

Travaux Dirigés
Lambda-calculs et Catégories

An equivalent formulation of adjunctions
Application to cartesian and to cartesian closed categories

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1 An equivalent formulation of adjunctions

§1. Suppose given a functor

$$R : \mathcal{B} \longrightarrow \mathcal{A}$$

between two categories \mathcal{A} and \mathcal{B} . Show that every map

$$\eta : A \longrightarrow R(L_A)$$

from an object A of the category \mathcal{A} into the image by R of an object noted L_A of the category \mathcal{B} induces a function

$$\varphi_B : \mathcal{B}(L_A, B) \longrightarrow \mathcal{A}(A, RB)$$

for every object B of the category \mathcal{B} .

§2. Show that the family of functions φ_B is natural in B in the sense that it defines a natural transformation

$$\varphi : \mathcal{B}(L_A, -) \Rightarrow \mathcal{A}(A, R-) : \mathcal{B} \longrightarrow \mathbf{Set}$$

between the set-valued functors

$$\mathcal{B}(L_A, -) = B \mapsto \mathcal{B}(L_A, B) \qquad \mathcal{A}(A, R-) = B \mapsto \mathcal{A}(A, RB)$$

from the category \mathcal{B} to the category \mathbf{Set} of sets and functions.

§3. One says that a pair (L_A, η) consisting of an object L_A of the category \mathcal{B} and of a map

$$\eta : A \longrightarrow R(L_A)$$

represents the set-valued functor

$$\mathcal{A}(A, R-) : \mathcal{B} \longrightarrow \mathbf{Set}$$

when the function φ_B defined in §1 is a bijection

$$\varphi_B : \mathcal{B}(L_A, B) \cong \mathcal{A}(A, RB)$$

for every object B of the category \mathcal{B} . Show that the pair (L_A, η) represents the set-valued functor $\mathcal{A}(A, R-)$ precisely when the following property holds: for every object B and for every map

$$f : A \longrightarrow RB$$

there exists a unique map

$$f^\dagger : L_A \longrightarrow B$$

such that the diagram below commutes:

$$\begin{array}{ccc} & & RB \\ & \nearrow f & \uparrow R(f^\dagger) \\ A & \xrightarrow{\eta} & R(L_A) \end{array}$$

§4. We suppose from now on that every object A of the category \mathcal{A} , there exists a pair (L_A, η_A) which represents the set-valued functor $\mathcal{A}(A, R-)$. For every map $f : A_1 \rightarrow A_2$ of the category \mathcal{A} , construct a map

$$L_f : L_{A_1} \longrightarrow L_{A_2}$$

of the category \mathcal{B} such that the diagram below commutes:

$$\begin{array}{ccc} A_2 & \xrightarrow{\eta_{A_2}} & R(L_{A_2}) \\ \uparrow f & & \uparrow R(L_f) \\ A_1 & \xrightarrow{\eta_{A_1}} & R(L_{A_1}) \end{array}$$

§5. Deduce from the construction in §4 that the function $A \mapsto L_A$ defines a functor

$$L : \mathcal{A} \longrightarrow \mathcal{B}$$

and a family of bijections

$$\phi_{A,B} : \mathcal{B}(LA, B) \cong \mathcal{A}(A, RB)$$

Show moreover that this family of bijections ϕ is natural in A and B .

§6. Conclude that given a functor $R : \mathcal{B} \rightarrow \mathcal{A}$, the existence of a pair (LA, η_A) representing the set-valued functor

$$\mathcal{A}(A, R-) : \mathcal{B} \longrightarrow \mathbf{Set}$$

for every object A of the category \mathcal{A} implies the existence of a left adjoint functor $L : \mathcal{A} \rightarrow \mathcal{B}$ to the functor $R : \mathcal{B} \rightarrow \mathcal{A}$.

§7. Conversely, show that whenever we have a pair of adjoint functors

$$L : \mathcal{A} \xrightleftharpoons[R]{L} \mathcal{B} : R$$

every object A of the category \mathcal{A} comes equipped with a pair (LA, η_A) which represents the set-valued functor

$$\mathcal{A}(A, R-) = B \mapsto \mathcal{A}(A, RB) : \mathcal{B} \longrightarrow \mathbf{Set}.$$

§8. Apply the results of §6 to establish that the forgetful functor

$$R = U \quad : \quad \mathbf{Mon} \longrightarrow \mathbf{Set}$$

from the category $\mathcal{B} = \mathbf{Mon}$ of monoids and homomorphisms to the category $\mathcal{A} = \mathbf{Set}$ of sets and functions has the free monoid functor

$$L = A \mapsto A^* \quad : \quad \mathbf{Set} \rightarrow \mathbf{Mon}$$

as left adjoint.

2 Application to cartesian closed categories

§1. Show that every adjoint pair

$$L : \mathcal{A} \rightleftarrows \mathcal{B} : R$$

where L is left adjoint to R induces an adjoint pair

$$R^{op} : \mathcal{B}^{op} \rightleftarrows \mathcal{A}^{op} : L^{op}$$

where the functor L^{op} is right adjoint to R^{op} .

§2. From this and exercise 1, deduce that a functor $L : \mathcal{A} \rightarrow \mathcal{B}$ has a right adjoint precisely when for every object B of the category \mathcal{C} there exists a pair (RB, ε_B) consisting of an object RB of the category \mathcal{A} and of a map

$$\varepsilon_B \quad : \quad L(RB) \longrightarrow B$$

such that the following property holds: for every object A of the category \mathcal{A} and for every map

$$f \quad : \quad LA \longrightarrow B$$

there exists a unique map

$$f^\dagger \quad : \quad A \longrightarrow RB$$

such that the diagram below commutes:

$$\begin{array}{ccc} L(RB) & \xrightarrow{\varepsilon_B} & B \\ \uparrow L(f^\dagger) & \nearrow f & \\ L(A) & & \end{array}$$

Terminology: one says in that case that the pair (RB, ε_B) *represents* the functor

$$\mathcal{B}(L-, B) \quad = \quad A \mapsto \mathcal{B}(LA, B) \quad : \quad \mathcal{A}^{op} \longrightarrow \mathbf{Set}.$$

§3. Apply this alternative formulation of adjunctions to the functor

$$L \quad = \quad B \mapsto A \times B \quad : \quad \mathcal{C} \longrightarrow \mathcal{C}$$

associated to an object A of a cartesian category \mathcal{C} with

- the object RB noted $A \Rightarrow B$
- the map $\varepsilon_B : L(RB) \rightarrow B$ noted $\text{eval}_B : A \times (A \Rightarrow B) \rightarrow B$

Show that one recovers in this way the equivalence between the two formulations of cartesian closed category given in the course.

3 Application to cartesian categories

As we explained during the course, the category $\mathbb{1}$ with one object $*$ and one map (=the identity map) is terminal in the category \mathbf{Cat} . This means that for every category \mathcal{C} , there exists a unique functor

$$! : \mathcal{C} \longrightarrow \mathbb{1}. \quad (1)$$

At the same time, every object A of the category \mathcal{C} gives rise to a functor, also noted

$$A : \mathbb{1} \longrightarrow \mathcal{C} \quad (2)$$

which transports the unique object $*$ of the category $\mathbb{1}$ to the object A .

§1. Show that an object A is terminal in the category \mathcal{C} if and only if the associated functor (2) is right adjoint to the canonical functor (1).

§2. Show that an object A is initial in the category \mathcal{C} if and only if the associated functor (2) is left adjoint to the canonical functor (1).

§3. Show that the operation $A \mapsto (A, A)$ which transports every object A of the category \mathcal{C} to the object (A, A) of the category $\mathcal{C} \times \mathcal{C}$ defines a functor

$$\Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}.$$

This functor Δ is called the diagonal functor of the category \mathcal{C} .

§4. Suppose given a pair of objects A, B in a category \mathcal{C} . Show that a triple

$$(A \times B, \pi_1, \pi_2)$$

consisting of an object $A \times B$ and of two maps

$$\pi_1 : A \times B \rightarrow A \quad \pi_2 : A \times B \rightarrow B$$

defines a cartesian product of A and B precisely when the pair $(A \times B, \pi)$ consisting of the object $A \times B$ and of the map in the category $\mathcal{C} \times \mathcal{C}$

$$\pi = (\pi_1, \pi_2) : \Delta(A \times B) \longrightarrow (A, B)$$

represents the functor

$$\mathcal{C} \times \mathcal{C}(\Delta -, (A, B)) : \mathcal{C}^{op} \longrightarrow \mathbf{Set}.$$

§5. From this and the exercise 2, deduce that a category \mathcal{C} is cartesian precisely when the two canonical functors

$$! : \mathcal{C} \longrightarrow \mathbb{1} \quad \Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$$

have a right adjoint.