Lambda calculs et catégories

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Synopsis of the lecture

1 – Categories and functors

2 – Natural transformations

2 – The 2-category of categories

3 – String diagrams

Categories and functors

A concise introduction

Categories

A category $\mathscr{C}$ is given by

$[0]$ a class of **objects**

$[1]$ a class $\mathbf{Hom}(A, B)$ of **morphisms**

$$f \quad : \quad A \quad \longrightarrow \quad B$$

for every pair of objects (A, B)

$[2]$ a **composition law** $\circ : \mathbf{Hom}(B, C) \times \mathbf{Hom}(A, B) \longrightarrow \mathbf{Hom}(A, C)$

$[2]$ an **identity** morphism

$$id_A \quad : \quad A \quad \longrightarrow \quad A$$

for every object A,

Categories

satisfying the following properties:

[3] the composition law $\circ$ is associative:

$$\begin{array}{l} \forall f \in \mathbf{Hom}(A,B) \\ \forall g \in \mathbf{Hom}(B,C) \\ \forall h \in \mathbf{Hom}(C,D) \end{array} \qquad f \circ (g \circ h) = (f \circ g) \circ h$$

[3] the morphisms id are neutral elements

$$\forall f \in \mathbf{Hom}(A,B) \qquad f \circ id_A \;=\; f \;=\; id_B \circ f$$

A hint of higher-dimensional wisdom

$$
\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "f \circ g"'] & & C
\end{tikzcd}
$$

The composition law hides a 2-dimensional simplex

A hint of higher-dimensional wisdom

$$
\begin{tikzcd}
 & B \arrow[r, "g"] & C \arrow[dr, "f"] & \\
A \arrow[ur, "h"] \arrow[urr, "g \circ h"] \arrow[rrr, "f \circ (g \circ h)"'] & & & D
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
 & B \arrow[r, "g"] \arrow[drr, "f \circ g"'] & C \arrow[dr, "f"] & \\
A \arrow[ur, "h"] \arrow[rrr, "(f \circ g) \circ h"'] & & & D
\end{tikzcd}
$$

The associativity rule hides a 3-dimensional simplex

Cartesian products

The **cartesian product** of two objects A and B in a category $\mathscr{C}$ is an object $A \times B$ equipped with two morphisms

$$\pi_1 : A \times B \longrightarrow A \qquad \pi_2 : A \times B \longrightarrow B$$

such that for every diagram

$$\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & B \end{tikzcd}$$

there exists a unique morphism $h : X \longrightarrow A \times B$ making the diagram

$$\begin{tikzcd} & & A \\ X \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, "h"] & A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & B \end{tikzcd}$$

commute.

Examples

1. The cartesian product in the category Set

2. The lub $a \wedge b$ of two elements a and b in an ordered set $(X, \leq)$

3. The cartesian product in the category Top of topological spaces and continuous functions

Terminal object

An object $\mathbf{1}$ is **terminal** in a category $\mathscr{C}$ when

$$\mathbf{Hom}(A, \mathbf{1})$$

is a singleton for all objects A.

One may consider $\mathbf{1}$ as a “nullary” product in $\mathscr{C}$.

Example 1. the singleton $\{*\}$ in the categories and ,

Example 2. the maximum of an ordered set $(X, \leq)$

Cartesian category

A **cartesian category** is a category $\mathscr{C}$ equipped with a product

$$A \times B$$

for all pairs A, B of objects, and of a terminal object

$$\mathbf{1}$$

Cartesian categories

In every cartesian category, one finds

▷ weakening maps $\epsilon_A : A \longrightarrow \mathbf{1}$,

▷ diagonal maps $\delta_A : A \longrightarrow A \times A$ obtained as

$$\begin{tikzcd} & & A \\ A \arrow[r, "\delta_A"] \arrow[urr, bend left, "id_A"] \arrow[drr, bend right, "id_A"'] & A \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & A \end{tikzcd}$$

▷ symmetry maps $\gamma_{A,B} : A \times B \longrightarrow B \times A$ obtained as

$$\begin{tikzcd} & & B \\ A \times B \arrow[r, "\gamma_{A,B}"] \arrow[urr, bend left, "\pi_2"] \arrow[drr, bend right, "\pi_1"'] & B \times A \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ & & A \end{tikzcd}$$

Functors

A **functor** between categories

$$F \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{D}$$

is defined as the following data:

[0] an object FA of $\mathscr{D}$ for every object A of $\mathscr{C}$,

[1] a function

$$F_{A,B} \quad : \quad \mathbf{Hom}_{\mathscr{C}}(A,B) \quad \longrightarrow \quad \mathbf{Hom}_{\mathscr{D}}(FA,FB)$$

for every pair of objects (A,B) of the category $\mathscr{C}$.

Functors

One requires moreover

[2] that F preserves composition

$$FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \qquad = \qquad FA \xrightarrow{F(g \circ f)} FC$$

[2] that F preserves the identities

$$FA \xrightarrow{Fid_A} FA \qquad = \qquad FA \xrightarrow{id_{FA}} FA$$

Illustration [*orders*]

Every ordered set

$$(X, \leq)$$

defines a category

$$[X, \leq]$$

- ▷ whose objects are the elements of X
- ▷ whose hom-sets are defined as

$$\mathbf{Hom}(x, y) \quad = \quad \begin{cases} \{*\} & \text{if} \quad x \leq y \\ \varnothing & \text{otherwise} \end{cases}$$

In this category, there exists at most one map between two objects

Illustration [*orders*]

Exercise: given two ordered sets

$$(X, \leq) \qquad (Y, \leq)$$

a functor

$$F \quad : \quad [X, \leq] \quad \longrightarrow \quad [Y, \leq]$$

is the same thing as a monotonic function

$$F \quad : \quad (X, \leq) \quad \longrightarrow \quad (Y, \leq)$$

between the underlying ordered sets.

Illustration [*monoids*]

A monoid $(M, \cdot, e)$ is a set M equipped with a binary operation

$$\cdot \quad : \quad M \times M \quad \longrightarrow \quad M$$

and a neutral element

$$e \quad : \quad \{*\} \quad \longrightarrow M$$

satisfying the two properties below:

Associativity law	$\forall x, y, z \in M,$	$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
Unit law	$\forall x \in M,$	$x \cdot e = x = e \cdot x.$

Illustration [*monoids*]

Key observation: there is a one-to-one relationship

$$(M, \cdot, e) \quad \mapsto \quad \Sigma\,(M, \cdot, e)$$

between

- ▷ monoids
- ▷ categories with one object $*$

obtained by defining $\Sigma(M, \cdot, e)$ as the category with unique hom-set

$$\Sigma(M, \cdot, e)\ (*, *) \quad = \quad M$$

and composition law and unit defined as

$$g \circ f \quad = \quad g \cdot f \qquad\qquad\qquad id_{*} = e$$

Illustration [*monoids*]

Key observation: given two monoids

$$(M, \cdot, e) \qquad\qquad (N, \bullet, u)$$

a functor

$$F \quad : \quad \Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \Sigma\,(N, \bullet, u)$$

is the same thing as a homomorphism

$$f \quad : \quad (M, \cdot, e) \quad \longrightarrow \quad (N, \bullet, u)$$

between the underlying monoids.

Recall that a homomorphism is a function f such that

$$\forall x, y \in M, \quad f(x \cdot y) \;=\; f(x) \bullet f(y) \qquad\qquad f(e) \;=\; u$$

Illustration [*actions*]

The action of a monoid

$$(M, \cdot, e)$$

on a set

$$X$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \mathbf{Set}$$

Illustration [*representations*]

The action of a monoid

$$(M, \cdot, e)$$

on a vector space

$$V$$

is the same thing as a functor

$$\Sigma\,(M, \cdot, e) \quad \longrightarrow \quad \mathbf{Vect}$$

Natural transformations

A notion of morphism between functors

Transformations

A **transformation**

$$\theta \quad : \quad F \stackrel{\cdot}{\longrightarrow} G$$

between two functors

$$F, G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$

is a family of morphisms

$$(\theta_A : FA \longrightarrow GA)_{A \in Obj(\mathscr{A})}$$

of the category $\mathscr{B}$ indexed by the objects of the category $\mathscr{A}$.

Vertical composition of transformations

The transformations compose vertically

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, "\Downarrow \theta_1 * \theta_2"] & \mathscr{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=4ex, "\Downarrow \theta_1"] \arrow[r, phantom, shift right=4ex, "\Downarrow \theta_2"] & \mathscr{B}
\end{tikzcd}$$

and thus define a category

$$\mathbf{Trans}(\mathscr{A}, \mathscr{B})$$

for all categories $\mathscr{A}$ and $\mathscr{B}$.

Left action

In the following situation:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C} \end{tikzcd}$$

the **left action** of the functor H on the transformation

$$\theta \quad : \quad F \quad \longrightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is defined as the transformation

$$H \circ_L \theta \quad : \quad H \circ F \quad \longrightarrow \quad H \circ G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{C}$$

whose instance at object A is defined as the morphism

$$H \circ F(A) \xrightarrow{\quad H(\theta_A) \quad} H \circ G(A).$$

Properties of the left action [1]

From a diagrammatic point of view, the two equations

$$H \circ_L (\theta_2 * \theta_1) \;=\; (H \circ_L \theta_2) * (H \circ_L \theta_1) \qquad\qquad H \circ_L 1_F \;=\; 1_{H\circ F}$$

mean that

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=U, to=M, "\theta_1"] \arrow[Rightarrow, from=M, to=D, "\theta_2"] & \mathscr{B} \arrow[r, "H"] & \mathscr{C}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M1}] \arrow[d, dotted, no head] \arrow[Rightarrow, from=U, to=M1, "\theta_1"] & \mathscr{B} \arrow[r, "H"] \arrow[d, dotted, no head] & \mathscr{C} \arrow[d, dotted, no head] \\
\mathscr{A} \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""{name=D, above}] \arrow[Rightarrow, from=M2, to=D, "\theta_2"] & \mathscr{B} \arrow[r, "H"'] & \mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{A} \arrow[r, "H"] & \mathscr{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "H\circ F"] \arrow[r, bend right=50, "H\circ F"'] \arrow[r, phantom, "\Downarrow 1_{H\circ F}"] & \mathscr{B}
\end{tikzcd}$$

Properties of the left action (2)

These two equations mean that

$$\boxed{\begin{array}{ccccc} H \circ_L - & : & \mathbf{Trans}(\mathscr{A}, \mathscr{B}) & \longrightarrow & \mathbf{Trans}(\mathscr{A}, \mathscr{C}) \\ & \theta & \mapsto & H \circ_L \theta & \end{array}}$$

defines a functor, while the two equations

$$(H_1 \circ H_2) \circ_L F = H_1 \circ_L (H_2 \circ_L F) \qquad id_{\mathscr{B}} \circ_L \theta = \theta$$

mean that $\circ_L$ defines an action.

Right action

In the following situation:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathscr{C} \end{tikzcd}$$

the functor H acts on the transformation

$$\theta \quad : \quad F \quad \longrightarrow \quad G \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{C}$$

and transports it into the transformation:

$$\theta \circ_R H \quad : \quad F \circ H \quad \longrightarrow \quad G \circ H \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{C}$$

whose instance at A is defined as the morphism

$$F \circ H(A) \quad \xrightarrow{\theta_{H(A)}} \quad G \circ H(A).$$

Properties of the right action (1)

From a diagrammatic point of view, the two equations

$$(\theta_2 * \theta_1) \circ_R H \;=\; (\theta_2 \circ_R H) * (\theta_1 \circ_R H) \qquad\qquad 1_F \circ_R H \;=\; 1_{F\circ H}$$

mean that

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D, above}] & \mathscr{C}
\arrow[from=U, to=M, phantom, "\Downarrow \theta_1"] \arrow[from=M, to=D, phantom, "\Downarrow \theta_2"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] \arrow[d, dotted, no head] & \mathscr{B} \arrow[r, bend left=60, ""{name=U, below}] \arrow[r, ""{name=M1}] \arrow[d, dotted, no head] & \mathscr{C} \arrow[d, dotted, no head] \\
\mathscr{A} \arrow[r, "H"'] & \mathscr{B} \arrow[r, ""{name=M2}] \arrow[r, bend right=60, ""{name=D, above}] & \mathscr{C}
\arrow[from=U, to=M1, phantom, "\Downarrow \theta_1"] \arrow[from=M2, to=D, phantom, "\Downarrow \theta_2"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A} \arrow[r, "H"] & \mathscr{B} \arrow[r, bend left=50, "F", ""{name=U, below}] \arrow[r, bend right=50, "F"', ""{name=D, above}] & \mathscr{B}
\arrow[from=U, to=D, phantom, "\Downarrow 1_F"]
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=50, "F\circ H", ""{name=U, below}] \arrow[r, bend right=50, "F\circ H"', ""{name=D, above}] & \mathscr{B}
\arrow[from=U, to=D, phantom, "\Downarrow 1_{F\circ H}"]
\end{tikzcd}$$

Properties of the right action (2)

The two equations mean that

$$\begin{array}{cccccc} - \circ_R H & : & \mathbf{Trans}(\mathscr{B}, \mathscr{C}) & \longrightarrow & \mathbf{Trans}(\mathscr{A}, \mathscr{C}) \\ & & \theta & \mapsto & \theta \circ_R H \end{array}$$

defines a functor, while the two equations

$$\theta \circ_R (H_2 \circ H_1) \quad = \quad (\theta \circ_R H_2) \circ_R H_1 \qquad\qquad \theta \circ_R id_{\mathscr{A}} \quad = \quad \theta$$

mean that $\circ_R$ defines an action.

Compatibility of the left and right actions

Last equation: in the situation

$$\begin{tikzcd} \mathscr{A}' \arrow[r, "H_1"] & \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathscr{B} \arrow[r, "H_2"] & \mathscr{B}' \end{tikzcd}$$

the order in which one makes the functors

$$H_1 \ : \ \mathscr{A}' \longrightarrow \mathscr{A} \qquad\qquad H_2 \ : \ \mathscr{B} \longrightarrow \mathscr{B}'$$

act on the transformation θ does not matter:

$$(H_2 \circ_L \theta) \circ_R H_1 \qquad = \qquad H_2 \circ_L (\theta \circ_R H_1)$$

Sesqui-category

A sesqui-category $\mathscr{D}$ is

$[0]$ a class of objects

$[1,2]$ equipped with a category

$$\mathscr{D}(A,B)$$

for every pair of objects (A,B) of the sesqui-category, where

the objects of $\mathscr{D}(A,B)$ = the morphisms from A to B

equipped with a pair of actions $\circ_L$ and $\circ_R$ satisfying...

Sesqui-categories

equipped with a pair of actions $\circ_L$ and $\circ_R$ satisfying the equations

$$
\begin{array}{rclcrcl}
h \circ_L (\theta_2 * \theta_1) & = & (h \circ_L \theta_2) * (h \circ_L \theta_1) & \qquad & h \circ_L 1_f & = & 1_{h \circ f} \\
(h_1 \circ h_2) \circ_L f & = & h_1 \circ_L (h_2 \circ_L f) & & id_{\mathscr{B}} \circ_L \theta & = & \theta \\
(\theta_2 * \theta_1) \circ_R h & = & (\theta_2 \circ_R h) * (\theta_1 *_R h) & & 1_f \circ_R h & = & 1_{f \circ h} \\
\theta \circ_R (h_2 \circ h_1) & = & (\theta \circ_R h_2) \circ_R h_1 & & \theta \circ_R id_{\mathscr{A}} & = & \theta
\end{array}
$$

$$
(h_2 \circ_L \theta) \circ_R h_1 = h_2 \circ_L (\theta \circ_R h_1)
$$

Theorem.
Categories, functors and transformations define a sesqui-category.

The sesqui-category of categories and transformations

Let θ_1 and θ_2 be two transformations in
$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathscr{C} \end{tikzcd}$$

In general, the transformation obtained by applying θ_1 then θ_2

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dotted, no head] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[d, dotted, no head] & \mathscr{C} \arrow[d, dotted, no head] \\ \mathscr{A} \arrow[r, bend right, "G_1"'] & \mathscr{B} \arrow[r, bend left] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & \mathscr{C} \end{tikzcd}$$

is not the same as the transformation obtained by applying θ_1 then θ_2:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F_1"] \arrow[d, dotted, no head] & \mathscr{B} \arrow[r, bend left, "F_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dotted, no head] & \mathscr{C} \arrow[d, dotted, no head] \\ \mathscr{A} \arrow[r, bend left] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & \mathscr{B} \arrow[r, bend right, "G_2"'] & \mathscr{C} \end{tikzcd}$$

Natural transformations

A transformation $$\theta \quad : \quad F \Rightarrow G \quad : \quad \mathscr{A} \longrightarrow \mathscr{B}$$
is **natural** when the diagram

$$\begin{tikzcd} FA \arrow[r, "\theta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\theta_B"] & GB \end{tikzcd}$$

commutes for every morphism $f : A \longrightarrow B$.

Notation. we write

$$\mathbf{Nat}(\mathscr{A}, \mathscr{B})$$

for the category of functors and natural transformations

$$\theta \quad : \quad F \quad \Rightarrow \quad G \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

Exchange law

A pair of 2-cells θ_1 and θ_2 in a sesqui-categorie $\mathscr{D}$

$$\begin{tikzcd} A \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C \end{tikzcd}$$

satisfy the exchange law when the equality

$$\begin{tikzcd} A \arrow[r, bend left, "F_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "F_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ A \arrow[r, bend right, "G_1"'] & B \arrow[r, bend left] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "F_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "F_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ A \arrow[r, bend left] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend right, "G_2"'] & C \end{tikzcd}$$

holds.

The order in which one applies θ_1 and θ_2 does not matter.

Definition

A 2-cell

$$\begin{tikzcd} B \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C \end{tikzcd}$$

is called **central on the left** when the exchange law

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "f_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ A \arrow[r, bend right, "g_1"'] & B \arrow[r, bend left] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C \end{tikzcd} = \begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[d, dotted, no head] & B \arrow[r, bend left, "f_2"] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta_2"] \arrow[d, dotted, no head] & C \arrow[d, dotted, no head] \\ A \arrow[r, bend left] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend right, "g_2"'] & C \end{tikzcd}$$

is satisfied for every 2-cell θ_1 of the sesqui-category $\mathscr{D}$.

Exercise

Show that in the sesqui-category with

- categories as objects
- functors as 1-cells
- transformations as 2-cells

the natural transformations are the 2-cells central on the left.

Deduce the existence of a functor

$$\mathbf{Nat}(\mathscr{B},\mathscr{C}) \times \mathbf{Nat}(\mathscr{A},\mathscr{B}) \quad \longrightarrow \quad \mathbf{Nat}(\mathscr{A},\mathscr{C})$$

The 2-category of categories

Categories, functors, natural transformations

2-categories

A 2-category $\mathscr{D}$ is a sesqui-category such that the **exchange law** is satisfied for every pair of 2-cells

$$\begin{tikzcd} A \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\theta_1"] & B \arrow[r, bend left, "f_2"] \arrow[r, bend right, "g_2"'] \arrow[r, phantom, "\Downarrow\theta_2"] & C \end{tikzcd}$$

2-categories (alternative definition)

A 2-category $\mathscr{D}$ is given by

$[0]$ a class of **objects**

$[1,2]$ a category $\mathscr{D}(A,B)$ for every pair of objects (A,B)

$[2,3,4]$ a **composition law** defined as a functor

$$\circ : \mathscr{D}(B,C) \times \mathscr{D}(A,B) \longrightarrow \mathscr{D}(A,C)$$

$[2,3,4]$ an **identity** defined as a functor

$$id_A : \mathbb{1} \longrightarrow \mathscr{D}(A,A)$$

this for all objects A,B,C of the 2-category,

2-categories (alternative definition)

1— such that the composition law $\circ$ is associative in the sense that

$$
\begin{tikzcd}
\mathscr{D}(C,D) \times \mathscr{D}(B,C) \times \mathscr{D}(A,B) \arrow[r, "\circ \times \mathscr{D}(A,B)"] \arrow[d, "\mathscr{D}(C,D) \times \circ"'] & \mathscr{D}(B,D) \times \mathscr{D}(A,B) \arrow[d, "\circ"] \\
\mathscr{D}(C,D) \times \mathscr{D}(A,C) \arrow[r, "\circ"] & \mathscr{D}(A,D)
\end{tikzcd}
$$

commutes.

2-categories (alternative definition)

2— such that id is a neutral element of $\circ$ in the sense that

$$\begin{tikzcd}
\mathscr{D}(A,B) \arrow[rr, equal] \arrow[d, "\cong"'] & & \mathscr{D}(A,B) \\
\mathscr{D}(A,B) \times \mathbb{1} \arrow[rr, "\mathscr{D}(A{,}B) \times id_A"] & & \mathscr{D}(A,B) \times \mathscr{D}(A,A) \arrow[u, "\circ"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{D}(A,B) \arrow[rr, equal] \arrow[d, "\cong"'] & & \mathscr{D}(A,B) \\
\mathbb{1} \times \mathscr{D}(A,B) \arrow[rr, "id_B \times \mathscr{D}(A{,}B)"] & & \mathscr{D}(B,B) \times \mathscr{D}(A,B) \arrow[u, "\circ"']
\end{tikzcd}$$

commute for all A and B.

Notation

One writes

$$\theta \quad : \quad f \quad \Rightarrow \quad g \quad : \quad A \longrightarrow B$$

when

$$\theta \quad : \quad f \longrightarrow g$$

is a morphism of the category $\mathscr{D}(A, B)$.

Godement law

In a 2-category

$$\mathscr{D}(\mathscr{A},\mathscr{B})$$

the two canonical ways to compose the 2-cells

$$\begin{tikzcd}
\mathscr{A} \arrow[r, bend left=80, "f_1"] \arrow[r, "g_1" description] \arrow[r, bend right=80, "h_1"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha_1"] \arrow[r, phantom, shift right=4, "\Downarrow\beta_1"] & \mathscr{B} \arrow[r, bend left=80, "f_2"] \arrow[r, "g_2" description] \arrow[r, bend right=80, "h_2"'] \arrow[r, phantom, shift left=4, "\Downarrow\alpha_2"] \arrow[r, phantom, shift right=4, "\Downarrow\beta_2"] & \mathscr{C}
\end{tikzcd}$$

coincide:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) \quad = \quad (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

Suspension

$\boxed{\text{The notion of monoidal category will be defined very soon.}}$

Every strict monoidal category $\mathscr{C}$ may be seen as the 2-category $\Sigma(\mathscr{C})$

- ▷ which contains only one 0-cell,
- ▷ whose 1-cells are the 0-cells of $\mathscr{C}$
- ▷ whose 2-cells are the 1-cells of $\mathscr{C}$

equipped with the induced composition laws.

A sesqui-category $\Sigma(\mathscr{C})$ with one object is
the same thing as a premonoidal category $(\mathscr{C}, \otimes, I)$.

Useful equality

In a 2-category $\mathscr{D}(\mathscr{A}, \mathscr{B})$, the two canonical ways to compose the 2-cells

$$\begin{tikzcd}[column sep=huge]
\mathscr{A} \arrow[r, bend left=70, "f_1"{name=F1}] \arrow[r, "g_1"{name=G1, description}] \arrow[r, bend right=70, "h_1"'{name=H1}] & \mathscr{B} \arrow[r, bend left=70, "f_2"{name=F2}] \arrow[r, "g_2"{name=G2, description}] \arrow[r, bend right=70, "h_2"'{name=H2}] & \mathscr{C}
\arrow[from=F1, to=G1, phantom, "\Downarrow\alpha_1"]
\arrow[from=G1, to=H1, phantom, "\Downarrow\beta_1"]
\arrow[from=F2, to=G2, phantom, "\Downarrow\alpha_2"]
\arrow[from=G2, to=H2, phantom, "\Downarrow\beta_2"]
\end{tikzcd}$$

commute:

$$(\beta_2 * \alpha_2) \circ (\beta_1 * \alpha_1) \quad = \quad (\beta_2 \circ \beta_1) * (\alpha_2 \circ \alpha_1)$$

The 2-category of sets and relations

The 2-category $\mathscr{Rel}$ is defined as follows:

▷ its 0-cells are the sets,

▷ its 1-cells are the relations between sets,

$$A \xrightarrow{f \cdot g} B \quad = \quad A \xrightarrow{f} B \xrightarrow{g} C$$

relationally composed:

$$a\,[f \cdot g]\,c \qquad \Longleftrightarrow \qquad \exists b \in B, \qquad a\,[f]\,b \quad \text{et} \quad b\,[g]\,c.$$

▷ its 2-cells are inclusions:

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd} \qquad \Longleftrightarrow \qquad f \subseteq g$$

In particular, the categories $\mathscr{Rel}(A, B)$ are order categories.

String diagrams

A notation introduced by Roger Penrose

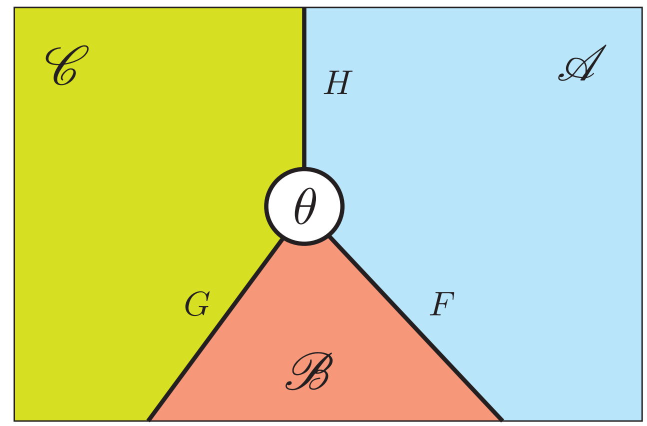
String diagrams

Two key ideas

1. apply the Poincaré duality on the original pasting diagrams:

$$\begin{tikzcd} & \mathscr{B} \arrow[dr, bend left, "G"] & \\ \mathscr{A} \arrow[ur, bend left, "F"] \arrow[rr, bend right, "H"'] \arrow[rr, phantom, shift left=4, "\Downarrow\theta"] & & \mathscr{C} \end{tikzcd}$$

is depicted as



$$\boxed{\theta \quad : \quad G \circ F \quad \Rightarrow \quad H}$$

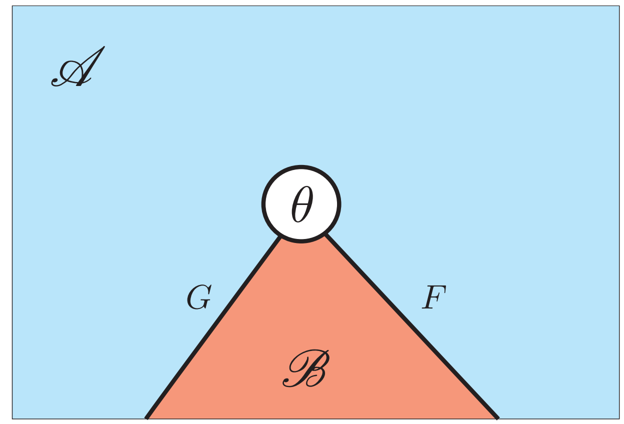
String diagrams

Two key ideas

2. hide the identity 1-cells in the picture:

$$
\begin{tikzcd}
 & \mathscr{B} \arrow[dr, bend left, "G"] & \\
\mathscr{A} \arrow[ur, bend left, "F"] \arrow[rr, bend right, "id"'] \arrow[rr, phantom, shift left=4ex, "\Downarrow\theta"] & & \mathscr{A}
\end{tikzcd}
$$

is depicted as



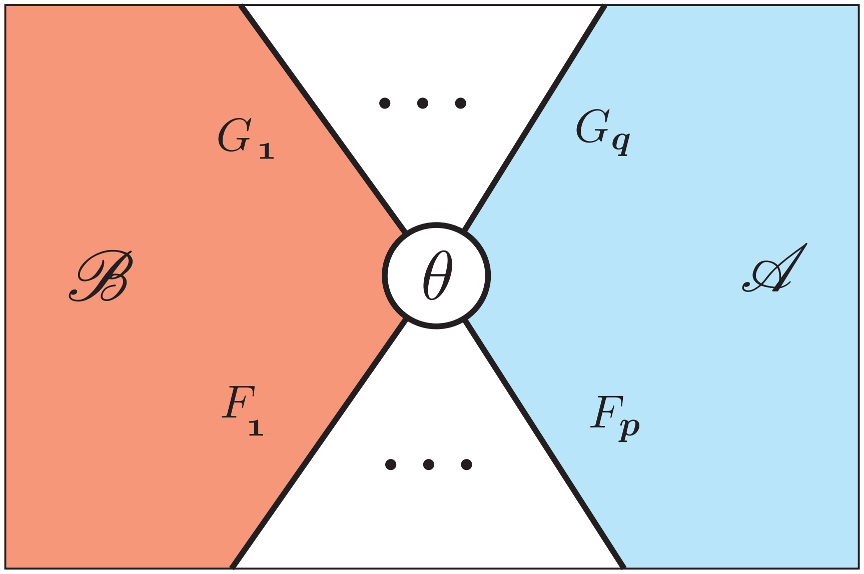
$$\boxed{\theta \quad : \quad G \circ F \quad \Rightarrow \quad id}$$

String diagrams

More generally, a 2-dimensional cell

$$\theta \quad : \quad F_1 \circ \cdots \circ F_p \quad \Rightarrow \quad G_1 \circ \cdots \circ G_q \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is depicted as



Exercise

Draw the exchange law and explain the connection to concurrency

Short bibliography of the course

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Springer Verlag 1995.

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A survey of graphical languages for monoidal categories.
New Structures for Physics
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