Comparing hierarchies of types in models of linear logic

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Abstract

We show that two models $\mathcal{M}$ and $\mathcal{N}$ of linear logic collapse to the same extensional hierarchy of types, when (1) their monoidal categories $\mathcal{C}$ and $\mathcal{D}$ are related by a pair of monoidal functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ and transformations $Id_{\mathcal{C}} \Rightarrow GF$ and $Id_{\mathcal{D}} \Rightarrow FG$, and (2) their exponentials $!^{\mathcal{M}}$ and $!^{\mathcal{N}}$ are related by distributive laws $\varrho : !^{\mathcal{N}} F \Rightarrow F\, !^{\mathcal{M}}$ and $\eta : !^{\mathcal{M}} G \Rightarrow G\, !^{\mathcal{N}}$ commuting to the promotion rule. The key ingredient of the proof is a notion of back-and-forth translation between the hierarchies of types induced by $\mathcal{M}$ and $\mathcal{N}$. We apply this result to compare (1) the qualitative and the quantitative hierarchies induced by the coherence (or hypercoherence) space model, (2) several paradigms of games semantics: error-free vs. error-aware, alternated vs. non-alternated, backtracking vs. repetitive, uniform vs. non-uniform.

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1 Introduction

Coherence spaces. Girard designed linear logic after his discovery of the coherence space model [19]. *Coherence space* is another name for "non-oriented graph", that is, a pair $(|A|, \coh_A)$ consisting of a *web* $|A|$ and a reflexive and symmetric relation $\coh_A$ over the elements of $|A|$. A *clique* f of A is a subset of the web $|A|$ such that:

$$\forall a, b \in f, \quad a \coh_A b.$$

The *negation* $A^\perp = (|A|, \incoh_A)$ of a coherence space $A = (|A|, \coh_A)$ is its dual graph, defined as

$$\forall a, b \in |A|, \quad a \incoh_A b \iff a = b \text{ or } \neg(a \coh_A b).$$

The *tensor product* of two coherence spaces $A = (|A|, \coh_A)$ and $B = (|B|, \coh_B)$ is their product as graphs: $A \otimes B = (|A| \times |B|, \coh_A \times \coh_B)$. The category $\mathcal{COH}$ has coherence spaces as objects, and cliques of $A \multimap B = (A \otimes B^\perp)^\perp$ as morphisms. Morphisms are composed as in the category of sets and relations. The resulting category $\mathcal{COH}$ is $*$-autonomous, and has finite products. As such, it is a model of multiplicative additive linear logic.

The exponential modality ! of linear logic may be interpreted in two different ways, inducing either a "qualitative" or a "quantitative" model of proofs:

- The qualitative exponential $!^{\mathsf{set}}$ is introduced in Girard's seminal article [19]. The commutative comonoid $!^{\mathsf{set}}A$ has the finite cliques of A as elements of the web, union of cliques as comultiplication, and the empty clique as counit. This defines a comonad $!^{\mathsf{set}}$ over the category $\mathcal{COH}$, which "linearizes" Berry's stable model of PCF, in the sense that the co-kleisli category associated to $!^{\mathsf{set}}$ embeds (as a model of PCF) in the category of dI-domains and stable functions.
- The quantitative exponential $!^{\mathsf{mset}}$ is formulated by Van de Wiele and Winskel (and possibly others) who establish — in harmony with Lafont's ideas in [25] — that the exponential $!^{\mathsf{mset}}$ is the free comonoidal construction in $\mathcal{COH}$. The commutative comonoid $!^{\mathsf{mset}}A$ has the finite *multi-cliques* of A as elements of the web, addition of multi-cliques as comultiplication, and the empty multi-clique as counit.

We recall briefly that a multiset w over a set E is a function $w : E \longrightarrow \mathbb{N}$ to the set of natural numbers. Its support is the subset

$$\mathsf{support}(w) = \{e \in E \mid w(e) > 0\}.$$

Every subset x of E induces the "characteristic" multiset

$$\mathsf{char}(x) : \begin{cases} e \mapsto 1 & \text{if } e \text{ is element of } x \\ e \mapsto 0 & \text{otherwise} \end{cases}$$

A multi-clique of a coherence space A is a multiset with support a clique of A. A multi-clique is finite (resp. empty) when its support is finite (resp. empty).
So, the category of coherence spaces induces a *qualitative* and a *quantitative* model of linear logic. Are the two models related in some way? The answer is positive: Barreiro and Ehrhard establish in [7] that the *extensional collapse* of the quantitative hierarchy is precisely the qualitative hierarchy. But their proof is difficult: what we call in french a *tour de force*. Here, we would like to prove the same result by another simpler route, starting from this elementary observation: For every coherence space A, there exists an embedding-retraction pair (η_A, ϱ_A) making the coherence space $!^{\mathsf{set}}A$ a retract of the coherence space $!^{\mathsf{mset}}A$:

$$!^{\mathsf{set}}A \xrightarrow{\eta_A} !^{\mathsf{mset}}A \xrightarrow{\varrho_A} !^{\mathsf{set}}A \quad = \quad !^{\mathsf{set}}A \xrightarrow{\mathbf{id}_{!^{\mathsf{set}}A}} !^{\mathsf{set}}A \qquad (1)$$

$$\eta_A = \{(\mathsf{support}(w), w) \mid w \text{ is a finite multiclique of } A\}$$
$$\varrho_A = \{(\mathsf{char}(x), x) \mid x \text{ is a finite clique of } A\}$$

The map η_A may be deduced from the fact that $!^{\mathsf{mset}}A$ is the free comonoid over A. It is the unique comonoidal morphism $!^{\mathsf{set}}A \longrightarrow !^{\mathsf{mset}}A$ making the diagram below commute:

$$\begin{tikzcd} !^{\mathsf{set}}A \arrow[rr, "\eta_A"] \arrow[dr, "\varepsilon^{\mathsf{set}}_A"'] & & !^{\mathsf{mset}}A \arrow[dl, "\varepsilon^{\mathsf{mset}}_A"] \\ & A & \end{tikzcd}$$

On the other hand, the projection map ϱ_A is not comonoidal in general, since the diagram below does not necessarily commute (take $A = \perp$ the singleton coherence space).

$$\begin{tikzcd} !^{\mathsf{mset}}A \arrow[r, "\varrho_A"] \arrow[d, "d^{\mathsf{mset}}_A"'] & !^{\mathsf{set}}A \arrow[d, "d^{\mathsf{set}}_A"] \\ !^{\mathsf{mset}}A \otimes !^{\mathsf{mset}}A \arrow[r, "\varrho_A \otimes \varrho_A"] & !^{\mathsf{set}}A \otimes !^{\mathsf{set}}A \end{tikzcd} \qquad (2)$$

Given a coherence space A and a clique $f : 1 \longrightarrow A$, let $f^{\mathsf{set}} : 1 \longrightarrow !^{\mathsf{set}}A$ and $f^{\mathsf{mset}} : 1 \longrightarrow !^{\mathsf{mset}}A$ denote the clique f promoted with respect to $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$

respectively. Remarkably, the maps η_A and ϱ_A commute to the promotion rules of $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$, in the sense that:

$$f^{\mathsf{set}} = \varrho_A \circ f^{\mathsf{mset}} \qquad \text{and} \qquad f^{\mathsf{mset}} = \eta_A \circ f^{\mathsf{set}}. \tag{3}$$

In particular,

$$d_A^{\mathsf{set}} \circ f^{\mathsf{set}} = f^{\mathsf{set}} \otimes f^{\mathsf{set}} = (\varrho_A \otimes \varrho_A) \circ (f^{\mathsf{mset}} \otimes f^{\mathsf{mset}}).$$

Thus, precomposing diagram (2) with the promoted map $f^{\mathsf{mset}} : 1 \longrightarrow !^{\mathsf{mset}} A$ induces a commutative diagram:

$$\begin{tikzcd}
1 \arrow[dr, "f^{\mathsf{mset}}"] \arrow[drr, bend left, "f^{\mathsf{set}}"] \arrow[ddr, bend right, "f^{\mathsf{mset}} \otimes f^{\mathsf{mset}}"'] & & \\
& !^{\mathsf{mset}} A \arrow[r, "\varrho_A"] \arrow[d, "d_A^{\mathsf{mset}}"'] & !^{\mathsf{set}} A \arrow[d, "d_A^{\mathsf{set}}"] \\
& !^{\mathsf{mset}} A \otimes\, !^{\mathsf{mset}} A \arrow[r, "\varrho_A \otimes \varrho_A"] & !^{\mathsf{set}} A \otimes\, !^{\mathsf{set}} A
\end{tikzcd} \tag{4}$$

To summarize, diagram (2) does not commute, but the object 1 *believes* that diagram (2) commutes. Now, the object 1 plays a very special role for the hierarchies $[-]^{\mathsf{set}}$ and $[-]^{\mathsf{mset}}$ which, we recall, are defined as hierarchies of *global elements* $1 \longrightarrow [T]^{\mathsf{set}}$ and $1 \longrightarrow [T]^{\mathsf{mset}}$ of the category $\mathcal{COH}$, for T a simple type. So, when it comes to hierarchies extracted from a model of linear logic, what really matters is what the object 1 believes in the underlying monoidal category! And indeed, as we will see in the course of the article, the equalities (3) are sufficient to deduce *diagrammatically* that the hierarchies $[-]^{\mathsf{set}}$ and $[-]^{\mathsf{mset}}$ collapse to the same extensional hierarchy: in that case, Berry's stable hierarchy $[-]^{\mathsf{set}}$.
This proves Barreiro and Ehrhard's collapse theorem by another route, and clarifies the situation. New translations (called *back-and-forth*) are exhibited between the qualitative and the quantitative hierarchies. These translations play a key role in our proof that the two hierarchies $[-]^{\mathsf{set}}$ and $[-]^{\mathsf{mset}}$ collapse to the same extensional hierarchy — see section 3 for details.

Game models. Many game models of (intuitionistic) linear logic have been introduced in the last decade, but they are still poorly connected. We are working here at building a "topography" which would connect these models in a dense network of (effective) translations.
We are guided by the idea that all the sequential game models live roughly in the same interactive universe, and differ only in the way the connectives (or constants) of linear logic are reflected in it. So, the translations we are looking for should be deduced algebraically from coercion laws between the various interpretations of the tensor product, the exponential modality, etc. in this universe.
Coherence spaces illustrate this idea perfectly: the qualitative and quantitative hierarchies differ only by their interpretation $!^{\mathsf{set}}$ or $!^{\mathsf{mset}}$ of the exponentials, and the translations between the two hierarchies follow mechanically from the coercion laws (1) between $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$.

We show in the last part of the article (section 7) that the same phenomenon occurs in games semantics, and that it explains many differences between the existing models of sequentiality. We restrict ourselves to sequential games played on decision trees [24,1,26,15,5] and leave the so-called arena games [22,32,3] for another study. So, a *sequential game* means here a triple $A = (M_A, \lambda_A, P_A)$ where (M_A, λ_A) is a polarized alphabet of moves, in which $\lambda_A : M_A \longrightarrow \{-1, +1\}$ assigns a polarity $+1$ (Player) or -1 (Opponent) to every move; and P_A is a non-empty prefix-closed set of finite strings over the alphabet M_A, called the plays of the game A. We will consider only "negative" games, in which a play is either empty, or starts by an Opponent move.
Every sequential game A is represented as a rooted tree, whose branches coincide with the plays of A. A play $s = m_1 \cdots m_k$ is called alternated when $\lambda_A(m_j) = (-1)^j$ for every $1 \leq j \leq k$. The sub-tree of alternated plays is denoted $\mathrm{alt}(A)$. It is a bipartite graph, whose nodes (=branches=plays) are assigned polarity $+1$ (Player) when the distance to the root (=the length of the branch) is even, and polarity -1 (Opponent) otherwise. Note that the root has polarity Player in a negative game.
Now, a strategy σ of A is defined as a subtree of $\mathrm{alt}(A)$ which branches only at Player nodes: that is, the moves m_1 and m_2 are equal when $s \in \sigma$ is of odd-length, and $s \cdot m_1 \in \sigma$ and $s \cdot m_2 \in \sigma$. This definition is more liberal than what one generally finds in the litterature, because it enables strategies to withdraw and play "error" (or rather: "I loose") at any point of the interaction. A strategy in the usual sense is just an error-free strategy, that is, a strategy σ in which every odd-length play $s \in \sigma$ may be extended to a (necessarily unique) even-length play $s \cdot m \in \sigma$, for m a Player move.
There exist several models of intuitionistic linear logic based on sequential games. We will organize them here according to a series of simple distinctions:

(1) error-aware vs. error-free: a strategy is allowed (error-aware model) or is not allowed (error-free model) to withdraw and play "error";
(2) alternated vs. non alternated: the interpretation $[T]$ of every formula T is alternated (ie. $[T] = \mathrm{alt}([T])$) or not necessarily alternated;
(3) backtracking vs. repetitive: Opponent repeats the same question to Player as many times as necessary (repetitive model) or Opponent remembers Player's answers, and thus does not need to repeat a question twice (backtracking model);
(4) uniform vs. non uniform: this distinction holds only in repetitive models: when Opponent asks Player the same question several times, Player always provides the same answer to Opponent (uniform model) or may vary his answers in the course of the interaction (non uniform model.) Note that every backtracking model may be called uniform in the sense that Player provides his answer once and for all.

Two remarkable models lie at both extremes of the spectrum:

- Lamarche [26] reformulates Berry and Curien sequential algorithm model of PCF [9] as an error-free, alternated, backtracking, uniform game model of intuitionistic linear logic. The interested reader will find a nice exposition of that work by Curien in [15,5]

- We indicate in section 7.5 that the less constrained of all arena game models, introduced by Abramsky, Honda and McCusker [2] is equivalent to an error-free, non alternated, repetitive, non uniform game model of intuitionistic linear logic.

Intermediate models were also considered in the litterature, most notably an alternated, repetitive, non uniform model by Hyland in [21]. We connect all these models by coercion laws in section 7; and deduce the following “topography” of models:

(a) All error-aware hierarchies are related by back-and-forth translations, and thus collapse extensionally to the same hierarchy: Berry and Curien sequential algorithm hierarchy with one error, what we call the *manifestly sequential* hierarchy after Cartwright, Curien and Felleisen [9,14].

(b) All error-free hierarchies are related by back-and-forth translations, and thus collapse extensionally to the same hierarchy: Bucciarelli and Ehrhard strongly stable hierarchy, by Ehrhard collapse theorem [17].

(c) All error-aware and error-free hierarchies are related by back-and-forth translations when erroes are not taken into account in the base types (using partial equivalence relations).

There is a recent thesis (defended by Longley [28] among others) that every sufficiently expressive *error-free* model of sequential computations collapses to the strongly stable hierarchy. After points (a)(b)(c), it is natural to *factorize* Longley’s thesis into:

(1) a thesis: every sufficiently expressive *error-aware* model of sequential computations collapses to the manifestly sequential hierarchy,

(2) a fact: the manifestly sequential hierarchy collapses to the strongly stable hierarchy when errors are not taken into account in the base types.

Diagrammatically:

$$\boxed{\begin{array}{c} \text{Any sufficiently expressive model of sequentiality with errors} \\ \Big\downarrow \text{ extensional collapse (1)} \\ \text{Manifestly sequential hierarchy} \\ \Big\downarrow \text{ extensional collapse (2)} \\ \text{Strongly stable hierarchy} \end{array}}$$

This sits the manifestly sequential hierarchy (with one or several errors) at a key position in the theory of sequentiality, and reveals at the same time its true nature: the extensional collapse of other (possibly more immediate) models of sequentiality.

Synopsis. In section 2, we deliver the necessary preliminaries on categorical models of linear logic, hierarchies of simple types, and extensional collapse. In section 3, we formulate the notion of *back-and-forth translation* between hierarchies of types, and prove that two hierarchies related by a back-and-forth translation collapse to the same extensional hierarchy. In section 4, we axiomatize the notion of linear coercion between models of linear logic. Our main theorem 15 appears in section 5. It states that two models related by a linear coercion, induce hierarchies related by a back-and-forth translation. In section 6, we illustrate the theorem by relating the qualitative and quantitative exponentials on coherence (and hypercoherence) space models ; we also analyze in detail the action of the back-and-forth translation at types $o \Rightarrow o$ and $(o \Rightarrow o) \Rightarrow o$. In section 7, we introduce the error-free and error-aware variants of two categories of sequential games, and compare three exponential structures on these categories: backtracking, repetitive uniform, and repetitive non uniform. We establish a series of linear coercions between the exponentials and models, and deduce from it that (1) all error-aware models collapse to the manifestly sequential hierarchy, and (2) all error-free models collapse to the strongly stable hierarchy.

Related works. T. Ehrhard [17] proves that the sequential algorithm hierarchy [9] collapses to the strongly stable hierarchy [13]. This result is important because it relates for the first time a *static* and a *dynamic* model of sequentiality. The theorem is proved another time by J. Van Oosten [36] and J. Longley [28] in a similar and somewhat indirect way: first, they establish that every finite strongly stable functional is equal to a PCF-term applied to some strongly stable functionals *of small order* (several of them of order 2 in [17], exactly one of order 3 in [28]) ; then they deduce Ehrhard's collapse theorem by denotational techniques.
After publishing his collapse theorem in [17], T. Ehrhard started studying other (possibly simpler) cases of extensional collapse, in order to extract general proof-techniques, which would lead ideally to a more direct proof of his theorem. For instance, T. Ehrhard establishes in collaboration with N. Barreiro [7] that the quantitative hierarchy of coherence spaces collapses to qualitative one, by exhibiting an heterogeneous relation between the two hierarchies, which is then shown to be *onto* for finite functionals. The same pattern of proof appears in A. Bucciarelli's work on bidomains [12]. One feels that a general proof-technique remains to be extracted, but the proof in [7] does not help much, because it requires a very precise and "anatomic" description of the extensional collapse, which seems difficult to generalize to other situations.
In a recent article inspired by concurrency [31], the author relates Lamarche sequential games and Ehrhard hypercoherence spaces; and delivers an "anatomic" proof of Ehrhard's collapse theorem based on games semantics. The present article results from the author's efforts to simplify the proof of [31] as much as possible: in particular, a back-and-forth translation between the sequential algorithm hierarchies on the *flat* and on the *lazy* natural numbers enables to decompose the proof of [31] in two steps: first, the finitely branching games are treated by a compactness argument (König's lemma); the result is then generalized to (possibly infinitely branching) games like the flat natural numbers, by exhibiting the back-and-forth translation and applying the results established in the present article.
Finally, recent discussions with J. Longley indicate that our definition of linear coercion between models of linear logic makes sense in (a linear and typed version

of) the 2-category of Partial Combinatory Algebra considered in [27]. This point deserves to be further investigated, because it could very well lead to a more conceptual proof of corollary 16 based on realisability.

2 Preliminaries

2.1 Monoidal closed categories

By monoidal closed category, we mean a monoidal category $\mathcal{C}$ in which the functor $(A \otimes -) : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint $(A \multimap -) : \mathcal{C} \longrightarrow \mathcal{C}$ for every object A of $\mathcal{C}$. Thanks to a theorem on adjunctions with parameters [29], the family of functors $(A \multimap -)$ may be seen as a bifunctor $\multimap: \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \longrightarrow \mathcal{C}$ for which there exists a family of bijections

$$\phi_{A,B,C} : \quad \mathcal{C}(A \otimes B, C) \;\; \cong \;\; \mathcal{C}(B, A \multimap C)$$

natural in A contravariantly, in B, C covariantly. In particular, every morphism $f \in \mathcal{C}(A, B)$ is in one-to-one relation with its *name* $\ulcorner f \urcorner \in \mathcal{C}(1, A \multimap B)$ defined as $\ulcorner f \urcorner = \phi_{A,1,B}(f \circ \rho_A^{-1})$.

Remark. We write 1 for the monoidal unit of the category $\mathcal{C}$, instead of the usual notation I. We follow here an habit of linear logic, dating back to the origin of the subject [19].

By (symmetric) monoidal functor between (symmetric) monoidal categories, we mean the *lax* definition, that is, a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ equipped with mediating natural transformations

$$m_{A,B} : F(A) \otimes_{\mathcal{N}} F(B) \longrightarrow F(A \otimes_{\mathcal{M}} B) \qquad\qquad m_{1_{\mathcal{M}}} : 1_{\mathcal{N}} \longrightarrow F(1_{\mathcal{M}})$$

making the usual diagrams commute. It is worth mentioning here a useful property of monoidal functors.

Lemma 1 *Suppose that $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a monoidal functor between monoidal closed categories. Then, there exist a family q of morphisms indexed by objects A, B of $\mathcal{C}$:*

$$q_{A,B} : F(A \multimap_{\mathcal{M}} B) \;\longrightarrow (FA \multimap_{\mathcal{N}} FB)$$

such that, for every morphism $f : A \longrightarrow B$, the diagram below commutes:

$$\begin{tikzcd}
F(1_{\mathcal{M}}) \arrow[r, "F(\ulcorner f \urcorner)"] & F(A \multimap_{\mathcal{M}} B) \arrow[d, "q_{A,B}"] \\
1_{\mathcal{N}} \arrow[u, "m_{1_{\mathcal{M}}}"] \arrow[r, "\ulcorner Ff \urcorner"] & (FA \multimap_{\mathcal{N}} FB)
\end{tikzcd} \tag{5}$$

where $\ulcorner f \urcorner$ and $\ulcorner Ff \urcorner$ are the names of the morphisms f in $\mathcal{C}$ and Ff in $\mathcal{D}$.

PROOF The morphism $q_{A,B}$ is defined as the unique morphism making the diagram below commute:

$$\begin{tikzcd}
FA \otimes_{\mathcal{N}} F(A \multimap_{\mathcal{M}} B) \arrow[r, "m_{A,A\multimap_{\mathcal{M}}B}"] \arrow[d, "FA\otimes q_{A,B}"'] & F(A \otimes_{\mathcal{M}} (A \multimap_{\mathcal{M}} B)) \arrow[d, "F(ev^{\mathsf{one}}_{A,B})"] \\
FA \otimes_{\mathcal{N}} (FA \multimap_{\mathcal{N}} FB) \arrow[r, "ev^{\mathsf{two}}_{FA,FB}"] & FB
\end{tikzcd}$$

Commutativity of diagram (5) follows easily. ∎

2.2 *Models of intuitionistic linear logic*

There exist several categorical definitions of what a model of intuitionistic linear logic should be. Instead of reviewing them here, which we do in [30], we will only indicate what properties of a model we need in this article. The reader interested in full definitions is advised to look at [21,10,30].

So, every model $\mathcal{M}$ of intuitionistic linear logic is given by (among other things) a symmetric monoidal closed category $\mathcal{C}$ equipped with what we call here an *exponential structure*, that is:

- a functor ! of the category $\mathcal{C}$ into itself,
- a morphism $\varepsilon_A : !A \longrightarrow A$ for every object A of the category,
- a morphism $f^{\mathsf{bang}} : 1 \longrightarrow !A$ for every morphism $f : 1 \longrightarrow A$ of the category $\mathcal{C}$, verifying:

$$1 \xrightarrow{\ f\ } A \quad = \quad 1 \xrightarrow{f^{\mathsf{bang}}} !A \xrightarrow{\varepsilon_A} A \tag{6}$$

  and making the diagram below commute for every morphism $g : A \longrightarrow B$:

$$\begin{tikzcd}
 & !A \arrow[d, "!g"] \\
1 \arrow[ur, bend left, "f^{\mathsf{bang}}"] \arrow[dr, bend right, "(g\circ f)^{\mathsf{bang}}"'] & !B
\end{tikzcd} \tag{7}$$

Remark. Another property which should be mentioned here, even if it is not used in the article, is that the endofunctor ! defines a comonad over the category $\mathcal{C}$, whose associated co-kleisli category happens to be cartesian closed.

2.3 *Hierarchies of types*

In this article, we consider the class of simple types T built over a fixed class K of constant types $\kappa \in K$, given by the grammar below:

$$T \ ::= \ \kappa \in K \mid T \Rightarrow T.$$

The typical example is $K = \{o, \iota\}$ where o and ι denote the boolean and the integer base types respectively.
A *hierarchy* $([-], \cdot, \sim)$ over K consists of:
(1) a family of sets $[T]$ indexed by simple type T,
(2) a family of functions indexed by simple types U, V:

$$\cdot_{UV} : [U \Rightarrow V] \times [U] \longrightarrow [V].$$

(3) a partial equivalence relation $\sim_T$ over the set $[T]$, for every simple type T, which verifies that, for every simple types U, V, and every elements $f, g \in [U \Rightarrow V]$:

$$f \sim_{U \Rightarrow V} g \iff (\forall x, y \in [U], x \sim_U y \Rightarrow f \cdot x \sim_V g \cdot y). \tag{8}$$

Given $f \in [U \Rightarrow V]$ and $x \in [U]$, we write $f \cdot_{UV} x$ or even $f \cdot x$ for the image of (f, x) by $\cdot_{UV}$ in $[V]$.

Remark. For expository reasons mainly, we add the family of partial equivalence relations (point 3 above) to the usual definition of a hierarchy $([-], \cdot)$. Let us clarify this. Property (8) implies that the family of partial equivalence relations is generated by the sub-family $(\sim_\kappa)_{\kappa \in K}$ of partial equivalence relations at constant types. So, a hierarchy in our sense is simply a hierarchy $([-], \cdot)$ in the usual sense, equipped with a partial equivalence relation $\sim_\kappa$ for every constant type $\kappa \in K$. We find convenient to integrate this family $(\sim_\kappa)_{\kappa \in K}$ in our definition, in order to discuss cases of extensional collapse in which the choice of $(\sim_\kappa)_{\kappa \in K}$ matters.

2.4 Models of linear logic over a class of constants

A model $\mathcal{M}$ of intuitionistic linear logic over a class K of constants, is a model of intuitionistic linear logic equipped, for every constant type $\kappa \in K$, with:
(1) an object X_κ of the underlying monoidal category $\mathcal{C}$,
(2) a partial equivalence relation $\sim_\kappa^{\mathcal{M}}$ over the set $\mathcal{C}(1, X_\kappa)$ of global elements of X_κ in the category $\mathcal{C}$.

Any such model $\mathcal{M}$ induces a hierarchy $([-], \cdot, \sim)$ over K, obtained by regarding every object $[T]$ of the category $\mathcal{C}$ as its set $\mathbf{Hom}_{\mathcal{C}}(1, [T])$ of global elements. The construction goes as follows. Every constant type $\kappa \in K$ is associated to the object $[\kappa] = X_\kappa$; and every simple type $T = U \Rightarrow V$ is associated to the object $[T]$ deduced from $[U]$ and $[V]$ by Girard's formula:

$$[U \Rightarrow V] \;=\; ![U] \multimap [V].$$

The function $\cdot_{UV} : [U \Rightarrow V] \times [U] \longrightarrow [V]$ associates to the pair $f : 1 \longrightarrow [U \Rightarrow V]$ and $x : 1 \longrightarrow [U]$ the composite $f \cdot x : 1 \longrightarrow [V]$ in the category $\mathcal{C}$:

$$1 \xrightarrow{f \cdot x} [V] \quad = \quad 1 \xrightarrow{x^{\mathsf{bang}}} ![U] \xrightarrow{\llcorner f \lrcorner} [V]$$

Here, the morphism $\llcorner f \lrcorner$ denotes the "co-name" of f, that is the unique morphism $![U] \longrightarrow [V]$ such that $f = \ulcorner \llcorner f \lrcorner \urcorner$.
The partial equivalence relation $\sim_T$ over the set of global elements $\mathbf{Hom}_{\mathcal{C}}(1, [T])$ is given by $\sim_\kappa^{\mathcal{M}}$ at a constant type $\kappa \in K$, and deduced from $\sim_U$ and $\sim_V$ by property (8) at a simple type $T = U \Rightarrow V$.

2.5 Extensional collapse

A hierarchy $([-], \cdot, \sim)$ is *extensional* when the partial equivalence relation $\sim_T$ is the equality at every simple type T. In that case, it follows from property (8) that, for every type $U \Rightarrow V$ and elements f, g of $[U \Rightarrow V]$, one has:

$$(\forall x \in [U], f \cdot x = g \cdot x) \quad \Rightarrow \quad f = g.$$

Every hierarchy $([-], \cdot, \sim)$ induces an extensional hierarchy $([-]_{\text{ext}}, \centerdot, =)$ called its *extensional collapse*. The construction goes as follows: $[T]_{\text{ext}}$ denotes the set $[T]/\sim_T$ of $\sim_T$-classes in $[T]$; while $\overline{f} \centerdot_{UV} \overline{a}$ denotes the $\sim_V$-class of $f \cdot_{UV} a$, for every two elements f of the $\sim_{U \Rightarrow V}$-class $\overline{f}$ and a of the $\sim_U$-class $\overline{a}$. We leave the reader check that the definition works, and induces an extensional hierarchy $([-]_{\text{ext}}, \centerdot, =)$.

3 Back-and-forth translations between hierarchies of types

In this section, we introduce the notion of *back-and-forth translation* between hierarchies of types, and show that two hierarchies related by such a translation collapse to the same extensional hierarchy (lemma 6).

3.1 The definition of back-and-forth translation

Definition 2 *A back-and-forth translation between two hierarchies of types*

$$([-], \cdot, \sim) \qquad \textit{and} \qquad ([\![-]\!], \cdot, \approx)$$

is the data of two families of (set-theoretic) functions

$$\phi_T : [T] \longrightarrow [\![T]\!] \qquad\qquad \psi_T : [\![T]\!] \longrightarrow [T]$$

indexed by simple types, such that

(1) the two functions ϕ_κ and ψ_κ preserve the partial equivalence relations at any base type $\kappa \in K$, that is:

$$\forall x, y \in [\kappa], \qquad x \sim_\kappa y \Rightarrow \phi_\kappa(x) \approx_\kappa \phi_\kappa(y),$$

$$\forall x, y \in [\![\kappa]\!], \qquad x \approx_\kappa y \Rightarrow \psi_\kappa(x) \sim_\kappa \psi_\kappa(y),$$

(2) the two functions ϕ_κ and ψ_κ are "weak inverse" at any base type $\kappa \in K$, that is:

$$\forall x, y \in [\kappa], \qquad x \sim_\kappa y \Rightarrow x \sim_\kappa \psi_\kappa(\phi_\kappa(y)),$$
$$\forall x, y \in [\![\kappa]\!], \qquad x \approx_\kappa y \Rightarrow x \approx_\kappa \phi_\kappa(\psi_\kappa(y)),$$

(3) for every types U, V, and elements $f \in [U \Rightarrow V]$ and $h \in [\![U]\!]$:

$$\phi_{U\Rightarrow V}(f) \cdot h \approx_V \phi_V(f \cdot \psi_U(h)), \tag{9}$$

(4) for every types U, V, and elements $f \in [\![U \Rightarrow V]\!]$ and $h \in [U]$:

$$\psi_{U\Rightarrow V}(f) \cdot h \sim_V \psi_V(f \cdot \phi_U(h)). \tag{10}$$

Remark. Our definition of back-and-forth translation may be weakened by requiring equivalence (9) only when $f \sim_{U\Rightarrow V} f$ and $h \approx_U h$, and similarly for equivalence (10). Our main result, lemma 6, still holds in that weaker situation — which we find for example in lemma 26.

Remark. Back-and-forth translations define a category between hierarchies, with obvious identities, and composition defined as follows. Suppose that families of functions:

$$\phi_T : [T] \rightleftarrows [\![T]\!] : \psi_T \qquad \phi'_T : [\![T]\!] \rightleftarrows [\![\![T]\!]\!] : \psi'_T$$

define back-and-forth translations between the hierarchies $[-]$ and $[\![-]\!]$ on one hand, and between the hierarchies $[\![-]\!]$ and $[\![\![-]\!]\!]$ on the other hand. Then, the families of functions obtained by composition:

$$\phi'_T \circ \phi_T : [T] \rightleftarrows [\![\![T]\!]\!] : \psi_T \circ \psi'_T$$

defines a back-and-forth translation between the hierarchies $[-]$ and $[\![\![-]\!]\!]$.

3.2 Back-and-forth translation and extensional collapse

Here, we prove that the existence of a back-and-forth translation between $[-]$ and $[\![-]\!]$ implies that the two hierarchies collapse to the same extensional hierarchy.

Lemma 3 (preservation) *$\sim_T$ and $\approx_T$ are preserved by translation. More precisely:*

$$\forall f, g \in [T], \qquad f \sim_T g \Rightarrow \phi_T(f) \approx_T \phi_T(g),$$
$$\forall f, g \in [\![T]\!], \qquad f \approx_T g \Rightarrow \psi_T(f) \sim_T \psi_T(g).$$

PROOF By induction on T. The property holds for every base type $\kappa \in K$ by definition of a back-and-forth translation, point (1). Then, suppose that the property is established for types U and V; and consider any two elements $f, g \in [U \Rightarrow V]$ such that $f \sim_{U\Rightarrow V} g$. We want to show that

$$\phi_{U\Rightarrow V}(f) \approx_{U\Rightarrow V} \phi_{U\Rightarrow V}(g). \tag{11}$$

To that purpose, we consider $h \approx_U h'$ and prove that

$$\phi_{U \Rightarrow V}(f) \cdot h \approx_V \phi_{U \Rightarrow V}(g) \cdot h'.$$

By definition of the back-and-forth translation, this reduces to

$$\phi_V(f \cdot \psi_U(h)) \approx_V \phi_V(g \cdot \psi_U(h')). \tag{12}$$

Let us prove claim (12). By induction hypothesis on U, and hypothesis $h \approx_U h'$:

$$\psi_U(h) \sim_U \psi_U(h').$$

From this, and hypothesis $f \sim_{U \Rightarrow V} g$, it follows:

$$f \cdot \psi_U(h) \sim_V g \cdot \psi_U(h').$$

We conclude that claim (12) holds by induction hypothesis on V. We have just proved (11). We prove

$$\forall f, g \in [\![U \Rightarrow V]\!], \qquad f \approx_{U \Rightarrow V} g \Rightarrow \psi_{U \Rightarrow V}(f) \sim_{U \Rightarrow V} \psi_{U \Rightarrow V}(g)$$

in a similar fashion. This concludes our proof by induction. ■

Lemma 4 (forth and back)

$$\forall f, g \in [T], \qquad f \sim_T g \Rightarrow f \sim_T \psi_T(\phi_T(g)).$$

PROOF By induction on T. The property holds for every base type $\kappa \in K$ by definition of a back-and-forth translation, point (2). Now, suppose that $f \sim_{U \Rightarrow V} g$. We prove that

$$f \sim_{U \Rightarrow V} \psi_{U \Rightarrow V}(\phi_{U \Rightarrow V}(g))$$

by establishing that, for every $h \sim_U h'$:

$$f \cdot h \sim_V \psi_{U \Rightarrow V}(\phi_{U \Rightarrow V}(g)) \cdot h'. \tag{13}$$

The right-hand side of the equivalence may be reformulated by definition of a back-and-forth translation:

$$\begin{aligned} \psi_{U \Rightarrow V}(\phi_{U \Rightarrow V}(g)) \cdot h' &\sim_V \psi_V(\phi_{U \Rightarrow V}(g) \cdot \phi_U(h')) \\ &\sim_V \psi_V(\phi_V(g \cdot \psi_U(\phi_U(h')))). \end{aligned}$$

Equation (13) follows by induction hypothesis on U and V, and hypothesis $f \sim_{U \Rightarrow V} g$. ■

Lemma 5 (back and forth)

$$\forall f, g \in [\![T]\!], \qquad f \approx_T g \Rightarrow f \approx_T \phi_T(\psi_T(g)).$$

Lemma 6 *Two hierarchies related by a back-and-forth translation, collapse to the same extensional hierarchy.*

4 Linear coercion between models of linear logic

In sections 4.2, 4.3 and 4.4, we define what we mean by a *linear coercion* between two models $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic over a class K of base types. Before that, in section 4.1, we focus on the particular case of two models $\mathcal{M}$ and $\mathcal{N}$ constructed over the same underlying monoidal category $\mathcal{C}$, and the same interpretation X_κ and $\sim_\kappa$ of the base types $\kappa \in K$.

Notation: in this section 4, as well as in section 5, we instantiate the notation f^{bang} introduced in section 2.2, and write

$$1_{\mathcal{M}} \xrightarrow{f^{\dagger}} !^{\mathcal{M}}A \qquad\qquad 1_{\mathcal{N}} \xrightarrow{g^{\dagger\dagger}} !^{\mathcal{N}}B$$

for the morphisms induced from the exponential structures of $\mathcal{M}$ and $\mathcal{N}$ applied on the morphism $f : 1_{\mathcal{M}} \longrightarrow A$ in $\mathcal{C}$ and $g : 1_{\mathcal{N}} \longrightarrow B$ in $\mathcal{D}$, respectively.

4.1 Linear coercion between exponential structures

We specialize our later definition of linear coercion (see section 4.4) to the particular case of two models $\mathcal{M}$ and $\mathcal{N}$ of linear logic with the same underlying monoidal category $\mathcal{C}$. In that case, the two models $\mathcal{M}$ and $\mathcal{N}$ are only distinguished by their respective exponential structures $!^{\mathcal{M}}$ and $!^{\mathcal{N}}$.

Definition 7 (linear coercion) *A* linear coercion *between two exponential structures* $!^{\mathcal{M}}$ *and* $!^{\mathcal{N}}$ *consists in two families* η *and* ϱ *of morphisms indexed by objects of the category* $\mathcal{C}$

$$!^{\mathcal{M}}A \xrightarrow{\eta_A} !^{\mathcal{N}}A \xrightarrow{\varrho_A} !^{\mathcal{M}}A$$

making the two diagrams below commute,

$$\begin{tikzcd} & !^{\mathcal{M}}A \arrow[d, "\eta_A"] \\ 1 \arrow[ur, bend left, "f^{\dagger}"] \arrow[dr, bend right, "f^{\dagger\dagger}"'] & \\ & !^{\mathcal{N}}A \end{tikzcd} \qquad\qquad \begin{tikzcd} & !^{\mathcal{N}}A \arrow[d, "\varrho_A"] \\ 1 \arrow[ur, bend left, "f^{\dagger\dagger}"] \arrow[dr, bend right, "f^{\dagger}"'] & \\ & !^{\mathcal{M}}A \end{tikzcd}$$

for every morphism $f : 1 \longrightarrow A$ *of the category* $\mathcal{C}$.

Definition 7 is an instance of a linear coercion between two models of intuitionistic linear logic over K, as formulated in section 4.4. More precisely, every choice of a family $(X_\kappa)_{\kappa\in K}$ of objects of the category, and of a family $(\sim_\kappa)_{\kappa\in K}$ of partial equivalence relations over their global elements, induces a model $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic over K. The linear coercion between $!^{\mathcal{M}}$ and $!^{\mathcal{N}}$ formulated in definition 7 induces a linear coercion between the two models $\mathcal{M}$ and $\mathcal{N}$ in the sense of definition 10. In particular, theorem 15 holds, and thus, for any choice of families $(X_\kappa)_{\kappa\in K}$ and $(\sim_\kappa)_{\kappa\in K}$, the two hierarchies deduced from $!^{\mathcal{M}}$ and $!^{\mathcal{N}}$ collapse to the same extensional hierarchy.

Definition 8 (monoidal elementwise transformation) *A monoidal elementwise transformation $\xi : F \Rightarrow G$ between two monoidal functors $(F, m) : \mathcal{C} \longrightarrow \mathcal{D}$ and $(G, n) : \mathcal{C} \longrightarrow \mathcal{D}$ is a family of morphisms $\xi_A : F(A) \longrightarrow G(A)$ indexed by objects of $\mathcal{C}$, making the two diagrams commute:*

$$
\begin{tikzcd}
 & F(1_{\mathcal{C}}) \arrow[dd, "\xi_{1_{\mathcal{C}}}"] \\
1_{\mathcal{D}} \arrow[ur, bend left, "m_{1_{\mathcal{C}}}"] \arrow[dr, bend right, "n_{1_{\mathcal{C}}}"'] & \\
 & G(1_{\mathcal{C}})
\end{tikzcd}
\qquad
\begin{tikzcd}
F(1_{\mathcal{C}}) \arrow[r, "F(f)"] \arrow[d, "\xi_{1_{\mathcal{C}}}"'] & F(A) \arrow[d, "\xi_A"] \\
G(1_{\mathcal{C}}) \arrow[r, "G(f)"] & G(A)
\end{tikzcd}
$$

for every morphism $f : 1_{\mathcal{C}} \longrightarrow A$.

Remark. Elementwise means that the naturality diagram commutes for every global elements $f : 1 \longrightarrow A$; and monoidal that the object 1 *believes* that the two coherence diagrams of monoidal natural transformations commute. Check in particular that, for every global element $f : 1_{\mathcal{C}} \longrightarrow A$ and $g : 1_{\mathcal{C}} \longrightarrow B$, the diagram below commutes:

$$
\begin{tikzcd}
F(A) \otimes F(B) \arrow[r, "m_{A,B}"] \arrow[d, "\xi_A \otimes \xi_B"'] & F(A \otimes B) \arrow[d, "\xi_{A \otimes B}"] \\
G(A) \otimes G(B) \arrow[r, "n_{A,B}"] & G(A \otimes B)
\end{tikzcd}
$$

when precomposed with the global element $(F(f) \otimes F(g)) \circ (m_{1_{\mathcal{C}}} \otimes m_{1_{\mathcal{C}}}) \circ \rho^{-1}_{1_{\mathcal{D}}}$.

Remark. In the particular case of two monoidal functors $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{D} \longrightarrow \mathcal{C}$, a monoidal elementwise transformation $\xi : Id_{\mathcal{C}} \Rightarrow GF$ (resp. $\zeta : Id_{\mathcal{D}} \Rightarrow FG$) is alternatively defined as a family of morphisms making the lefthand (resp. righthand) diagram below commute:

$$
\begin{tikzcd}
1_{\mathcal{C}} \arrow[r, "f"] \arrow[d, "n_{1_{\mathcal{D}}}"'] & A \arrow[dd, "\xi_A"] \\
G(1_{\mathcal{D}}) \arrow[d, "G(m_{1_{\mathcal{C}}})"'] & \\
GF(1_{\mathcal{C}}) \arrow[r, "GF(f)"] & GF(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathcal{D}} \arrow[r, "g"] \arrow[d, "m_{1_{\mathcal{C}}}"'] & B \arrow[dd, "\zeta_B"] \\
F(1_{\mathcal{C}}) \arrow[d, "F(n_{1_{\mathcal{D}}})"'] & \\
FG(1_{\mathcal{D}}) \arrow[r, "FG(g)"] & FG(B)
\end{tikzcd}
\qquad (14)
$$

for every pair of global elements $f : 1_{\mathcal{C}} \longrightarrow A$ and $g : 1_{\mathcal{D}} \longrightarrow B$.

4.3 Distributive law

Suppose given two models $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic, and a monoidal functor $(F, m) : (\mathcal{C}, \otimes_{\mathcal{M}}, 1_{\mathcal{M}}) \longrightarrow (\mathcal{D}, \otimes_{\mathcal{N}}, 1_{\mathcal{N}})$ between their underlying monoidal categories $\mathcal{C}$ and $\mathcal{D}$.

Definition 9 (distributive law) *A distributive law*

$$\varrho : \ !^{\mathcal{N}} F \ \Rightarrow \ F \ !^{\mathcal{M}}$$

is a family of morphisms (ϱ_A) of $\mathcal{D}$ indexed by objects of $\mathcal{C}$, making the diagram below commute for every morphism $f : 1_{\mathcal{M}} \longrightarrow A$ of the category $\mathcal{C}$:

$$\begin{tikzcd}
1_{\mathcal{N}} \arrow[r, "(F(f) \circ m_{1_{\mathcal{M}}})^{\dagger\dagger}"] \arrow[d, "m_{1_{\mathcal{M}}}"'] & !^{\mathcal{N}} F(A) \arrow[d, "\varrho_A"] \\
F(1_{\mathcal{M}}) \arrow[r, "F(f^{\dagger})"] & F(\ !^{\mathcal{M}} A)
\end{tikzcd} \tag{15}$$

Remark. In every model of intuitionistic linear logic, the functor ! defines a *monoidal* comonad, see [21,10,30]. So, a condition stronger than commutativity of diagram (15) would be to require that ϱ is a *monoidal* natural transformation ϱ : $!^{\mathcal{N}} F \stackrel{\cdot}{\longrightarrow} F \ !^{\mathcal{M}}$. Commutativity of diagram (15) would then follow from commutativity of the diagram below, which follows from monoidality (lefthand-side) and naturality (righthand-side) of ϱ. Note that $\underline{m}_{1_{\mathcal{M}}}$ and $\underline{n}_{1_{\mathcal{N}}}$ denote the monoidal coercions of $!^{\mathcal{M}}$ and $!^{\mathcal{N}}$ respectively.

$$\begin{tikzcd}
1_{\mathcal{N}} \arrow[r, "\underline{n}_{1_{\mathcal{N}}}"] \arrow[d, "m_{1_{\mathcal{M}}}"'] & !^{\mathcal{N}} 1_{\mathcal{N}} \arrow[r, "!^{\mathcal{N}} m_{1_{\mathcal{M}}}"] & !^{\mathcal{N}} F(1_{\mathcal{M}}) \arrow[r, "!^{\mathcal{N}} F(f)"] \arrow[d, "\varrho_{1_{\mathcal{M}}}"] & !^{\mathcal{N}} F(A) \arrow[d, "\varrho_A"] \\
F(1_{\mathcal{M}}) \arrow[rr, "F(\underline{m}_{1_{\mathcal{M}}})"] & & F(\ !^{\mathcal{M}} 1_{\mathcal{M}}) \arrow[r, "F(\ !^{\mathcal{M}} f)"] & F(\ !^{\mathcal{M}} A)
\end{tikzcd}$$

We choose definition 9 instead of this more conceptual definition, for practical reasons. In the introduction, we exhibit a family of morphisms $\varrho_A : !^{\mathsf{mset}} A \longrightarrow !^{\mathsf{set}} A$ in the category of coherence spaces, see (1). This family defines a distributive law in our sense (definition 9) but at the same time, is *not* natural in A. Indeed, if $\Delta : A \longrightarrow A \& A$ denotes the diagonal morphism induced by the cartesian product $\&$, the diagram below does not necessarily commute, for similar reasons as diagram (2) (again, take $A = \bot$ the singleton coherence space).

$$\begin{tikzcd}
!^{\mathsf{mset}} A \arrow[r, "\varrho_A"] \arrow[d, "!^{\mathsf{mset}} \Delta_A"'] & !^{\mathsf{set}} A \arrow[d, "!^{\mathsf{set}} \Delta_A"] \\
!^{\mathsf{mset}} (A \& A) \arrow[r, "\varrho_{A \& A}"] & !^{\mathsf{set}} (A \& A)
\end{tikzcd}$$

Remark. Our definition differs also from Hyland and Schalk's definition [23] of a linearly distributive law $\lambda : !^{\mathcal{N}} F \ \Rightarrow \ F \ !^{\mathcal{N}}$ as a distributive law in the sense of Beck [8,34] respecting the comonoid structure, that is, making the diagram below commute:

$$
\begin{tikzcd}
1_{\mathcal{N}} \arrow[dd, "m_{1_{\mathcal{M}}}"'] & {!^{\mathcal{N}} F(A)} \arrow[l, "e^{\mathcal{N}}_{FA}"'] \arrow[r, "d^{\mathcal{N}}_{F(A)}"] \arrow[dd, "\lambda_A"] & {!^{\mathcal{N}} F(A) \otimes_{\mathcal{N}} !^{\mathcal{N}} FA} \arrow[d, "\lambda_A \otimes_{\mathcal{N}} \lambda_A"] \\
 & & {F(\,!^{\mathcal{M}} A) \otimes_{\mathcal{N}} F(\,!^{\mathcal{M}} A)} \arrow[d, "m_{\,!^{\mathcal{M}} A,\,!^{\mathcal{M}} A}"] \\
F(1_{\mathcal{M}}) & {F(\,!^{\mathcal{M}} A)} \arrow[l, "F(e^{\mathcal{M}}_A)"'] \arrow[r, "F(d^{\mathcal{M}}_A)"] & {F(\,!^{\mathcal{M}} A \otimes_{\mathcal{M}} !^{\mathcal{M}} A)}
\end{tikzcd}
\tag{16}
$$

for every object A of the category $\mathcal{C}$. This definition implies that the functor F lifts to a functor between the kleisli category of cofree coalgebras — which does not necessarily happen with our notion of distributivity. Again, we choose a less conceptual definition for practical reasons: diagram (16) specializes as diagram (2) when applied to the category $\mathcal{COH}$ equipped with the qualitative and quantitative exponentials $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$, and this diagram (2) does not commute generally.

4.4 Linear coercion between models of linear logic

In this section, we consider two models $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic over a class K of constants, as formulated in section 2.4. Their underlying monoidal categories are denoted $\mathcal{C}$ and $\mathcal{D}$, and their families of constants $(X_\kappa, \sim_\kappa)_{\kappa \in K}$ and $(Y_\kappa, \approx_\kappa)_{\kappa \in K}$ respectively.

Definition 10 (linear coercion) *A linear coercion between $\mathcal{M}$ and $\mathcal{N}$ is given by:*

(1) a pair of monoidal functors $(F, m) : \mathcal{C} \longrightarrow \mathcal{D}$ and $(G, n) : \mathcal{D} \longrightarrow \mathcal{C}$,

(2) a pair of monoidal elementwise transformations $\xi : Id_{\mathcal{C}} \Rightarrow GF$ and $\zeta : Id_{\mathcal{D}} \Rightarrow FG$,

(3) a pair of distributive laws $\eta : !^{\mathcal{M}} G \Rightarrow G\, !^{\mathcal{N}}$ and $\varrho : !^{\mathcal{N}} F \Rightarrow F\, !^{\mathcal{M}}$,

(4) for every constant $\kappa \in K$, a pair of morphisms $\overline{\phi}_\kappa : F(X_\kappa) \longrightarrow Y_\kappa$ and $\overline{\psi}_\kappa : G(Y_\kappa) \longrightarrow X_\kappa$ making the two diagrams below commute modulo $\approx_\kappa$ and $\sim_\kappa$ respectively, when the two morphisms $x, y : 1_{\mathcal{M}} \longrightarrow X_\kappa$ verify $x \sim_\kappa y$:

$$
\begin{tikzcd}
 & F(1_{\mathcal{M}}) \arrow[r, "F(x)"] & F(X_\kappa) \arrow[dr, "\overline{\phi}_\kappa"] & \\
1_{\mathcal{N}} \arrow[ur, "m_{1_{\mathcal{M}}}"] \arrow[dr, "m_{1_{\mathcal{M}}}"'] \arrow[rrr, phantom, "\approx_\kappa"] & & & Y_\kappa \\
 & F(1_{\mathcal{M}}) \arrow[r, "F(y)"'] & F(X_\kappa) \arrow[ur, "\overline{\phi}_\kappa"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{\mathcal{M}} \arrow[r, "x"] \arrow[d, "n_{1_{\mathcal{N}}}"'] & X_\kappa \\
G(1_{\mathcal{N}}) \arrow[d, "G(m_{1_{\mathcal{M}}})"'] \arrow[r, phantom, "\sim_\kappa"] & G(Y_\kappa) \arrow[u, "\overline{\psi}_\kappa"'] \\
GF(1_{\mathcal{M}}) \arrow[r, "GF(y)"'] & GF(X_\kappa) \arrow[u, "G(\overline{\phi}_\kappa)"']
\end{tikzcd}
$$

and making the two diagrams below commute modulo $\sim_\kappa$ and $\approx_\kappa$ respectively, *when the two morphisms $x, y : 1_{\mathcal{N}} \longrightarrow Y_\kappa$ verify $x \approx_\kappa y$:*

$$\begin{tikzcd} & G(1_{\mathcal{N}}) \arrow[r, "G(x)"] & G(Y_\kappa) \arrow[dr, "\overline{\psi}_\kappa"] & \\ 1_{\mathcal{M}} \arrow[ur, "n_{1_{\mathcal{N}}}"] \arrow[dr, "n_{1_{\mathcal{N}}}"'] \arrow[rrr, phantom, "\sim_\kappa"] & & & X_\kappa \\ & G(1_{\mathcal{N}}) \arrow[r, "G(y)"'] & G(Y_\kappa) \arrow[ur, "\overline{\psi}_\kappa"'] & \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{N}} \arrow[rr, "x"] \arrow[d, "m_{1_{\mathcal{M}}}"'] & & Y_\kappa \\ F(1_{\mathcal{M}}) \arrow[d, "F(n_{1_{\mathcal{N}}})"'] \arrow[rr, phantom, "\approx_\kappa"] & & F(X_\kappa) \arrow[u, "\overline{\phi}_\kappa"'] \\ FG(1_{\mathcal{N}}) \arrow[rr, "FG(y)"'] & & FG(Y_\kappa) \arrow[u, "F(\overline{\psi}_\kappa)"'] \end{tikzcd}$$

Remark. It is not difficult to show that, given a linear coercion between $\mathcal{M}$ and $\mathcal{N}$, the diagrams below commute for every pair of morphisms $f : 1_{\mathcal{M}} \longrightarrow A$ in the category $\mathcal{C}$ and $g : 1_{\mathcal{N}} \longrightarrow B$ in the category $\mathcal{D}$:

$$\begin{tikzcd} 1_{\mathcal{M}} \arrow[rr, "f^\dagger"] \arrow[dd, "n_{1_{\mathcal{N}}}"'] & & !^{\mathcal{M}}A \arrow[d, "!^{\mathcal{M}}\xi_A"] \\ & & !^{\mathcal{M}}GF(A) \arrow[d, "\eta_{F(A)}"] \\ G(1_{\mathcal{N}}) \arrow[rr, "G\left(F(f) \circ m_{1_{\mathcal{M}}} \right)^{\dagger\dagger}"] & & G\,!^{\mathcal{N}}F(A) \end{tikzcd} \qquad \begin{tikzcd} 1_{\mathcal{N}} \arrow[rr, "g^{\dagger\dagger}"] \arrow[dd, "m_{1_{\mathcal{M}}}"'] & & !^{\mathcal{N}}B \arrow[d, "!^{\mathcal{M}}\zeta_B"] \\ & & !^{\mathcal{N}}FG(B) \arrow[d, "\varrho_{G(B)}"] \\ F(1_{\mathcal{M}}) \arrow[rr, "F\left(G(g) \circ n_{1_{\mathcal{N}}} \right)^{\dagger}"] & & F\,!^{\mathcal{M}}G(B) \end{tikzcd} \tag{17}$$

Point (2) of definition 10 is slightly enigmatic. It is mainly here to ensure the existence of morphisms $A \longrightarrow G\,!^{\mathcal{N}}F(A)$ and $B \longrightarrow F\,!^{\mathcal{M}}G(B)$ making the diagrams (17) commute. In fact, we could very well remove point (2) of definition 10 and forget the two transformations ξ and ζ, but at a heavy price: we need to replace the distributive laws of point (3) by the (slightly unorthodox) laws $!^{\mathcal{M}} \Rightarrow F\,!^{\mathcal{N}}G$ and $!^{\mathcal{N}} \Rightarrow G\,!^{\mathcal{N}}F$; and we must require accordingly that the straightforward variant of diagram (17) commutes. If we do so, our main result (theorem 15 in section 5) still holds.

5 From linear coercions to back-and-forth translations

We prove our main result here (theorem 15). Given two models $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic over a class K of constants, we proceed as in section 2.4, and derive their respective hierarchies $([-], \cdot, \sim)$ and $([\![-]\!], \cdot, \approx)$. Theorem 15 states that there exists a back-and-forth translation between the hierarchies $([-], \cdot, \sim)$ and $([\![-]\!], \cdot, \approx)$ when there exists a linear coercion between the two models $\mathcal{M}$ and $\mathcal{N}$. So, we suppose from now on that the two models $\mathcal{M}$ and $\mathcal{N}$ are related by a linear coercion, with same notations as in section 4.4. Our first step is to extend to every simple type T the families of coercion maps $(\overline{\phi}_\kappa)_{\kappa \in K}$ and $(\overline{\psi}_\kappa)_{\kappa \in K}$ given at constant types in definition 10.

Definition 11 (coercion maps at every type (1)) *The two families of morphisms below*

$$F([T]) \xrightarrow{\overline{\phi}_T} [\![T]\!] \qquad\qquad G([\![T]\!]) \xrightarrow{\overline{\psi}_T} [T]$$

indexed by simple types T, are defined by structural induction:

$$\overline{\phi}_{U\Rightarrow V} \;=\; \Big(\big(F(\,!^{\mathcal{M}}\overline{\psi}_U) \circ \varrho_{G[\![U]\!]} \circ\, !^{\mathcal{N}}\zeta_{[\![U]\!]}\big) \multimap_{\mathcal{N}} \overline{\phi}_V\Big) \;\circ\; q^F_{!^{\mathcal{M}}[U],[V]}$$

$$\overline{\psi}_{U\Rightarrow V} \;=\; \Big(\big(G(\,!^{\mathcal{N}}\overline{\phi}_U) \circ \eta_{F[U]} \circ\, !^{\mathcal{M}}\xi_{[U]}\big) \multimap_{\mathcal{M}} \overline{\psi}_V\Big) \;\circ\; q^G_{!^{\mathcal{N}}[\![U]\!],[\![V]\!]}$$

Definition 12 (coercion maps at every type (2)) *For every element $f \in [T]$, the element $\phi_T(f) \in [\![T]\!]$ is defined as follows:*

$$1_{\mathcal{N}} \xrightarrow{m_{1_{\mathcal{M}}}} F(1_{\mathcal{M}}) \xrightarrow{F(f)} F([T]) \xrightarrow{\overline{\phi}_T} [\![T]\!]$$

Similarly, for every element $f \in [\![T]\!]$, the element $\psi_T(f) \in [T]$ is defined as follows:

$$1_{\mathcal{M}} \xrightarrow{n_{1_{\mathcal{N}}}} G(1_{\mathcal{N}}) \xrightarrow{G(f)} G([\![T]\!]) \xrightarrow{\overline{\psi}_T} [T]$$

Lemma 13 *For every element $f \in [U \Rightarrow V]$ and $h \in [\![U]\!]$,*

$$\phi_{U\Rightarrow V}(f) \;\cdot\; h \;=\; \phi_V(f \;\cdot\; \psi_U(h)).$$

PROOF Consider two elements $f \in [U \Rightarrow V]$ and $h \in [\![U]\!]$.
It is worth recalling that the element $\phi_{U\Rightarrow V}(f) \;\cdot\; h \in [\![V]\!]$ is defined in section 2.4 as the composite:

$$1_{\mathcal{N}} \xrightarrow{h^{\dagger\dagger}} ![\![U]\!] \xrightarrow{\llcorner\!\llcorner \phi_{U\Rightarrow V}(f) \lrcorner\!\lrcorner} [\![V]\!]$$

where $\llcorner\!\llcorner \phi_{U\Rightarrow V}(f) \lrcorner\!\lrcorner$ denotes the morphism of name $\phi_{U\Rightarrow V}(f) \in [\![U \Rightarrow V]\!]$. Now, let

$$\llcorner f \lrcorner : \,!^{\mathcal{M}}[U] \longrightarrow [V]$$

denote the morphism of name $f = \ulcorner \llcorner f \lrcorner \urcorner$ in the category $\mathcal{C}$; and let

$$\ulcorner\!\ulcorner F \llcorner f \lrcorner \urcorner\!\urcorner : 1_{\mathcal{N}} \longrightarrow F(\,!^{\mathcal{M}}[U]) \multimap_{\mathcal{N}} F([V])$$

denote the name of the morphism $F\llcorner f \lrcorner$ in the category $\mathcal{D}$.
By lemma 1, the diagram below commutes:

$$\begin{tikzcd}
F(1_{\mathcal{M}}) \arrow[rr, "Ff"] & & F(\,!^{\mathcal{M}}[U] \multimap_{\mathcal{M}} [V]) \arrow[d, "q^F_{!^{\mathcal{M}}[U],[V]}"] \\
1_{\mathcal{N}} \arrow[u, "m_{1_{\mathcal{M}}}"] \arrow[rr, "\ulcorner\!\ulcorner F \llcorner f \lrcorner \urcorner\!\urcorner"] & & (F\,!^{\mathcal{M}}[U]) \multimap_{\mathcal{N}} F([V])
\end{tikzcd}$$

From this and definitions 11 and 12, it follows that $\phi_{U\Rightarrow V}(f)$ is equal to the morphism

$$1_{\mathcal{N}} \xrightarrow{\ulcorner\!\ulcorner F \llcorner f \lrcorner \urcorner\!\urcorner} F(\,!^{\mathcal{M}}[U]) \multimap_{\mathcal{N}} F([V])$$

post-composed with

$$\Big(F(\,!^{\mathcal{M}}\overline{\psi}_U) \circ \varrho_{G[\![U]\!]} \circ\, !^{\mathcal{N}}\zeta_{[\![U]\!]}\Big) \multimap_{\mathcal{N}} \overline{\phi}_V .$$

From this, and naturality in A and B of the bijection

$$\mathcal{D}(A,B) \cong \mathcal{D}(1_{\mathcal{N}}, A \multimap_{\mathcal{N}} B)$$

the diagram below commutes:

$$\begin{tikzcd}
!^{\mathcal{N}}[\![U]\!] \arrow[r, "\llcorner\!\llcorner \phi_{U\Rightarrow V}(f) \lrcorner\!\lrcorner"] \arrow[d, "!^{\mathcal{N}}\zeta_{[\![U]\!]}"] & {[\![V]\!]} \\
!^{\mathcal{N}}FG[\![U]\!] \arrow[d, "\varrho_{G[\![U]\!]}"] & \\
F\,!^{\mathcal{M}}G[\![U]\!] \arrow[d, "F\,!^{\mathcal{M}}\overline{\psi}_U"] & \\
F\,!^{\mathcal{M}}[U] \arrow[r, "F\llcorner f \lrcorner"] & F[V] \arrow[uuu, "\overline{\phi}_V"']
\end{tikzcd} \tag{18}$$

Now, we show that diagram (19) commutes. Diagram a. commutes by the property of exponential structures recalled in section 2.2. Diagram b. commutes by definition 8 of a monoidal elementwise transformation $\zeta : Id_{\mathcal{C}} \Rightarrow FG$. Diagram c. commutes by definition of the distributive law ρ. Finally, diagram d. commutes by definition 12 of $\psi_U(h)$ as the composite

$$\psi_U(h) \;=\; \overline{\psi}_U \circ G(h) \circ m_{1_{\mathcal{N}}}$$

and functoriality of F. We conclude that diagram (19) commutes.

$$\begin{tikzcd}
1_{\mathcal{N}} \arrow[r, "h^{\dagger\dagger}"] \arrow[d, equal] \arrow[dr, phantom, "a."] & !^{\mathcal{N}}[\![U]\!] \arrow[r, "\llcorner\!\llcorner \phi_{U\Rightarrow V}(f) \lrcorner\!\lrcorner"] \arrow[d, "!^{\mathcal{N}}\zeta_{[\![U]\!]}"] & {[\![V]\!]} \\
1_{\mathcal{N}} \arrow[r, "(\zeta_{[\![U]\!]} \circ h)^{\dagger\dagger}"] \arrow[d, equal] \arrow[dr, phantom, "b."] & !^{\mathcal{N}}FG[\![U]\!] \arrow[d, equal] & \\
1_{\mathcal{N}} \arrow[r, "\left(F(G(h) \circ n_{1_{\mathcal{N}}}) \circ m_{1_{\mathcal{M}}}\right)^{\dagger\dagger}"] \arrow[d, "m_{1_{\mathcal{M}}}"'] \arrow[dr, phantom, "c."] & !^{\mathcal{N}}FG[\![U]\!] \arrow[d, "\varrho_{G[\![U]\!]}"] & \\
F(1_{\mathcal{M}}) \arrow[r, "F\left(\,G(h) \circ n_{1_{\mathcal{N}}}\,\right)^{\dagger}"] \arrow[d, equal] \arrow[dr, phantom, "d."] & F\,!^{\mathcal{M}}G[\![U]\!] \arrow[d, "F\,!^{\mathcal{M}}\overline{\psi}_U"] & \\
F(1_{\mathcal{M}}) \arrow[r, "F\left(\,\psi_U(h)\,\right)^{\dagger}"] & F\,!^{\mathcal{M}}[U] \arrow[r, "F\llcorner f \lrcorner"] & F[V] \arrow[uuuu, "\overline{\phi}_V"']
\end{tikzcd} \tag{19}$$

It follows that $\phi_{U\Rightarrow V}(f) \,\cdot\, h$ is equal to the composite

$$1_{\mathcal{N}} \xrightarrow{m_{1_{\mathcal{M}}}} F(1_{\mathcal{M}}) \xrightarrow{F(f\,\cdot\,\psi_U(h))} F[V] \xrightarrow{\overline{\phi}_V} [\![V]\!]$$

which is precisely the element $\phi_V(f \cdot \psi_U(h))$. This concludes the proof. ■

Lemma 14 *For every elements $f \in [\![U \Rightarrow V]\!]$ and $h \in [U]$,*

$$\psi_{U \Rightarrow V}(f) \cdot h = \psi_V(f \cdot \phi_U(h)).$$

PROOF As for lemma 13. ■

Theorem 15 (main result) *Suppose that two models of intuitionistic linear logic over a class K of constant types $\mathcal{M}$ and $\mathcal{N}$ are related by a linear coercion. Then, their associated hierarchies $([-], \cdot, \sim)$ and $([\![-]\!], \cdot, \approx)$ are related by a back-and-forth translation.*

In that case, it follows from lemma 6 that:

Corollary 16 *The two hierarchies $([-], \cdot, \sim)$ and $([\![-]\!], \cdot, \approx)$ collapse to the same extensional hierarchy.*

6 Application 1: coherence and hypercoherence spaces

6.1 A linear coercion between the qualitative and the quantitative exponentials

In the introduction, we exhibit a family of embedding-retraction pairs (1) in the category $\mathcal{COH}$ of coherence spaces:

$$\eta_A : !^{\mathsf{set}}A \longrightarrow !^{\mathsf{mset}}A, \qquad \varrho_A : !^{\mathsf{mset}}A \longrightarrow !^{\mathsf{set}}A.$$

We claim that the families η and ϱ define a linear coercion (in the sense of definition 7) between the exponentials $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$. Indeed, consider any morphism $f : 1 \longrightarrow A$, or equivalently any clique f of A. The cliques f^{set} of $!^{\mathsf{set}}A$ and f^{mset} of $!^{\mathsf{mset}}A$ are defined as follows:

$$f^{\mathsf{set}} = \{x \in |\,!^{\mathsf{set}}A| \;\big|\; x \subset f\}, \qquad f^{\mathsf{mset}} = \{w \in |\,!^{\mathsf{mset}}A| \;\big|\; \mathsf{support}(w) \subset f\}.$$

The equality $f^{\mathsf{set}} \circ \eta_A = f^{\mathsf{mset}}$ holds by definition of f^{mset}; while the equality $f^{\mathsf{mset}} \circ \varrho_A = f^{\mathsf{set}}$ holds because for every element $x \in |\,!^{\mathsf{set}}A|$,

$$x \subset f \iff \mathsf{support}(\mathsf{char}(x)) \subset f.$$

So, theorem 15 implies:

Corollary 17 (Barreiro-Ehrhard) *The qualitative hierarchy over coherence spaces, (also called the stable hierarchy) is the extensional collapse of the quantitative one.*

Theorem 15 applied in a similar fashion to the hypercoherence space model introduced in [16], shows that:

Corollary 18 *The qualitative hierarchy over hypercoherence spaces (also called the strongly stable hierarchy) is the extensional collapse of the quantitative one.*

Remark. The interested reader will find theorem 15 applied in Boudes' PhD thesis [11] to relate refinements of the quantitative and qualitative strongly stable hierarchies.

6.2 An illustration at types $o \Rightarrow o$ and $(o \Rightarrow o) \Rightarrow o$

In our proof of corollary 17, we exhibit for every type T an embedding-retraction pair

$$[T]^{\mathsf{set}} \xrightarrow{\quad \phi_T \quad} [T]^{\mathsf{mset}} \xrightarrow{\quad \psi_T \quad} [T]^{\mathsf{set}}$$

between the qualitative and quantitative interpretations $[T]^{\mathsf{set}}$ and $[T]^{\mathsf{mset}}$ in the category of coherence spaces. The morphism ψ_T transports any clique in $[T]^{\mathsf{mset}}$ to its "extensional content" in $[T]^{\mathsf{set}}$, while ϕ_T transports any function in $[T]^{\mathsf{set}}$ to a "canonical representative" in $[T]^{\mathsf{mset}}$. By construction, the composite $\psi_T \circ \phi_T$ is the identity on $[T]^{\mathsf{set}}$, and the composite $p = \phi_T \circ \psi_T$ transports every clique $f \in [T]^{\mathsf{mset}}$ to a "canonical form" $p(f) \in [T]^{\mathsf{mset}}$. In order to illustrate this, let us compute the canonical form of a clique, for the types $T = o \Rightarrow o$ and $T = (o \Rightarrow o) \Rightarrow o$ of the hierarchy over the boolean base type (that is: $K = \{o\}$). We recall that the coherence space $X_o = 1 \oplus 1$ representing the booleans has exactly two elements true and false in its web, which are incoherent.
When $T = (o \Rightarrow o)$, the elements $f \in [o \Rightarrow o]^{\mathsf{mset}}$ are of five possible forms:

(1) f is empty,

(2) $f = \{([-], b)\}$ is constant,

(3) $f = \{([\overbrace{\mathsf{true}, ..., \mathsf{true}}^{k}], b)\}$,

(4) $f = \{([\overbrace{\mathsf{false}, ..., \mathsf{false}}^{k}], b)\}$,

(5) $f = \{([\overbrace{\mathsf{true}, ..., \mathsf{true}}^{k}], b), ([\overbrace{\mathsf{false}, ..., \mathsf{false}}^{k'}], b')\}$.

for $b, b' \in \{\mathsf{true}, \mathsf{false}\}$ and $k, k' \geq 1$. The canonical form $p(f)$ is computed as follows:

- $p(f) = f$ when f is empty, or constant,
- otherwise, $p(f)$ is f in which every element $([b, ..., b], b') \in f$ is altered into the element $([b], b') \in p(f)$, for $b = \mathsf{true}$ and $b = \mathsf{false}$.

Intuitively, transforming f into $p(f)$ amounts to replacing the "stuttering" f by the clique $p(f)$ which "asks its questions only once".
When $T = (o \Rightarrow o) \Rightarrow o$, a clique $f \in [T]^{\mathsf{mset}}$ contains elements of five possible forms only:

(1) $\big([-], b\big)$

(2) $\Big(\Big[\overbrace{([-], b), ..., ([-], b)}^{j}\Big], b'\Big)$

(3) $\Big(\Big[\overbrace{([\mathsf{true}^k], b), ..., ([\mathsf{true}^k], b)}^{j}\Big], b'\Big)$

(4) $\Big(\Big[\overbrace{([\mathsf{false}^k], b), ..., ([\mathsf{false}^k], b)}^{j}\Big], b'\Big)$

(5) $\Big(\Big[\overbrace{([\mathsf{true}^k], b), ..., ([\mathsf{true}^k], b)}^{j}, \overbrace{([\mathsf{false}^{k'}], b'), ..., ([\mathsf{false}^{k'}], b')}^{j'}\Big], b''\Big)$.

for $b, b', b'' \in \{\mathsf{true}, \mathsf{false}\}$ and $j, j', k, k' \geq 1$. Here, $[\mathsf{true}^k]$ and $[\mathsf{false}^k]$ are shorter notations for the multi-sets $[\mathsf{true}, ..., \mathsf{true}]$ and $[\mathsf{false}, ..., \mathsf{false}]$ of cardinality k.

The translation of $f \in [T]^{\mathsf{mset}}$ into $p(f) \in [T]^{\mathsf{mset}}$ proceeds as follows:

- step a: if $\Big([-], b\Big)$ is element of f, keep it in $p(f)$,
- step b: translate every element in f of the form (2) into the element $\Big([([-], b)], b'\Big)$ in $p(f)$,
- step c: remove from f every element of the form (3,4,5) in which $k > 1$ or $k' > 1$;
- step d: translate every remaining element x of the form (3,4,5) in f, into *all* elements of the corresponding form (3,4,5) for every pair of integers $k, k' \geq 1$:

(3) $\Big(\Big[([\underbrace{\mathsf{true}, ..., \mathsf{true}}_{k}], b)\Big], b'\Big)$,

(4) $\Big(\Big[([\underbrace{\mathsf{false}, ..., \mathsf{false}}_{k}], b)\Big], b'\Big)$,

(5) $\Big(\Big[([\underbrace{\mathsf{true}, ..., \mathsf{true}}_{k}], b), ([\underbrace{\mathsf{false}, ..., \mathsf{false}}_{k'}], b')\Big], b''\Big)$.

So, intuitively, transforming f into $p(f)$ at type $(o \Rightarrow o) \Rightarrow o$ amounts to:

- step a: keep the constants,
- step b: replace every Player's "stuttering questions" by a "single question",
- step c: remove every check by Player of Opponent's "stuttering questions",
- step d: expand every check by Player of an Opponent's "single question", by a check on all equivalent Opponent's "stuttering questions".

We would like to illustrate this transformation with an example. Consider the clique Φ of $[((o \Rightarrow o) \Rightarrow o]^{\mathsf{mset}}$ introduced by Barreiro and Ehrhard in [7]:

$$\Phi = \{([[\mathsf{true}], \mathsf{true}], \mathsf{true}), ([[\mathsf{true}, \mathsf{true}], \mathsf{true}], \mathsf{false})\}.$$

The clique Φ "tastes" whether a "function" $h \in [o \Rightarrow o]^{\mathsf{mset}}$ requires its argument true once or twice, before answering true. Since the two cliques $\{([\mathsf{true}], \mathsf{true})\}$ and $\{([\mathsf{true}, \mathsf{true}], \mathsf{true})\}$ are equivalent at type $o \Rightarrow o$, the taster Φ which separates them, is not equivalent to itself modulo $\approx_{(o\Rightarrow o)\Rightarrow o}$.

Now, observe that the clique Φ is transported by $\psi_{(o\Rightarrow o)\Rightarrow o}$ to the element $\Psi \in [(o \Rightarrow o) \Rightarrow o]^{\mathsf{set}}$ below:

$$\Psi = \{\{(\{\mathsf{true}\}, \mathsf{true})\}, \mathsf{true})\}.$$

Part of the information has disappeared in the translation. Recall that the qualitative hierarchy $[-]^{\mathsf{set}}$ is extensional. So, $\sim_{(o\Rightarrow o)\Rightarrow o}$ is just the equality, and the singleton Ψ is therefore equivalent to itself modulo $\sim_{(o\Rightarrow o)\Rightarrow o}$. The function ψ_T transports Ψ back to the canonical element $p(\Phi)$ of $[(o \Rightarrow o) \Rightarrow o]^{\mathsf{mset}}$:

$$p(\Phi) = \{([([\underbrace{\mathsf{true}, ..., \mathsf{true}}_{k}], \mathsf{true})], \mathsf{true}) \quad | \quad k \geq 1\}$$

It follows from lemma 3 that $p(\Phi)$ is equivalent to itself modulo $\approx_{(o\Rightarrow o)\Rightarrow o}$. This illustrates the fact that the embedding-retraction between $[T]^{\mathsf{set}}$ and $[T]^{\mathsf{mset}}$ defines a procedure which "repairs" cliques of $[T]^{\mathsf{mset}}$ by pruning out their non-extensional behaviours.

Remark. The choice of the projection map ϱ_A is somewhat arbitrary. For instance, we may have chosen any of the alternative family of cliques

$$\varrho_A^n = \{(n \times \mathsf{char}(x), x) \mid x \text{ is a finite clique of } A\}$$

to play the role of $\varrho = \varrho^1$. To clarify our notation, $n \times \mathsf{char}(x)$ denotes here the characteristic function of the set x, multiplied by the integer $n \geq 1$: that is, the multiset of support x in which every element is repeated n times. Any of the ϱ_A^n defines with η_A a linear coercion (and even embedding-projection pair) between $!^{\mathsf{set}}$ and $!^{\mathsf{mset}}$. Observe that the projection p explicated above is already altered at types $o \Rightarrow o$ and $(o \Rightarrow o) \Rightarrow o$, by a choice of coercion ϱ^n different from ϱ. For instance, $p(\Phi)$ is replaced by

$$p'(\Phi) = \{([([\underbrace{\mathsf{true}, ..., \mathsf{true}}_{k}], \mathsf{true})], \mathsf{false}) \quad | \quad k \geq 1\}$$

when $n = 2$.

Remark. In their proof that the quantitative hierarchy collapses to the stable hierarchy, Barreiro and Ehrhard deliver an interesting "anatomy" of the extensional collapse, quite far from what we explain here. It would be instructive to understand how the two analysis are precisely related.

7 Application 2: sequential games

The definitions of sequential game $A = (M_A, \lambda_A, P_A)$ and of sequential strategy σ are given in the introduction, and we do not recall them. We only mention that a strategy σ of A is alternatively defined as a set of alternated plays of A verifying that, for every play s and moves m, n_1, n_2:

(1) σ is nonempty: the empty play ϵ is element of σ,
(2) σ is closed under prefix: if $s \cdot m \in \sigma$, then $s \in \sigma$,
(3) σ is deterministic: if $s \cdot m \cdot n_1 \in \sigma$ and $s \cdot m \cdot n_2 \in \sigma$ and $\lambda_A(n_1) = \lambda_A(n_2) = +1$, then $s \cdot m \cdot n_1 = s \cdot m \cdot n_2$.

As already indicated, this definition enables a strategy to withdraw at any point of the interaction, and play "error". The usual definition of *error-free* strategy is given in definition 20.

Definition 19 (deadlock,error,fixpoint) *We suppose below that σ is a strategy.*

- *a play s is called maximal in σ when $s \in \sigma$ and $\forall m \in M_A, s \cdot m \notin \sigma$,*
- *a deadlock of σ is an odd-length play $s \cdot m$ such that $s \cdot m \notin \sigma$ but $s \in \sigma$,*
- *an error of σ is an odd-length play $s \cdot m$ maximal in σ,*
- *a fixpoint of σ is an error or an even-length play of σ.*

Notation: We write P_A^{even}, P_A^{odd} and P_A^{alt} for the even-length, odd-length and alternated plays of a sequential game A. We write $\sigma : A$ when σ is a strategy of A, and

$\text{even}(\sigma)$ and $\text{error}(\sigma)$ and $\text{fix}(\sigma) = \text{even}(\sigma) \cup \text{error}(\sigma)$ for the sets of even-length plays, errors and fixpoints of σ respectively.

Definition 20 (error-free strategy) *A strategy $\sigma : A$ is error-free when* $\text{error}(\sigma) = \emptyset$*, or equivalently, when:*

$$\forall s \in P_A^{\text{odd}}, \quad s \in \sigma \;\Rightarrow\; \exists m \in M_A,\; s \cdot m \in \sigma.$$

Remark. Every strategy σ may be recovered from $\text{fix}(\sigma)$ by the equality below:

$$\sigma = \text{fix}(\sigma) \cup \{s \in P_A, \exists m \in M_A, s \cdot m \in \text{fix}(\sigma)\}. \tag{20}$$

In particular, every error-free strategy is characterized by the set $\text{even}(\sigma)$ which coincides with $\text{fix}(\sigma)$ in that case.

7.1 The category $\mathcal{G}^{\text{err}}$ of sequential games (error-aware)

The category $\mathcal{G}^{\text{err}}$ is a negative and error-aware variant of the category of Conway games formulated by Joyal in [24]. By negative, we mean that all games start by an Opponent move; and by error-aware, that the strategies possibly admit errors.

The category $\mathcal{G}^{\text{err}}$ has sequential games as objects and strategies of $A \multimap B$ as morphisms $A \longrightarrow B$. Given two sequential games A, B, the sequential game $A \multimap B$ is defined by reversing the polarities of the moves of A, and interleaving the plays of A and B:

- $M_{A \multimap B} = M_A + M_B$ and $\lambda_{A \multimap B} = [-\lambda_A, \lambda_B]$,
- a play s of $A \multimap B$ is a string over the alphabet $M_{A \multimap B}$ such that (1) the projection $s_{|A}$ over M_A is a play of A and (2) the projection $s_{|B}$ over M_B is a play of B and (3) s starts by a move of B if non empty.

Composition is defined in $\mathcal{G}^{\text{err}}$ by *sequential composition + hiding,* identities by *copycat* strategies, in the usual fashion, see e.g. [1,21]. In the presence of errors, the composition and identity laws are better defined on sets of fixpoints, rather than on strategies — just as in concurrent games [4]. Typically, the identity of A has fixpoints;

$$\text{fix}(\mathbf{id}_A) = \{s \in P_{A \multimap A}^{\text{even}}, \forall t \in P_{A \multimap A}^{\text{even}}, t \text{ is prefix of } s \Rightarrow t_{|A_1} = t_{|A_2}\}$$

where the indices $1, 2$ indicate on which component of $A_1 \multimap A_2$ the play t is projected. The composite of two strategies $\sigma : A \multimap B$ and $\tau : B \multimap C$ is the strategy $\tau \circ \sigma : A \multimap C$ whose set of fixpoints $\text{fix}(\tau \circ \sigma)$ is given by:

$$\{s \in P_{A \multimap C}^{\text{alt}} \mid \exists t \in (M_A + M_B + M_C)^*, t_{|A,B} \in \text{fix}(\sigma), t_{|B,C} \in \text{fix}(\tau), t_{|A,C} = s\}$$

where $(M_A + M_B + M_C)^*$ denotes the set of finite strings (=words) on the alphabet $M_A + M_B + M_C$.

The category $\mathcal{G}^{\text{err}}$ is symmetric monoidal closed, with tensor product $A \otimes B$ of two sequential games A, B defined as the sequential game obtained by "freely interleaving" the plays of A and B:

- $M_{A\otimes B} = M_A + M_B$ and $\lambda_{A\otimes B} = [\lambda_A, \lambda_B]$,
- a play of $A \otimes B$ is a string of moves in $M_{A\otimes B}$ such that $s_{|A} \in P_A$ and $s_{|B} \in P_B$.

The monoidal unit 1 is the game with an empty set of moves.

7.2 The category $\mathcal{A}^{\mathrm{err}}$ of alternated games (error-aware)

The category $\mathcal{A}^{\mathrm{err}}$ is an error-aware variant of the category of negative alternated games generally considered in the litterature, typically in [1,26,15,21,5]. The category $\mathcal{A}^{\mathrm{err}}$ is defined as the full subcategory of alternated games in $\mathcal{G}^{\mathrm{err}}$. The resulting category $\mathcal{A}^{\mathrm{err}}$ is not a submonoidal category of $\mathcal{G}^{\mathrm{err}}$, since the tensor product of two alternated games in $\mathcal{A}^{\mathrm{err}}$ may not be alternated. But fortunately, the category $\mathcal{A}^{\mathrm{err}}$ is the "intersection" of a reflective subcategory and a co-reflective subcategory of $\mathcal{G}^{\mathrm{err}}$, and the monoidal structure of $\mathcal{A}^{\mathrm{err}}$ may be deduced from that. Let us explain this point below.

Call a sequential game OP-alternated (resp. PO-alternated) when only Player (resp. Opponent) may play two successive moves in a play of the game A. The full subcategory of OP-alternated games is reflective in $\mathcal{G}^{\mathrm{err}}$: every strategy $A \longrightarrow B$ to an OP-alternated game B factorizes as $A \longrightarrow T(A) \longrightarrow B$ in a unique way, where $T(A)$ is the OP-alternated game obtained from A by removing every play containing two successive Opponent moves, and $A \longrightarrow T(A)$ is the obvious error-free copycat strategy. Dually, the full subcategory of PO-alternated games is coreflective, with counit $D(A) \longrightarrow A$ the copycat strategy between A and the PO-alternated game obtained by removing every play containing two successive Player moves in A.

The category $\mathcal{A}^{\mathrm{err}}$ is symmetric monoidal closed, with tensor and closed structure deduced from their counterpart in $\mathcal{G}^{\mathrm{err}}$, as follows. Let A and B denote two alternated games:
- their tensor product $A \otimes_{\mathsf{alt}} B$ in the category $\mathcal{A}^{\mathrm{err}}$ is the alternated game $T(A \otimes B)$,
- their closed structure $A \multimap_{\mathsf{alt}} B$ is the alternated game $D(A \multimap B)$,
- the monoidal units of $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{A}^{\mathrm{err}}$ coincide.

There is certainly more to say about the categorical situation: for instance, the monad T distributes over the comonad D in the sense of [35,33], the distributive law $\lambda : TD \Rightarrow DT$ being just the identity; and the category $\mathcal{A}^{\mathrm{err}}$ is precisely the category of λ-bialgebras. An axiomatic account in the vein of [6] would be interesting, but beyond the scope of this article. We indicate only what is needed to build a linear coercion between $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{A}^{\mathrm{err}}$.

We write $U : \mathcal{A}^{\mathrm{err}} \longrightarrow \mathcal{G}^{\mathrm{err}}$ for the inclusion functor and $\mathrm{alt} : \mathcal{G}^{\mathrm{err}} \longrightarrow \mathcal{A}^{\mathrm{err}}$ for the functor which transports every morphism $f : A \longrightarrow B$ to the morphism $DT(f) : DT(A) \longrightarrow DT(B)$. These two functors define monoidal functors (U, m) and (alt, n) with mediating natural transformations:
- $m_{A,B} : A \otimes B \longrightarrow A \otimes_{\mathsf{alt}} B$ is the unit of T at instance $A \otimes B$; and m_1 is the identity of $1 = U(1)$;
- $n_{A,B} : \mathrm{alt}(A) \otimes_{\mathsf{alt}} \mathrm{alt}(B) \longrightarrow \mathrm{alt}(A \otimes B)$ is the obvious error-free copycat strategy restricted to the plays of $\mathrm{alt}(A) \otimes_{\mathsf{alt}} \mathrm{alt}(B)$; and n_1 is the identity of $1 = \mathrm{alt}(1)$.

Every morphism $\sigma : 1 \longrightarrow B$ in the category $\mathcal{G}^{\mathrm{err}}$ is a strategy of B, thus a set of alternated plays of B. It follows that the diagram below commutes:

$$\begin{tikzcd} 1 \arrow[r, "\sigma"] \arrow[dr, "\mathrm{alt}(\sigma)"'] & B \arrow[d, "\zeta_B"] \\ & \mathrm{alt}(B) \end{tikzcd}$$

for $\zeta_B : B \longrightarrow \mathrm{alt}(B)$ the obvious error-free copycat strategy. On the other hand, the functor $(\mathrm{alt} \circ U)$ coincides with the identity functor of the category $\mathcal{A}^{\mathrm{err}}$. Thus, the family $(\xi_A) = (\mathbf{id}_A)$ of identities indexed by alternated games, and the family (ζ_B) indexed by sequential games, define two monoidal elementwise transformations $\xi : Id \Rightarrow \mathrm{alt} \circ U$ and $\zeta : Id \Rightarrow U \circ \mathrm{alt}$ in the sense of definition 8 — see also diagram (14).

7.3 The categories $\mathcal{G}$ and $\mathcal{A}$ of sequential and alternated games (error-free)

We write $\mathcal{G}$ and $\mathcal{A}$ for the subcategories of error-free strategies in the categories $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{A}^{\mathrm{err}}$ respectively. The two categories $\mathcal{G}$ and $\mathcal{A}$ are symmetric monoidal closed, their structure being inherited in each case from the surrounding category $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{A}^{\mathrm{err}}$.

7.4 Three models on alternated games (error-aware + error-free)

Each category $\mathcal{A}^{\mathrm{err}}$ and $\mathcal{A}$ gives rise to three models of intuitionistic linear logic, which differ only in their interpretation of the exponential modality, either as the backtracking $!^{\mathsf{btk}}$, the repetitive non uniform $!^{\mathsf{rpt}}$ or the repetitive uniform $!^{\mathsf{unif}}$ exponential. Each exponential structure $!^{\mathsf{btk}}$ and $!^{\mathsf{rpt}}$ and $!^{\mathsf{unif}}$ expresses a particular memory or uniformity paradigm, which we recall briefly now.

The backtracking exponential $!^{\mathsf{btk}}$ is defined by Lamarche [26] on the category $\mathcal{A}$, but is easily adapted to the error-aware setting of $\mathcal{A}^{\mathrm{err}}$. The reader is advised to follow the presentation of Lamarche's work by Curien [15,5]. The model of intuitionistic linear logic induced by $\mathcal{A}$ and $!^{\mathsf{btk}}$ *linearizes* the sequential algorithm model of PCF [9], in the sense that the co-kleisli category associated to the comonad $!^{\mathsf{btk}}$ embeds (as a model of PCF) in the category of concrete data structures and sequential algorithms. Similarly, the model of intuitionistic linear logic based on $\mathcal{A}^{\mathrm{err}}$ and $!^{\mathsf{btk}}$ linearizes an error-aware variant of the sequential algorithm model, already formulated by Cartwright, Curien and Felleisen in [14]: the *manifestly sequential function* model of PCF — with exactly one error. The associated hierarchy of types — which we call the *manifestly sequential* hierarchy — is *extensional*. This important fact reappears in corollary 22.

The repetitive non uniform exponential $!^{\mathsf{rpt}}$ is defined by Hyland in his course notes on game semantics [21]. Like the exponential $!^{\mathsf{btk}}$, the exponential $!^{\mathsf{rpt}}$ is defined on the category $\mathcal{A}$ but is easily adapted to the error-aware setting of $\mathcal{A}^{\mathrm{err}}$. In the sequential game $!^{\mathsf{btk}}A$ defined by Lamarche, Opponent has some kind of "memory" of the past, and thus does not need to ask Player the same question twice in the course of the interaction. Instead, Opponent simply *backtracks* to Player's previous answer to the question. In contrast, in the sequential game $!^{\mathsf{rpt}}A$, Opponent does not memorize Player's answer, and thus asks Player the same question as many times as necessary. This "repetitive" style enables "non-uniform" behaviours by Player, in which the same answer is not necessarily given to the same question repeated by Opponent. Technically, the plays of the alternated game $!^{\mathsf{rpt}}A$ are defined in [21] as the finite alternated strings over the alphabet $M_A \times \mathbb{N}$ such that (i) every projection over $i \in \mathbb{N}$ is a play in A, and (ii) the first move in the $(i+1)$-th copy is made after the first move in the i-th copy. The resulting game models are closer to arena games: in section 7.5, we observe that, once adapted to non-alternated games, the exponential $!^{\mathsf{rpt}}$ linearizes a well-known arena game model of the litterature.

The repetitive uniform exponential $!^{\mathsf{unif}}$ is a variant of the exponential $!^{\mathsf{rpt}}$ in which copies are regulated by a "uniformity" principle. A play of $!^{\mathsf{rpt}}A$ is called *uniform* when there exists a strategy σ of A, such that every projection $s_{|i} \in P_A$ is element of σ. The alternated game $!^{\mathsf{unif}}A$ is simply defined as the game $!^{\mathsf{rpt}}A$ restricted to its uniform plays.

Linear coercions between the exponentials $!^{\mathsf{btk}}$ **and** $!^{\mathsf{rpt}}$ **and** $!^{\mathsf{unif}}$ may be exhibited in each category $\mathcal{A}^{\mathrm{err}}$ and $\mathcal{A}$, inducing in each case two families of embedding-retraction pairs indexed by alternated games A:

$$!^{\mathsf{btk}}A \xrightarrow{\eta_A} !^{\mathsf{unif}}A \xrightarrow{\varrho_A} !^{\mathsf{btk}}A, \qquad !^{\mathsf{unif}}A \xrightarrow{\eta'_A} !^{\mathsf{rpt}}A \xrightarrow{\varrho'_A} !^{\mathsf{unif}}A. \tag{21}$$

It follows from this and theorem 15 that in the error-aware setting:

Lemma 21 *The backtracking, the repetitive non uniform and the repetitive uniform error-aware sequential hierarchies are related by back-and-forth translations.*

As already noted, the backtracking sequential hierarchy is the manifestly sequential hierarchy formulated by Cartwright, Curien and Felleisen in [14]. This hierarchy is extensional, and it follows from lemma 6 that:

Corollary 22 *The three error-aware hierarchies collapse to the manifestly sequential hierarchy.*

It also follows from the linear coercions (21) and Ehrhard's collapse theorem [17] that in the error-free setting:

Lemma 23 *The backtracking, the repetitive non uniform and the repetitive uniform error-free sequential hierarchies are related by back-and-forth translations, and thus collapse to the strongly stable hierarchy.*

Remark. Because (21) exhibits embedding-retraction pairs and not just linear coercions, the resulting back-and-forth translations are *embedding-retraction* pairs; that is, both morphisms

$$[T]^{\mathsf{btk}} \longrightarrow [T]^{\mathsf{unif}} \longrightarrow [T]^{\mathsf{btk}} \qquad \text{and} \qquad [T]^{\mathsf{unif}} \longrightarrow [T]^{\mathsf{rpt}} \longrightarrow [T]^{\mathsf{unif}}$$

compose as identities. It is worth indicating briefly the action of the associated projection maps $p = p \circ p$ and $q = q \circ q$ on the elements of $[T]^{\mathsf{rpt}}$ and $[T]^{\mathsf{unif}}$. The projection map $p : [T]^{\mathsf{rpt}} \longrightarrow [T]^{\mathsf{rpt}}$ prunes out all "non-uniform" plays from the

	$!^{\mathsf{rpt}}\mathsf{bool}$	$\longrightarrow$	bool
(1)			$*$
(2)	$(*, 0)$		
(3)	$(\mathsf{true}, 0)$		
(4)	$(*, 1)$		
(5)	$(\mathsf{false}, 1)$		
(6)			false

Fig. 1. A “non-uniform” play in the interpretation $[M]^{\mathsf{rpt}}$

	$!^{\mathsf{unif}}\mathsf{bool}$	$\longrightarrow$	bool
(1)			$*$
(2)	$(*, 0)$		
(3)	$(\mathsf{true}, 0)$		
(4)	$(*, 1)$		
(5)	$(\mathsf{true}, 1)$		
(6)			true

Fig. 2. A “stuttering” play in the interpretation $[N]^{\mathsf{unif}}$

strategies of $[T]^{\mathsf{rpt}}$. For instance, the play of figure 1 disappears after applying p to the interpretation $[M]^{\mathsf{rpt}}$ of the PCF-term:

M `= if` b `then (if` b `then true else false) else true.`

Similarly, the projection map $q : [T]^{\mathsf{unif}} \longrightarrow [T]^{\mathsf{unif}}$ prunes out all ”stuttering” plays (as in figure 2) from the interpretation $[N]^{\mathsf{unif}}$ of the PCF-term N.

N `= if` b `then (if` b `then true else true) else true.`

Finally, combining the action of the two projection maps p and q transports the interpretation of M and N in $[o \Rightarrow o]^{\mathsf{rpt}}$ to the interpretation $[P]^{\mathsf{rpt}}$ of the PCF-term:

P `= if` b `then true else true.`

Note that these projections p and q are very similar to the projections on cliques described in our section 6.2 on coherence spaces.

7.5 Two models on sequential games (error-aware + error-free)

It is not difficult to adapt the two exponentials $!^{\mathsf{rpt}}$ and $!^{\mathsf{unif}}$ defined on alternated games in section 7.4 to two exponentials $!^{\mathsf{rpt}}$ and $!^{\mathsf{unif}}$ on general sequential games.

In that way, each category $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{G}$ gives rise to a so-called *uniform* and *non-uniform* model of intuitionistic linear logic. Note that the two exponential structures $!^{\mathsf{rpt}}$ and $!^{\mathsf{unif}}$ are related by a linear coercion in each category $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{G}$, in the same way as in section 7.4.

Notations: For clarity's sake, we write $!^{\mathsf{alt}}$ for the exponential $!^{\mathsf{rpt}}$ in the categories $\mathcal{A}^{\mathrm{err}}$ and $\mathcal{A}$, and keep the notation $!^{\mathsf{rpt}}$ for the categories $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{G}$. The notation $!^{\mathsf{unif}}$ is retained in the four categories $\mathcal{A}^{\mathrm{err}}$, $\mathcal{G}^{\mathrm{err}}$, $\mathcal{A}$ and $\mathcal{G}$.

Remark. It is worth stressing that the error-free category $\mathcal{G}$ of Conway games equipped with the repetitive non uniform exponential $!^{\mathsf{rpt}}$ linearizes a well-known and particularly simple arena game model. Arena game models were introduced in order to characterize PCF sequentiality by two constraints on strategies, called innocence and well-bracketedness [22,32] . In a series of subsequent papers, Abramsky and McCusker demonstrated that many programming mechanisms, like ground-type reference, are captured in a fully abstract way, by relaxing some of these constraints, see [3] for a survey. Eventually, by relaxing all these constraints but *single-threadedness*, Abramsky, Honda and McCusker [2] obtain a fully abstract model of a programming language with general reference à la ML, see also [20]. This model is precisely the arena game model linearized by the category $\mathcal{G}$ and the exponential $!^{\mathsf{rpt}}$. We establish below (lemma 24) that the single-threaded hierarchy collapses to the strongly stable hierarchy, and that its error-aware variant collapses to the manifestly sequential hierarchy.

We carry on our topography of models, and establish linear coercions between the two models of sequential games based on $\mathcal{G}^{\mathrm{err}}$ and $\mathcal{G}$ described above, and the three models of alternated games described in section 7.4. Instead of treating all models, we focus on the two error-aware models $\mathcal{M}$ and $\mathcal{N}$ of intuitionistic linear logic over a class K of constants, built respectively from the categories $\mathcal{A}^{\mathrm{err}}$ and $\mathcal{G}^{\mathrm{err}}$ and the exponentials $!^{\mathsf{alt}}$ and $!^{\mathsf{rpt}}$. To fix notations, every constant type $\kappa \in K$ is interpreted:
- in $\mathcal{M}$ as an alternated game X_κ and a partial equivalence relation $\sim_\kappa$ over the set of strategies $\mathcal{A}^{\mathrm{err}}(1, X_\kappa)$,
- in $\mathcal{N}$ as a sequential game Y_κ and a partial equivalence relation $\approx_\kappa$ over the set of strategies $\mathcal{G}^{\mathrm{err}}(1, Y_\kappa)$.

We defined in section 7.2 two symmetric monoidal functors $(U, m) : \mathcal{A}^{\mathrm{err}} \longrightarrow \mathcal{G}^{\mathrm{err}}$ and $(\mathrm{alt}, n) : \mathcal{G}^{\mathrm{err}} \longrightarrow \mathcal{A}^{\mathrm{err}}$ related by monoidal elementwise transformations $\xi : Id \Rightarrow \mathrm{alt} \circ U$ and $\zeta : Id \Rightarrow U \circ \mathrm{alt}$. For every alternated game A and sequential game B, we let:

$$\eta_A : !^{\mathsf{rpt}} U(A) \longrightarrow U(!^{\mathsf{alt}} A) \qquad\qquad \varrho_B : !^{\mathsf{alt}} \mathrm{alt}(B) \longrightarrow \mathrm{alt}(!^{\mathsf{rpt}} B)$$

denote the error-free copycat strategies restricted to the plays of $U(!^{\mathsf{alt}} A)$ and $!^{\mathsf{alt}} \mathrm{alt}(B)$ respectively. We let the reader check that each family η and ϱ defines a distributive law in the sense of section 4.3, that is, that the two diagrams below commute, for every pair of strategies $\sigma : A$ and $\tau : B$.

$$
\begin{tikzcd}
 & {!^{\mathsf{rpt}}U(A)} \arrow[d, "\eta_A"] \\
1 \arrow[ur, bend left, "(U(\tau))^{\mathsf{rpt}}"] \arrow[r, bend right, "U(\tau^{\mathsf{alt}})"'] & U(\,!^{\mathsf{alt}}A)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {!^{\mathsf{alt}}\mathrm{alt}(B)} \arrow[d, "\varrho_B"] \\
1 \arrow[ur, bend left, "(\mathrm{alt}(\sigma))^{\mathsf{alt}}"] \arrow[r, bend right, "\mathrm{alt}(\sigma^{\mathsf{rpt}})"'] & \mathrm{alt}(\,!^{\mathsf{rpt}}B)
\end{tikzcd}
$$

We need to be more careful here about the constant types $\kappa \in K$ than in section 7.4 because the monoidal categories underlying the models $\mathcal{M}$ and $\mathcal{N}$ are different.
Suppose that for every $\kappa \in K$, $X_\kappa = \mathrm{alt}(Y_\kappa)$ and that the two partial equivalence relations $\sim_\kappa$ and $\approx_\kappa$ are the identity relations on $\mathcal{A}^{\mathrm{err}}(1, X_\kappa) = \mathcal{G}^{\mathrm{err}}(1, Y_\kappa)$. Define the morphism $\overline{\phi}_\kappa : X_\kappa \longrightarrow Y_\kappa$ as the strategy with same plays as the identity on X_κ, and the morphism $\overline{\psi}_\kappa : \mathrm{alt}(Y_\kappa) \longrightarrow X_\kappa$ as the identity on X_κ. In that case, one obtains a linear coercion between the two models $\mathcal{M}$ and $\mathcal{N}$. This implies that:
Lemma 24 *The error-aware single-threaded hierarchy collapses to the manifestly sequential hierarchy.*
Similar results are established in the uniform case, as well as in the error-free uniform and non-uniform cases.

7.6 Error-free vs. error-aware models

We have established that all our game models collapse to exactly two extensional hierarchies: the manifestly sequential hierarchy for the error-aware models and the strongly stable hierarchy for the error-free models. There remains to connect the two extensional hierarchies, by establishing that the manifestly sequential hierarchy collapses to the strongly stable hierarchy when errors are not taken into account in the base types.
To that purpose, we consider two models $\mathcal{M}$ and $\mathcal{N}$ built respectively from the categories $\mathcal{A}$ and $\mathcal{A}^{\mathrm{err}}$ equipped with the backtracking exponential $!^{\mathsf{btk}}$. We suppose that every constant $\kappa \in K$ is interpreted in the two models as the same alternated game $X_\kappa = Y_\kappa$ equipped with the partial equivalence relations defined as:
- $\sim_\kappa$ is the identity over $\mathcal{A}(1, X_\kappa)$,
- $\approx_\kappa$ relates two strategies $\sigma, \tau \in \mathcal{A}^{\mathrm{err}}(1, X_\kappa)$ exactly when $\mathrm{even}(\sigma) = \mathrm{even}(\tau)$.

We write $F : \mathcal{A} \longrightarrow \mathcal{A}^{\mathrm{err}}$ for the inclusion functor, and $G : \mathcal{A}^{\mathrm{err}} \longrightarrow \mathcal{A}$ for the functor which transports every strategy $\sigma : A \longrightarrow B$ to the error-free strategy $G(\sigma) : A \longrightarrow B$ defined as: $\mathrm{fix}(G(\sigma)) = \mathrm{even}(\sigma)$. Note that every simple type T is interpreted by the "same" alternated game in the two models $\mathcal{M}$ and $\mathcal{N}$, what we may write: $F([T]) = [\![T]\!]$ and that $G([\![T]\!]) = [T]$.
One difficulty now is that the pair of functors F and G (equipped with identities as mediating morphisms) does not define a linear coercion in the sense of definition 10. More precisely, points 1, 3, 4 of definition 10 are verified, but not point 2 when it comes to the definition of ζ. Indeed, one would like to define ζ_A as the identity $A \longrightarrow F \circ G(A)$ for every alternated game $A = F \circ G(A)$. Unfortunately, this does not define an elementwise transformation $\zeta : Id \Rightarrow F \circ G$, since the diagram

below commutes in the category $\mathcal{A}^{\mathrm{err}}$ only when the strategy $\sigma : A$ is error-free:

$$\begin{tikzcd} & A \arrow[d, "\zeta_A = \mathbf{id}_A"] \\ 1 \arrow[ur, bend left, "\sigma"] \arrow[dr, bend right, "F \circ G(\sigma)"'] & A \end{tikzcd} \tag{22}$$

So, we need to proceed in another way: we show directly that the pair of monoidal functors F and G defines a back-and-forth coercion between the two hierarchies. We prove slightly more in fact. The definitions of $\sim_\kappa$ and $\approx_\kappa$ imply that for every constant type $\kappa \in K$ and strategies $\sigma, \tau \in \mathcal{A}^{\mathrm{err}}(1, X_\kappa)$:

$$\sigma \approx_\kappa \tau \iff G(\sigma) \sim_\kappa G(\tau). \tag{23}$$

We show below (lemma 25) that the equivalence (23) generalizes at every simple type T in fact. Before starting the proof, we indicate two useful equations (24) and (25) verified at every simple type $T = U \Rightarrow V$. First, for every strategies $\sigma \in [\![T]\!]$ and $\nu \in [\![U]\!]$, we have the equality:

$$G(\sigma \cdot \nu) = G(\sigma) \cdot G(\nu). \tag{24}$$

Then, by instantiating ν by $F(\mu)$ in (24) and by observing that $G \circ F(\mu) = \mu$, we obtain the equality below for every strategies $\sigma \in [\![T]\!]$ and $\mu \in [U]$:

$$G(\sigma) \cdot \mu = G(\sigma \cdot F(\mu)). \tag{25}$$

Using these equations, we prove that for every simple type T:
Lemma 25 $\forall \sigma, \tau \in [\![T]\!], \quad \sigma \approx_T \tau \iff G(\sigma) \sim_T G(\tau).$
PROOF By structural induction on the simple type T. We have already indicated in (23) that the assertion holds at every base type $\kappa \in K$. Suppose now that the assertion holds at instance U and V, and that $T = U \Rightarrow V$. We establish that the assertion holds at instance T in two steps: we prove first the implication $(\Rightarrow)$ then the implication $(\Leftarrow)$.
$(\Rightarrow)$ Suppose that $\sigma \approx_T \tau$ and consider any $\sim_U$-equivalent pair of strategies $\mu, \mu' \in [U]$. The strategies $G \circ F(\mu)$ and $G \circ F(\mu')$ are equal to μ and μ' respectively, and thus $\sim_U$-equivalent. It follows by induction hypothesis $(\Leftarrow)$ on U, that the error-free strategies $F(\mu)$ and $F(\mu')$ are $\approx_U$-equivalent. Thus,

$$\begin{array}{rll} G(\sigma) \cdot \mu = & G(\sigma \cdot F(\mu)) & \text{by equation (25) on } \sigma \text{ and } \mu, \\ \sim_V & G(\tau \cdot F(\mu')) & \text{by } \sigma \approx_T \tau,\ F(\mu) \approx_U F(\mu'), \text{ induction hyp } (\Rightarrow) \text{ on } V, \\ = & G(\tau) \cdot \mu' & \text{by equation (25) on } \tau \text{ and } \mu'. \end{array}$$

We conclude that $G(\sigma) \cdot \mu$ and $G(\tau) \cdot \mu'$ are $\sim_V$-equivalent for every pair of $\sim_U$-equivalent strategies $\mu, \mu' \in [U]$. Thus, $G(\sigma) \sim_T G(\tau)$.

($\Leftarrow$) Suppose that two strategies $\sigma, \tau \in [\![T]\!]$ verify $G(\sigma) \sim_T G(\sigma)$, and consider any pair of $\approx_U$-equivalent strategies $\nu, \nu' \in [\![U]\!]$. The equivalence $G(\nu) \sim_U G(\nu')$ follows from our induction hypothesis ($\Rightarrow$) on U. We have:

$$\begin{array}{rll} G(\sigma \cdot \nu) = & G(\sigma) \cdot G(\nu) & \text{by equation (24) on } \sigma \text{ and } \nu, \\ \sim_V & G(\tau) \cdot G(\nu') & \text{by definition of } G(\sigma) \sim_T G(\tau) \text{ and } G(\nu) \sim_U G(\nu'), \\ = & G(\tau \cdot \nu') & \text{by equation (24) on } \tau \text{ and } \nu'. \end{array}$$

We conclude by induction hypothesis ($\Leftarrow$) on V that $\sigma \cdot \nu \approx_V \tau \cdot \nu'$ for every pair of $\approx_U$-equivalent strategies $\nu, \nu' \in [\![U]\!]$. Thus, $\sigma \approx_T \tau$. This concludes our proof by induction. ■

When added to the fact that the function $\sigma \mapsto G(\sigma)$ is onto from the set of error-aware strategies $[\![T]\!]$ to the set of error-free strategies $[T]$, lemma 25 implies that the two hierarchies $[-]$ and $[\![-]\!]$ collapse to the same extensional hierarchy. This is the result we were aiming at in the section. But there is another interesting fact. Equation (24) together with the equality $G \circ F = Id_{\mathcal{A}}$ implies the equality below for every strategies $\sigma \in [T]$ and $\nu \in [\![T]\!]$:

$$G(F(\sigma) \cdot \nu) = \sigma \cdot G(\nu).$$

and thus:

$$G(F(\sigma) \cdot \nu) = G(F(\sigma \cdot G(\nu))).$$

We deduce easily from lemma 25 that for every strategies $\sigma \in [T]$ and $\nu \in [\![T]\!]$:

$$\sigma \sim_T \sigma \text{ and } \nu \approx_T \nu \Rightarrow F(\sigma) \cdot \nu \approx_T F(\sigma \cdot G(\nu)). \tag{26}$$

Now, we conclude from equations (25) and (26) that, if we shift to the *weaker* definition of back-and-forth translation indicated after definition 2 (section 3.1):

Lemma 26 *The hierarchies* $([-], \sim)$ *and* $([\![-]\!], \approx)$ *induced by* $\mathcal{M}$ *and* $\mathcal{N}$ *are related by a back-and-forth translation.*

We deduce from lemma 26, or more directly from lemma 25, what we claimed at the beginning of the section:

Corollary 27 *The manifestly sequential hierarchy collapses to the strongly stable hierarchy when errors are not taken into account in the base types.*

8 Conclusion

We formulate a series of categorical axioms which ensures that two models of intuitionistic linear logic collapse to the same extensional hierarchy. We illustrate our axiomatization on two families of models:

- clique models based on either coherence or hypercoherence spaces, and their qualitative or quantitative exponentials,

- sequential games, based on either error-free or error-aware strategies, and on either a backtracking, or a repetitive uniform, or a repetitive non uniform treatment of exponential modality.

In the case of sequential games, we deduce a “topography” of models in which:

- all error-aware models collapse to the manifestly sequential hierarchy [14],
- all error-free models collapse to the strongly stable hierarchy [17],
- the manifestly sequential hierarchy collapses to the strongly stable hierarchy when errors are not taken into account in the base types.

The topography enables to revisit and possibly refine the so-called Longley’s thesis [28] that every sufficiently expressive model of sequential computations collapses to the strongly stable hierarchy. More, by revealing that the manifestly sequential hierarchy is an artefact deduced by “extensional collapse” from other (more immediate) models of sequentiality, the topography provides a precious hint in the ongoing quest for concurrency in games semantics: the exploration should probably start from somewhere else.

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