger claim

$$\mathcal{G}_\alpha \equiv \mathcal{G}_\beta \wedge \mathcal{G}_\alpha{\restriction}k \equiv \mathcal{G}_\beta{\restriction}k' \wedge a = a' \implies \mathcal{G}_{\alpha(k+1,a)} \equiv \mathcal{G}_{\beta(k'+1,a')},$$

where $\mathcal{G}_\alpha{\restriction}k$ is the restriction of $\mathcal{G}_\alpha$ with k as its maximal element. The equivalence class of $\mathcal{G}_\alpha{\restriction}k$ would be the automaton state assigned to k. (Notice that $\mathcal{G}_\alpha{\restriction}k$ is a *LZ*-graph, but $(k+1,a)$ is not a *LZ*-compressed word.) The problem is that duplicator's winning strategies for $G_r(\mathcal{G}_\alpha, \mathcal{G}_\beta)$ and $G_r(\mathcal{G}_\alpha{\restriction}k, \mathcal{G}_\beta{\restriction}k')$ may pick *different* elements to answer spoilers playing of some element of, say, $\mathcal{G}_\alpha{\restriction}k$. For example, if spoiler plays element k in some round $< r$, duplicator has to answer with k' in the second game, but not necessarily so in the first game.

Thus, unlike in the case of strings or trees, for compound compressed words $\alpha(k,a)$ we have component *LZ*-graphs $\mathcal{G}_\alpha$ and $\mathcal{G}_\alpha{\restriction}k$ that are not disjoint, and winning strategies in games for these do not combine to winning strategies for composed *LZ*-graphs.

From this we conclude that we cannot use a Büchi-Myhill-Nerode construction to obtain from the $\equiv_r^{MSO}$-classes a finite sequential automaton for *LZ*-graphs, and likewise for *LZ*-tries.

4 *LZ*-trie-automata

If we view *LZ*-tries as trees with an additional edge *Succ* between nodes, we obtain directed acyclic graphs of a special kind: the successor child may be equal to some decendant with respect to the $\xleftarrow{a}$-child-relations. Since these are still very close to trees, it is natural to use a variation of tree-automata as an approximative notion of *LZ*-automaton for checking properties of *LZ*-tries.

Definition 3. *Let $n \in G$ be a node in a graph (G, E). For $m \in \mathbb{N}$, the* sphere of radius m around n*, $s_m(n)$, is the set of nodes $k \in G$ such that there is a E-path of length $\leq m$ from n to k or vice versa. The* hemisphere of radius m around n*, $hs_m(n)$, is the set of nodes k such that there is an E-path of length $\leq m$ from n to k.*

Definition 4. *Let $n \in T$ be a node in the LZ-trie $\mathcal{T}$. The* bottom-up *LZ*-hemisphere of radius m around n*, $bu\text{-}hs^{\mathcal{T}}_m(n)$, is the restriction of $\mathcal{T}$ to the m-hemisphere around n in the graph (T, E), where*

$$E := \bigcup \{ \xleftarrow{a} \mid a \in \Sigma\} \cup \{Succ\}.$$

The top-down *LZ*-hemisphere of radius m around n*, $td\text{-}hs^{\mathcal{T}}_m(n)$, is the restriction of $\mathcal{T}$ to the m-hemisphere around n in the graph $(T, \breve{E})$, where $\breve{E}$ is the converse of E.*

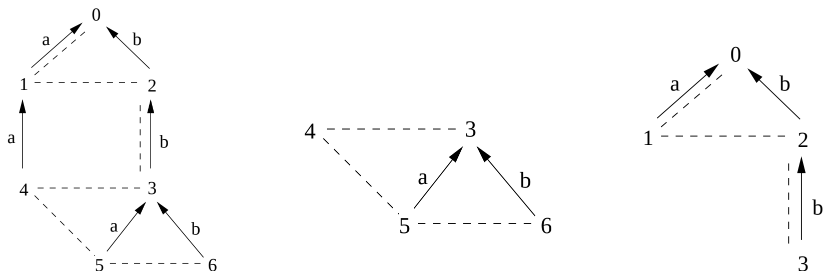
An *LZ*-hemisphere *is an LZ-hemisphere of some radius around some node in some LZ-trie $\mathcal{T}$.*

Definition 5. *A finite* bottom-up (resp. top-down) m-*LZ*-automaton *$\mathcal{A} = (Q, \Sigma, \delta, q_{in}, F)$ consists of a finite set Q of* states*, sets $I, F \subseteq Q$ of* initial *and* final *states, a finite* alphabet *Σ, a finite* transition relation *δ consisting of pairs (P, q), written $P \to q$, where $q \in Q$ and P is a bottom-up (resp. top-down) LZ-hemisphere of radius m whose nodes except the root are labelled by elements of Q.*

A run *of $\mathcal{A}$ on an LZ-trie $\mathcal{T}$ is a function $r : T \to Q$ where $r(\max) \in I$ (resp. $r(0) \in I$) and for each $n \in T$ there is some $(P, q) \in \delta$ such that $bu\text{-}hs^{\mathcal{T}}_m(n)$ (resp. $td\text{-}hs^{\mathcal{T}}_m(n)$), expanded by the labelling of nodes given by r, is isomorphic to P with label q at its root. $\mathcal{A}$* accepts *$\mathcal{T}$ if there is a run r of $\mathcal{A}$ on $\mathcal{T}$ such that $r(0) \in F$ (resp. $r(\max) \in F$). Let $L(\mathcal{A}) := \{\mathcal{T} \mid \mathcal{A}$ accepts $\mathcal{T}\}$ be* the class of *LZ*-tries accepted by $\mathcal{A}$.

$\mathcal{A}$ is deterministic *if $|I| = 1$ and $q = q'$ when $(P, q), (P, q') \in \delta$.*

Example 1 (Cont.). An *LZ*-trie and the bottom-up resp. top-down 2-hemispheres of node 3 (with dashed edges for *Succ* resp. *Pred*):



While an m-*LZ*-automaton $\mathcal{A}$ sequentially follows the enumeration of a trie $\mathcal{T}_\alpha$, it can access the states reached at suffixes (resp. prefixes) of α. Strictly speaking, it does not have a 'finite memory'.

Proposition 2. *For every m-LZ-automaton $\mathcal{A}$ there is an $\exists$-MSO-sentence $\varphi_{\mathcal{A}}$ defining the class of LZ-tries accepted by $\mathcal{A}$.*

Proof. (Sketch) Consider states q of the automaton as monadic predicates on nodes of tries. The condition "there is an accepting $\mathcal{A}$-run" can be expressed by a sentence $\exists \boldsymbol{q} \psi$. Here $\psi(\boldsymbol{q})$ is a FO-formula saying that the $\boldsymbol{q}$-labelled m-hemispheres of the nodes are isomorphic to the tiles allowed by δ, and that the acceptance condition holds.

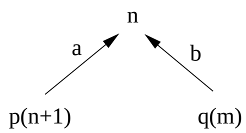
4.1 Bottom-up *LZ*-trie-automata

A 1-*LZ*-automaton working bottom-up the *LZ*-trie towards the root has transitions that determine the state at a node from the states at the node's Σ-children and successor. But it also has to distinguish which of the Σ-children is the successor of the node, if any.

Example 2. Consider the class $\mathcal{K}$ of *LZ*-tries over $\Sigma = \{a, b\}$ which have a node whose successor and a-child agree, i.e. which satisfy the sentence $\varphi := \exists x \exists y\ [y = x + 1 \land (x \xleftarrow{a} y)]$. We give a bottom-up 1-*LZ*-automaton $\mathcal{A}$ accepting $\mathcal{K}$. We write a transition in the form

$$(q_a, q_b, q_{succ}, i) \to p,$$

where q_a, q_b, q_{succ} are the states of the a-, b- and *Succ*-child or $\perp$, when there is no such child, and $i \in \{1, 2, 3, \perp\}$ says which of the children is equal to the successor node, if any. Thus, $(p, q, p, 1) \to q'$ corresponds to the transition $P \to q'$ where P is



$\mathcal{A}$ has a final state q_1, which is assigned to all ancestors of the root of a subtrie satisfying φ, and an initial state q_0, which is assigned to all other nodes of the input trie. Letting q, p, q' range over $\{q_0, q_1\}$, the transition table is

$$\begin{array}{llll} a) & (\bot,\bot,\bot,\bot) \to q_0 & e) & (q,p,q',3) \to q' \\ b) & (q,\bot,q,1) \to q_1 & f) & (q,\bot,q',3) \to q' \\ c) & (q,p,q,1) \to q_1 & g) & (\bot,q,q',3) \to q' \\ d) & (q,p,q',2) \to q' & h) & (\bot,\bot,q',3) \to q'. \end{array}$$

Rule $a)$ means that if there is no successor-node, $\mathcal{A}$ is in state q_0. Rules $b)$ and $c)$ say that if the successor-node is the a-child, then $\mathcal{A}$ goes to q_1 as we just saw the pattern φ. Rules $d) - f)$ say that if the successor node differs from the a-child, $\mathcal{A}$ remains in the state of the successor node. Similar for $g)$ and $h)$, which cover the case when there is no a-child.

Theorem 2. *For every m, there is an MSO-sentence defining a class of LZ-tries that is not accepted by any b.u. m-LZ-automaton.*

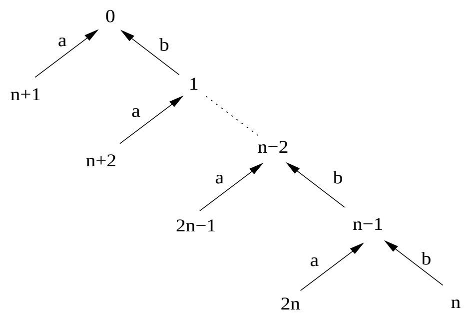
Proof. For $m = 1$, let $L = \{b^1 b^2 \cdots b^n a b^1 a b^2 a \cdots b^{n-1} a \mid n \in \mathbb{N}\} \subseteq \{a,b\}^+$. Each $w \in L$ has a LZ-block decomposition as indicated by

$$w_n = b^1.b^2.\cdots.b^n.a.b^1a.b^2a.\cdots.b^{n-1}a$$

and a compression

$$LZ(w_n) = (0,b)(1,b)\cdots(n-1,b)(0,a)(1,a)\cdots(n-1,a).$$

The corresponding tries $\mathcal{T}_{LZ(w_n)}$ look like



(2)

where the successor relation is given by the node numbers.

The class of enumerated tries in $LZ(L)$ can be defined by the MSO-sentence φ saying that the set B of nodes that are the root 0 or a b-child, has the following properties (of which (i) is neither $\exists$-MSO nor $\forall$-MSO):

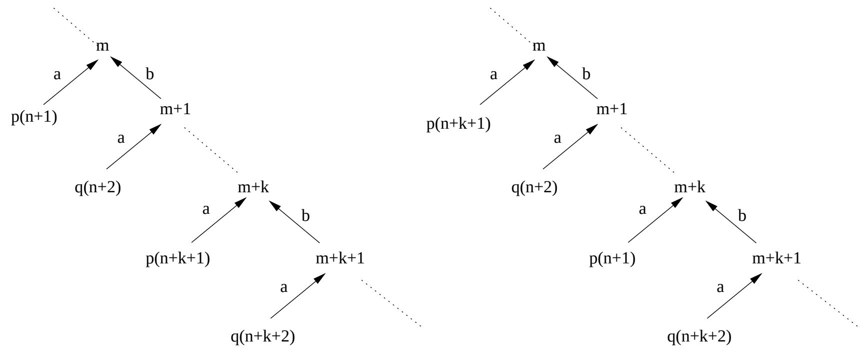
(i) $B \supseteq \{0\}$ is the smallest set being closed under b-children,
(ii) a node is not in B iff it is an a-child of a node in B,
(iii) for all nodes x, y, we have $y = x+1$ iff one of the following holds:
  (a) y is the b-child of x,
  (b) y is the a-child of the b-child y' of some x' whose a-child is x,
  (c) y is the a-child of 0 and x the member of B that has no b-child.

Claim. $LZ(L)$ is not accepted by a 1-LZ-automaton.

Suppose that $\mathcal{A}$ is a 1-LZ-automaton that accepts the class defined by φ. Let $m > |Q|$ and $w = w_{2m}$. An accepting run of $\mathcal{A}$ on $\mathcal{T}_{LZ(w)}$ assigns the same state, say q, to at least two different a-childs, say nodes $n+k+2$ and $n+2$. The state of their predecessors is determined by a rule

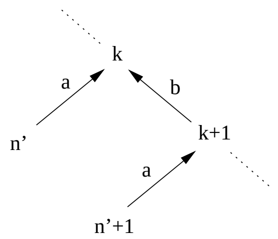
$$(\perp, \perp, q, 3) \to p.$$

Let $\mathcal{T}'$ be like $\mathcal{T}_{LZ(w)}$, except that nodes $n+1$ and $n+k+1$ are switched in the ordering. Then $\mathcal{A}$ will also accept the modified structure, as indicated by the states assigned to nodes as follows:

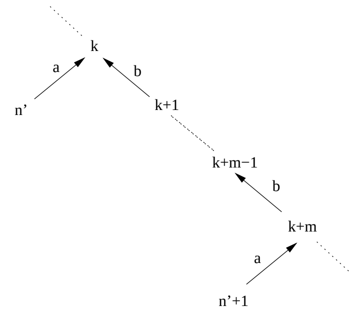


But the enumeration of a-children in $\mathcal{T}'$ does not conform to φ, so $\mathcal{T}' \in T(\mathcal{A}) \setminus LZ(L)$.

For the case $m > 1$, we modify the example as follows: along the B-part, between two nodes that have both an a-child and a b-child, we add $m-1$ nodes that have no a-child, i.e. the subgraphs



are replaced by



The *LZ*-tries arising this way are the compressions of the words

$$w_{n,k} = b^1.b^2.b^3.\cdots.b^{nm+k}.a.b^m a.b^{2m}a.\cdots.b^{nm}a.$$

Then $\{\mathcal{T}_{LZ(w_{n,k})} \mid n, k \in \mathbb{N}, k \le m\}$ is accepted by an $(m+1)$-*LZ*-automaton, hence MSO-definable. But it is not accepted by an m-*LZ*-automaton: intuitively, the m-hemisphere of a node k does not tell whether the path from k following a and successor and the path following $b^m a$ end at the same node.

4.2 Graph acceptors and ∃-MSO for *LZ*-tries

We now compare LZ-automata with the graph acceptors for directed acyclic graphs presented by W. Thomas [7].

On graphs $G = (V, E)$, a *tile* δ over a set Q is an m-neighbourhood of a node with a labelling in Q, i.e. a finite node-labelled graph. A *graph acceptor* is a triple $\mathcal{A} = (Q, \Delta, Occ)$ where Δ is a finite set of tiles over the finite set Q and Occ is a boolean combination of conditions: *there are* $\ge p$ *occurences of tile* $\delta \in \Delta$. A *run* of $\mathcal{A}$ on G is a function $r : V \to Q$ such that each m-neighbourhood of G becomes a tile $\delta \in \Delta$. Then $\mathcal{A}$ *accepts* a graph G if there exists a run of $\mathcal{A}$ on G whose tiles satisfy Occ.

Notice that the existence of a run is non-constructive, it could be exponentially hard to find an accepting run. The main result of [7] says that a class of graphs of bounded degree is definable in ∃-MSO iff it is accepted by a graph acceptor.

Theorem 3. *A class $\mathcal{K}$ of LZ-tries is ∃-MSO-definable iff for some m, $\mathcal{K}$ is accepted by an m-LZ-automaton.*

Proof. Note that each *LZ*-trie $\mathcal{T}$ satisfies the following conditions:

(i) each node of $\mathcal{T}$ has a degree $\leq |\Sigma| + 1$,

(ii) $\mathcal{T}$ is an acyclic (directed) graph with a designated out-edge (the successor) for each node,

(iii) $\mathcal{T}$ has a node that is reachable from any node by a path.

Using (i), by Theorem 3 of [7], $\mathcal{K}$ is $\exists$-MSO-definable iff $\mathcal{K}$ is recognizable by a graph acceptor. Moreover, from (i)-(iii) and Proposition 6 of [7], it follows that $\mathcal{K}$ is recognizable by a graph acceptor iff it is recognizable by a graph acceptor without occurrence constraints.

$\Leftarrow$: Suppose $\mathcal{K}$ is accepted by some m-*LZ*-automaton $\mathcal{A}$. We may assume that final states of $\mathcal{A}$ do not occur in the m-hemispheres P of transitions (P, q) of $\mathcal{A}$. Consider the transitions of $\mathcal{A}$ as a tiling system. Then an accepting run of $\mathcal{A}$ is a tiling that uses at least one of the tiles (P, q) where $q \in F$. Thus, $\mathcal{K}$ has a graph acceptor.

$\Rightarrow$: By the above remarks, $\mathcal{K}$ is recognizable by a graph acceptor without occurrence constraints. On the *LZ*-tries, the tiles have a root node, so we can view each tile as a transition rule saying that the automaton enters a state at the root depending on the m-sphere of the root and the states at the descendants in the sphere. Let each state be final; then the automaton accepts iff there is a tiling.

Remark 3. It may seem to follow that every FO-definable property of strings can be checked on *LZ*-tries by an m-*LZ*-automaton. But $<$-conditions on strings translate to conditions involving $Succ^+$ on the tries, and the transitive closure is not $\exists$-MSO-definable. (Yet, $\exists$-FO-formulas on strings translate to $\exists$-MSO-formulas on *LZ-graphs*.)

4.3 Colorability and deterministic acceptors

A graph $G = (V, E)$ is *k-colorable* if there is a partitioning of the set V of nodes into at most k classes (colors) such that any two adjacent nodes belong to different classes.

Clearly, k-colorability can be expressed by an $\exists$-MSO sentence, and hence has a graph acceptor. By reduction from the 3-satisfiablility problem, 3-colorability on arbitrary graphs is an NP-complete problem. However, on *LZ*-tries it is not:

Proposition 3. *Every LZ-trie is 3-colorable.*

Proof. By induction on the size of the *LZ*-trie. Suppose the prefix trie without the maximal node is 3-colorable. The maximal node m is connected to at most two nodes: to its predecessor, having some color B, and to a node k such that $k \xleftarrow{a} m$, for some $a \in \Sigma$, having color A, say. Choose a third color C different form A, B to color m.

Potthoff e.a. [6] have shown that on the class of directed acyclic graphs whose edges are uniquely labelled with a bounded number of labels, 2-colorability is *not* recognizable by a deterministic graph acceptor, i.e. one with at most one accepting run on each graph. For the subclass of LZ-tries, however, one gets:

Proposition 4. *On the class of LZ-tries, 2-colorability is recognizable by a deterministic graph acceptor.*

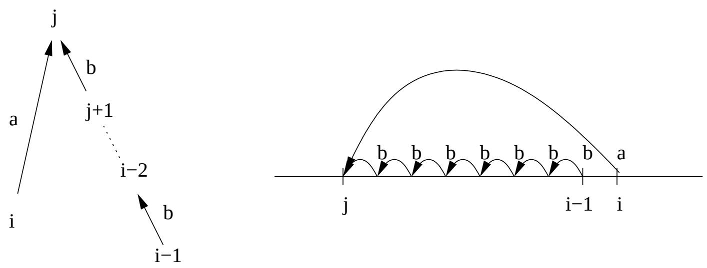
Proof. We construct a deterministic bottom-up 1-*LZ*-automaton $\mathcal{A}$ with states $q_0 = (\textit{odd}, \textit{accept})$, $q_1 = (\textit{odd}, \textit{reject})$, $q_3 = (\textit{even}, \textit{accept})$, $q_4 = (\textit{even}, \textit{reject})$, where the first components correspond to the two colors. Every node n of an LZ-trie has at most $1 + |\Sigma|$ children, its successor node $n+1$ and nodes $k_a > n$ such that $n \xleftarrow{a} k_a$, for $a \in \Sigma$. Let q_a resp. q_{succ} be the states assigned to these nodes by a run from the maximal node up to node n, beginning in state q_0. Then assign the following state q to node n: the first component of q is *odd* (resp. *even*) if the first component of q_{succ} is *even* (resp. *odd*). The second component of q is *accept* if the second component of q_{succ} is *accept*, i.e. if the suffix-trie starting at node $n+1$ is 2-colorable, and the first components of q_{succ} and all the q_a's coincide, i.e. the corresponding nodes have the same color. Otherwise, the second component of q is *reject*. The automaton accepts the input LZ-trie if it assigns one of the final states q_0, q_3 to the initial node 0.

Theorem 4. *There is a property of LZ-tries that is recognized by a non-deterministic bottom-up* 1*-LZ-automaton but not by any deterministic bottom-up m-LZ-automaton.*

Proof. Let $\Sigma = \{a, b, c\}$ and consider the following property φ of enumerated Σ-tries:

> There are two subsequent nodes $i-1$ and i such that some node j is both the a-predecessor of i and a b-ancestor of $i-1$.

An LZ-trie satisfies φ iff it contains nodes linked as follows (in the trie and in the LZ-graph, respectively), where $j < i - 1$:



(3)

In the appendix we prove that φ can be checked by a non-deterministic 1-LZ-automaton, but not by a deterministic m-LZ-automaton.

By the theorem it seems likely that the class of acceptable LZ-tries is not closed under complement, and hence:

Conjecture 1. On the class of all LZ-tries, not every MSO-formula is equivalent to an $\exists$-MSO-formula. (Can the non-existence of a subgraph of the form (3) be expressed in $\exists$-MSO?)

It seems to us that deterministic bottom-up-LZ-automata are too weak to check properties of strings on the LZ-compressed strings:

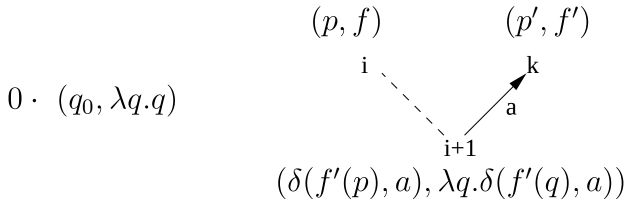
Conjecture 2. There is no deterministic bottom-up m-LZ-automaton that can check on $LZ(w)$ whether $w \in \{a, b\}^+$ has a subword ab.

4.4 Top-down LZ-trie-automata

G.Navarro [5] has shown how to do regular expression search on LZ-78-compressed texts by simulating an automaton reading the original text, beating the decompression-and-search approach by a factor of 2. Actually, this simulation is a deterministic top-down 1-LZ-automaton on the LZ-compressed text:

Theorem 5. *The LZ-compression $\{\mathcal{T}_{LZ(w)} \mid w \in R\}$ of any regular set $R \subseteq \Sigma^+$ is accepted by a deterministic top-down 1-LZ-automaton.*

Proof. Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, F)$ be a deterministic finite automaton accepting R. Define a deterministic top-down 1-LZ-automaton $\mathcal{A}' = (Q', \Sigma, \delta', q'_{in}, F')$ by $Q' := Q \times (Q \to Q)$, $q'_0 := (q_0, \lambda q.q)$, $F' := \{(q, f) \mid q \in F\}$ and δ' according to the following transitions:



for each $a \in \Sigma$ (including the case $i = k$).

Suppose r' is a run of $\mathcal{A}'$ on $\mathcal{T}_{LZ(w)}$ for a compressed word $LZ(w) = p_0 \cdots p_{m-1}$ where p_k represents block B_k of $w = B_0 \cdots B_{m-1}$. By induction we see that for each $k < m$,

$$r'(k) = (\delta(q_0, B_0 \cdots B_{k-1}), \lambda p.\delta(p, B_{k-1})). \tag{4}$$

Hence $\mathcal{A}'$ accepts $\{\mathcal{T}_{LZ(w)} \mid w \in R\}$, because

$$w \in R \iff \delta(q_0, B_0 \cdots B_{m-1}) \in F \iff r'(m) \in F'.$$

Corollary 1. *Every property of strings which is definable in MSO on strings is definable in ∃-MSO on LZ-tries.*

Proof. By Büchi's theorem and Proposition 2.

For the case of sentences, this improves on the translation given in Theorem 1, at the price of destroying the structure of the sentence.

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5 Appendix

Proof (of Theorem 4). Claim 1 The property φ can be checked by a non-deterministic bottom-up 1-*LZ*-automaton $\mathcal{A}$.

Proof. $\mathcal{A}$ has four states $p_a, p_b, p, accept$ and transition rules as follows:

(i) (*Delaying rule*) For 1-hemispheres P in which the root has no successor node or its successor node is marked by state p, $\mathcal{A}$ has the rule $P \to p$.

(ii) (*Guessing rule* 1) For any 1-hemisphere P in which the root has no successor node or its successor node is marked by state p, $\mathcal{A}$ has a rule $P \to p_a$.

(iii) (*Guessing rule* 2) For each 1-hemisphere P in which the root has a successor-child having state p_a, $\mathcal{A}$ has the rule $P \to p_b$.

(iv) (*Propagating rule* 1) For each 1-hemisphere P in which the root has a successor with state p_b which also is the b-child of the root, but does not have an a-child having state p_a, $\mathcal{A}$ has the rule $P \to p_b$.

(v) (*Checking rule*) For any 1-hemisphere P in which the root has a successor with state p_b which also is the b-child of the root, and does have an a-child having state p_a, $\mathcal{A}$ has the rule $P \to accept$.

(vi) (*Propagating rule* 2) For any 1-hemisphere P in which the root has a successor node with state *accept*, $\mathcal{A}$ has the rule $P \to accept$.

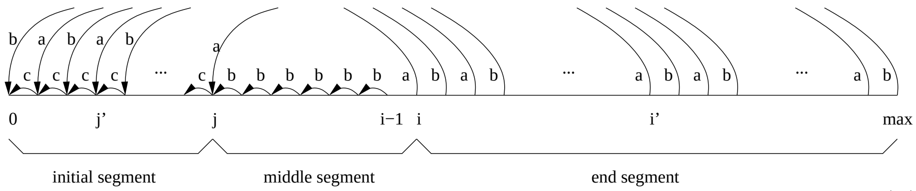
When a given *LZ*-trie has the property φ, then by (i) $\mathcal{A}$ may assign state p to nodes $\max, \dots, i+1$. By (ii), $\mathcal{A}$ can assign state p_a to node i, then by (iii) state p_b to node $i-1$. The state p_b can be propagated along the predecessors back to the node $j+1$ by (iv), and by (v) $\mathcal{A}$ can assign *accept* to node j and then propagate *accept* to the initial node 0 by (vi).

But $\mathcal{A}$ cannot accept an *LZ*-trie that does not satisfy φ: the rules ensure that p_a is assigned to at most one node, so the first time that *accept* is assigned to some node of the trie, its a-child is the (only) node i having state p_a and its b-successor chain leads through nodes in state p_b to node $i-1$.

Claim 2 The property φ cannot be checked by any deterministic bottom-up m-*LZ*-automaton.

Intuitively speaking, the m-hemisphere of a node j is not big enough to see if the predecessor of its a-child is one of its b-descendants.

Proof. Suppose a deterministic m-LZ-automaton $\mathcal{B}$ recognizes the set of LZ-tries satisfying φ. Depending on $\mathcal{B}$ we shall construct two words w, w' whose graphs look like



(5)

such that in the trie of $LZ(w)$, we have edges $j \xleftarrow{a} i$ and $j' \xleftarrow{a} i'$, so the pattern (3) occurs and φ holds, while in the trie of $LZ(w')$ we have $j' \xleftarrow{a} i$ and $j \xleftarrow{a} i'$, so that φ does not hold. To be specific, the tries have the following edges:

(i) In the end segment from node i to node $\max$, there are only outgoing edges, alternatingly labelled with a or b.
The a- and b-edges leaving nodes $i, \ldots, \max$ go to nodes in the initial segment from node 0 to node j, such that the a-edges leaving $\{i, i'\}$ go to $\{j', j\}$ and the remaining ones go, say, from $\max$ to 0, from $\max -1$ to 1, etc.
In the trie of w, there are edges $j \xleftarrow{a} i$ and $j' \xleftarrow{a} i'$, while in the trie of w' these nodes are connected by edges $j' \xleftarrow{a} i$ and $j \xleftarrow{a} i'$.

(ii) The nodes of the middle segment from j to i have no other incoming or outgoing edges except the ones shown, i.e. node j has an outgoing c-edge and incoming a- and b-edges, node i has an outgoing a-edge, and nodes j to $i-1$ are related by b-edges that connect nodes in the predecessor relation.

(iii) The nodes $0, \ldots, j$ of the initial segment have c-edges between nodes in the predecessor relation, and incoming a- and b-edges from nodes in the end segment, as described in (i).

Clearly, the trie of w has the property φ while the trie of w' does not. We now show that for suitable i, j, i', j' and $\max$, $\mathcal{B}$ accepts both tries, contradicting the assumption. Recall that $\mathcal{B}$ is a bottom-up trie-automaton, so it visits the nodes in reverse order, beginning at node max.

The m-hemispheres of nodes $k \in \{i, \ldots, \max - m\}$ consist of m nodes ordered by the successor and labelled by states of $\mathcal{B}$. Since B

is finite, there are only finitely many such m-hemispheres. Hence, if max is large, there are two points i, i' having the same m-hemisphere P, where $i + m < i'$. We may assume that that i and i' have an outgoing edge labelled a.

Since the m-hemispheres of i and i' agree, the m-hemishperes of $i - 1, \ldots, j, \ldots, j', \ldots, 0$ in the tries of w and w' agree as well. Since $\mathcal{B}$ is deterministic, it has to assign the same state to the initial node of the two tries.