Travaux Dirigés Lambda-calculs et Catégories

Distributivity laws between functors and monads Grothendieck construction and set-theoretic colimits

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1 Distributivity laws between functors and monads

§1. Suppose given two categories \mathscr{A} and \mathscr{B} , each of them equipped with a monad

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A} \qquad (T, \mu_T, \eta_T) : \mathscr{B} \longrightarrow \mathscr{B}$$

A homomorphism

$$(F,\lambda) : (\mathscr{A},S) \longrightarrow (\mathscr{B},T)$$
 (1)

is defined as a functor $F: \mathscr{A} \to \mathscr{B}$ equipped with distributivity law

$$\lambda : T \circ F \Rightarrow F \circ S$$

making the diagrams of natural transformations below commute:

$$T \circ T \circ F \xrightarrow{T \circ \lambda} T \circ F \circ S \xrightarrow{\lambda \circ S} F \circ S \circ S \qquad F$$

$$\downarrow^{\mu_T \circ F} \qquad (a) \qquad \downarrow^{F \circ \mu_S} \qquad \eta_T \qquad (b) \qquad \eta_S$$

$$T \circ F \xrightarrow{T \circ \lambda} F \circ S \qquad T \circ F \xrightarrow{\lambda} F \circ S$$

- §1. Formulate the two commutative diagrams (a) and (b) as families of commutative diagrams between maps living in the category \mathcal{B} .
- §2. Depict the commutative diagrams (a) and (b) in the language of string diagrams.
- §3. Show that every homomorphism (F, λ) as in (1) induces a functor

$$\widetilde{F} \quad : \quad \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram below commute:

$$\begin{array}{ccc}
\mathbf{Alg}(S) & \xrightarrow{\widetilde{F}} & \mathbf{Alg}(T) \\
U_S \downarrow & & (*) & \downarrow U_T \\
\mathscr{A} & \xrightarrow{F} & \mathscr{B}
\end{array}$$

where U_S and U_T are the forgetful functors associated to the monads S and T, respectively.

§4. Conversely, show that every functor

$$\widetilde{F}$$
 : $\mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$

making the diagram (*) commute induces a distributivity law $\lambda: T \circ F \Rightarrow F \circ S$ making the two diagrams (a) and (b) commute.

§5. Conclude that a homomorphism $(F,\lambda):(\mathscr{A},S)\to(\mathscr{B},T)$ between two monads may be equivalently defined as a pair (F,\widetilde{F}) of functors

$$F: \mathscr{A} \longrightarrow \mathscr{B} \hspace{1cm} \widetilde{F}: \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram (*) commute.

- §6. Deduce that there is a category Mon of monads and homomorphisms between them.
- §7. Describe the free abelian group functor $F : \mathbf{Sets} \to \mathbf{Sets}$ which transports every set A to the free abelian group FA generated by the set A.
- §8. Consider the free monoid monad $T : \mathbf{Sets} \to \mathbf{Sets}$ and construct a family of functions

$$\lambda_A : TF(A) \longrightarrow FT(A)$$

parametrized by an object $A \in \mathscr{A}$ and check that the family λ is natural in A and makes the diagrams (a) and (b) commute.

§9. From this, deduce the existence of a functor

$$\widetilde{F}$$
 : Monoid \longrightarrow Monoid

from the category of monoids and homomorphisms, making the diagram below commute:

$$egin{aligned} \mathbf{Monoid} & \longrightarrow \widetilde{F} & \mathbf{Monoid} \ U & (*) & \downarrow U \ \mathbf{Sets} & \longrightarrow & \mathbf{Sets} \end{aligned}$$

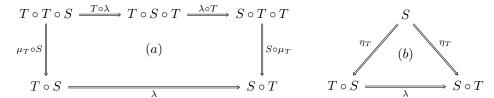
- §10. Describe the natural transformations μ_F and η_F equipping the functor F with a monad structure (F, μ_F, η_F) .
- §11. A distributivity law

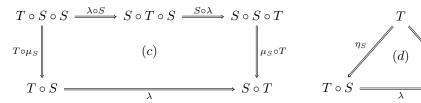
$$\lambda : T \circ S \Rightarrow S \circ T$$

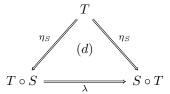
between two monads on the same category

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A}$$
 $(T, \mu_T, \eta_T) : \mathscr{A} \longrightarrow \mathscr{A}$

is a natural transformation making the diagrams below commute







Depict the commutative diagrams (c) and (d) in the language of string diagrams.

- §12. Show that every distributive law $\lambda: T \circ S \Rightarrow S \circ T$ between two monads S and T on the same category $\mathscr A$ induces a monad structure on the composite functor $S \circ T : \mathscr A \to \mathscr A$.
- §13. Show that the natural transformation λ defined in §7. defines a distributivity law between the monads S = F and T.
- §14. Show that the monad $S \circ T : \mathbf{Sets} \to \mathbf{Sets}$ associated to the distributivity law $\lambda :$ $T \circ S \Rightarrow S \circ T$ coincides with the free algebra monad (here, by algebra, we mean \mathbb{Z} -algebra).

Grothendieck construction and colimits computed in the category of sets and functions

We recall that a contravariant presheaf on a small category \mathscr{C} is a functor

$$\varphi : \mathscr{C}^{op} \longrightarrow \mathbf{Sets}.$$

The purpose of this exercise is to compute the colimit of this functor, seen as a diagram in the category Sets. Every contravariant presheaf φ induces a category $\operatorname{Groth}[\varphi]$ together with a projection functor

$$\pi[\varphi] : \operatorname{Groth}[\varphi] \longrightarrow \mathscr{C}.$$
 (2)

The objects of the category are the pairs (c, x) with c an object of $\mathscr C$ and x an element of $\varphi(x)$; the maps

$$(c,x) \longrightarrow (d,y)$$

of the category are maps $f: c \to d$ of the underlying category $\mathscr C$ such that

$$\varphi(f)(y) = x.$$

- §1. Show that these data define a category $Groth[\varphi]$ together with a functor (2).
- §2. Show that every natural transformation

$$\theta : \varphi \Rightarrow \varphi : \mathscr{C}^{op} \longrightarrow \mathbf{Sets}$$

induces a functor

$$\operatorname{\mathbf{Groth}}[\theta] : \operatorname{\mathbf{Groth}}[\varphi] \longrightarrow \operatorname{\mathbf{Groth}}[\psi]$$

making the diagram below commute

$$\mathbf{Groth}[\varphi] \xrightarrow{\mathbf{Groth}[\theta]} \mathbf{Groth}[\psi]$$

$$\pi[\varphi] \qquad \qquad \pi[\psi]$$

§3. Conversely, show that every functor

$$F : \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\mathbf{Groth}[\varphi] \xrightarrow{F} \mathbf{Groth}[\psi]$$

is of the form $F = \mathbf{Groth}[\theta]$ for a unique natural transformation

$$\theta : \varphi \Rightarrow \psi : \mathscr{C}^{op} \longrightarrow \mathbf{Sets}$$

§4. For every object c of the category \mathscr{C} , construct a function

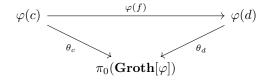
$$\theta_c : \varphi(c) \longrightarrow \pi_0(\mathbf{Groth}[\varphi])$$

where the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

denotes the set of connected components of the category $Groth[\varphi]$, defined as the connected components of the underlying graph.

§5. Show that the diagram below commutes



for every map $f:c\to d$ in the category $\mathscr C.$ Deduce from this that the family θ defines a natural transformation

$$\theta : \varphi \Rightarrow \pi_0(\mathbf{Groth}[\varphi])$$

and thus a cone.

§5. Show that the cone is a colimiting cone, and thus that the colimit of the diagram

$$\varphi : \mathscr{C}^{op} \longrightarrow \mathbf{Sets}$$

coincides with the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

of connected components of the Grothendieck category $\mathbf{Groth}[\varphi]$. From this, deduce that the category Sets has all small colimits.