Categorical models of linear logic revisited

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Abstract

In this survey article, we review the existing categorical axiomatizations of linear logic, with a special emphasis on Seely and Lafont presentations. In a first part, we explain why Benton, Bierman, de Paiva and Hyland had to replace Seely categories with a more complicated axiomatization, and how a while later, Benton managed to simplify this axiomatization. In a second part, we show how Lafont axiomatization may be relaxed, in order to admit more exponential interpretations than just the free comonoidal one. Finally, we illustrate with a few examples what categorical models can teach us about linear logic and its models.

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1 Introduction

Seely's axiomatization and related categorical models

A year after Girard published his seminal work on linear logic [20] two alternative definitions of categorical model of intuitionistic linear logic (ILL) were already formulated. The first axiomatization by Lafont [27] was simple and elegant, based on a free construction of the exponentials. Unfortunately, it did not encompass key models, in particular Girard's coherence space model [20] where the exponential modality ! is interpreted in the set-theoretic fashion, see section 7.1.
The second axiomatization by Seely [39] was a bit more complicated, but since it captures all existing models of intuitionistic linear logic, most authors chose to promote this definition [7,29,1,4]. In Seely's axiomatization, linear logic is explicitly reduced to a decomposition of intuitionistic logic. To quote Seely in [39]: "what is really wanted [of a model of intuitionistic linear logic] is that the kleisli category associated to [the comonad] $(!, \delta, \epsilon)$ be cartesian closed, so the question is: what is the minimal condition on $(!, \delta, \epsilon)$ that guarantees this — ie. can we axiomatize this condition satisfactorily?"
A few years later, four authors: Benton, Bierman, de Paiva and Hyland [9,23] reconsidered Seely's axiomatization from the point of view of linear logic and its cut-elimination procedure, instead of intuitionistic logic. Surprisingly, they discovered that something was missing in Seely's picture. In [13,14], Bierman points out that Seely's axiomatization is *not* sound. This means that two ILL proofs π and π' equivalent by cut-elimination may be interpreted as *different* morphisms $[\pi]$ and $[\pi']$ in a Seely category, that is a category verifying Seely's axiomatization. In fact, key commutative diagrams are missing in the axiomatization, crucially the diagram below which interprets the duplication of an exponential box $!g \circ \delta_A : !A \longrightarrow !B$ inside a proof-net $h \circ \underline{d}_B \circ !g \circ \delta_A \circ f : \Gamma \longrightarrow !C$.

$$\begin{array}{ccccccc} \Gamma & \xrightarrow{f} & !A & \xrightarrow{\delta_A} & !!A & \xrightarrow{!g} & !B \\ & & \downarrow{\scriptstyle \underline{d}_A} & & & & \downarrow{\scriptstyle \underline{d}_B} \\ & & !A \otimes !A & \xrightarrow{\delta_A \otimes \delta_A} & !!A \otimes !!A & \xrightarrow{!g \otimes !g} & !B \otimes !B \xrightarrow{h} C \end{array} \qquad (1)$$

At this point, it is worth recalling Seely's definition, as Bierman formulates it in [13,14]:

Definition 1 (Seely) *A* **Seely category** $\mathcal{C}$ *consists of*

(1) a symmetric monoidal closed category with finite products $(\mathcal{C}, \otimes, 1, \&, \top)$ *together with a comonad* $(!, \delta, \epsilon)$*,*

(2) for each object A *of* $\mathcal{C}$*, a comonoid* $(!A, \underline{d}_A, \underline{e}_A)$ *with respect to the tensor product,*

(3) two natural isomorphisms $\underline{n} : !A \otimes !B \cong !(A \& B)$ *and* $\underline{p} : 1 \cong !\top$*,*

(4) the functor ! *takes the comonoid structure of the cartesian product to the comonoid structure of the tensor product.*

The main point of Seely's axiomatization is to ensure that the kleisli category $\mathcal{C}_!$ derived from the comonad $(!, \delta, \epsilon)$ is cartesian closed. As a cartesian category, the category $\mathcal{C}_!$ is symmetric monoidal. Now, consider the canonical adjunction $U \dashv F$ relating the symmetric monoidal category $\mathcal{C}$ to its kleisli category $\mathcal{C}_!$.

$$\mathcal{C}_! \underset{F}{\overset{U}{\rightleftarrows}} \mathcal{C} \qquad (2)$$

Bierman shows that a Seely category is *sound* when the canonical adjunction $U \dashv F$ is *symmetric monoidal*. This motivates the definition.

Definition 2 (Bierman) *A Seely category is called a* **new-Seely category** *when its canonical adjunction (2) is symmetric monoidal.*

Soundness of new-Seely categories is proved by relating them to *linear categories*, another axiomatization of intuitionistic linear logic introduced by Benton, Bierman, de Paiva and Hyland in [9], see also [14,23].

Definition 3 (Benton,Bierman,de Paiva,Hyland) *A* **linear category** $\mathcal{C}$ *consists of*

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1)$ *together with:*

(2) a symmetric monoidal comonad $(!, \delta, \epsilon, \underline{m}_{A,B}, \underline{m}_1)$*, such that*

a. for every free !-coalgebra $(!A, \delta_A)$ *there are two distinguished monoidal natural transformations with components* $\underline{e}_A : !A \longrightarrow 1$ *and* $\underline{d}_A : !A \longrightarrow !A \otimes !A$ *which form a commutative comonoid and are coalgebra morphisms,*

b. whenever $f : (!A, \delta_A) \longrightarrow (!B, \delta_B)$ *is a coalgebra morphism between free coalgebras, then it is also a comonoid morphism.*

Soundness of linear categories, which is the cornerstone of the theory, is proved in [9,13].

Lemma 4 (Benton,Bierman,de Paiva,Hyland) *Every linear category is a sound model of ILL.*

For instance, one shows that diagram (1) commutes in every linear category, by introducing the morphism $\delta_A : !A \longrightarrow !!A$ in the diagram, and using the fact that, as coalgebra morphisms between free coalgebras, the two morphisms $\delta_A : !A \longrightarrow !!A$ and $!g : !!A \longrightarrow !B$ are *comonoidal.*

$$\begin{array}{ccccccccc}
\Gamma & \xrightarrow{f} & !A & \xrightarrow{\delta_A} & !!A & \xrightarrow{!g} & !B & & \\
 & & \downarrow{\scriptstyle \underline{d}_A} & & \downarrow{\scriptstyle \underline{d}_{!A}} & & \downarrow{\scriptstyle \underline{d}_B} & & \\
 & & !A \otimes !A & \xrightarrow{\delta_A \otimes \delta_A} & !!A \otimes !!A & \xrightarrow{!g \otimes !g} & !B \otimes !B & \xrightarrow{h} & C
\end{array}$$

Soundness of new-Seely categories follows from their characterization as linear categories with finite products.

Lemma 5 (Bierman) *Every new-Seely category is a linear category, and every linear category with finite products is a new-Seely category.*

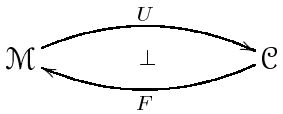
Linear, and new-Seely categories are sophisticated but *defensive* axiomatizations of linear logic. They protect us from unsound models, but their formulation is complex and can be hard to establish in practice. Very fortunately, inspired by discussions with Martin Hyland and Gordon Plotkin, Nick Benton [10] was able to simplify a lot the axiomatization of linear categories. Here, we follow Barber and Plotkin [5,6] and do not mention the cartesian *closedness* condition found in Benton's original paper [10].

Definition 6 (Benton) *A* **Linear-Non-Linear (LNL) category** *consists of:*

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1)$,

(2) a category $(\mathcal{M}, \times, e)$ *with finite products,*

(3) an adjunction $U \dashv F$ *between functors* $U : \mathcal{M} \longrightarrow \mathcal{C}$ *and* $F : \mathcal{C} \longrightarrow \mathcal{M}$*:*



(4) isomorphisms $m_{A,B} : U(A \times B) \longrightarrow U(A) \otimes U(B)$ *and* $m_e : 1 \longrightarrow U(e)$ *making* $(U, m_{A,B}, m_e)$ *a strong symmetric monoidal functor from* $(\mathcal{M}, \times, e)$ *to* $(\mathcal{C}, \otimes, 1)$.

Remark. The definition of linear-non-linear categories articulates a wonderfully simple axiomatics of linear logic, which deserves to be better known. This is one reason for writing this survey. The author must confess that, being unaware of Benton's definition, he had to rediscover it for himself (as well as Kelly's characterization of monoidal adjunctions) before tracing the axiomatization back to [10].

Remark. Kelly characterization theorem (recalled in section 3.2) states that an adjunction $U \dashv F$ between monoidal categories is monoidal iff the left adjoint functor U is strongly monoidal. Thus, linear-non-linear categories may be reformulated as a *monoidal* adjunction between a cartesian category $\mathcal{M}$, and a symmetric monoidal closed category $\mathcal{C}$.

In particular, every new-Seely category is a linear-non-linear category — because the kleisli category $\mathcal{C}_!$ in (2) has finite products, inherited from $\mathcal{C}$. Conversely, every linear-non-linear category in which the category $\mathcal{C}$ has finite products, induces a new-Seely category. This is an instance of Street's 2-categorical description of monads [41] which shows that the monoidal adjunction between $\mathcal{C}$ and $\mathcal{M}$ induces a monoidal adjunction between $\mathcal{C}$ and its kleisli category $\mathcal{C}_!$.

In section 3 we prove in all details that linear and linear-non-linear categories are equivalent axiomatizations of intuitionistic linear logic.

Lafont's axiomatization

To summarize, Seely's early axiomatization, because it is unsound, induced a series of later axiomatizations inspired by the theory of monoidal categories (recalled briefly in section 2.)

(1) new-Seely categories [which correct Seely categories]
(2) linear categories [which are sound]
(3) linear-non-linear categories [a simple account of linear categories].

Unfortunately, these axiomatics are often difficult to check, especially in the case of game-theoretic models. New-Seely and linear categories require to construct a comonad on the monoidal category $\mathcal{C}$. And linear-non-linear categories, which look simpler, require to introduce the cartesian category $\mathcal{M}$ at the same time as the monoidal category $\mathcal{C}$. The two methods are often counter-intuitive (and a risky business: see section 7.4 for an illustration.)

For that reason, we will study another kind of axiomatics, in which the coherence diagrams of monoidality are replaced by a simple *universality* principle making everything work technically: soundness, etc... This principle was formulated for the first time by Lafont in his PhD thesis [27].

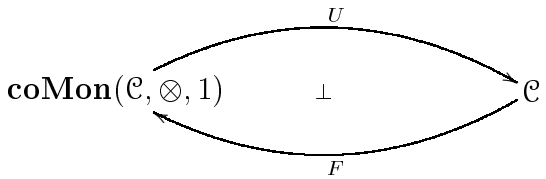
Instead of starting from the comonad $(!, \delta, \epsilon)$ like Seely does, Lafont starts from the category $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ of commutative comonoids over $(\mathcal{C}, \otimes, 1)$ and the forgetful functor $U : \mathbf{coMon}(\mathcal{C}, \otimes, 1) \longrightarrow \mathcal{C}$. This approach is closer to the original spirit of linear logic, which is more concerned with linear vs. comonoidal formulas, than with monoidal functors and monoidal adjunctions.

By construction, the category $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ has finite products (given by the $\otimes$-product and unit 1) and the functor U is strictly monoidal. Therefore, according to definition 6, there only remains to check whether U is left-adjoint to a functor $F : \mathcal{C} \longrightarrow \mathbf{coMon}(\mathcal{C}, \otimes, 1)$. This leads to the definition below.

Definition 7 (Lafont) *A* **Lafont category** *consists of*

(1) a symmetric monoidal closed category with finite products $(\mathcal{C}, \otimes, 1, \&, \top)$,
(2) for each object A *of* $\mathcal{C}$*, the object* $!A$ *is the free commutative comonoid generated by* A.

Point 2. means that the forgetful functor $U : \mathbf{coMon}(\mathcal{C}, \otimes, 1) \longrightarrow \mathcal{C}$ has a right adjoint F, and that $! = UF$ is the comonad over $\mathcal{C}$ of this adjunction.



Bierman proves in his PhD thesis (oddly enough, the result does not appear in his TLCA paper) that

Lemma 8 (Bierman) *Every Lafont category is a linear category (with finite products.)*

Remark. Of course, lemma 8 is obvious when linear categories are reformulated as linear-non-linear categories. The point is that Bierman *did not know* this reformulation at the time.

It follows from lemmas 4 and 8 that Lafont categories are *sound* models of ILL. Unfortunately, Lafont axiomatics compels to interpret the exponential modality as the *free* comonoidal construction. This means that the axiomatization rejects key models, like Girard's original coherence space model [20], see also [27,42,21], and section 7.2; as well as many game models, in which several exponentials generally coexist, each of them expressing a particular memory management or uniformity paradigm — and at most one of them free, see [15,23] and the discussion in [33].
For that reason, it is worth relaxing the definition of Lafont categories, to overcome these unnecessary technical limitations — what we do in section 5. Because the resulting axiomatization is still a bit complicated, we carry on and produce in section 6 a new axiomatization cross-breeding Lafont and Seely formulations. The axioms we obtain (definition 26) are particularly simple to check on most models of linear logic — and especially on game models.

Remark. Linear categories, linear-non-linear categories, and new-Lafont categories are models of ILL without products, presented in figure 1. as a sequent calculus. On the other hand, new-Seely, Lafont and Seely-Lafont categories are models of ILL with products, whose sequent calculus admits the rules of figure 2 in addition.
In a model of ILL with products, the kleisli category associated to the comonad is cartesian closed. In a model of ILL without product, one does not deduce a cartesian closed category, but only a cartesian category $(\mathcal{M}, \times, e)$ equipped with an *exponential ideal* [2], formulated here as: (1) a category $\mathcal{C}$ and a functor $F : \mathcal{C} \longrightarrow \mathcal{M}$, (2) a functor $\Rightarrow : \mathcal{M}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ (3) and a natural bijection:

$$\mathcal{M}(A \times B, F(C)) \;\cong\; \mathcal{M}(A, F(B \Rightarrow C)) \tag{3}$$

This is enough to interpret the simply-typed λ-calculus (without products). In a linear category, the base types are the objects of $\mathcal{C}$; the functor $\Rightarrow$ is defined as $A \Rightarrow B = U(A) \multimap B$; and a sequent $A_1, ..., A_n \vdash M : B$ is interpreted as a morphism $f : F(A_1) \times ... \times F(A_n) \longrightarrow F(B)$ in the category $\mathcal{M}$. It is worth noting that an exponential ideal may be obtained in $\mathcal{M}$ without requiring that the category $\mathcal{C}$ is monoidal closed. This happens in models of continuations, formulated using polarized linear logic [30]. The interested reader will look at the conclusion of the article, and at [34] for details.

Remark. In their categorical axiomatization of linear logic, most authors limit themselves to describing *intuitionistic* linear logic, instead of *full* linear logic. This is only an apparent limitation, since duality at the monoidal level — that is starting from a $*$-autonomous category $\mathcal{C}$ instead of just a symmetric monoidal closed category — ensures that a linear category is a sound model

axiom	$\dfrac{}{A \vdash A}$	cut	$\dfrac{\Delta \vdash A \qquad \Gamma, A \vdash B}{\Gamma, \Delta \vdash B}$
$\otimes$ left	$\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$	$\otimes$ right	$\dfrac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$
$\multimap$ left	$\dfrac{\Delta \vdash A \qquad \Gamma, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$	$\multimap$ right	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$
1 left	$\dfrac{\Gamma \vdash A}{\Gamma, \mathbf{1} \vdash A}$	1 right	$\dfrac{}{\vdash \mathbf{1}}$
dereliction	$\dfrac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$	promotion	$\dfrac{!\Gamma \vdash A}{!\Gamma \vdash !A}$
weakening	$\dfrac{\Gamma \vdash B}{\Gamma, !A \vdash B}$	contraction	$\dfrac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$
	exchange	$\dfrac{\Gamma, A_1, A_2, \Delta \vdash B}{\Gamma, A_2, A_1, \Delta \vdash B}$	

Fig. 1. Sequent calculus of intuitionistic linear logic (ILL)

& left-1	$\dfrac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}$	& left-2	$\dfrac{\Gamma, B \vdash C}{\Gamma, A \& B \vdash C}$
& right	$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}$	true	$\dfrac{}{\Gamma \vdash \top}$

Fig. 2. Addendum to figure 1: ILL with products

of full linear logic, see [13,14].

Synopsis. After the necessary preliminaries on monoidal categories in section 2, we reformulate linear categories as linear-non-linear categories in section 3. This leads to another formulation of new-Seely categories in section 4. Then, we relax Lafont categories in section 5, and cross-breed the axiomatization with Seely categories in section 6. Finally, we illustrate the difference of spirit between Seely and Lafont axiomatizations with coherence space models, and discuss a deficient "exponential construction" on a relational model, in section 7.

2 Preliminaries on monoidal categories and monoidal functors

We recall briefly the standard definitions of symmetric monoidal closed category, of symmetric (strong) monoidal functor, of monoidal natural transformation, of commutative comonoid, of comonoidal morphism. We refer the reader to [31] for the definitions of a comonad, a coalgebra, and a coalgebraic morphism.
A monoidal category is a category $\mathcal{C}$ equipped with a functor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, an object 1 of $\mathcal{C}$, and natural isomorphisms

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \longrightarrow (A \otimes B) \otimes C \qquad \lambda_A : 1 \otimes A \longrightarrow A \qquad \rho_A : A \otimes 1 \longrightarrow A$$

making the two diagrams below commute:

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & & \xrightarrow{\quad\alpha\quad} & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow{\scriptstyle A\otimes\alpha} & & & & \downarrow{\scriptstyle \alpha} \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha\otimes D} & ((A \otimes B) \otimes C) \otimes D \end{array}$$

$$\begin{array}{ccc} A \otimes (1 \otimes B) & \xrightarrow{\alpha} & (A \otimes 1) \otimes B \\ \downarrow{\scriptstyle A\otimes\lambda} & & \downarrow{\scriptstyle \rho\otimes B} \\ A \otimes B & = & A \otimes B \end{array}$$

and such that

$$\lambda_1 = \rho_1 \ : \ 1 \otimes 1 \longrightarrow 1$$

A symmetry for a monoidal category $(\mathcal{C}, \otimes, 1)$ is a natural isomorphism

$$\gamma_{A,B} : \ A \otimes B \longrightarrow B \otimes A$$

verifying that $\gamma_{B,A} \circ \gamma_{B,A} = \mathbf{id}_{A\otimes B}$, and making the two diagrams below commute:

$$\begin{array}{ccccc} A \otimes (B \otimes C) & \xrightarrow{A\otimes\gamma_{B,C}} & A \otimes (C \otimes B) & \xrightarrow{\alpha_{A,C,B}} & (A \otimes C) \otimes B \\ \downarrow{\scriptstyle \alpha_{A,B,C}} & & & & \downarrow{\scriptstyle \gamma_{A,C}\otimes B} \\ (A \otimes B) \otimes C & \xrightarrow{\gamma_{A\otimes B,C}} & C \otimes (A \otimes B) & \xrightarrow{\alpha_{C,A,B}} & (C \otimes A) \otimes B \end{array}$$

$$\begin{array}{ccccc} A \otimes 1 & & \xrightarrow{\gamma_{A,1}} & & 1 \otimes A \\ & {\scriptstyle \rho_A} \searrow & & \swarrow {\scriptstyle \lambda_A} & \\ & & A & & \end{array}$$

A *closed structure* on a (symmetric) monoidal category is given by a bifunctor $(- \multimap -) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$ together with an isomorphism

$$\mathcal{C}(A \otimes B, C) \ \cong \ \mathcal{C}(B, A \multimap C)$$

natural in A, B and C. A *symmetric monoidal closed category* is a monoidal category equipped with a symmetry and a closed structure.

Suppose that $(\mathcal{M}, \times, e)$ and $(\mathcal{C}, \otimes, 1)$ are symmetric monoidal categories. A *symmetric monoidal functor* is a functor $U : \mathcal{M} \longrightarrow \mathcal{C}$ equipped with mediating natural transformations

$$m_{A,B} : U(A) \otimes U(B) \longrightarrow U(A \times B) \qquad m_e : 1 \longrightarrow U(e)$$

making the diagrams commute:

$$\begin{array}{ccc} UA \otimes (UB \otimes UC) & \xrightarrow{\alpha_{\mathcal{C}}} & (UA \otimes UB) \otimes UC \\ \downarrow{\scriptstyle UA \otimes m_{B,C}} & & \downarrow{\scriptstyle m_{A,B} \otimes UC} \\ UA \otimes U(B \times C) & & U(A \times B) \otimes UC \\ \downarrow{\scriptstyle m_{A,B \times C}} & & \downarrow{\scriptstyle m_{A \times B, C}} \\ U(A \times (B \times C)) & \xrightarrow{U(\alpha_{\mathcal{M}})} & U((A \times B) \times C) \end{array}$$

$$\begin{array}{ccc} UA \otimes 1 & \xrightarrow{\rho_{\mathcal{C}}} & UA \\ \downarrow{\scriptstyle UA \otimes m_e} & & \uparrow{\scriptstyle U(\rho_{\mathcal{M}})} \\ UA \otimes Ue & \xrightarrow{m_{A,e}} & U(A \times e) \end{array} \qquad \begin{array}{ccc} 1 \otimes UB & \xrightarrow{\lambda_{\mathcal{C}}} & UB \\ \downarrow{\scriptstyle m_e \otimes UB} & & \uparrow{\scriptstyle U(\lambda_{\mathcal{M}})} \\ Ue \otimes UB & \xrightarrow{m_{e,B}} & U(e \times B) \end{array}$$

$$\begin{array}{ccc} UA \otimes UB & \xrightarrow{\gamma_{\mathcal{C}}} & UB \otimes UA \\ \downarrow{\scriptstyle m_{A,B}} & & \downarrow{\scriptstyle m_{B,A}} \\ U(A \times B) & \xrightarrow{U(\gamma_{\mathcal{M}})} & U(B \times A) \end{array}$$

A symmetric monoidal functor is *strong* when m_e and every $m_{A,B}$ are isomorphisms. It is *strict* when they are identities.
A *monoidal natural transformation*

$$\theta : (U, m_{A,B}, m_e) \stackrel{\cdot}{\longrightarrow} (F, n_{A,B}, n_e) : (\mathrm{M}, \times, e) \longrightarrow (\mathcal{C}, \otimes, 1)$$

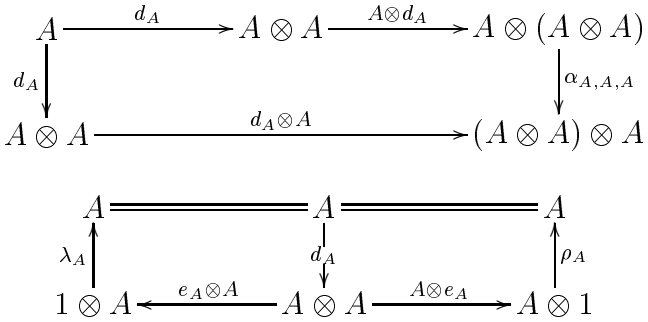
between symmetric monoidal functors is a natural transformation between the underlying functors $\theta : U \longrightarrow F$ making the two diagrams below commute:

$$\begin{array}{ccc} UA \otimes UB & \xrightarrow{m_{A,B}} & U(A \times B) \\ \downarrow{\scriptstyle \theta_A \otimes \theta_B} & & \downarrow{\scriptstyle \theta_{A \times B}} \\ FA \otimes FB & \xrightarrow{n_{A,B}} & F(A \times B) \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{m_e} & Ue \\ \| & & \downarrow{\scriptstyle \theta_e} \\ 1 & \xrightarrow{n_e} & Fe \end{array}$$

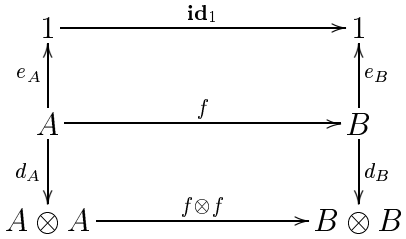
Lemma 9 *Symmetric monoidal categories, symmetric monoidal functors and monoidal natural transformations form a 2-category.*
A commutative comonoid in a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ is a triple (A, d_A, e_A) consisting of an object A and two morphisms

$$A \otimes A \xleftarrow{d_A} A \xrightarrow{e_A} 1$$

making the diagrams below commute:



Suppose that (A, d_A, e_A) and (B, d_B, e_B) are two commutative comonoids. A morphism $f : A \longrightarrow B$ is *comonoidal* when the diagrams below commute:



3 Reformulating linear categories as LNL categories (Benton)

The section is devoted to Nick Benton's reformulation as *linear-non-linear categories* [6] of Benton, Bierman, de Paiva and Hyland's *linear categories*. We establish that the two axiomations of linear logic are equivalent, in six steps:

(1) every linear category defines a linear-non-linear category (section 3.1)
(2) we recall and establish in full details Kelly's characterization of monoidal adjunctions (section 3.2)
(3) we show that strong monoidal functors preserve commutative comonoids (section 3.3)
(4) we introduce the commutative comonoid $(UFA, \underline{d}_A, \underline{e}_A)$ induced by a linear-non-linear category (section 3.4)
(5) we show that every coalgebraic morphism between free coalgebras is a comonoidal morphism (section 3.5)
(6) we conclude from steps 2,3,4 and 5 that every linear-non-linear category induces a linear category (section 3.6.)

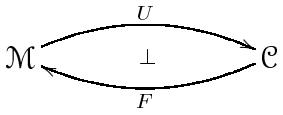
3.1 Linear-non-linear categories

We recall the definition of a linear-non-linear category.

Definition 10 (Benton) *A* **linear-non-linear category** *consists of:*

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1)$,
(2) a category $(\mathcal{M}, \times, e)$ *with finite products,*

(3) an adjunction $U \dashv F$ between functors $U : \mathcal{M} \longrightarrow \mathcal{C}$ and $F : \mathcal{C} \longrightarrow \mathcal{M}$:



(4) isomorphisms $m_{A,B} : U(A \times B) \longrightarrow U(A) \otimes U(B)$ and $m_e : 1 \longrightarrow U(e)$ making $(U, m_{A,B}, m_e)$ a strong symmetric monoidal functor from $(\mathcal{M}, \times, e)$ to $(\mathcal{C}, \otimes, 1)$.

It is not difficult to see that

Lemma 11 *Every linear category defines a linear-non-linear category, where $(\mathcal{M}, \times, e)$ is the category of coalgebras of the comonad $(!, \delta, \epsilon)$.*

Proof The category of coalgebras $(\mathcal{M}, \times, e)$ is shown to have finite products in [13], with tensor product of coalgebras as cartesian product

$$(A \xrightarrow{h_A} !A) \times (B \xrightarrow{h_B} !B) = (A \otimes B \xrightarrow{h_A \otimes h_B} !A \otimes !B \xrightarrow{m_{A,B}} !(A \otimes B))$$

and the coalgebra $m_1 : 1 \longrightarrow !1$ as terminal object. Symmetric monoidality of the "forgetful" functor

$$(U, \mathbf{id}_{A \otimes B}, \mathbf{id}_1) : (\mathcal{M}, \times, e) \longrightarrow (\mathcal{C}, \otimes, 1)$$

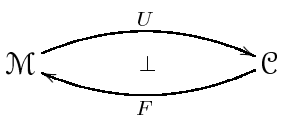
follows quite immediately. ∎

The main task of section 3 is to establish the converse property: that every linear-non-linear category induces a linear category. The property is established in section 3.6 (theorem 21) after the series of preliminary lemmas of sections 3.2 — 3.5. At this point only, linear and linear-non-linear categories will appear as equivalent notions.

3.2 From adjunctions to monoidal adjunctions (Kelly)

First, we recall a nice characterization lemma [1] by Kelly [25,24]. We establish Kelly's lemma in full details, mainly for pedagogical reasons. The diagrams we obtain should be compared with similar diagrams appearing in Bierman's PhD [13].

Lemma 12 (Kelly) *Suppose that $(\mathcal{M}, \times, e)$ and $(\mathcal{C}, \otimes, 1)$ are monoidal categories, and that $U \dashv F$ is an adjunction between the categories $\mathcal{M}$ and $\mathcal{C}$.*



[1] Kelly proves also a converse to lemma 12: an adjunction $U \dashv F$ is monoidal iff U is strong. This shows that every new-Seely category defines a linear-non-linear category with $(\mathcal{M}, \times, e) = (\mathcal{C}_!, \&, \top)$.

Suppose also that the functor $(U, m_{A,B}, m_e)$ *is strongly monoidal from* $(\mathcal{M}, \times, e)$ *to* $(\mathcal{C}, \otimes, 1)$. *Then, the adjunction* $U \dashv F$ *is monoidal.*

PROOF By universality, the adjunction $U \dashv F$ defines two morphisms in $\mathcal{M}$

$$FA \times FB \xrightarrow{p_{A,B}} F(A \otimes B) \qquad\qquad e \xrightarrow{p_1} F1$$

making the diagrams commute in $\mathcal{C}$

$$\begin{array}{ccc} U(FA \times FB) & \xrightarrow{U(p_{A,B})} & UF(A \otimes B) \\ {\scriptstyle m^{-1}_{FA,FB}}\downarrow & & \downarrow{\scriptstyle \epsilon_{A\otimes B}} \\ UFA \otimes UFB & \xrightarrow{\epsilon_A \otimes \epsilon_B} & A \otimes B \end{array} \qquad\qquad \begin{array}{ccc} Ue & \xrightarrow{U(p_1)} & UF1 \\ {\scriptstyle m_e^{-1}}\downarrow & & \downarrow{\scriptstyle \epsilon_1} \\ 1 & \xrightarrow{\mathbf{id}_1} & 1 \end{array}$$

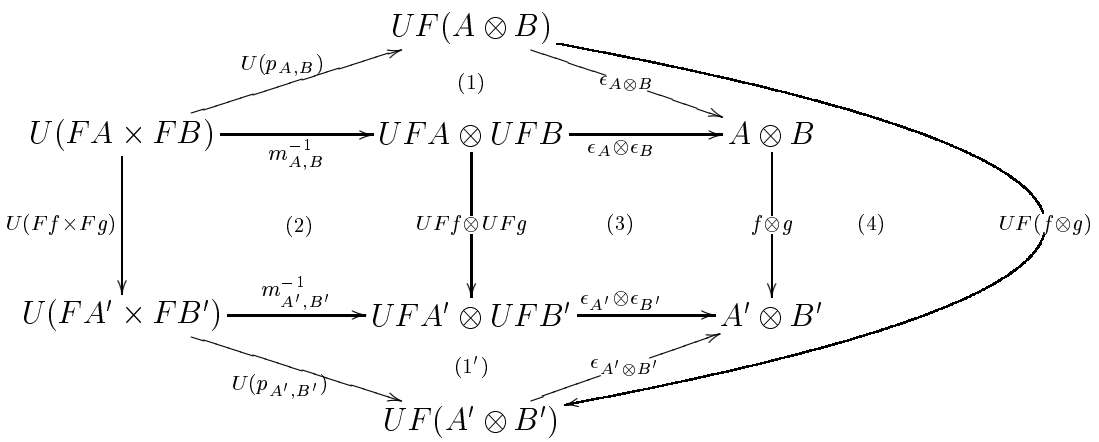
Naturality of $p_{A,B}$ in A and B, that is commutation in $\mathcal{M}$ of the diagram, for any morphisms $f : A \longrightarrow A'$ and $g : B \longrightarrow B'$ in $\mathcal{C}$:

$$\begin{array}{ccc} FA \times FB & \xrightarrow{p_{A,B}} & F(A \otimes B) \\ {\scriptstyle Ff \times Fg}\downarrow & & \downarrow{\scriptstyle F(f \otimes g)} \\ FA' \times FB' & \xrightarrow{p_{A',B'}} & F(A' \otimes B') \end{array}$$

follows from equality of in $\mathcal{C}$:

$$\epsilon_{A',B'} \circ U(F(f \otimes g) \circ p_{A,B}) \quad = \quad \epsilon_{A',B'} \circ U(p_{A',B'} \circ (Ff \times Fg))$$

established by the diagram:



(1, 1') by definition of $p_{A,B}$ and $p_{A',B'}$ (2) by naturality of m

(3) by naturality of $\epsilon \otimes \epsilon$ (4) by naturality of ϵ

This proves that p_1 and $p_{A,B}$ define natural transformations

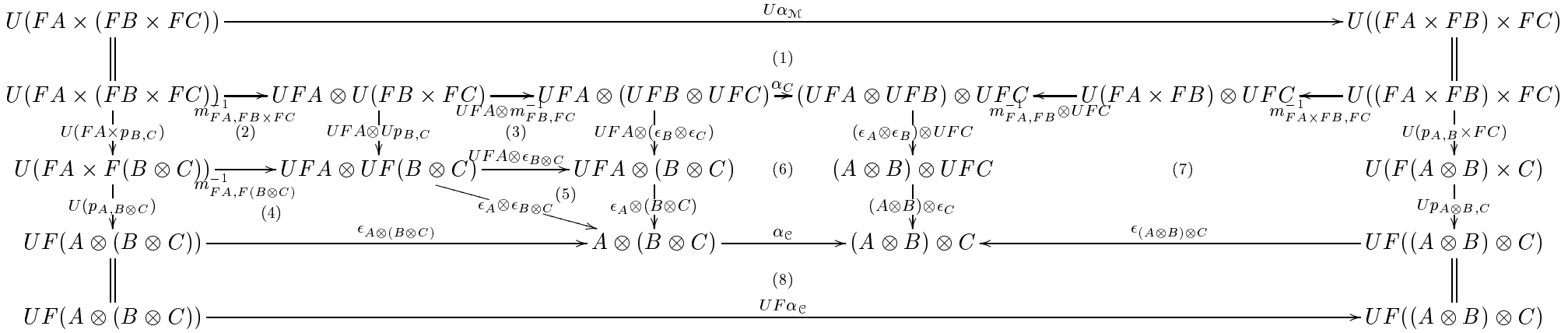
$$p_1 : e \xrightarrow{\cdot} F(1) \qquad\qquad p_{A,B} : F(A \times B) \xrightarrow{\cdot} FA \otimes FB$$

Now, we verify in three diagrams that $(F, p_{A,B}, p_1)$ is a monoidal functor.

The first commutative diagram

$$\begin{array}{ccc}
FA \times (FB \times FC) & \xrightarrow{\alpha_{\mathcal{M}}} & (FA \times FB) \times FC \\
\downarrow{\scriptstyle FA \times p_{B,C}} & & \downarrow{\scriptstyle p_{A,B} \times FC} \\
FA \times F(B \otimes C) & & F(A \otimes B) \times FC \\
\downarrow{\scriptstyle p_{FA,F(B\otimes C)}} & & \downarrow{\scriptstyle p_{F(A\otimes B),FC}} \\
F(A \otimes (B \otimes C)) & \xrightarrow{F\alpha_{\mathcal{C}}} & F((A \otimes B) \otimes C)
\end{array}$$

follows from the commutative diagrams below:



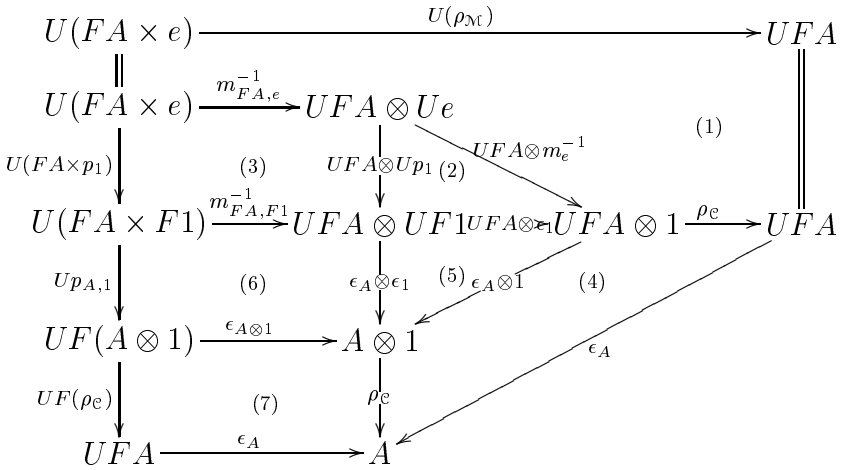
(1) monoidality of U (2) naturality of m^{-1} (3,4) definition of p

(5) bifunctoriality of $\otimes$ (6) naturality of $\alpha_{\mathcal{C}}$ (7) symmetric to (2–6) (8) naturality of ϵ

The second and third commutative diagrams in $\mathcal{M}$

$$\begin{array}{ccc} FA \times e & \xrightarrow{\rho_{\mathcal{M}}} & FA \\ {\scriptstyle FA\times p_1}\downarrow & & \uparrow{\scriptstyle F\rho_{\mathcal{C}}} \\ FA \times F1 & \xrightarrow{p_{A,1}} & F(A \otimes 1) \end{array} \qquad \begin{array}{ccc} e \times FB & \xrightarrow{\lambda_{\mathcal{M}}} & FB \\ {\scriptstyle p_1\times FB}\downarrow & & \uparrow{\scriptstyle F\lambda_{\mathcal{C}}} \\ F1 \times FB & \xrightarrow{p_{1,B}} & F(1 \otimes B) \end{array}$$

follow from the commutative diagram below (and its analogue for λ) in $\mathcal{C}$:



(1)	monoidality of $(U, m_{A,B}, m_e)$	(5)	bifonctoriality of $\otimes$
(2)	definition of p_1	(6)	definition of $p_{A,1}$
(3)	naturality of m^{-1}	(7)	naturality of ϵ
(4)	naturality of $\rho_{\mathcal{C}}$		

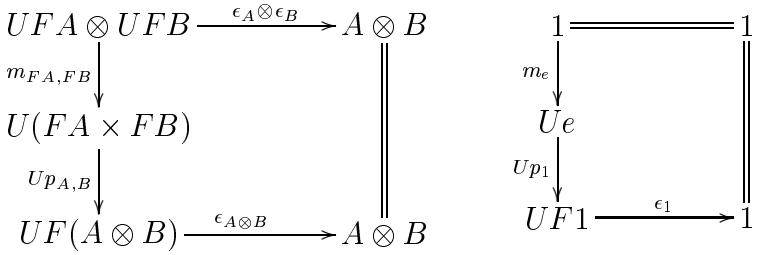
This proves that $(F, p_{A,B}, p_1)$ is a monoidal functor. There remains to prove monoidality of the two natural transformations

$$\epsilon : UF \stackrel{\cdot}{\longrightarrow} Id_{\mathcal{C}} \qquad \eta : Id_{\mathcal{M}} \stackrel{\cdot}{\longrightarrow} FU$$

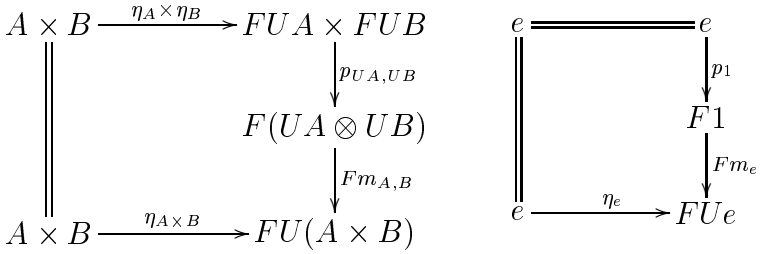
As composites of monoidal functors, the two functors $UF : \mathcal{C} \longrightarrow \mathcal{C}$ and $FU : \mathcal{M} \longrightarrow \mathcal{M}$ are monoidal, with mediating natural transformations:

$$\begin{array}{ccccc} UFA \otimes UFB & \xrightarrow{m_{FA,FB}} & U(FA \times FB) & \xrightarrow{Up_{A,B}} & UF(A \otimes B) \\ 1 & \xrightarrow{m_e} & Ue & \xrightarrow{Up_1} & UF1 \\ FUA \times FUB & \xrightarrow{p_{UA,UB}} & F(UA \otimes UB) & \xrightarrow{Fm_{A,B}} & FU(A \times B) \\ e & \xrightarrow{p_1} & F1 & \xrightarrow{Fm_e} & FUe \end{array}$$

Monoidality of $\epsilon : Id_{\mathcal{C}} \stackrel{\cdot}{\longrightarrow} UF$ translates as commutativity of the diagrams



which follows from the definition of the natural transformations $p_{A,B}$ and p_1. Monoidality of $\eta : Id_{\mathcal{M}} \stackrel{\cdot}{\longrightarrow} FU$ translates as commutativity of the diagrams:

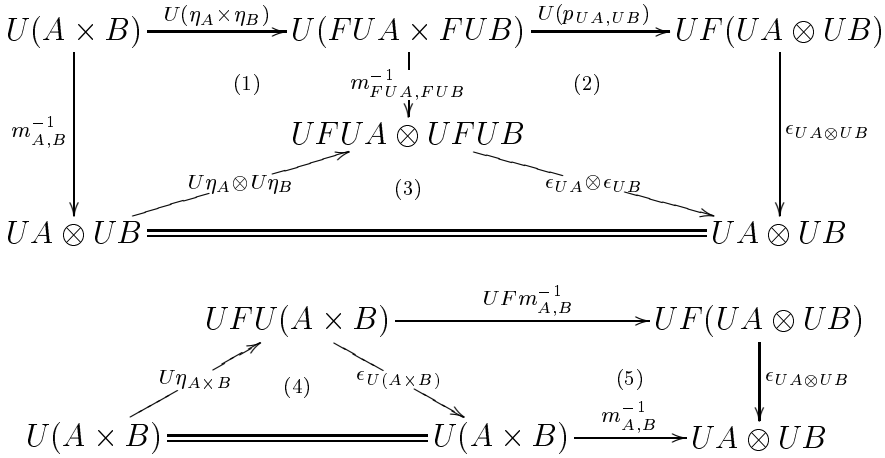


(4)

Commutativity of left-hand side follows from equality in $\mathcal{C}$

$$\epsilon_{UA\otimes UB} \circ U(p_{UA,UB} \circ (\eta_A \times \eta_B)) \;=\; \epsilon_{UA\otimes UB} \circ U(Fm^{-1}_{A,B} \circ \eta_{A\times B})$$

the two morphisms $U(A \times B) \longrightarrow UA \otimes UB$ being shown equal to $m^{-1}_{A,B}$ by the commutative diagrams below:

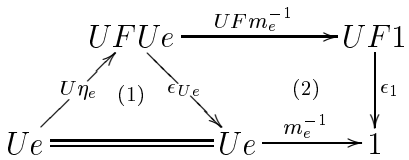


(1) naturality of m^{-1} (2) definition of $p_{UA,UB}$

(3) property of adjunction (4) property of adjunction

(5) naturality of ϵ

Similarly, commutativity of right-hand side of equation (4) follows from definition of p_1 as $\epsilon_1 \circ U(p_1) = m_e^{-1}$, and equality $\epsilon_1 \circ U(Fm_e^{-1} \circ \eta_e) = m_e^{-1}$ established in the diagram



(1) property of adjunction

(2) naturality of ϵ

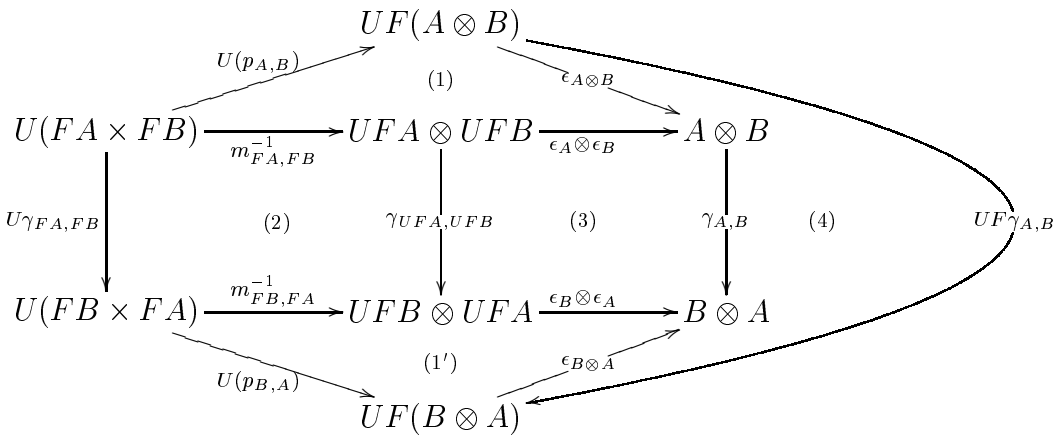
■

Corollary 13 *Suppose that in lemma 12, the strong monoidal functor* $(U, m_{A,B}, m_e)$ *is symmetric between symmetric monoidal categories* $(\mathcal{M}, \times, e)$ *and* $(\mathcal{C}, \otimes, 1)$*. Then, the adjunction* $U \dashv F$ *is symmetric monoidal.*

Proof We only have to check that the functor $(F, p_{A,B}, p_1)$ is symmetric when the functor $(U, m_{A,B}, m_e)$ is symmetric. In other words, we have to prove that the diagram below commutes in $\mathcal{M}$

$$\begin{array}{ccc} FA \times FB & \xrightarrow{p_{A,B}} & F(A \otimes B) \\ {\scriptstyle \gamma_{FA,FB}}\downarrow & & \downarrow{\scriptstyle F\gamma_{A,B}} \\ FB \times FA & \xrightarrow{p_{B,A}} & F(B \otimes A) \end{array}$$

This follows from commutation of the diagram below in $\mathcal{C}$:



(1, 1') by definition of $p_{A,B}$ and $p_{B,A}$

(2) by symmetric monoidality of $(U, m_{A,B}, m_1)$

(3) by naturality of c

(4) by naturality of ϵ

■

Corollary 14 *In every linear-non-linear category, the category* $\mathcal{C}^{FG}$ *of* UF*-coalgebras is symmetric monoidal.*

Proof This a well-known fact of symmetric monoidal adjunctions. The tensor product of two coalgebras

$$A \xrightarrow{h_A} UFA \qquad\qquad B \xrightarrow{h_B} UFB$$

is defined as the coalgebra

$$A \otimes B \xrightarrow{h_A \otimes h_B} UFA \otimes UFB \xrightarrow{m_{FA,FB}} U(FA \times FB) \xrightarrow{Up_{A,B}} UF(A \otimes B)$$

The unit coalgebra is defined as

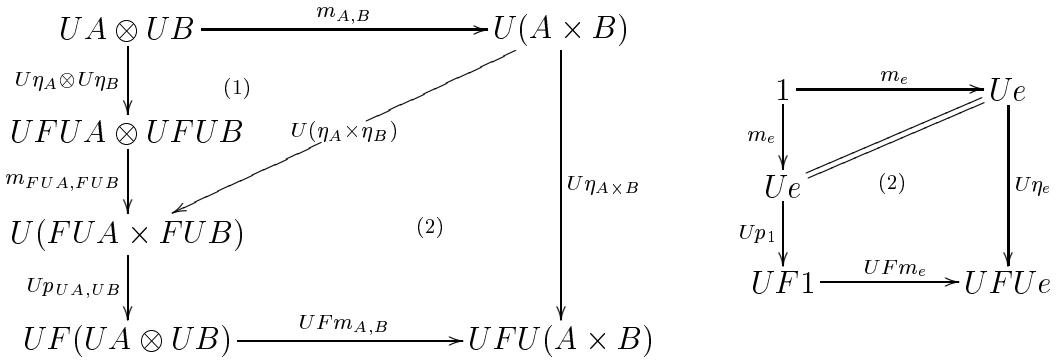
$$1 \xrightarrow{m_e} U1 \xrightarrow{Up_e} UF1$$

The category $\mathcal{C}^{FG}$ defines a symmetric monoidal category with the same structural isomorphisms (associativity, left and right neutrality, symmetry) as in $(\mathcal{C}, \otimes, 1)$. This is established by a series of elementary diagrams, that we omit here. ■

Corollary 15 (lift) *In every linear-non-linear category, the adjunction $U \dashv F$ "lifts" to an adjunction $\tilde{U} \dashv \tilde{F}$ where $(\mathcal{C}, \otimes, 1)$ is replaced by its category $(\mathcal{C}^{FG}, \otimes, 1)$ of UF-coalgebras. The functor $\tilde{U} : (\mathcal{M}, \times, e) \longrightarrow (\mathcal{C}^{FG}, \otimes, 1)$ is strong symmetric monoidal.*

Proof The adjunction $\tilde{U} \dashv \tilde{F}$ between $\tilde{U} : \mathcal{M} \longrightarrow \mathcal{C}^{FG}$ and $\tilde{F} : \mathcal{C}^{FG} \longrightarrow \mathcal{M}$ follows from basic category theory, see for instance chapters IV and VI in [31]. By construction, the functor $\tilde{U}$ transports an object A to the coalgebra $U\eta_A : UA \longrightarrow UFUA$ and a morphism $f : A \longrightarrow B$ to the coalgebraic morphism $Uf : (U\eta_A) \longrightarrow (U\eta_B)$.

Corollary 14 tells us that the category $(\mathcal{C}^{FG}, \otimes, 1)$ is symmetric monoidal, with the same structural isomorphisms (associativity, left and right neutrality, symmetry) as in $(\mathcal{C}, \otimes, 1)$. We prove now that the functor $\tilde{U}$ equipped with the mediating maps $m_{A,B} : \tilde{U}A \otimes \tilde{U}B \longrightarrow \tilde{U}(A \times B)$ and $m_e : 1 \longrightarrow \tilde{U}e$ is strong symmetric monoidal. Symmetric monoidality reduces to proving that the morphisms $m_{A,B}$ and m_e are coalgebraic: their mere existence in $\mathcal{C}^{FG}$ ensures that the coherence diagrams commuting in $(\mathcal{C}, \otimes, 1)$ commute also in $(\mathcal{C}^{FG}, \otimes, 1)$. Coalgebraicity of $m_{A,B}$ and m_e is established in the diagrams below.



(1) by naturality of $m_{A,B}$

(2) by monoidality of the natural transformation η.

We conclude by observing that the morphisms $m_{A,B}$ and m_e are isomorphisms in $\mathcal{C}^{FG}$ as well as in $\mathcal{C}$. ■

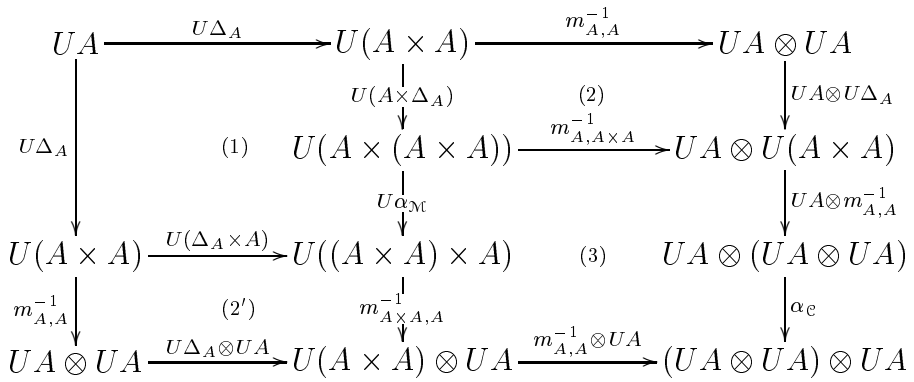
3.3 Strong monoidal functors preserve commutative comonoids

We prove that

Lemma 16 (preservation of commutative comonoids) *Consider a strong monoidal functor* $(U, m_{A,B}, m_e) : (\mathcal{M}, \times, e) \longrightarrow (\mathcal{C}, \otimes, 1)$ *between symmetric monoidal categories* $(\mathcal{M}, \times, e)$ *and* $(\mathcal{C}, \otimes, 1)$*, and a commutative comonoid* $(A, \Delta_A, \Upsilon_A)$ *in* $(\mathcal{M}, \times, e)$*. Then, the object* UA *equipped with the maps* d_A *and* e_A *below defines a commutative comonoid in* $(\mathcal{C}, \times, e)$*.*

$$\begin{aligned} d_A &= UA \xrightarrow{U\Delta_A} U(A \times A) \xrightarrow{m^{-1}_{A,A}} UA \otimes UA \\ e_A &= UA \xrightarrow{U\Upsilon_A} Ue \xrightarrow{m_e^{-1}} 1 \end{aligned} \tag{5}$$

Proof This follows from the series a diagram chasing below, one for associativity:

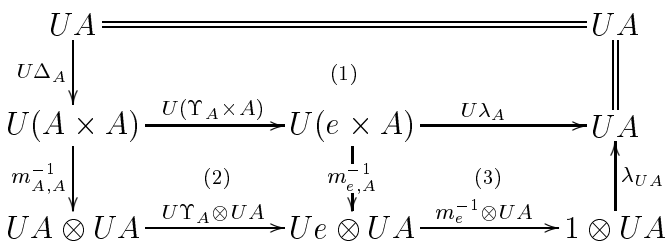


(1) comonoidality of $(A, \Delta_A, \Upsilon_A)$

(2,2') naturality of m^{-1}

(3) monoidality of $(U, m_{A,B}, m_e)$

one for left neutrality (a similar diagram is required for right neutrality):

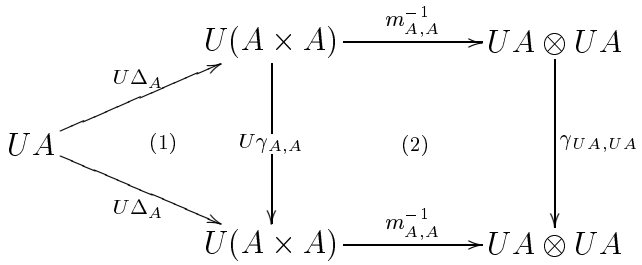


(1) comonoidality of $(A, \Delta_A, \Upsilon_A)$

(2) naturality of m^{-1}

(3) monoidality of $(U, m_{A,B}, m_e)$

and finally, one diagram for symmetry:



(1) symmetry of the comonoid $(A, \Delta_A, \Upsilon_A)$

(2) symmetry of $(U, m_{A,B}, m_e)$.

■

Remark. Lemma 16 may be strengthened to a functorial property, stating that the functor U lifts to a monoidal functor between the categories of commutative comonoids of the respective categories $\mathcal{M}$ and $\mathcal{C}$.

3.4 The two morphisms $\underline{d}_A : UFA \longrightarrow UFA \otimes UFA$ and $\underline{e}_A : UFA \longrightarrow 1$

In this subsection, we establish that the series of properties required of the morphisms $\underline{d}_A$ and $\underline{e}_A$ in the axiomatics of linear categories, holds in every linear-non-linear category. But first of all, we need to define the two morphisms.
In every linear-non-linear category, every object A of $(\mathcal{M}, \times, e)$ defines a commutative comonoid

$$A \times A \xleftarrow{\Delta_A} A \xrightarrow{\Upsilon_A} e$$

generated by the cartesian structure. In particular, every image FA in $\mathcal{M}$ of an object A in $\mathcal{C}$. For every object A of $\mathcal{C}$, the morphisms $\underline{d}_A$ and $\underline{e}_A$ are defined below:

$$\begin{aligned} \underline{d}_A &= UFA \xrightarrow{U\Delta_{FA}} U(FA \times FA) \xrightarrow{m^{-1}_{FA,FA}} UFA \otimes UFA \\ \underline{e}_A &= UFA \xrightarrow{U\Upsilon_{FA}} Ue \xrightarrow{m_e^{-1}} 1 \end{aligned} \tag{6}$$

Lemma 17 (commutative comonoid) *In every linear-non-linear category, $(UFA, \underline{d}_A, \underline{e}_A)$ defines a commutative comonoid.*
Proof Simply apply lemma 16. ■

We take the opportunity to say even a bit more.
Lemma 18 (coalgebraic) *The morphisms $\underline{d}_A$ and $\underline{e}_A$ are coalgebraic.*
Proof Apply lemma 16 on the "lifted" adjunction $\widetilde{U} \dashv \widetilde{F}$ defined in corollary 15. ■

Now, we check that

Lemma 19 (monoidal naturality) *In every linear-non-linear category, the morphisms* $\underline{d}_A : UFA \longrightarrow UFA \otimes UFA$ *and* $\underline{e}_A : UFA \longrightarrow 1$ *define monoidal natural transformations.*

PROOF Every object A of $(\mathcal{M}, \times, e)$ is equipped with the comonoidal structure

$$A \times A \xleftarrow{\Delta_A} A \xrightarrow{\Upsilon_A} e$$

induced by cartesian product. It follows from universality and an elementary diagram chasing that the two families $(\Delta_A)_{A\in\mathcal{M}}$ and $(\Upsilon_A)_{A\in\mathcal{M}}$ define *monoidal* natural transformations. Consequently, the families $(U\Delta_{FA})_{A\in\mathcal{C}}$ and $(U\Upsilon_A)_{A\in\mathcal{C}}$ define monoidal natural transformations

$$(U\Delta_{FA})_{A\in\mathcal{C}} \ : \ UFA \dot{\longrightarrow} U(FA \times FA) \qquad (U\Upsilon_{FA})_{A\in\mathcal{C}} \ : \ UFA \dot{\longrightarrow} Ue \quad (7)$$

We may prove this directly by diagram chasing, or remember that symmetric monoidal functors and monoidal natural transformations form a 2-category, and that the functor $(U, m_{A,B}, m_e)$ is symmetric monoidal.
Now, the families $(\underline{d}_A)_{A\in\mathcal{C}}$ and $(\underline{e}_A)_{A\in\mathcal{C}}$ are obtained by postcomposing (7) with the monoidal natural transformations

$$(m^{-1}_{UFA,UFA})_{A\in\mathcal{C}} \ : \ U(FA\times FA) \dot{\longrightarrow} UFA\otimes UFA \qquad (m_e^{-1})_{A\in\mathcal{C}} \ : \ Ue \dot{\longrightarrow} 1$$

Thus, the families $(\underline{d}_A)_{A\in\mathcal{C}}$ and $(\underline{e}_A)_{A\in\mathcal{C}}$ are monoidal natural transformations. We conclude. ■

3.5 Every coalgebraic morphism between free coalgebras is a comonoidal morphism

Here follows the last property required by the axiomatics of linear categories.

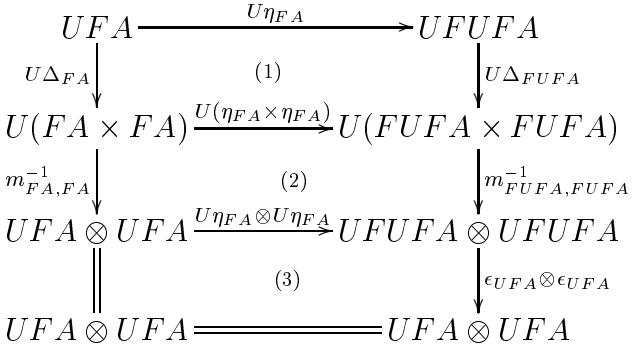
Lemma 20 *Every coalgebraic morphism between free coalgebras*

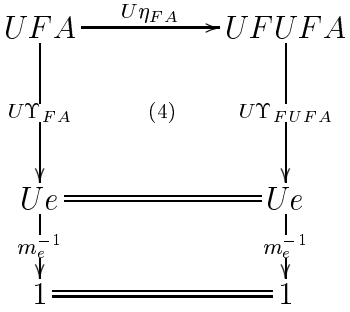
$$(UFA \xrightarrow{U\eta_{FA}} UFUFA) \quad \longrightarrow \quad (UFB \xrightarrow{U\eta_{FB}} UFUFB)$$

is a comonoidal morphism

$$(UFA \otimes UFA \xleftarrow{\underline{d}_A} UFA \xrightarrow{\underline{e}_A} 1) \quad \longrightarrow \quad (UFB \otimes UFB \xleftarrow{\underline{d}_B} UFB \xrightarrow{\underline{e}_B} 1)$$

PROOF In order to understand why the coalgebraic morphisms are also comonoidal, it is useful to reformulate the morphisms $\underline{d}_A$ and $\underline{e}_A$ as below:

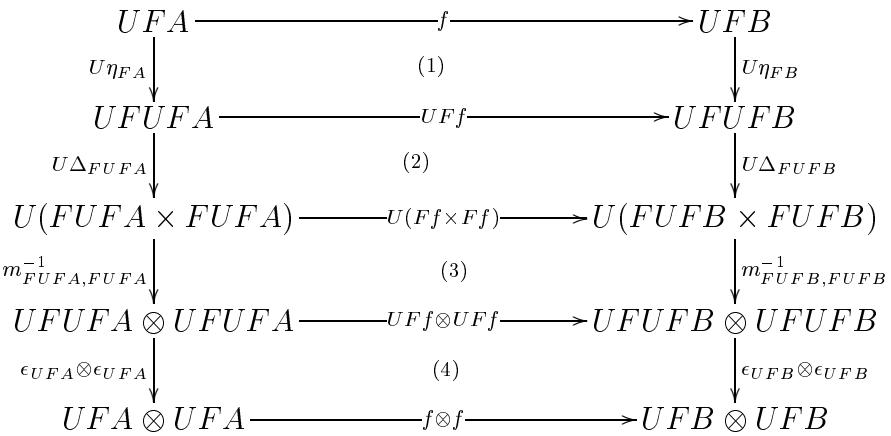




(1) naturality of Δ (2) naturality of $m_{A,B}$

(3) comonad rule of $(UF, \epsilon, U\eta_F)$ (4) naturality of Υ

Then, comonoidality of a coalgebraic morphism $f : UFA \longrightarrow UFB$ follows from the diagram below ; we omit the similar diagram proving that f and $\underline{e}_A$ commute:



(1) coalgebraicity of f (2) naturality of Δ

(3) naturality of m^{-1} (4) naturality of ϵ

■

3.6 Main result

Now, we are ready to establish theorem 21, the converse of lemma 11.

Theorem 21 (Benton) *Every linear-non-linear category is a linear category.*

PROOF Follows from the lemmas established in sections 3.2, 3.3, 3.4 and 3.5. The symmetric monoidal comonad $(!, \delta, \epsilon, \underline{m}_{A,B}, \underline{m}_1)$ is defined as the comonad

$(UF, U\eta_F, \epsilon)$ associated to the symmetric monoidal adjunction

$$(U, m_{A,B}, m_e) \dashv (F, p_{A,B}, p_1)$$

By lemma 19, the families of morphisms $\underline{d}_A : UFA \longrightarrow UFA \otimes UFA$ and $\underline{e}_A : UFA \longrightarrow 1$ defined at the beginning of section 3.4, form monoidal natural transformations. Every triple $(UFA, \underline{d}_A, \underline{e}_A)$ is a commutative comonoid in $(\mathcal{C}, \otimes, 1)$ by lemma 17, in the category $(\mathcal{C}^{FG}, \otimes, 1)$ of UF-coalgebras by lemma 18. To conclude, every coalgebraic morphism between free UF-coalgebras is comonoidal, by lemma 20. ■

Remark. Consider a linear category whose underlying category $\mathcal{C}$ has finite products, or equivalently, a new-Seely category. Then, the kleisli category associated to the comonad ! over $\mathcal{C}$, is cartesian closed. In the absence of finite products in $\mathcal{C}$, it is the full subcategory of "exponentiable" objects in the category $\mathcal{C}^{FG}$ of coalgebras, see [9,13]. By the way, Hyland showed that this category is the whole category $\mathcal{C}^{FG}$ of coalgebras when $\mathcal{C}^{FG}$ has equalizers of coreflexive pairs.

4 Reformulating new-Seely categories

Here, we mention briefly another more explicit formulation of new-Seely categories, resulting from lemma 12 and 13.

Lemma 22 *A Seely category is a new-Seely category precisely when the diagrams below commute, for every objects A, B, C of $\mathcal{C}$:*

$$\begin{array}{ccc}
!A \otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A \& B) \\
{\scriptstyle \delta_A \otimes \delta_B}\downarrow & & \downarrow{\scriptstyle \delta_{A\&B}} \\
!!A \otimes !!B & & !!(A \& B) \\
 & \xrightarrow{\underline{n}_{!A,!B}} & \downarrow{\scriptstyle !\langle !\pi_1, !\pi_2\rangle} \\
 & & !(!A \& !B)
\end{array} \qquad (8)$$

$$\begin{array}{ccc}
!A \otimes (!B \otimes !C) & \xrightarrow{\alpha} & (!A \otimes !B) \otimes !C \\
{\scriptstyle !A \otimes \underline{n}_{B,C}}\downarrow & & \downarrow{\scriptstyle \underline{n}_{A,B} \otimes !C} \\
!A \otimes !(B \& C) & & !(A \& B) \otimes !C \\
{\scriptstyle \underline{n}_{A,B\&C}}\downarrow & & \downarrow{\scriptstyle \underline{n}_{A\&B,C}} \\
!(A \& (B \& C)) & \xrightarrow{!\alpha} & !((A \& B) \& C)
\end{array}$$

$$\begin{array}{ccc}
!A \otimes 1 & \xrightarrow{\rho} & !A \\
{\scriptstyle !A \otimes \underline{p}_\top}\downarrow & & \uparrow{\scriptstyle !\rho} \\
!A \otimes !\top & \xrightarrow{\underline{n}_{B,1}} & !(A \& \top)
\end{array} \qquad
\begin{array}{ccc}
1 \otimes !B & \xrightarrow{\lambda} & !B \\
{\scriptstyle \underline{p}_\top \otimes !B}\downarrow & & \uparrow{\scriptstyle !\lambda} \\
!\top \otimes !B & \xrightarrow{\underline{n}_{\top,B}} & !(\top \& B)
\end{array}$$

$$\begin{array}{ccc} !A\otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A\&B) \\ {\scriptstyle \gamma_{!A,!B}}\downarrow & & \downarrow{\scriptstyle !\gamma_{A,B}} \\ !B\otimes !A & \xrightarrow{\underline{n}_{B,A}} & !(B\&A) \end{array}$$

PROOF ($\Leftarrow$) We show that every Seely category in which the five diagrams above commute, is a new-Seely category. In definition 1 of a Seely category, the family of isomorphisms $\underline{n}_{A,B}$ is required to be natural wrt. the morphisms of $\mathcal{C}$. It is not difficult to deduce from diagram (8) that the family of isomorphisms $\underline{n}_{A,B}$ is also natural wrt. the morphisms of $\mathcal{C}_!$. In diagram (8), the morphism $\langle !\pi_1, !\pi_2\rangle : !(A\&B) \longrightarrow !A\&!B$ is the cartesian pair of cartesian projections $\pi_1 : A\&B \longrightarrow A$ and $\pi_2 : A\&B \longrightarrow B$. The four remaining diagrams indicate that the functor $(U, \underline{n}_{A,B}, \underline{p}_\top) : (\mathcal{C}_!, \&, \top) \longrightarrow (\mathcal{C}, \otimes, 1)$ is symmetric monoidal. We conclude by lemma 12 and 13 that the adjunction (2) itself is symmetric monoidal.
($\Rightarrow$) We prove that the five diagrams above commute in every new-Seely category. Every new-Seely category is an instance of linear-non-linear category, with $(U, \underline{n}_{A,B}, \underline{p}_\top)$ as strong monoidal functor from $(\mathcal{C}_!, \&, \top)$ to $(\mathcal{C}, \otimes, 1)$. So, corollary 15 applies, and states that the morphism $\underline{n}_{A,B} : !A\otimes !B \longrightarrow !(A\&B)$ is coalgebraic for every A, B. Let $f, g : !A\otimes !B \longrightarrow !(!A\&!B)$ denote respectively the upper and lower side of diagram (8):

$$f = !\langle !\pi_1, !\pi_2\rangle \circ \delta_{A\&B} \circ \underline{n}_{A,B} \qquad g = \underline{n}_{!A,!B} \circ (\delta_A \otimes \delta_B)$$

Both morphisms f, g are coalgebraic. It is not difficult to deduce from theorem 21 that the series of equality holds between the morphisms $!A\otimes !B \longrightarrow !A\&!B$:

$$\epsilon_{!A\&!B} \circ f = !\langle !\pi_1, !\pi_2\rangle \circ \underline{n}_{A,B} = \langle \rho_A \circ (A \otimes \underline{e}_B), \lambda_B \circ (\underline{e}_A \otimes B)\rangle = \epsilon_{!A\&!B} \circ g.$$

The equality $\epsilon_{!A\&!B} \circ f = \epsilon_{!A\&!B} \circ g$ implies that the coalgebraic morphisms f and g are equal; alternatively, that diagram (8) commutes. The four remaining diagrams commute because the functor $(U, \underline{n}_{A,B}, \underline{p}_\top) : (\mathcal{C}_!, \&, \top) \longrightarrow (\mathcal{C}, \otimes, 1)$ is symmetric monoidal. We conclude. ■

5 Relaxing Lafont categories

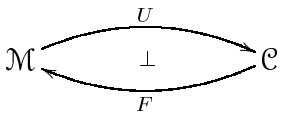
In that section, we relax Lafont categories in order to enscope key examples, like Girard's original coherence space model, or the various game-theoretic exponentials exposed in [15,23,33].

Definition 23 *A* **new-Lafont category** *consists of*

(1) a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1)$*,*

(2) a full submonoidal category $\mathcal{M}$ *of the category* $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ *of commutative comonoids over* $(\mathcal{C}, \otimes, 1)$*,*

(3) the forgetful functor $U : \mathcal{M} \longrightarrow \mathcal{C}$ *has a right adjoint* $F : \mathcal{C} \longrightarrow \mathcal{M}$



Point 2. reduces to the fact that $\mathcal{M}$ contains the unit comonoid 1, as well as the tensor product $A \otimes B$ of any two comonoids A and B of $\mathcal{M}$.

Remark. A new-Lafont category is said to have *finite products* when its underlying category $\mathcal{C}$ has finite products given by & and a terminal object $\top$. A new-Lafont category with finite products, defines a model of ILL+product, see figures 1 and 2. A Lafont category is just a new-Lafont category with finite products, in which $\mathcal{M}$ is the whole category $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ of commutative comonoids.

Lemma 24 *Every new-Lafont category is a linear category.*

Proof The category $(\mathcal{M}, \otimes, 1)$ has finite products, and the forgetful functor $U : (\mathcal{M}, \otimes, 1) \longrightarrow (\mathcal{C}, \otimes, 1)$ is symmetric monoidal and strong. ■

6 Cross-breeding Lafont and Seely categories

The axiomatization of new-Lafont categories requires to define a subcategory $\mathcal{M}$ of commutative comonoids, closed under tensor products. Unfortunately, the category $\mathcal{M}$ may be difficult to explicate in some cases, in another way than formally, see section 7.2 for a discussion. Thus, we introduce here a useful tradeoff between Lafont and Seely axiomatizations, which avoids mentioning the category $\mathcal{M}$.

Definition 25 (exponential structure) *An exponential structure over a symmetric monoidal category* $(\mathcal{C}, \otimes, 1)$ *with finite products given by* $(\&, \top)$ *is the following data:*

(1) for each object A*, a commutative comonoid* $(!A, \underline{d}_A, \underline{e}_A)$ *with respect to the tensor product,*

(2) for every object A*, a morphism* $\epsilon_A : !A \longrightarrow A$*, such that for each morphism*

$$f : !A \longrightarrow B$$

there exists a unique comonoidal morphism

$$f^\dagger : (!A, \underline{d}_A, \underline{e}_A) \longrightarrow (!B, \underline{d}_B, \underline{e}_B)$$

satisfying

$$!A \xrightarrow{\ f\ } B \qquad = \qquad !A \xrightarrow{\ f^\dagger\ } !B \xrightarrow{\ \epsilon_B\ } B$$

(3) for every objects A, B *of* $\mathcal{C}$*, two (comonoidal) isomorphisms between the*

commutative comonoids:

$$\underline{n}_{A,B} : (!A, \underline{d}_A, \underline{e}_A) \otimes (!B, \underline{d}_B, \underline{e}_B) \cong (!A\&B, \underline{d}_{A\&B}, \underline{e}_{A\&B})$$

$$\underline{p}_\top : (1, \rho_1^{-1} = \lambda_1^{-1}, \mathbf{id}_1) \cong (!\top, \underline{d}_\top, \underline{e}_\top)$$

Remark. Point (2) makes ! an endofunctor of $\mathcal{C}$, transporting every morphism $f : A \longrightarrow B$ to the morphism $(f \circ \epsilon_A)^\dagger : !A \longrightarrow !B$. Besides, we will see in the proof of lemma 27, that the comonoidal morphism $\underline{n}_{A,B}$ and $\underline{p}_\top$ may be chosen, in such a way that the diagrams below commute:

$$\begin{array}{ccc} !A\otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A\&B) \\ {\scriptstyle !A\otimes \underline{e}_B}\downarrow & & \downarrow{\scriptstyle \epsilon_{A\&B}} \\ !A \otimes 1 & & A\&B \\ {\scriptstyle \rho_{!A}}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ !A & \xrightarrow{\epsilon_A} & A \end{array} \qquad \begin{array}{ccc} !A\otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A\&B) \\ {\scriptstyle \underline{e}_A\otimes !B}\downarrow & & \downarrow{\scriptstyle \epsilon_{A\&B}} \\ 1\otimes !B & & A\&B \\ {\scriptstyle \lambda_{!B}}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ !B & \xrightarrow{\epsilon_B} & B \end{array} \tag{9}$$

Moreover, this choice of $\underline{n}_{A,B}$ and $\underline{p}_\top$ is unique, since by (2) the identity $\mathbf{id}_{!(A\&B)}$ (resp. $\mathbf{id}_\top$) is the unique comonoidal morphism f of $!(A\&B)$ (resp. $!\top$) such that $f \circ \epsilon_{A\&B} = \epsilon_{A\&B}$ (resp. $f \circ \epsilon_\top = \epsilon_\top$).
Diagrams (9) make $(!, \underline{n}_{A,B}, \underline{p}_\top)$ a symmetric monoidal functor from $(\mathcal{C}, \&, \top)$ to the cartesian category $\mathbf{coMon}(\mathcal{C}, \otimes, \mathbf{1})$ of commutative comonoids over $(\mathcal{C}, \otimes, \mathbf{1})$.
Another equivalent way to express the coherence diagrams (9), is to require that the diagrams below commute, for every objects A, B of the category $\mathcal{C}$:

$$\begin{array}{ccc} !A\otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A\&B) \\ {\scriptstyle !A\otimes \underline{e}_B}\downarrow & & \downarrow{\scriptstyle !\pi_1} \\ !A \otimes 1 & \xrightarrow{\rho_{!A}} & !A \end{array} \qquad \begin{array}{ccc} !A\otimes !B & \xrightarrow{\underline{n}_{A,B}} & !(A\&B) \\ {\scriptstyle \underline{e}_A\otimes !B}\downarrow & & \downarrow{\scriptstyle !\pi_2} \\ 1\otimes !B & \xrightarrow{\lambda_{!B}} & !B \end{array} \tag{10}$$

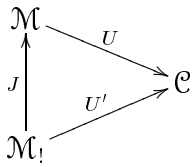
Among other things, this implies the equality:

$$!A \xrightarrow{\underline{d}_A} !A\otimes !A \xrightarrow{\underline{n}_{A,A}} !(A\&A) \quad = \quad !A \xrightarrow{!\Delta_A} !(A\&A)$$

in which $\Delta_A : A \longrightarrow A\&A$ denotes the diagonal morphism induced by universality of the cartesian product &.
Definition 26 *A Lafont-Seely category is a symmetric monoidal closed category $(\mathcal{C}, \otimes, 1, \multimap)$ with finite products $(\&, \top)$ equipped with an exponential structure.*
Lemma 27 *Every Lafont-Seely category is a new-Lafont category with finite products, and conversely, every new-Lafont category with finite products is a Lafont-Seely category.*
Proof Consider a Lafont-Seely category, and define $\mathcal{M}$ as the full subcategory of $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ consisting of the commutative comonoids of the form

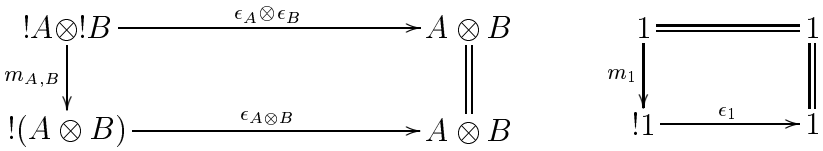
$$X = 1 \mid (!A, \underline{d}_A, \underline{e}_A) \mid X \otimes X$$

The category is obviously closed under tensor products. We claim that the forgetful functor $U : \mathcal{M} \longrightarrow \mathcal{C}$ has a right adjoint. The proof proceeds by comparing $\mathcal{M}$ to its full subcategory $\mathcal{M}_!$ of objects of the form $(!A, \underline{d}_A, \underline{e}_A)$, for some object A of $\mathcal{C}$. By point (3) of definition 25, every object of $\mathcal{M}$ is isomorphic (in $\mathcal{M}$) to an object of $\mathcal{M}_!$. In particular, the inclusion functor $J : \mathcal{M}_! \longrightarrow \mathcal{M}$ is an equivalence of categories. Now, let $U' : \mathcal{M}_! \longrightarrow \mathcal{C}$ be the functor U restricted to $\mathcal{M}_!$. The situation may be illustrated with a commutative diagram:



By point (3) of definition 25, every morphism $\epsilon_A : !A \longrightarrow A$ is universal from U' to A. It follows that the functor U' has a right adjoint $F' : \mathcal{C} \longrightarrow \mathcal{M}_!$, see chapter IV of [31] for instance. Because J is an equivalence of categories, the functor F' lifts to a functor $F : \mathcal{M} \longrightarrow \mathcal{C}$ which is right adjoint to the functor U. It follows that every Lafont-Seely category defines a new-Lafont category. Conversely, every new-Lafont category induces a Lafont-Seely category, by defining the symmetric comonoid $(!A, \underline{d}_A, \underline{e}_A)$ as $UF(A)$, for every object A of $\mathcal{C}$. Point (2) of definition 25 follows from the adjunction $U \dashv F$ and from the fact that $\mathcal{M}$ is a full subcategory of $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$. Point (3) follows from the fact that the right adjoint F transports the finite products $(\&, \top)$ of $\mathcal{C}$ to the finite products $(\otimes, 1)$ of $\mathcal{M}$. Moreover, the family of isomorphisms $\underline{n}_{A,B}$ and $\underline{n}_1$ verify the two coherence diagrams (10) which are equivalent to the coherence diagrams (9). ■

Remark. Given an exponential structure ! over a category $\mathcal{C}$, there exists unique comonoidal morphisms $m_{A,B}$ and m_1 making the diagrams below commute, for every pair of objects A, B of $\mathcal{C}$:



The functor ! equipped with the mediating morphisms $m_{A,B} : !A \otimes !B \longrightarrow !(A \otimes B)$ and $m_1 : 1 \longrightarrow !1$ defines a symmetric monoidal functor from $(\mathcal{C}, \otimes, 1)$ to itself.

Remark. Point (3) may be replaced by the property that for every finite (possibly empty) family $(A_i)_{i \in I}$ of objects of $\mathcal{C}$, and morphism $f : \otimes_{i \in I} !A_i \longrightarrow B$, there exists a unique comonoidal morphism $f^\dagger : \otimes_{i \in I} !A_i \longrightarrow !B$, such that $f = \epsilon_B \circ f^\dagger$. A variant of lemma 27 relates this weaker notion of Lafont-Seely category, to general new-Lafont category, possibly without finite products.

7 What do we learn from the categorical models?

7.1 Coherence spaces (in Seely's style)

Girard introduces in [19] the category $\mathcal{STAB}$ of qualitative domains and stable functions, a remarkably simple submodel of Berry's stable model [11]. A few months later, Girard reformulates the category $\mathcal{STAB}$ as a kleisli construction over the category $\mathcal{COH}$ of coherence spaces, and a so-called *exponential* comonad $!_{\mathsf{set}}$. We recall this construction which leads to linear logic, in Seely's fashion, see [4].

Qualitative domains and stable maps
A *qualitative domain* is a pair $(|X|, D(X))$ consisting of a set $|X|$ called the *web* of X and a set $D(X)$ of finite subsets of $|X|$, called the *domain* of X, satisfying that every subset x of an element $y \in D(X)$ is an element $x \in D(X)$. A *stable map* $f : X \longrightarrow Y$ between qualitative domains is a function $D(X) \longrightarrow D(Y)$ satisfying

$$\text{monotonicity} \qquad x \subset x' \in D(X) \Rightarrow f(x) \subset f(x')$$

$$\text{stability} \qquad x, x' \subset x'' \in D(X) \Rightarrow f(x \cap x') = f(x) \cap f(x')$$

A stable map $f : X \longrightarrow Y$ is *linear* when, moreover,

$$\text{linearity} \quad x, x' \subset x'' \in D(X) \Rightarrow f(x \cup x') = f(x) \cup f(x')$$

The category $\mathcal{STAB}$ of qualitative domains and stable maps has finite products given by $X \times Y = (|X| + |Y|, D(X) \times D(Y))$ and $e = (\emptyset, \{\emptyset\})$.

Coherence spaces and cliques
A *coherence space* is a pair $A = (|A|, \frown_A)$ consisting of a set $|A|$ called the *web* of A, and a reflexive binary relation $\frown_A$ on the elements of $|A|$, called the *coherence* of A. A *clique* of A is a set of pairwise coherent elements of $|A|$. Every coherence space X has a *dual* coherence space $A^\perp$ with same web $|A|$ and coherence relation

$$a \frown_{A^\perp} b \iff a = b \text{ or } \neg(a \frown_A b)$$

The coherence space $A \multimap B$ has web $|A \multimap B| = |A| \times |B|$ and coherence relation

$$(a, b) \frown_{A \multimap B} (a', b') \iff \begin{cases} a \frown_A a' \Rightarrow b \frown_B b' \\ \text{and} \\ b \frown_{B^\perp} b' \Rightarrow a \frown_{A^\perp} a' \end{cases}$$

The category $\mathcal{COH}$ has coherence spaces as objects, cliques of $A \multimap B$ as morphisms $A \longrightarrow B$. Morphisms are composed as relations, and identities are given by $\mathbf{id}_A = \{(a, a) \mid a \in |A|\}$. The category is symmetric monoidal closed (in fact, $*$-autonomous) and has finite products.

A linear-non-linear model of linear logic

The categories $\mathcal{STAB}$ and $\mathcal{COH}$ induce a linear-non-linear category, as follows. First, a "forgetful" functor $U : \mathcal{STAB} \longrightarrow \mathcal{COH}$ transports

- every qualitative domain $X = (|X|, D(X))$ to the coherence space $U(X) = (D(X), \frown_{D(X)})$ whose coherence relation $\frown_{D(X)}$ is defined as the usual order-theoretic coherence:

$$\forall x, x' \in D(X), \quad x \frown_{D(X)} x' \iff \exists x'' \in D(X), \quad x, x' \subset x''$$

- every stable function $f : X \longrightarrow Y$ to the clique $U(f) : U(X) \longrightarrow U(Y)$

$$U(f) = \left\{ (x, y) \;\middle|\; \begin{array}{l} y \subset f(x) \text{ and} \\ \forall x' \in D(X), x' \subset x \text{ and } y \subset f(x') \quad \Rightarrow \quad x = x' \end{array} \right\}$$

The functor U is a *strict* monoidal functor from $(\mathcal{STAB}, \times, e)$ to $(\mathcal{COH}, \otimes, 1)$, and is left adjoint to the functor $F : \mathcal{COH} \longrightarrow \mathcal{STAB}$ which transports

- every coherence space $A = (|A|, \frown_A)$ to the qualitative domain $F(A) = (|A|, D(A))$ with domain $D(A)$ the set of finite cliques of A,
- every clique $f : A \multimap B$ to the linear map $F(f) : F(A) \longrightarrow F(B)$:

$$x \in D(A) \quad \mapsto \quad \{\, b \in |B| \mid \exists a \in x, (a, b) \in f \,\} \;\in\; D(B)$$

The adjunction $U \dashv F$ is the categorical way to formulate the theory of *traces* developped by Berry in his PhD thesis [11], see also [4]. This shows that the adjunction $U \dashv F$ defines a linear-non-linear category, thus a model of linear logic.

Remark. It is well-known that the functor F is an isomorphism between the category $\mathcal{COH}$ and the subcategory $\mathcal{LINEAR}$ of *coherent* qualitative domains and *linear* maps of the category $\mathcal{STAB}$. We recall that a qualitative domain $(|X|, D(X))$ is *coherent* when, for every $x, y, z \in D(X)$:

$$x \cup y \in D(X), y \cup z \in D(X), x \cup z \in D(X) \;\Rightarrow\; x \cup y \cup z \in D(X)$$

Less known is that the functor U factors as

$$\mathcal{STAB} \xrightarrow{\quad K \quad} \mathbf{coMon}(\mathcal{C}, \otimes, 1) \xrightarrow{\quad V \quad} \mathcal{COH}$$

where V is the forgetful functor from the category $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$ of commutative comonoids over $(\mathcal{COH}, \otimes, 1)$, to the category $\mathcal{COH}$; and where K is full and faithfull embedding of $\mathcal{STAB}$ into $\mathbf{coMon}(\mathcal{C}, \otimes, 1)$. Indeed, for every qualitative domain X, the coherence space $U(X)$ defines a commutative comonoid $K(X) = (D(X), d_X, e_X)$ equipped with the comultiplication $d_X : K(X) \longrightarrow K(X) \otimes K(X)$ and counit $e_X : K(X) \longrightarrow 1$ below:

$$\underline{d}_X = \{(x, y, z) \in D(X) \mid x = y \cup z\} \qquad \underline{e}_X = \{(\emptyset, *)\} \tag{11}$$

Moreover, every stable map $f : X \longrightarrow Y$ induces a *comonoidal* morphism $U(f) : U(X) \longrightarrow U(Y)$ from $K(X)$ to $K(Y)$; and conversely, every comonoidal morphism $K(X) \longrightarrow K(Y)$ is the image of a unique stable function $f : X \longrightarrow Y$.

Remark. The comonad $!_{\mathsf{set}}$ over $\mathcal{COH}$ induced by the adjunction $U \dashv F$ transports every coherence space A to the commutative comonoid $!_{\mathsf{set}}A$ below:
- its web is the set of finite *cliques* of A,
- two cliques are coherent when their union is a clique,
- coproduct is union of clique, and counit is the empty set.

7.2 Coherence spaces (in Lafont's style)

Seely's approach of section 7.1 is fine when one starts by the "domain-theoretic" (=kleisli) category $\mathcal{STAB}$ and deduces from it the "graph-theoretic" (=linear) category $\mathcal{COH}$, and exponential construction $!_{\mathsf{set}}$. However, in many cases, it is easier (1) to start with a symmetric monoidal closed category, (2) equip it with an exponential construction, (3) check that it defines a model of linear logic, and then only (4) explicate the meaning of the induced kleisli category.

Free comonoids in coherence spaces (Van de Wiele)
This is precisely what happened with the *free* comonoidal construction $!_{\mathsf{mset}}$ over coherence spaces, which is characterized by Van de Wiele in [42]. The commutative comonoid $!_{\mathsf{mset}}A$ freely co-generated by a coherence space A is the coherence space $!_{\mathsf{mset}}A$ below:
- its web is the set of finite *multicliques* of A,
- two multicliques are coherent when their sum is a multiclique,
- coproduct is sum of multiclique, and counit is the empty multiset.

Here, we call *multiclique* a multiset whose support is a clique of A. The free construction $!_{\mathsf{mset}}$ defines a Lafont category, thus a model of linear logic. The meaning of the induced kleisli category was explicated by Barreiro and Ehrhard, only years after the model was introduced: it is the category $\mathcal{CONV}$ of convex and multiplicative functions [8].

Diagonal comonoids and the exponential $!_{\mathsf{set}}$
It is possible to show that the exponential $!_{\mathsf{set}}$ defines a model of linear logic, without mentioning the category $\mathcal{STAB}$. A commutative comonoid X over $\mathcal{COH}$ is called *diagonal* when the clique $\underline{d}_X : X \longrightarrow X \otimes X$ contains the diagonal $\{(x,(x,x)) \mid x \in |X|\}$. Van de Wiele establishes in [42] that $!_{\mathsf{set}}A$ is the free commutative *diagonal* comonoid generated by A. Since diagonal comonoids are closed by tensor product, the construction $!_{\mathsf{set}}$ defines a new-Lafont category, thus a model of linear logic.

7.3 Coherence spaces (in Lafont-Seely's style)

In sections 7.1 and 7.2, we illustrate how Seely and Lafont axiomatizations work for coherence spaces and exponentials $!_{\mathsf{set}}$ and $!_{\mathsf{mset}}$. We believe however that the alternative axiomatics delivered in section 6 (Lafont-Seely categories) is simpler. We show here how to check this axiomatics for $!_{\mathsf{set}}$. Suppose that A, B are coherence spaces. Then,

- $!_{\mathsf{set}}A$ defines a commutative comonoid in $\mathcal{COH}$, because set-theoretic union is associative and commutative, and has the empty set as unit,
- the comonoidal isomorphism $!_{\mathsf{set}}A\&B \longrightarrow !_{\mathsf{set}}A \otimes !_{\mathsf{set}}B$ is given by the clique $\{(x,(x_A,x_B)), x_A = x \cap |A|, x_B = x \cap |B|\}$,
- the comonoidal isomorphism $!_{\mathsf{set}}\top \longrightarrow 1$ is given by the clique $\{(\emptyset,*)\}$.

There remains to show that the dereliction morphism

$$\epsilon_A^{\mathsf{set}} = \{(\{a\},a), a \in |A|\}$$

verifies the universal property (2) of definition 25. This amounts to characterizing the comonoidal morphisms $!_{\mathsf{set}}A \longrightarrow !_{\mathsf{set}}B$. A morphism $g :!_{\mathsf{set}}A \longrightarrow !_{\mathsf{set}}B$ is comonoidal iff it verifies the four properties below:

- unit (forth): $(\emptyset, y) \in g$ implies $y = \emptyset$,
- unit (back): $(x,\emptyset) \in g$ implies $x = \emptyset$,
- product (forth): $(x_1,y_1) \in g$ and $(x_2,y_2) \in g$ and $x = x_1 \cup x_2$ is a clique, implies that $(x, y_1 \cup y_2) \in g$,
- product (back): $(x, y_1 \cup y_2) \in g$ implies that there exists two cliques x_1, x_2 such that $x = x_1 \cup x_2$ and $(x_1,y_1) \in g$ and $(x_2,y_2) \in g$.

In particular, when g is comonoidal, x is a clique of A and $y = \{b_1,...,b_n\}$ is a clique of B, then

$(x,\{b_1,...,b_n\}) \in g$ iff x decomposes as $x = x_1 \cup ... \cup x_n$ where $(x_i,\{b_i\}) \in g$.

So, every comonoidal morphism $g :!_{\mathsf{set}}A \longrightarrow !_{\mathsf{set}}B$ is characterized by the composite $\epsilon_B^{\mathsf{set}} \circ g :!_{\mathsf{set}}A \longrightarrow B$. Conversely, every morphism $f :!_{\mathsf{set}}A \longrightarrow B$ induces a comonoidal morphism g such that $f = \epsilon_B^{\mathsf{set}} \circ g$. This point is further discussed in section 7.4. The correspondence between f and g is one-to-one, and we conclude.

7.4 The relational non-model (observed by Ehrhard)

The category $\mathcal{COH}$ may be replaced by the $*$-autonomous category $\mathcal{REL}$ of sets and relations, equipped with the set-theoretic product as tensor product. The category $\mathcal{REL}$ has finite products, and enjoys a free (commutative) comonoidal construction $!_{\mathsf{mset}}$ similar to the construction in $\mathcal{COH}$ discussed in section 7.2. So, $\mathcal{REL}$ and $!_{\mathsf{mset}}$ define together a Lafont category, thus a model of linear logic.

It is therefore tempting to adapt the "set-theoretic" interpretation of exponentials discussed in section 7.1. Indeed, every object A of $\mathcal{REL}$ defines a

commutative comonoid $(!_{\mathsf{set}}A, \underline{d}_A, \underline{e}_A)$

$$!_{\mathsf{set}}A = \{x \mid x \subset_{\mathsf{fin}} A\} \qquad \underline{d}_A = \{(x \cup y, (x,y)) \mid x, y \subset_{\mathsf{fin}} A\} \qquad \underline{e}_A = \{(\emptyset, *)\}$$

where $x \subset_{\mathsf{fin}} A$ means that x is a finite subset of A ; as well as a "dereliction" morphism

$$\epsilon_A = (\{a\}, a) \mid a \in |A|\} : !A \longrightarrow A$$

However, Ehrhard observed that this "set-theoretic" interpretation of exponentials fails to define a model of linear logic. Indeed, reasoning in Seely's framework, Ehrhard points out that the dereliction family $(\epsilon_A)_{A \in \mathcal{REL}}$ is not natural. For instance, the naturality diagram below does not commute from $A = \{a_1, a_2\}$ to $B = \{b\}$, for the relation $f = \{(a_1, b), (a_2, b)\}$.

$$\begin{array}{ccc} !A & \xrightarrow{\quad !f \quad} & !B \\ {\scriptstyle \epsilon_A}\downarrow & & \downarrow{\scriptstyle \epsilon_B} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \qquad \begin{array}{l} ((a_1, a_2), b) \in \epsilon_B \circ !f \\ ((a_1, a_2), b) \notin f \circ \epsilon_B \end{array} \tag{12}$$

This lack of commutation shows that all coherence diagrams of a linear/new-Seely category should be checked carefully every time a new model is introduced. This justifies to introduce simpler categorical axiomatics, like Lafont-Seely categories in section 6.
We explicate below how Lafont-Seely categories explain that the exponential $!_{\mathsf{set}}$ does not define a model on the category $\mathcal{REL}$ of sets and relations.
Every object A of $\mathcal{REL}$ defines a "codiagonal" morphism

$$\mathbf{codiag}_A = \{((a,a),a) \mid x \in |A|\} : A \otimes A \longrightarrow A$$

Now, observe that the diagram below commutes in $\mathcal{REL}$, for every set B:

$$\begin{array}{ccc} !_{\mathsf{set}}B & \xrightarrow{\quad \epsilon_B \quad} & B \\ {\scriptstyle \underline{d}_B}\downarrow & & \uparrow{\scriptstyle \mathbf{codiag}_B} \\ !_{\mathsf{set}}B \otimes !_{\mathsf{set}}B & \xrightarrow{\quad \epsilon_B \otimes \epsilon_B \quad} & B \otimes B \end{array}$$

Thus, every morphism $f : !_{\mathsf{set}}A \longrightarrow B$ lifting as a comonoidal morphism $f^\dagger : !_{\mathsf{set}}A \longrightarrow !_{\mathsf{set}}B$ such that $f = \epsilon_B \circ f^\dagger$, makes the diagram below commute:

$$\begin{array}{ccc} !_{\mathsf{set}}A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle \underline{d}_A}\downarrow & & \uparrow{\scriptstyle \mathbf{codiag}_B} \\ !_{\mathsf{set}}A \otimes !_{\mathsf{set}}A & \xrightarrow{\quad f \otimes f \quad} & B \otimes B \end{array}$$

The diagram translates as the following closure property on every morphism $f : !_{\mathsf{set}}A \longrightarrow B$:

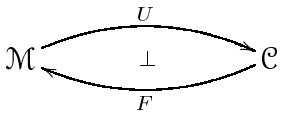
$$\text{if } (x,b) \in f \text{ and } (y,b) \in f \text{ then } (x \cup y, b) \in f. \tag{13}$$

Obviously, the property is not valid in $\mathcal{REL}$; it follows that the set-theoretic exponential does not define a model of linear logic.

Remark. The pitfall mentioned in (13) is common to all "relational models" equipped with a "set-theoretic" exponential. It is rewarding to see how the coherence space model avoids (13): if (x,b) and (y,b) and $x \neq y$, then x and y are incompatible — thus, $x \cup y$ does not appear in the web of $!_{\mathsf{set}}A$.
Besides coherence, another solution appears in [43] where Winskel develops a relational model based on *set-theoretic exponentials*, see also [16]. The kleisli category is equivalent to Scott's model of prime algebraic lattices. Spaces are partial orders whose cliques are downward subsets. There, property (13) is verified because the element $(x \cup y, b)$ is always smaller than the elements (x,b) and (y,b) in the ordered space $!_{\mathsf{set}}A \multimap B$.

8 Conclusion

Let us briefly summarize the article. We have seen that a model of intuitionistic linear logic is given by (1) a category $\mathcal{M}$ with finite products (2) a symmetric monoidal closed category $\mathcal{C}$ (3) a monoidal adjunction $U \dashv F$ between them:



Besides, by Kelly characterisation lemma, a monoidal adjunction reduces to an adjunction $U \dashv F$ between the underlying categories, in which the functor U is monoidal strict between the monoidal categories. Thus, besides the adjunction between the categories $\mathcal{M}$ and $\mathcal{C}$, one only requires that the functor U transports the cartesian structure of $\mathcal{M}$ to the monoidal structure of $\mathcal{C}$.
When the category $\mathcal{C}$ has finite products, then the kleisli category associated to the comonad over $\mathcal{C}$, is cartesian closed. Otherwise, one only gets a cartesian category $(\mathcal{M}, \times, e)$ and an exponential ideal in it, which is enough to interpret the simply-typed λ-calculus. We recall that an exponential ideal, see [2], is defined here as:
(1) a category $\mathcal{C}$ and a functor $F : \mathcal{C} \longrightarrow \mathcal{M}$,
(2) a functor $\Rightarrow : \mathcal{M}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}$,
(3) a natural bijection $\mathcal{M}(A \times B, F(C)) \cong \mathcal{M}(A, F(B \Rightarrow C))$.
We would like to conclude the article by exposing two recent variations on the axiomatizations studied in this article.

Non commutative exponential and Basic Syntactic Control of Interference
Reddy [38] introduces a coherence space model of Basic Syntactic Control of Interference (SCI), an affine fragment of ALGOL. The model is based on a *non-commutative* version $!_{\mathsf{sci}}$ of $!_{\mathsf{mset}}$, in the sense that the web of $!_{\mathsf{sci}}A$ contains all finite *sequences* of pairwise coherent elements of the web of A, instead of the

finite multicliques of $!_{\mathsf{mset}}A$. In particular, the comonoid $!_{\mathsf{sci}}A$ is generally *non commutative* in $\mathcal{COH}$. In fact, the comonad $!_{\mathsf{sci}}$ is induced by an adjunction $U \dashv F$ between $\mathcal{COH}$ and its category $\mathcal{MONCOH}$ of (not necessarily commutative) comonoids, and comonoidal morphisms:

$$\mathcal{MONCOH} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{COH} \tag{14}$$

The category $\mathcal{MONCOH}$ inherits its monoidal structure from $\mathcal{COH}$; and the functor U is strongly monoidal from $(\mathcal{MONCOH}, \otimes, 1)$ to $(\mathcal{REL}, \otimes, 1)$. Unfortunately, the tensor product does not define a cartesian product in $\mathcal{MONCOH}$, in contrast to what happens in its subcategory of *commutative* comonoids. So, the adjuction $F \dashv U$ does not define a linear category.
All in all, we obtain a symmetric monoidal category $(\mathcal{MONCOH}, \otimes, 1)$ equipped with a *monoidal* variant of exponential ideal, that is,
(1) a category $\mathcal{COH}$ and a functor $F : \mathcal{COH} \longrightarrow \mathcal{MONCOH}$,
(2) a functor $\Rightarrow : \mathcal{MONCOH}^{op} \times \mathcal{COH} \longrightarrow \mathcal{COH}$,
(3) a natural bijection

$$\mathcal{MONCOH}(A \otimes B, F(C)) \;\cong\; \mathcal{MONCOH}(A, F(B \Rightarrow C)).$$

McCusker [17] reconstructs Reddy's model, and shows that it is fully abstract for Basic SCI. More precisely, McCusker replaces the category $\mathcal{COH}$ by the category $\mathcal{REL}$ of sets and relations; and defines a full subcategory $\mathcal{MONREL}$ of the category of (not necessarily commutative) comonoids over $\mathcal{REL}$. The category $\mathcal{MONREL}$ is a closed under tensor product, and contains the comonoid $!_{\mathsf{sci}}A$, for every set A. Thus, one obtains an adjunction similar to (14)

$$\mathcal{MONREL} \underset{F}{\overset{U}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{REL}$$

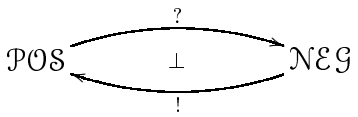
and its induced *monoidal* exponential ideal in $\mathcal{MONREL}$. One novelty of McCusker's presentation is to formulate the category $\mathcal{MONREL}$ as (the dual of) the kleisli category induced by a powerset monad over the category of (usual set-theoretic) monoids.

Category of continuations and polarized linear logic
We start with a category $\mathcal{POS}$ with distributive finite products and coproducts, equipped with an exponentiable object R. By exponentiable object, we mean a contravariant functor $\mathcal{POS}^{op} \longrightarrow \mathcal{POS}$ noted $A \mapsto R^A$, and a bijection natural in A and B:

$$\mathcal{POS}(A \times B, R) \cong \mathcal{POS}(B, R^A)$$

When the canonical morphism $A \longrightarrow R^{R^A}$ is monic for all A, the category $\mathcal{POS}$ is known as a *response category* in the litterature, see [28,22] for instance.

In [34], we call the category $\mathcal{POS}$ *positive* and its dual category $\mathcal{NEG}$ *negative*, and observe that every response category induces an adjunction $? \dashv !$ between them:



(15)

Here, ? and ! stand for the functor $R : \mathcal{NEG} \longrightarrow \mathcal{POS}$ and its dual. This polarized presentation of response categories is inspired by Girard LC [21] and Laurent polarized linear logic (LLP) [30].
The adjunction (15) is very similar to a linear-non-linear category, in which $\mathcal{POS}$ plays the role of the cartesian category $\mathcal{M}$, and $\mathcal{NEG}$ the role of the symmetric monoidal closed category $\mathcal{C}$. Of course, $\mathcal{NEG}$ is not required to be monoidal, nor closed. However, we show that the *category of continuations* $R^{\mathcal{POS}}$ considered in [3,28,22,40] is precisely the co-kleisli category induced by the adjunction (15). Its objects are the objects of $\mathcal{POS}$, and its morphisms $A \longrightarrow B$ are the morphisms $R^{R^A} \longrightarrow B$ of $\mathcal{NEG}$, that is, the morphisms $R^A \longrightarrow R^B$ of $\mathcal{POS}$.
It is remarkable that, as in the case of a linear-non-linear category, the co-kleisli category $R^{\mathcal{POS}}$ induced by the adjunction (15) is *cartesian-closed.* In fact, $R^{\mathcal{POS}}$ is also a control category, in the sense of [40], thus a model of Parigot's $\lambda\mu$-calculus [36].

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References

[1] **S. Abramsky, R. Jagadeesan**. New foundations for the geometry of interaction, *Information and Computation,* 111(1):53-119, 1994. Conference version appeared in LiCS '92.

[2] **S. Abramsky and G. McCusker**. Game Semantics. In *Computational Logic.* U. Berger and H. Schwichtenberg eds, Springer Verlag, 1999.

[3] **S. Abramsky and E. Moggi.** Declarative continuations and monads. Unpublished draft, 1991.

[4] **R. Amadio and P.-L. Curien**. *Domains and Lambda-Calculi.* Cambridge University Press, 1998.

[5] **A. Barber**. *Linear type theories, semantics and action calculi.* PhD thesis ECS-LFCS-97-371, University of Edinburgh, 1997.

[6] **A. Barber, P. Gardner, M. Hasegawa, G. Plotkin**. *From action calculi to linear logic.* In Annual Conference of the European Association for Computer Science Logic (CSL'97). Springer LNCS 1414, pages 78-97, 1998.

[7] **M. Barr**. $*$-autonomous categories and linear logic. *Mathematical Structures in Computer Science*, 1(2):159-178, 1991.

[8] **N. Barreiro and T. Ehrhard.** Quantitative semantics revisited. *Proceedings of the Typed Lambda-Calculi and Applications conference*, Lecture Notes in Computer Science, Springer Verlag, 1581, 40-53, 1999.

[9] **P.N. Benton, G.M. Bierman, V.C.V. de Paiva and J.M.E. Hyland**. *Term assignment for intuitionistic linear logic.* Technical Report 262, Computer Laboratory, University of Cambridge, 1992.

[10] **P.N. Benton**. *A mixed linear and non-linear logic: proofs, terms and models.* Proceedings of Computer Science Logic '94, Springer Verlag, LNCS 933, 1994.

[11] **G. Berry**. Stable models of typed lambda-calculi. Proceedings of the 5th International Colloquium on Automatas, Languages and Programming, number 62 in Lecture Notes in Computer Science. Springer Verlag 1978.

[12] **G. Berry and P.-L. Curien**. Sequential algorithms on concrete data structures. *Theoretical Computer Science*, 20:265-321, 1982.

[13] **G. Bierman**. *On intuitionistic linear logic.* PhD thesis. University of Cambridge, August 1994.

[14] **G. Bierman**. What is a categorical model of intuitionistic linear logic? *Typed Lambda-Calculus and Applications*, 1995.

[15] **P.-L. Curien**. On the symmetry of sequentiality. In *Proceedings of Mathematical Foundations of Programming Semantics 1993*, LNCS 802, Springer Verlag, 1993.

[16] **P.-L. Curien, G. Plotkin, and G. Winskel**. Bidomains and linear logic. *Festschrift for Robin Milner*, MIT Press.

[17] **G. McCusker**. A fully abstract relational model of Syntactic Control of Interference. *Computer Science Logic*, 2002.

[18] **T. Ehrhard**. A relative definability result for strongly stable functions and some corollaries. *Information and Computation*, 1997.

[19] **J.-Y. Girard.** The system F of variable types fifteen years later. *Theoretical Computer Science*, 45, 1986.

[20] **J.-Y. Girard.** Linear logic. *Theoretical Computer Science*, 50: 1-102, 1987.

[21] **J.-Y. Girard.** A new constructive logic: classical logic. *Mathematical Structures in Computer Science*, 1:255-296, 1991.

[22] **M. Hofmann and T. Streicher.** Continuation models are universal for $\lambda\mu$-calculus. In *Proceedings of Logic in Computer Science*, 1997.

[23] **M. Hyland.** Game Semantics. In *Semantics and logics of computation.* A. Pitts and P. Dybjer editors. Publications of the Newton Institute, Cambridge University Press, 1997.

[24] **G. B. Im and M. Kelly.** A universal property of the convolution monoidal structure, *J. Pure Appl. Algebra* 43, pp. 75-88, 1986.

[25] **M. Kelly.** Doctrinal adjunction. *Lecture Notes in Math.* 420, pp. 257-280, 1974.

[26] **A. Kock.** Strong functors and monoidal monads. *Various Publications, Series 11*, Aarhus Universitet, 1970.

[27] **Y. Lafont** *Logiques, catégories et machines.* PhD thesis, Université Paris 7, 1988.

[28] **Y. Lafont, B. Reus, and T. Streicher** Continuations semantics or expressing implication by negation. Technical Report 93-21, Universit of Munich, 1993.

[29] **F. Lamarche**. Sequentiality, games and linear logic. *Manuscript*, 1992.

[30] **O. Laurent**. Polarized proof-nets and lambda-mu calculus. *Theoretical Computer Science,* 290(1):161-188. Décembre 2002

[31] **S. Mac Lane**. *Categories for the Working Mathematician.* Springer Verlag, 1971.

[32] **M. Maietti, P. Maneggia, V. de Paiva, E. Ritter** Relating categorical semantics for intuitionistic linear logic. Technical Report, University of Birmingham, CSR-01-07-2001.

[33] **P-A. Melliès.** Comparing hierarchies of types by forth-and-back translations. *Submitted.* 2002.

[34] **P-A. Melliès, P. Selinger.** Polarized categories and games. Talk at the Linear Logic workshop, Boston, 2001. *In preparation,* 2003.

[35] **H. Nickau**. Hereditarily Sequential Funtionals. In *Proc. Symp. Logical Foundations of Computer Science: Logic at St. Petersburg*, Eds. A. Nerode and Yu. V. Matiyasevich, Lecture Notes in Computer Science, volume 813, pages 253-264, Springer-Verlag, 1994.

[36] **M. Parigot**. $\lambda\mu$-calculus: an algorithmic interpretation of classical natural deduction. *Proceedings of International Conference on Logic Programming and Automated Reasoning*, Springer LNCS 624, 1992.

[37] **J. Power and E. Robinson**. Premonoidal categories and notions of computation. *Mathematical Structures in Computer Science 7*, 1997, 453-468.

[38] **U. S. Reddy**. Global state considered unnecessary: Object-based semantics for interference-free imperative programs. *Lisp and Symbolic Computation*, 9(1), 1978.

[39] **R. Seely**. Linear logic, $*$-autonomous categories and cofree coalgebras. *Applications of categories in logic and computer science*, Contemporary Mathematics, 92, 1989.

[40] **P. Selinger**. Control Categories and Duality: on the Categorical Semantics of the Lambda-Mu Calculus. *Mathematical Structures in Computer Science* 11:207-260, 2001.

[41] **R. Street.** The formal theory of monads. *Journal of Pure and Applied Algebra*, 2:149-168, 1972.

[42] **J. van de Wiele**. Manuscript, 1987.

[43] **G. Winskel**. A linear metalanguage for concurrency. *Proceedings of Algebraic Methodology and Software Technology, AMAST '98*, 1998.