Travaux Dirigés
Lambda-calculs et Catégories

Distributivity laws between functors and monads
Grothendieck construction and set-theoretic colimits

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1 Distributivity laws between functors and monads

§1. Suppose given two categories $\mathscr{A}$ and $\mathscr{B}$, each of them equipped with a monad

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad (T, \mu_T, \eta_T) : \mathscr{B} \longrightarrow \mathscr{B}$$

A homomorphism

$$(F, \lambda) \quad : \quad (\mathscr{A}, S) \longrightarrow (\mathscr{B}, T) \tag{1}$$

is defined as a functor $F : \mathscr{A} \to \mathscr{B}$ equipped with distributivity law

$$\lambda \quad : \quad T \circ F \Rightarrow F \circ S$$

making the diagrams of natural transformations below commute:

$$\begin{tikzcd}
T \circ T \circ F \arrow[r, Rightarrow, "T\circ\lambda"] \arrow[d, Rightarrow, "\mu_T \circ F"'] & T \circ F \circ S \arrow[r, Rightarrow, "\lambda \circ S"] \arrow[d, phantom, "(a)"] & F \circ S \circ S \arrow[d, Rightarrow, "F \circ \mu_S"] \\
T \circ F \arrow[rr, Rightarrow, "T\circ\lambda"'] & & F \circ S
\end{tikzcd}
\qquad
\begin{tikzcd}
& F \arrow[dl, Rightarrow, "\eta_T"'] \arrow[dr, Rightarrow, "\eta_S"] \arrow[d, phantom, "(b)"] & \\
T \circ F \arrow[rr, Rightarrow, "\lambda"'] & {} & F \circ S
\end{tikzcd}$$

§1. Formulate the two commutative diagrams (a) and (b) as families of commutative diagrams between maps living in the category $\mathscr{B}$.

§2. Depict the commutative diagrams (a) and (b) in the language of string diagrams.

§3. Show that every homomorphism (F, λ) as in (1) induces a functor

$$\widetilde{F} \quad : \quad \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram below commute:

$$\begin{tikzcd}
\mathbf{Alg}(S) \arrow[r, "\widetilde{F}"] \arrow[d, "U_S"'] \arrow[dr, phantom, "(*)"] & \mathbf{Alg}(T) \arrow[d, "U_T"] \\
\mathscr{A} \arrow[r, "F"'] & \mathscr{B}
\end{tikzcd}$$

where U_S and U_T are the forgetful functors associated to the monads S and T, respectively.

§4. Conversely, show that every functor

$$\widetilde{F} \quad : \quad \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram $(*)$ commute induces a distributivity law $\lambda : T \circ F \Rightarrow F \circ S$ making the two diagrams (a) and (b) commute.

§5. Conclude that a homomorphism $(F, \lambda) : (\mathscr{A}, S) \to (\mathscr{B}, T)$ between two monads may be equivalently defined as a pair $(F, \widetilde{F})$ of functors

$$F : \mathscr{A} \longrightarrow \mathscr{B} \qquad\qquad \widetilde{F} : \mathbf{Alg}(S) \longrightarrow \mathbf{Alg}(T)$$

making the diagram $(*)$ commute.

§6. Deduce that there is a category **Mon** of monads and homomorphisms between them.

§7. Describe the free abelian group functor $F : \mathbf{Sets} \to \mathbf{Sets}$ which transports every set A to the free abelian group FA generated by the set A.

§8. Consider the free monoid monad $T : \mathbf{Sets} \to \mathbf{Sets}$ and construct a family of functions

$$\lambda_A \quad : \quad TF(A) \longrightarrow FT(A)$$

parametrized by an object $A \in \mathscr{A}$ and check that the family λ is natural in A and makes the diagrams (a) and (b) commute.

§9. From this, deduce the existence of a functor

$$\widetilde{F} \quad : \quad \mathbf{Monoid} \longrightarrow \mathbf{Monoid}$$

from the category of monoids and homomorphisms, making the diagram below commute:

$$\begin{tikzcd}
\mathbf{Monoid} \arrow[r, "\widetilde{F}"] \arrow[d, "U"'] \arrow[dr, phantom, "(*)"] & \mathbf{Monoid} \arrow[d, "U"] \\
\mathbf{Sets} \arrow[r, "F"'] & \mathbf{Sets}
\end{tikzcd}$$

§10. Describe the natural transformations μ_F and η_F equipping the functor F with a monad structure (F, μ_F, η_F).

§11. A distributivity law

$$\lambda \quad : \quad T \circ S \Rightarrow S \circ T$$

between two monads on the same category

$$(S, \mu_S, \eta_S) : \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad (T, \mu_T, \eta_T) : \mathscr{A} \longrightarrow \mathscr{A}$$

is a natural transformation making the diagrams below commute

$$\begin{tikzcd}
T \circ T \circ S \arrow[r, Rightarrow, "T\circ\lambda"] \arrow[d, Rightarrow, "\mu_T \circ S"'] & T \circ S \circ T \arrow[r, Rightarrow, "\lambda\circ T"] \arrow[d, phantom, "(a)"] & S \circ T \circ T \arrow[d, Rightarrow, "S\circ\mu_T"] \\
T \circ S \arrow[rr, Rightarrow, "\lambda"'] & & S \circ T
\end{tikzcd}
\qquad
\begin{tikzcd}
& S \arrow[dl, Rightarrow, "\eta_T"'] \arrow[dr, Rightarrow, "\eta_T"] \arrow[d, phantom, "(b)"] & \\
T \circ S \arrow[rr, Rightarrow, "\lambda"'] & {} & S \circ T
\end{tikzcd}$$

$$\begin{tikzcd}
T \circ S \circ S \arrow[r, Rightarrow, "\lambda \circ S"] \arrow[d, Rightarrow, "T \circ \mu_S"'] & S \circ T \circ S \arrow[r, Rightarrow, "S \circ \lambda"] \arrow[d, phantom, "(c)"] & S \circ S \circ T \arrow[d, Rightarrow, "\mu_S \circ T"] \\
T \circ S \arrow[rr, Rightarrow, "\lambda"'] & & S \circ T
\end{tikzcd}
\qquad
\begin{tikzcd}
& T \arrow[dl, Rightarrow, "\eta_S"'] \arrow[dr, Rightarrow, "\eta_S"] \arrow[d, phantom, "(d)"] & \\
T \circ S \arrow[rr, Rightarrow, "\lambda"'] & & S \circ T
\end{tikzcd}$$

Depict the commutative diagrams (c) and (d) in the language of string diagrams.

§12. Show that every distributive law $\lambda : T \circ S \Rightarrow S \circ T$ between two monads S and T on the same category $\mathscr{A}$ induces a monad structure on the composite functor $S \circ T : \mathscr{A} \to \mathscr{A}$.

§13. Show that the natural transformation λ defined in §7. defines a distributivity law between the monads $S = F$ and T.

§14. Show that the monad $S \circ T : \mathbf{Sets} \to \mathbf{Sets}$ associated to the distributivity law $\lambda : T \circ S \Rightarrow S \circ T$ coincides with the free algebra monad (here, by algebra, we mean $\mathbb{Z}$-algebra).

2 Grothendieck construction and colimits computed in the category of sets and functions

We recall that a contravariant presheaf on a small category $\mathscr{C}$ is a functor

$$\varphi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}.$$

The purpose of this exercise is to compute the colimit of this functor, seen as a diagram in the category Sets. Every contravariant presheaf φ induces a category $\mathbf{Groth}[\varphi]$ together with a projection functor

$$\pi[\varphi] \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathscr{C}. \tag{2}$$

The objects of the category are the pairs (c, x) with c an object of $\mathscr{C}$ and x an element of $\varphi(x)$; the maps

$$(c, x) \longrightarrow (d, y)$$

of the category are maps $f : c \to d$ of the underlying category $\mathscr{C}$ such that

$$\varphi(f)(y) = x.$$

§1. Show that these data define a category $\mathbf{Groth}[\varphi]$ together with a functor (2).

§2. Show that every natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \phi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}$$

induces a functor

$$\mathbf{Groth}[\theta] \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\begin{tikzcd}
\mathbf{Groth}[\varphi] \arrow[rr, "\mathbf{Groth}[\theta]"] \arrow[dr, "{\pi[\varphi]}"'] & & \mathbf{Groth}[\psi] \arrow[dl, "{\pi[\psi]}"] \\
& \mathscr{C} &
\end{tikzcd}$$

§3. Conversely, show that every functor

$$F \quad : \quad \mathbf{Groth}[\varphi] \longrightarrow \mathbf{Groth}[\psi]$$

making the diagram below commute

$$\begin{tikzcd} \mathbf{Groth}[\varphi] \arrow[rr, "F"] \arrow[dr, "{\pi[\varphi]}"'] & & \mathbf{Groth}[\psi] \arrow[dl, "{\pi[\psi]}"] \\ & \mathscr{C} & \end{tikzcd}$$

is of the form $F = \mathbf{Groth}[\theta]$ for a unique natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \psi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}$$

§4. For every object c of the category $\mathscr{C}$, construct a function

$$\theta_c \quad : \quad \varphi(c) \longrightarrow \pi_0(\mathbf{Groth}[\varphi])$$

where the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

denotes the set of connected components of the category $\mathbf{Groth}[\varphi]$, defined as the connected components of the underlying graph.

§5. Show that the diagram below commutes

$$\begin{tikzcd} \varphi(c) \arrow[rr, "\varphi(f)"] \arrow[dr, "\theta_c"'] & & \varphi(d) \arrow[dl, "\theta_d"] \\ & \pi_0(\mathbf{Groth}[\varphi]) & \end{tikzcd}$$

for every map $f : c \to d$ in the category $\mathscr{C}$. Deduce from this that the family θ defines a natural transformation

$$\theta \quad : \quad \varphi \Rightarrow \pi_0(\mathbf{Groth}[\varphi])$$

and thus a cone.

§5. Show that the cone is a colimiting cone, and thus that the colimit of the diagram

$$\varphi \quad : \quad \mathscr{C}^{\,op} \longrightarrow \mathbf{Sets}$$

coincides with the set

$$\pi_0(\mathbf{Groth}[\varphi])$$

of connected components of the Grothendieck category $\mathbf{Groth}[\varphi]$. From this, deduce that the category $\mathbf{Sets}$ has all small colimits.