Travaux Dirigés Lambda-calculs et Catégories

An equivalent formulation of adjunctions Application to cartesian and to cartesian closed categories

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1 An equivalent formulation of adjunctions

§1. Suppose given a functor

 $R \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$

between two categories \mathscr{A} and \mathscr{B} . Show that every map

$$\eta : A \longrightarrow R(L_A)$$

from an object A of the category $\mathscr A$ into the image by R of an object noted L_A of the category $\mathscr B$ induces a function

$$\varphi_B : \mathscr{B}(L_A, B) \longrightarrow \mathscr{A}(A, RB)$$

for every object B of the category \mathcal{B} .

§2. Show that the family of functions φ_B is natural in *B* in the sense that it defines a natural transformation

$$\varphi$$
 : $\mathscr{B}(L_A, -) \Rightarrow \mathscr{A}(A, R-)$: $\mathscr{B} \longrightarrow \mathbf{Set}$

between the set-valued functors

$$\mathscr{B}(L_A, -) = B \mapsto \mathscr{B}(L_A, B)$$
 $\mathscr{A}(A, R-) = B \mapsto \mathscr{A}(A, RB)$

from the category \mathcal{B} to the category Set of sets and functions.

§3. One says that a pair (L_A, η) consisting of an object L_A of the category \mathscr{B} and of a map

 $\eta : A \longrightarrow R(L_A)$

represents the set-valued functor

$$\mathscr{A}(A, R-)$$
 : $\mathscr{B} \longrightarrow \mathbf{Set}$

when the function φ_B defined in §1 is a bijection

$$\varphi_B : \mathscr{B}(L_A, B) \cong \mathscr{A}(A, RB)$$

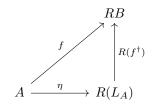
for every object B of the category \mathscr{B} . Show that the pair (L_A, η) represents the set-valued functor $\mathscr{A}(A, R-)$ precisely when the following property holds: for every object B and for every map

$$f : A \longrightarrow RB$$

there exists a unique map

$†$
 : $L_A \longrightarrow B$

such that the diagram below commutes:



§4. We suppose from now on that every object A of the category \mathscr{A} , there exists a pair (L_A, η_A) which represents the set-valued functor $\mathscr{A}(A, R-)$. For every map $f: A_1 \to A_2$ of the category \mathscr{A} , construct a map

$$L_f : L_{A_1} \longrightarrow L_{A_2}$$

of the category $\mathcal B$ such that the diagram below commutes:

f

$$\begin{array}{c} A_2 & \xrightarrow{\eta_{A_2}} & R(L_{A_2}) \\ \uparrow & & \uparrow \\ A_1 & \xrightarrow{\eta_{A_1}} & R(L_{A_1}) \end{array}$$

§5. Deduce from the construction in §4 that the function $A \mapsto L_A$ defines a functor

 $L : \mathscr{A} \longrightarrow \mathscr{B}$

and a family of bijections

$$\phi_{A,B}$$
 : $\mathscr{B}(LA,B) \cong \mathscr{A}(A,RB)$

Show moreover that this family of bijections ϕ is natural in *A* and *B*.

§6. Conclude that given a functor $R : \mathscr{B} \to \mathscr{A}$, the existence of a pair (LA, η_A) representing the set-valued functor

$$\mathscr{A}(A, R-)$$
 : $\mathscr{B} \longrightarrow \mathbf{Set}$

for every object A of the category \mathscr{A} implies the existence of a left adjoint functor $L: \mathscr{A} \to \mathscr{B}$ to the functor $R: \mathscr{B} \to \mathscr{A}$.

§7. Conversely, show that whenever we have a pair of adjoint functors

$$L:\mathscr{A} \xrightarrow{L} \mathscr{B}: R$$

every object A of the category \mathscr{A} comes equipped with a pair (LA,η_A) which represents the set-valued functor

$$\mathscr{A}(A, R-) \quad = \quad B \mapsto \mathscr{A}(A, RB) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathbf{Set}.$$

§8. Apply the results of §6 to establish that the forgetful functor

$$R = U \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Set}$$

from the category $\mathscr{B} = Mon$ of monoids and homomorphisms to the category $\mathscr{A} =$ Set of sets and functions has the free monoid functor

$$L = A \mapsto A^*$$
 : **Set** \to **Mon**

as left adjoint.

2 Application to cartesian closed categories

§1. Show that every adjoint pair

 $L:\mathscr{A} \longleftrightarrow \mathscr{B}: R$

where L is left adjoint to R induces an adjoint pair

 $R^{op}:\mathscr{B}^{op} \xleftarrow{\hspace{0.5cm}} \mathscr{A}^{op}: L^{op}$

where the functor L^{op} is right adjoint to R^{op} .

§2. From this and exercise 1, deduce that a functor $L : \mathscr{A} \to \mathscr{B}$ has a right adjoint precisely when for every object B of the category \mathscr{C} there exists a pair (RB, ε_B) consisting of an object RB of the category \mathscr{A} and of a map

$$\varepsilon_B$$
 : $L(RB) \longrightarrow B$

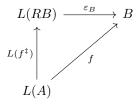
such that the following property holds: for every object A of the category $\mathscr A$ and for every map

$$f$$
 : $LA \longrightarrow B$

there exists a unique map

$$f^{\ddagger}$$
 : $A \longrightarrow RB$

such that the diagram below commutes:



Terminology: one says in that case that the pair (RB, ε_B) represents the functor

$$\mathscr{B}(L-,B) = A \mapsto \mathscr{B}(LA,B) : \mathscr{A}^{op} \longrightarrow \mathbf{Set}$$

§3. Apply this alternative formulation of adjunctions to the functor

 $L \hspace{.1 in} = \hspace{.1 in} B \mapsto A \times B \hspace{.1 in} : \hspace{.1 in} \mathscr{C} \hspace{.1 in} \longrightarrow \hspace{.1 in} \mathscr{C}$

associated to an object A of a cartesian category \mathscr{C} with

—	the object	RB	noted	$A \Rightarrow B$
-	the map	$\varepsilon_B : L(RB) \to B$	noted	$\operatorname{eval}_B : A \times (A \Rightarrow B) \to B$

Show that one recovers in this way the equivalence between the two formulations of cartesian closed category given in the course.

3 Application to cartesian categories

As we explained during the course, the category 1 with one object * and one map (=the identity map) is terminal in the category Cat. This means that for every category \mathscr{C} , there exists a unique functor

$$! : \mathscr{C} \longrightarrow \mathbb{1}. \tag{1}$$

At the same time, every object A of the category \mathscr{C} gives rise to a functor, also noted

$$A : \mathbb{1} \longrightarrow \mathscr{C}$$
 (2)

which transports the unique object * of the category 1 to the object A.

§1. Show that an object A is terminal in the category \mathscr{C} if and only if the associated functor (2) is right adjoint to the canonical functor (1).

§2. Show that an object A is initial in the category \mathscr{C} if and only if the associated functor (2) is left adjoint to the canonical functor (1).

§3. Show that the operation $A \mapsto (A, A)$ which transports every object A of the category \mathscr{C} to the object (A, A) of the category $\mathscr{C} \times \mathscr{C}$ defines a functor

 $\Delta \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C} \times \mathscr{C}.$

This functor Δ is called the diagonal functor of the category \mathscr{C} .

§4. Suppose given a pair of objects A, B in a category \mathscr{C} . Show that a triple

$$(A \times B, \pi_1, \pi_2)$$

consisting of an object $A \times B$ and of two maps

$$\pi_1: A \times B \to A \qquad \qquad \pi_2: A \times B \to B$$

defines a cartesian product of A and B precisely when the pair $(A \times B, \pi)$ consisting of the object $A \times B$ and of the map in the category $\mathscr{C} \times \mathscr{C}$

$$\pi = (\pi_1, \pi_2) \quad : \quad \Delta(A \times B) \quad \longrightarrow \quad (A, B)$$

represents the functor

$$\mathscr{C} \times \mathscr{C}(\Delta -, (A, B))$$
 : $\mathscr{C}^{op} \longrightarrow \mathbf{Set}$

§5. From this and the exercise 2, deduce that a category $\mathscr C$ is cartesian precisely when the two canonical functors

$$! : \mathscr{C} \longrightarrow \mathbb{1} \qquad \qquad \Delta \, : \, \mathscr{C} \, \longrightarrow \, \mathscr{C} \times \mathscr{C}$$

have a right adjoint.