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A topological correctness criterion for multiplicative non-commutative logic

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Abstract

We formulate Girard's long trip criterion for multiplicative linear logic (MLL) in a topological way, by associating a ribbon diagram to every switching, and requiring that it is homeomorphic to the disk. Then, we extend the well-known planarity criterion for multiplicative cyclic linear logic (McyLL) to multiplicative non-commutative logic (MNL) and show that the resulting planarity criterion is equivalent to Abrusci and Ruet's original long trip criterion for MNL.

1.1 Introduction

In his seminal article [7] on linear logic, Jean-Yves Girard develops two alternative notations for proofs:

- a *sequential* syntax where proofs are expressed as derivation trees in a sequent calculus,
- a *parallel* syntax where proofs are expressed as bipartite graphs called *proof-nets*.

The proof-net notation plays the role of natural deduction in intuitionistic logic. It exhibits more of the intrinsic structure of proofs than the derivation tree notation, and is closer to denotational semantics. Typically, a derivation tree defines a unique proof-net, while a proof-net may represent several derivation trees, each derivation tree witnessing a particular order of *sequentialization* of the proof-net.

The parallel notation requires to separate "real proofs" (proof-nets) from "proof alikes" (called *proof-structures*) using a *correctness criterion.* Intuitively, the criterion reveals the "geometric" essence of the logic, beyond its "grammatical" presentation as a sequent calculus. In the case of MLL, the (unit-free) multiplicative fragment of (commutative) linear logic, Girard introduces a "long trip condition" which characterizes proof-nets among proof-structures. The criterion is then extended to full linear logic in [9].

The article is divided in two parts. In part one, we recall Girard's long trip criterion (section 1.2) reformulate the criterion topologically (section 1.3) and relate it to an alternative formulation by Vincent Danos and Laurent Regnier (section 1.4). In part two, we shift from commutative to non-commutative logic. So, we start by reformulating carefully the well-known planarity criterion for multiplicative cyclic logic (McyLL) (section 1.5). And we recall multiplicative non-commutative logic (MNL) (section 1.6) as well as the long trip criterion devised for MNL by V. Michele Abrusci and Paul Ruet [3] (section 1.7). Finally, we generalize to MNL the "planarity" criterion for McyLL (section 1.8) and show that the criterion is equivalent to Abrusci-Ruet "long trip" criterion (section 1.9). We conclude the article with an appendix discussing the topological status of logics like MLL, McyLL or MNL (section 1.10).

1.2 Girard's long trip correctness criterion

We recall below the long trip correctness criterion, which appears in [7], and characterizes the proofs of the (unit-free) multiplicative fragment of linear logic (MLL).

MLL formulas and negation. — An MLL *formula* is a tree with leaves p, q, r, ... and $p^\perp$, $q^\perp$, $r^\perp$,... called *atoms*, and binary connectives $\otimes$ and $⅋$. The *negation* $A^\perp$ of a formula A is the formula defined inductively by so-called de Morgan laws:

$$(A\otimes B)^\perp = B^\perp ⅋ A^\perp, \quad (A ⅋ B)^\perp = B^\perp \otimes A^\perp, \quad (p)^\perp = p^\perp, \quad (p^\perp)^\perp = p.$$

It follows that $(A^\perp)^\perp = A$ for every formula A.

MLL sequent calculus. — An MLL sequent is a finite sequence of formulas, noted $\vdash A_0, ..., A_{k-1}$. We usually write formulas as latin letters A, B, C, and finite sequences of formulas as greek letters Γ, Δ. A *derivation tree* is a tree with a sequent at each node, constructed inductively by the five rules below.

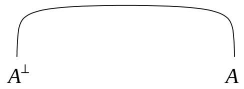
$$(\textsc{Ax})\ \frac{}{\vdash A^{\perp}, A} \qquad (\textsc{Cut})\ \frac{\vdash \Gamma, A \qquad \vdash A^{\perp}, \Delta}{\vdash \Gamma, \Delta}$$

$$(\otimes)\ \frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta} \qquad (\parr)\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}$$

$$(\textsc{Exch})\ \frac{\vdash \Gamma, A, B, \Delta}{\vdash \Gamma, B, A, \Delta}$$

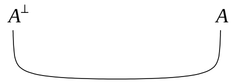
MLL links. — An MLL *link* is a graph of the following form, whose vertices are labelled with MLL formulas:

(i) Axiom link



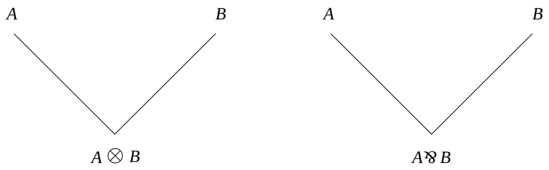
with two conclusions A and $A^{\perp}$, and no premise,

(ii) Cut link



with two premises A and $A^{\perp}$, and no conclusion,

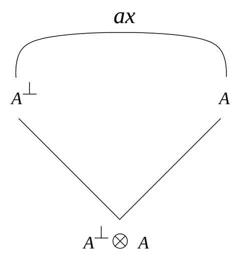
(iii) $\otimes$ and $\parr$ links



where the formula A is the *first* premise, the formula B is the *second* premise, and $A \otimes B$ (or $A \parr B$) is the conclusion.

MLL proof-structures. — A *proof-structure* Θ is a graph constructed with links such that every (occurrence of) formula is the conclusion of one link, and the premise of at most one link. We define a *conclusion* of Θ as a formula which is not the premise of any link. A link of Θ is *terminal* when its conclusion is a conclusion of Θ.

Every derivation tree defines a proof-structure, but conversely, not every proof-structure is deduced from a derivation tree. The simplest example is the proof-structure:



So, which proof-structures exactly are obtained from a derivation tree? Here follows Girard's remarkable answer, the so-called *long trip* criterion.

Decorated formulas. — Call decorated formula a couple $(A, \uparrow)$ or $(A, \downarrow)$ where A is an MLL formula and $\uparrow$ or $\downarrow$ is a tag. We write $A^\uparrow$ and $A^\downarrow$ for the decorated formulas $(A, \uparrow)$ and $(A, \downarrow)$. Now, for each axiom, cut, $\otimes$ or $\parr$ link l, we define two sets l^{in} and l^{out} of decorated formulas, as follows:

- l^{in} is the set of all decorated formulas $A^\downarrow$ where A is a premise of l, and all decorated formulas $A^\uparrow$ where A is a conclusion of l;
- l^{out} is the set of all decorated formulas $A^\uparrow$ where A is a premise of l, and all decorated formulas $A^\downarrow$ where A is a conclusion of l.

Switching positions. — For every link l, a set $\mathcal{S}(l)$ of functions from l^{in} to l^{out} is defined, called the *switching positions* of l:

- if l is an axiom link $[A^\perp, A]$, then $\mathcal{S}(l) = \{ax\}$ where
$$ax : (A^\perp)^\uparrow \mapsto A^\downarrow, A^\uparrow \mapsto (A^\perp)^\downarrow;$$
- if l is a cut link $[A^\perp, A]$, then $\mathcal{S}(l) = \{cut\}$ where
$$cut : (A^\perp)^\downarrow \mapsto A^\uparrow, A^\downarrow \mapsto (A^\perp)^\uparrow;$$
- if l is a $\otimes$-link $[A, A \otimes B, B]$, then $\mathcal{S}(l) = \{\otimes_R, \otimes_L\}$ where
$$\otimes_R : A^\downarrow \mapsto B^\uparrow, B^\downarrow \mapsto (A \otimes B)^\downarrow, (A \otimes B)^\uparrow \mapsto A^\uparrow,$$
$$\otimes_L : A^\downarrow \mapsto (A \otimes B)^\downarrow, B^\downarrow \mapsto A^\uparrow, (A \otimes B)^\uparrow \mapsto B^\uparrow;$$
- if l is a $\parr$-link $[A, A \parr B, B]$, then $\mathcal{S}(l) = \{\parr_R, \parr_L\}$ where
$$\parr_R : A^\downarrow \mapsto A^\uparrow, B^\downarrow \mapsto (A \otimes B)^\downarrow, (A \otimes B)^\uparrow \mapsto B^\uparrow,$$

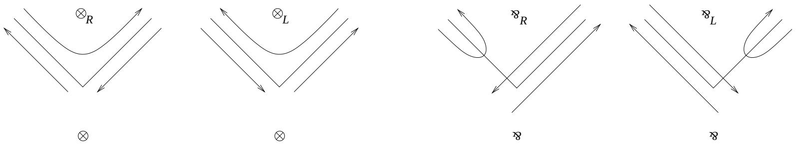


Fig. 1.1. Girard switching positions for tensor and par

$$\parr_L : A^{\downarrow} \mapsto (A \otimes B)^{\downarrow}, B^{\downarrow} \mapsto B^{\uparrow}, (A \otimes B)^{\uparrow} \mapsto A^{\uparrow}.$$

Long trip criterion. — A *switching* s of an MLL proof-structure Θ is a function which associates a switching position $s(l) \in \mathcal{S}(l)$ to every link l of Θ. The *switched proof-structure* $\mathbf{trip}(\Theta, s)$ is the oriented graph with vertices the decorated formulas labelling Θ, and with an edge from A^x to B^y iff $B^y = s(l)A^x$, for some link l in Θ, or $A^x = C^{\downarrow}$ and $B^y = C^{\uparrow}$, for some conclusion C of Θ.

Definition 1.2.1 (Girard) *A* Girard proof-net *is a proof-structure* Θ *such that every switched proof-structure* $\mathbf{trip}(\Theta, s)$ *contains a unique cycle. This unique cycle is called* the long trip.

Intuitively, every switching s defines a trajectory for a particle visiting the proof. Each $\otimes$ and $\parr$ link is visited according to one switching position of figure 1.1; the particle rebounces on axioms, cuts and conclusions. A proof-structure is a proof-net when the particle visits every part, without being captured into a cycle, this for every switching.

Three important properties are established in [7].

- (i) *soundness:* every MLL derivation tree translates as a Girard proof-net.
- (ii) *sequentialization:* every Girard proof-net is the translation of an MLL derivation tree. The proof is based on the notions of (maximal) empire, and splitting tensor.
- (iii) *cut-elimination:* MLL enjoys cut-elimination.

1.3 Our topological reformulation

The characterization of proofs provided by Girard's criterion is not only "geometric", it is also "computational". Expressed in game semantics, the criterion characterizes proofs as uniform strategies which do not deadlock during communication, and which interact with every part of

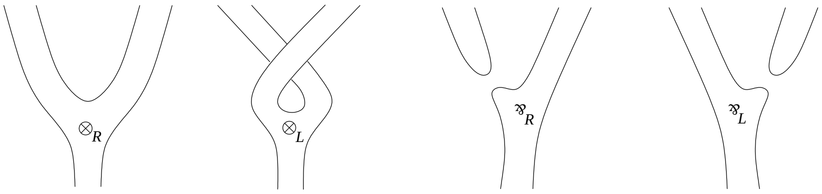
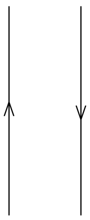
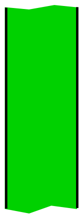


Fig. 1.2. Ribbon version of figure 1.1

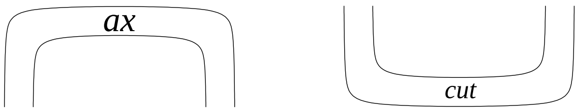
the formula, see [1]. In fact, switchings should be understood as counter-proofs in an extended "para-logic", see [10].

One technical point is that long trips are oriented in Girard's criterion. However, the orientation may be avoided by reformulating the criterion topologically. The idea is to replace oriented edges by *ribbons*, and to apply the convention below.

Convention. —

Replace two oriented edges:  by a ribbon: 

According to the convention, the $\otimes$ and $⅋$ switching positions of figure 1.1 are replaced by the ribbon diagrams of figure 1.2, while the (switching position of) axiom and cut links are replaced by simple ribbons:



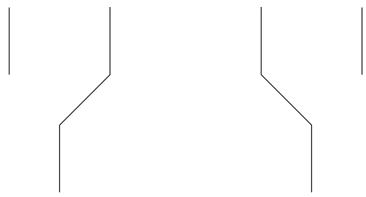
Similarly, each conclusion C is replaced by a 2-dimensional "cul-de-sac":

Now, every proof-structure Θ and every switching s induces a surface $\mathbf{ribbon}(\Theta, s)$ obtained by replacing every switched link and conclusion of Θ by its ribbon diagram, and pasting all diagrams together. This enables us to reformulate Girard's long trip criterion below, see also section 1.4 for a proof that the two formulations are equivalent (lemma 1.4.2.)

Definition 1.3.1 (topological proof-net) *A* topological proof-net *is a proof-structure* Θ *such that the surface* $\mathbf{ribbon}(\Theta, s)$ *is homeomorphic to the disk, for every Girard switching* s*.*

1.4 Danos and Regnier correctness criterion

Many alternative formulations of Girard's long trip criterion are possible. We recall here the "tree" criterion formulated by Vincent Danos and Laurent Regnier in [5]. A Danos-Regnier switching for an MLL proof-structure Θ is the data for every $\parr$-link of a switching position chosen among $\parr_R$ and $\parr_L$:

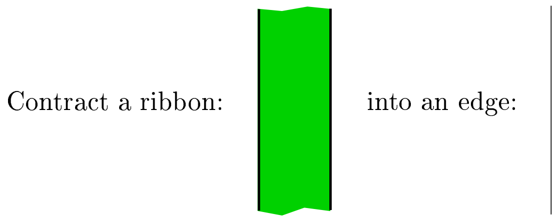


$\parr_R$ $\parr_L$

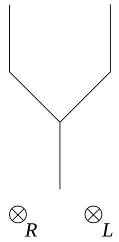
Given a Danos-Regnier switching s, the switched graph $\mathbf{graph}(\Theta, s)$ is defined by replacing every $\parr$-link in Θ by the corresponding switching position. Danos and Regnier's formulation of the criterion follows.

Definition 1.4.1 (Danos-Regnier) *A* Danos-Regnier proof-net *is a proof-structure whose all switching graphs are trees, ie. connected and acyclic graphs.*

Herebelow, we establish that the three formulations of proof-net (Girard, Danos-Regnier, topological) are equivalent. The proof is not really difficult, but informative enough to appear here. We will consider the "shrink" operation contracting ribbons into one-dimensional edges, like this:



It is worth observing that the operation "shrinks" the ribbon diagrams of figure 1.2 into the Danos-Regnier switching positions above. In particular, the operation contracts the two $\otimes_R$ and $\otimes_L$ positions into the "invisible" Danos-Regnier switching position $\otimes_R = \otimes_L$:



Every Danos-Regnier proof-net is a topological proof-net. — Consider a Danos-Regnier proof-net Θ. Every (topological) switching s defines a surface $\mathbf{ribbon}(\Theta, s)$ which "retracts" as the tree $\mathbf{graph}(\Theta, s)$. Thus, the surface is a "thick tree" homeomorphic to the disk. We conclude.

Every topological proof-net is a Girard proof-net. — Consider a topological proof-net Θ. Every (topological = Girard) switching s defines a surface $\mathbf{ribbon}(\Theta, s)$ homeomorphic to the disk. Its border $\mathbf{trip}(\Theta, s)$ is unique, therefore a long trip. We conclude.

Every Girard proof-net is a Danos-Regnier proof-net. — This is the only delicate step of our series of equivalence. We proceed by contradiction. Suppose that Θ is a Girard proof-net, and not a Danos-Regnier proof-net. By definition, there exists a Danos-Regnier switching s such that $\mathbf{graph}(\Theta, s)$ is not a tree. The difficult point is to define a (topological = Girard) switching s' inducing a surface $\mathbf{ribbon}(\Theta, s')$ with two borders at least. When $\mathbf{graph}(\Theta, s)$ is not connected, we take $s' = s$.

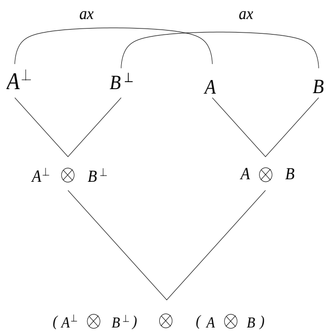
When $\mathbf{graph}(\Theta, s)$ contains a cycle $\mathcal{C}$, it is always possible to alter the switching positions of the $\otimes$-links visited by $\mathcal{C}$ in $\mathbf{ribbon}(\Theta, s)$ in such a way that the altered switching s' verifies $\mathbf{graph}(\Theta, s) = \mathbf{graph}(\Theta, s')$ and that the cycle $\mathcal{C}$ "lifts" to a border of $\mathbf{ribbon}(\Theta, s')$. Note that the resulting surface $\mathbf{ribbon}(\Theta, s')$ has two borders at least. Each such border induces a cycle in $\mathbf{trip}(\Theta, s')$. It follows that $\mathbf{trip}(\Theta, s')$ is not a long trip, and we conclude.

Lemma 1.4.2 *The three formulations of MLL proof-net are equivalent.*

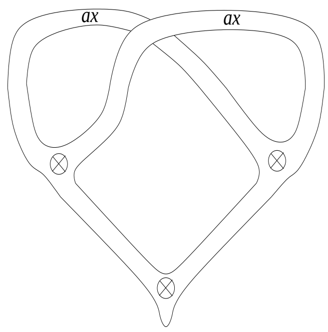
Intuitively, the topological criterion stands halfway between Girard and Danos-Regnier criteria, keeping the best of both worlds. For instance, the switching position $\otimes_L$ is necessary to test a proof-structure in the long trip criterion; but not in the Danos-Regnier and topological formulations.

Lemma 1.4.3 *In definition 1.3.1, switchings may be replaced by $\otimes_L$-free switchings.*

This point is best illustrated by the proof-structure (1.1) pointed out by Abrusci and Ruet [3]. Switching every $\otimes$-link as $\otimes_R$ is enough to show that Θ is not a topological proof-net — since the induced switching surface is not planar. On the other hand, the surface has a unique border... So, it takes one switching position $\otimes_L$ at least to detect that (1.1) is not a Girard proof-net.



switched as



(1.1)

This is the advantage of thinking topologically: the long trip criterion counts the number of borders of $\mathbf{ribbon}(\Theta, s)$ while the topological criterion takes also into account its planarity and genus.

1.5 A planarity correctness criterion for cyclic linear logic

Suggested by Girard in [8] expounded by Yetter in [22] cyclic linear logic (cyLL) is the variant of linear logic obtained by limiting the exchange rule Exch to *cyclic* permutations:

$$(\textsc{cyExch})\ \frac{\vdash A_0, ..., A_{k-1}}{\vdash A_{\xi(0)}, ..., A_{\xi(k-1)}}\ \text{where } \xi \text{ is a cyclic permutation.}$$

In this section, we consider McyLL, the multiplicative (unit-free) fragment of cyclic linear logic. As in [3], we use the notations $\odot$ for "next" and $\triangledown$ for "sequential" to distinguish the cyclic connectives from their commutative counterparts $\otimes$ and $⅋$. The definitions of formula, sequent and proof-structure are the same in McyLL as in MLL, with the only difference that the connectives $\odot$ and $\triangledown$ replace $\otimes$ and $⅋$ everywhere, respectively. Negation is defined as in MLL:

$$(A \odot B)^\perp = B^\perp \triangledown A^\perp, \qquad (A \triangledown B)^\perp = B^\perp \odot A^\perp.$$

Except for the restriction on the exchange rule, the rules of McyLL are the same as in MLL:

$$(\textsc{Ax})\ \frac{}{\vdash A^\perp, A} \qquad\qquad (\textsc{Cut})\ \frac{\vdash \Gamma, A \qquad\qquad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}$$

$$(\odot)\ \frac{\vdash \Gamma, A \qquad\qquad \vdash B, \Delta}{\vdash \Gamma, A \odot B, \Delta} \qquad\qquad (\triangledown)\ \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \triangledown B}$$

It is worth noting that the formula $(A \odot B) \multimap\!\!\bullet (B \odot A)$ is not provable in McyLL, where $A \multimap\!\!\bullet B$ is notation for $A^\perp \triangledown B$. This is the reason why the logic is called non-commutative.

Today, three correctness criteria are available for McyLL.

(i) A "planarity" criterion characterizes McyLL proof-nets as *planar* MLL proof-nets. This criterion was observed by Girard at the very first days of cyclic linear logic, and is well-known today. It appears explicitly in [4, 16, 17]. François Métayer delivers an alternative but equivalent characterization of the logic in his simplicial presentation [14].

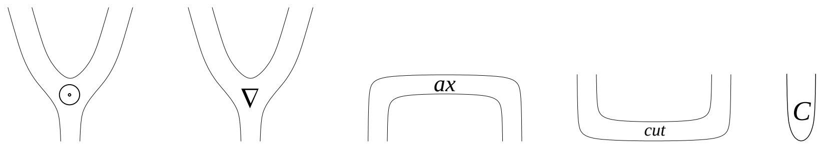
(ii) A "long trip" criterion by V. Michele Abrusci adapts Girard's correctness criterion for MLL, by (1) limiting $\odot$ to the switching position $\otimes_R$ and (2) adding a new position $\triangledown_3$ to the switching positions of $\triangledown$. The criterion is formulated for McyLL in [2] and extended to non-commutative logic (MNL) in [3]. The criterion is exposed in section 1.7 where we also discuss a recent version of the criterion by Virgil Mogbil and Quentijn Puite [15].

(iii) A recent "seaweed" criterion by Roberto Maieli [12] formulates a criterion for McyLL and MNL in the fashion of Danos and Regnier criterion for MLL. The idea is to replace trees by series-parallel order varieties (seaweed).

We formulate very carefully the "planarity" criterion for McyLL, which is not as straightforward as it seems. The first part of the criterion requires that an McyLL proof-net Θ translates as an MLL proof-net Θ^*.

Definition 1.5.1 (commutative translation) *The commutative translation Θ^* of an McyLL proof-structure Θ is the MLL proof-structure obtained as the result of replacing every $\odot$ and $\triangledown$ link by $\otimes$ and $⅋$, respectively.*

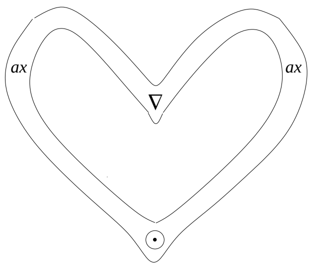
The second part of the criterion requires "planarity" of Θ, or more precisely planarity of the (orientable) surface **ribbon**(Θ) obtained as in section 1.3, by replacing every $\{\odot$, $\triangledown$, axiom, cut$\}$-link and conclusion in Θ by the associated ribbon diagram



The unexpected point is that planarity of **ribbon**(Θ) is *not* sufficient to characterize McyLL proofs among McyLL proof-structures. Typically, the McyLL proof-structure Θ of conclusion

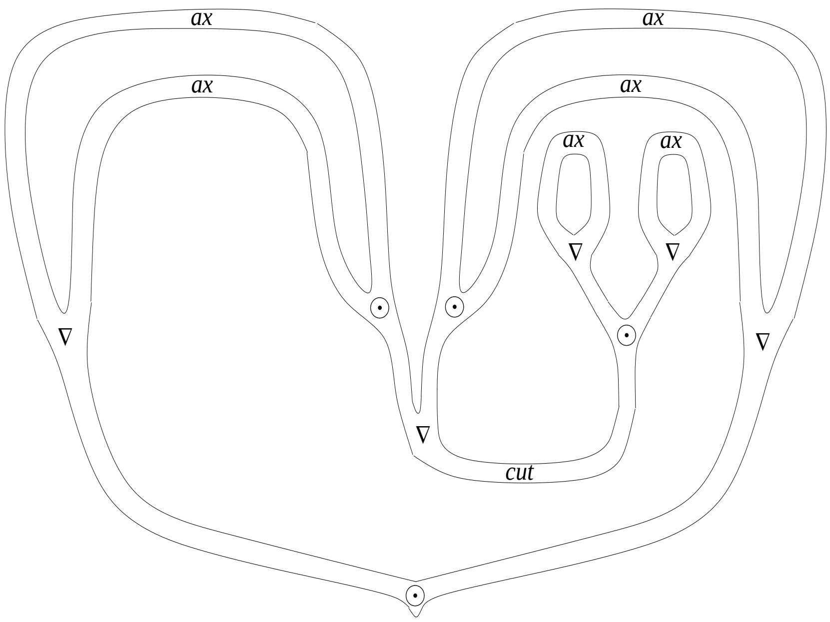
$$\vdash (A^{\perp} \triangledown B^{\perp}), (A \odot B)$$

is not sequentializable in McyLL, but its surface **ribbon**(Θ) is planar:



(1.2)

So, how should one characterize McyLL proof-nets? One possible answer is to require that all conclusions of Θ lie on the same border of **ribbon**(Θ). It is not very complicated to prove that this requirement added to planarity characterizes all *cut-free* proofs among *cut-free* proof-structures. Unfortunately, the criterion is too weak to characterize proofs with cuts, as witnessed by the example below of a non-sequentializable McyLL proof-structure, with a unique conclusion.



(1.3)

Remark. — The proof-structure (1.3) is interpreted as a disk in Métayer's simplicial presentation. This explains why Métayer's sequentialization theorem for McyLL [14] is limited to *cut-free* proof-nets.

Planar logic. — At this point, it is tempting to define a conservative logic over McyLL, which would capture exactly the idea of "planarity". Let us call it *planar logic*. Its formulas are McyLL formulas, and its sequents are finite sets of (occurrences of) McyLL sequents, written

$$\vdash \Gamma_1 \big| \cdots \big| \Gamma_n$$

Each McyLL sequent Γ_i is called a *component* of the sequent. Two sequents $\vdash \Gamma_1 \big| ... \big| \Gamma_n \big| \Delta$ and $\vdash \Gamma_1 \big| ... \big| \Gamma_n$ of the logic are generally identified when Δ is the empty component. Planar logic enables general exchange between components:

$$(\textsc{Exch})\ \frac{\vdash \cdots \mid \Gamma \mid \Delta \mid \cdots}{\vdash \cdots \mid \Delta \mid \Gamma \mid \cdots}$$

and cyclic permutations ξ inside a component:

$$(\textsc{cyExch})\ \frac{\vdash \cdots \mid A_0, ..., A_{k-1}}{\vdash \cdots \mid A_{\xi(0)}, ..., A_{\xi(k-1)}}$$

The remaining rules of planar logic follow:

$$(\textsc{Ax})\ \frac{}{\vdash A^\perp, A} \qquad\qquad (\textsc{Cut})\ \frac{\vdash \cdots \big| \Gamma, A \qquad\qquad \vdash A^\perp, \Delta \big| \cdots}{\vdash \cdots \big| \Gamma, \Delta \big| \cdots}$$

$$(\nabla)\ \frac{\vdash \cdots \mid \Gamma, A, \Delta, B}{\vdash \cdots \mid \Gamma, A \nabla B \mid \Delta} \qquad (\odot)\ \frac{\vdash \cdots \big| \Gamma, A \qquad\qquad \vdash B, \Delta \big| \cdots}{\vdash \cdots \big| \Gamma, A \odot B, \Delta \big| \cdots}$$

Every proof π of $\vdash \Gamma_1 \big| \cdots \big| \Gamma_m$ of planar logic defines a McyLL proof-structure Θ whose translation Θ^* is a MLL proof-net, and whose surface $\mathbf{ribbon}(\Theta)$ is planar with $m+n$ borders $\sigma_1, ..., \sigma_m$ and $\tau_1, ..., \tau_n$; each border σ_i visits the formulas of Γ_i in the order in which they appear in the component; none of the remaining borders τ_j visits a conclusion of Θ.

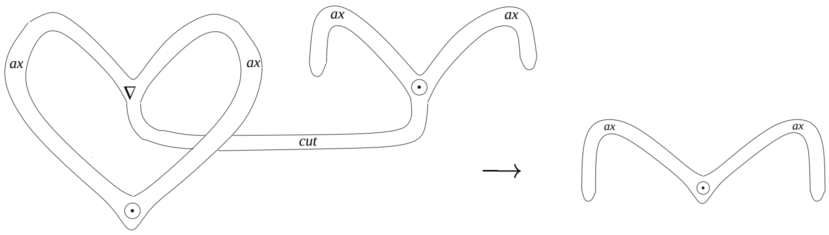
Conversely, every McyLL proof-structure Θ whose translation Θ^* is a MLL proof-net, and whose surface $\mathbf{ribbon}(\Theta)$ is planar, sequentializes as a proof π of planar logic. Typically, the "twist" proof-structure (1.2) sequentializes as the proof

$$\dfrac{\dfrac{\vdash A^\perp, A \qquad \vdash B, B^\perp}{\vdash A^\perp, A \odot B, B^\perp}\,\odot}{\vdash A^\perp \nabla B^\perp \big| A \odot B}\,\nabla$$

But (1.2) does not sequentialize as a proof of $\vdash A^\perp \nabla B^\perp, A \otimes B$. In a similar way, the proof-structure (1.3) sequentializes as a derivation tree of planar logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash A, A^\perp \qquad \vdash A, A^\perp}{\vdash A, A^\perp \odot A, A^\perp}\,\odot}{\vdash A^\perp \odot A, A^\perp \nabla A}\,\nabla \qquad \dfrac{\dfrac{\vdash B, B^\perp \qquad \vdash B, B^\perp}{\vdash B, B^\perp \odot B, B^\perp}\,\odot}{\vdash B^\perp \nabla B, B^\perp \odot B}\,\nabla}{\vdash A^\perp \odot A, (A^\perp \nabla A) \odot (B^\perp \nabla B), B^\perp \odot B}\,\odot}{\vdash (A^\perp \nabla A) \odot (B^\perp \nabla B) | (A^\perp \odot A) \nabla (B^\perp \odot B)}\,\nabla \qquad \dfrac{\dfrac{\vdash B^\perp, B}{\vdash B^\perp \nabla B}\,\nabla \qquad \dfrac{\vdash A^\perp, A}{\vdash A^\perp \nabla A}\,\nabla}{(B^\perp \nabla B) \odot (A^\perp \nabla A)}\,\odot}{\vdash (A^\perp \nabla A) \odot (B^\perp \nabla B)}\,\mathrm{cut}$$

It is worth noting that cut-elimination preserves the planarity of proof-structures, but generally reduces the number of borders of the surface. Typically:



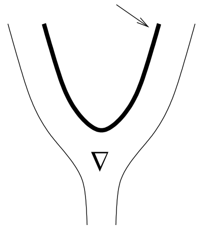
Accordingly, planar logic enjoys the following cut-elimination property: if π is a proof of $\vdash \Gamma_1 \big| \cdots \big| \Gamma_m$ in planar logic, and π' is a proof obtained after a series of cut-elimination steps applied to π, then π' is a proof of a sequent $\vdash \Delta_1 \big| \cdots \big| \Delta_n$ which reduces to the sequent $\vdash \Gamma_1 \big| \cdots \big| \Gamma_m$ by applying a series of "divide" rules:

$$(\textsc{Divide})\ \frac{\vdash \cdots \big|\ \Gamma, \Delta\ \big|\ \cdots}{\vdash \cdots \big|\ \Gamma\ \big|\ \Delta\ \big| \cdots}$$

Conservativity of planar logic over McyLL follows from this and the cut-elimination property of McyLL, established in corollary 1.5.5. Indeed, the *cut-free* proofs of a McyLL sequent $\vdash \Gamma$ are the same in McyLL and in planar logic.

Planar logic seems interesting for itself. But from now on, we stick to cyclic linear logic, and characterize its sequentializable proof-structures, notwithstanding the difficulties.

Index. Internal and external borders. — Given an McyLL proof-structure Θ, and a border σ of $\mathbf{ribbon}(\Theta)$, we shall count the number of ∇-links visited by the border σ on their thick side, see (1.4). We call this number the *index* of σ. A border of index 0 is called *external*, and a border of index more than 1 is called *internal*.



(1.4)

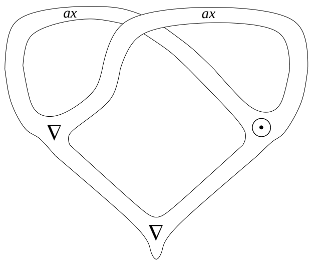
Conversely, the border of $\mathbf{ribbon}(\Theta)$ which visits the thick side of a given ∇-link of Θ, is called the *internal border* of this link.

The correctness criterion. — Example (1.2) and (1.3) suggest to reinforce the definition of McyLL proof-net as follows.

Definition 1.5.2 (McyLL proof-net) *An* McyLL proof-net *is an McyLL proof-structure* Θ *such that,*

(i) *its commutative translation* Θ^* *is an MLL proof-net,*
(ii) *its surface* $\mathbf{ribbon}(\Theta)$ *is planar with a unique external border* σ*,*
(iii) σ *contains all the conclusions.*

The criterion rejects the proof-structures (1.2) and (1.3) because one of their conclusions lies on an internal border. The criterion rejects the proof-structure (1.5) of conclusion $\vdash (B \odot A) \multimap\!\!\bullet (A \odot B)$ as well, because it is not planar.



(1.5)

Remark. — The criterion implies that every internal border is of index exactly one in $\mathbf{ribbon}(\Theta)$, when Θ is a McyLL proof-net. Indeed, by condition 1, the surface $\mathbf{ribbon}(\Theta)$ defines a surface homeomorphic to the disk, when every ∇-link is replaced by a switching position $⅋_L$ or $⅋_R$. Consequently, the planar surface $\mathbf{ribbon}(\Theta)$ has $n+1$ borders, where n is the number of ∇-links appearing in Θ. Since there exists only one external border, each of the remaining n internal borders of $\mathbf{ribbon}(\Theta)$ visits exactly one ∇-link.

Soundness. — It is not difficult to show by induction that the criterion is *sound.* At each step, one proves that the McyLL derivation tree of $\vdash A_1, ..., A_k$ translates as an McyLL proof-net whose external border visits the conclusions $A_1, ..., A_k$ in the clockwise order (here, one assumes implicitly that the surface is oriented.)

Planarity 1. — We recall one elementary property of planar surfaces, which we shall use in our proof of sequentialization. If one pastes (with glue) the two borders σ_1 and σ_2 of a planar surface S, on disjoints segments A of σ_1 and B of σ_2, in such a way that orientation of S is preserved, one obtains a surface S' which is:

- planar when $\sigma_1 = \sigma_2$,
- not planar when $\sigma_1 \neq \sigma_2$.

In the next lemma, the concept of splitting $\odot$-link or cut-link is adapted from [7, 9].

Lemma 1.5.3 *Suppose that Θ is an McyLL proof-structure whose MLL translation Θ^* is an MLL proof-net, and whose surface* **ribbon**(Θ) *is planar. Then, either Θ is the axiom link, or every external border of* **ribbon**(Θ) *visits one of the following:*

- *the conclusion of a terminal $\triangledown$-link of Θ,*
- *a splitting $\odot$-link of Θ,*
- *a splitting cut-link of Θ.*

Proof By induction on the size of Θ. We suppose that every McyLL proof-structure Λ strictly smaller than Θ verifies the property. Consider an external border σ of Θ. We proceed by case analysis.

[**A**] Suppose that Θ contains a terminal $\triangledown$-link l of conclusion $A \triangledown B$. Remove the $\triangledown$-link l from Θ. The resulting McyLL proof-structure Λ translates as an MLL proof-net Λ^* and has a planar surface **ribbon**(Λ). We proceed by case analysis.

1. either the external border σ visits the conclusion of the terminal $\triangledown$-link l of Θ, and we are done,

2. or the external border σ does not visit the conclusion of the $\triangledown$-link l. Since σ is not the internal border of l either, σ is the residual of an external border σ' of **ribbon**(Λ) which does not visit the conclusions A and B of Λ. This shows already that Λ is not the axiom-link. By induction hypothesis on Λ, two cases may occur. Either the external border σ' visits the conclusion of a terminal $\triangledown$-link m of Λ. In that case, the $\triangledown$-link remains terminal in Θ, and σ visits the conclusion of m: we are done. Or the external border σ' visits a splitting $\odot$-link (or cut-link) m of Λ, which splits Λ in two McyLL proof-structures Λ_1 and

Λ_2. Since the border σ visits the link m in Θ, the proof reduces to showing that m is splitting in Θ. The surface **ribbon**(Θ) is the result of glueing together the two conclusions A and B of **ribbon**(Λ). Planarity of **ribbon**(Θ) implies that the two conclusions A and B appear on the same border σ'' of **ribbon**(Λ). This border σ'' cannot be σ' because σ' does not visit the formulas A and B. Since the border σ' is the unique border of **ribbon**(Λ) visiting both Λ_1 and Λ_2, the border σ'' is either a border of **ribbon**(Λ_1) or border of **ribbon**(Λ_2). In the former case, A and B are conclusions of Λ_1, in the latter case, A and B are conclusions of Λ_2. In both cases, the link m remains splitting in Θ, and we are done.

[**B**] Suppose that Θ does not contain any terminal $\triangledown$-link. In that case, Θ^* is an MLL proof-net with no terminal $⅋$-link, and it follows that the proof-structure Θ contains a splitting $\odot$-link or cut-link l, see [7, 9]. Remove the link l from Θ. The two resulting McyLL proof-structures Λ_1 and Λ_2 translate as MLL proof-nets Λ_1^* and Λ_2^* and define planar surfaces **ribbon**(Λ_1) and **ribbon**(Λ_2). Either σ visits both Λ_1 and Λ_2: in that case, we are done, because σ visits the splitting link l. Or the border σ visits Λ_1 only, or Λ_2 only. Suppose that we are in the first situation. It follows by induction hypothesis on Λ_1, which cannot be the axiom-link, that the external border σ visits either the conclusion of a terminal $\triangledown$-link m of Λ_1, or a splitting $\odot$-link m of Λ_1, or a splitting cut-link m of of Λ_1. In the two last cases, we are done, because the link m remains splitting in Λ. In the first case, note that the conclusion of m is not the premise in Θ of the splitting link l. Thus, the $\triangledown$-link m is a terminal link in Θ, whose conclusion is visited by σ. We conclude. □

Sequentialization. — We prove that every McyLL proof-net sequentializes as an McyLL derivation tree, theorem 1.5.4. The proof is not really complicated, except for the cut-link case, which requires the preliminary lemma 1.5.3.

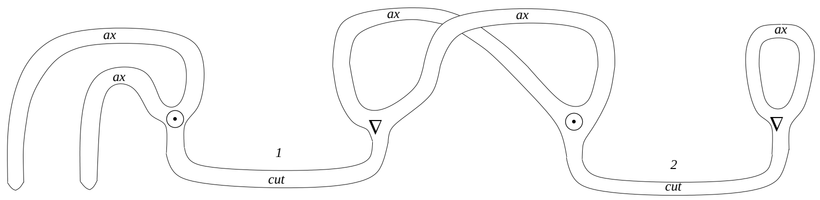
Theorem 1.5.4 (McyLL sequentialization) *Every McyLL proof-net is the translation of an McyLL derivation tree.*

Proof We show by induction on the number of connectives in Θ, that there exists an McyLL derivation tree π sequentializing the McyLL proof-net Θ.

Suppose that Θ contains a terminal $\triangledown$-link of conclusion $A \triangledown B$. Remove this $\triangledown$-link l from Θ. The resulting McyLL proof-structure Λ is an McyLL proof-net. By induction hypothesis, there exists an McyLL derivation tree π sequentializing Λ of, say, conclusion $\vdash A_0, ..., A_{n-1}$. Let i and j be the two indices $0 \leq i, j \leq n-1$ such that $A = A_i$ and $B = A_j$. We claim that $j = i + 1$ modulo n. Suppose not. Then, the conclusions $A_{i+1}, ..., A_{j-1}$ appear on the segment of border between A and B in Λ, thus on the internal border of a $\triangledown$-link l in Θ. This contradicts the hypothesis that Θ is a McyLL proof-net. We conclude that $j = i + 1$ modulo n. The McyLL derivation tree π' sequentializing Θ follows immediately from π, and we are done.

Suppose now that Θ contains no terminal $\triangledown$-link. We are done when Θ is an axiom link. Otherwise, Θ^* is an MLL proof-net without terminal $\parr$-link, and thus, there exists a splitting $\odot$-link or cut-link l in Θ, see [7, 9]. Obviously, when l is a $\odot$-link, it connects two McyLL proof-nets Λ_1 and Λ_2, and we conclude by a simple induction argument.

The remaining case, when there are only splitting cut-links, and no splitting $\odot$-link, is more delicate. Indeed, removing an arbitrary splitting cut-link l from Θ induces two McyLL proof-structures Λ_1 and Λ_2; and one of them, say Λ_1, may not be a McyLL proof-net. This case happens when Λ_2 has a unique conclusion A, whose dual formula $A^\perp$ appears on an internal border of the surface **ribbon**(Λ_1). Note that in this "pathological" case, the cut-link l is visited by an *internal* border of **ribbon**(Θ). The situation is illustrated by the cut-link number 2 in the McyLL proof-net below:



(1.6)

In other words, we need to *choose* which splitting cut-link should be removed first from a McyLL proof-net, if we want to sequentialize it. Typically, the cut-link number 1 must be removed before the cut-link number 2 in the McyLL proof-net (1.6). Fortunately, there is always a correct choice, induced by lemma 1.5.3. By hypothesis, the proof-net Θ does not contain any terminal $\triangledown$-link, nor splitting $\odot$-link; moreover, by definition of a McyLL proof-net, its translation Θ^* is planar. It follows by lemma 1.5.3 that the unique external border of Θ visits one splitting cut-link l at least. We choose to remove this cut-link l from Θ first, and avoid in this way the "pathological" case. So, we obtain two McyLL proof-nets Λ_1 and Λ_2 and conclude by a simple induction argument. □

Planarity 2. — We recall another elementary property of planar surfaces, that we shall use in our proof of cut-elimination. If one cuts (with scisors) a planar surface S which is connected, from a border σ_1 to a border σ_2 of S, one obtains a surface S' with:

- two connected components when $\sigma_1 = \sigma_2$,
- one connected component and one border less than S, when $\sigma_1 \neq \sigma_2$.

Cut-elimination. — The planarity criterion, definition 1.5.2, enables to prove cut-elimination of McyLL in a simple and intuitive way.

Corollary 1.5.5 *McyLL enjoys cut-elimination.*

Proof We prove that McyLL proof-nets are preserved by cut-elimination. Let Θ be an McyLL proof-net containing a cut-elimination pattern R. We prove that the McyLL proof-structure Λ obtained after rewriting the pattern R, is an McyLL proof-net. Cut-elimination in MLL ensures already that Λ translates as an MLL proof-net Λ^*. There remains to show that $\mathbf{ribbon}(\Lambda)$ is planar, and has a unique external border visiting all conclusions of Λ.

Topologically, cut-elimination consists in cutting (with scisors) the surface separating two borders σ_1 and σ_2 of $\mathbf{ribbon}(\Theta)$. One border, say σ_1, visits the internal border of the ∇-link l of R, while the other border σ_2 visits the $\odot$-link. Planarity of $\mathbf{ribbon}(\Lambda)$ follows. Besides, the surface $\mathbf{ribbon}(\Lambda)$ is connected because Λ translates as an MLL proof-net Λ^*. We conclude that the two borders σ_1 and σ_2 are different in $\mathbf{ribbon}(\Theta)$.

Let σ_3 denote the border of $\mathbf{ribbon}(\Lambda)$ obtained by "merging" the two borders σ_1 and σ_2 of $\mathbf{ribbon}(\Theta)$. We mentioned that every internal border of $\mathbf{ribbon}(\Theta)$ has index one, for a McyLL proof-net like Θ, see the remark after definition 1.5.2. In particular, the ∇-link l is the unique ∇-link visited internally by σ_1. Since cut-elimination removes this ∇-link l, the index of σ_2 and σ_3 are equal.

It follows that $\mathbf{ribbon}(\Lambda)$ has a unique external border σ. This border σ is the border σ_3 when the border σ_2 is external, and the residual of the external border of $\mathbf{ribbon}(\Theta)$ when the border σ_2 is internal. In each case, the border σ visits all conclusions of Λ. We conclude that the proof-structure Λ is a McyLL proof-net.

We have just proved that McyLL proof-nets are preserved by cut-elimination. The end of the proof is easy. Suppose that π_1 and π_2 are McyLL derivation trees of conclusion $\vdash \Gamma, A$ and $\vdash A^{\perp}, \Delta$. By soundness, the derivation trees π_1 and π_2 define McyLL proof-nets Λ_1 and Λ_2, respectively. Now, connect Λ_1 to Λ_2 by a cut-link between the conclusions A and $A^{\perp}$. This defines a McyLL proof-net Θ which reduces by cut-elimination to a cut-free McyLL proof-net Θ'. The proof-net Θ' sequentializes to a cut-free McyLL derivation tree π', by theorem 1.5.4. The derivation tree π' has conclusion $\vdash \Gamma, \Delta$. We conclude that McyLL enjoys cut-elimination. $\square$

Remark. — The proof-structure (1.3) appears independently in Robert Schneck's work on non-symmetric linearly distributive categories [21]. Motivated by this example, Schneck strengthens the planarity criterion for negation-free multiplicative linear logic, and formulates a new criterion, in a similar way as we do above.

1.6 Non commutative logic

Non commutative logic (NL) was introduced by Paul Ruet in his PhD thesis [19] and developped with collaborators in a series of articles [3, 20, 13]. It is a conservative extension of both commutative linear logic (LL) and cyclic linear logic (cyLL). The idea is to equip every sequent $\vdash A_0, ..., A_{k-1}$ with additional information on the relative positions of the conclusions, provided by an *order variety* on the set of (occurrences of) formulas $A_1, ..., A_k$.

Order varieties. — An order variety α on a set X is a ternary relation which is:

(i) cyclic: $\forall x, y, z \in E, \alpha(x, y, z) \Rightarrow \alpha(y, z, x)$,
(ii) anti-reflexive: $\forall x, y \in E, \neg\alpha(x, x, y)$,
(iii) transitive: $\forall x, y, z, t \in E, \alpha(x, y, z) \wedge \alpha(x, z, t) \Rightarrow \alpha(x, y, t)$,
(iv) spreading: $\forall x, y, z, t \in E, \alpha(x, y, z) \Rightarrow \alpha(t, y, z) \vee \alpha(x, t, z) \vee \alpha(x, y, t)$.

The three first properties define a *cyclic order*, as introduced by Novak in [18]. A cyclic order is *total* when it verifies the additional property:

$$\forall x,y,z \in E, x \neq y \neq z \neq x \Rightarrow \alpha(x,y,z) \vee \alpha(x,z,y)$$

A total cyclic order is often called *oriented cycle* on X, because, at least when X is finite, it can be described by a graph $(X,\to)$ which relates $x \to y$ when there exists no $z \in X$ such that $\alpha(x,z,y)$. This graph contains a unique cycle, and $\alpha(x,y,z)$ simply means in that case that "y stands between x and z".

Order varieties generalize total cyclic orders, like partial orders generalize total orders. Every order variety on X becomes a *partial* order on $X-\{x\}$ once an origin x is fixed in X — in a reversible way, in the sense that the order variety on X may be reconstructed from the partial order on $X-\{x\}$. The following properties are established in [3, 20].

Focusing. — Given an order variety α on X and an element $x \in X$, define the partial order α_x on $X-\{x\}$, called *focus* of α on x, by:

$$\forall y,z \in X-\{x\}, \qquad \alpha_x(y,z) \iff \alpha(x,y,z)$$

Conversely, given a partial order $\omega = (X,<)$ on X and an element $z \in X$, define the binary relation on X:

$$x \overset{z}{<} y \iff x<y \text{ and } z \text{ is comparable with neither } x \text{ nor } y.$$

Then, the order variety $\overline{\omega}$ on X, the *closure* of ω on X, is defined as the ternary relation $\overline{\omega}(x,y,z)$ on X:

$$x<y<z \text{ or } y<z<x \text{ or } z<x<y \text{ or } x \overset{z}{<} y \text{ or } y \overset{x}{<} z \text{ or } z \overset{y}{<} x.$$

Parallel and series. — Given two partial orders ω on X and ω' on Y, define the partial orders $\omega|\omega'$ (called ω *parallel* ω') and $\omega<\omega'$ (called ω *series* ω') on $X+Y$.

$$x(\omega|\omega')y \iff \begin{cases} x\in X \text{ and } y \in X \text{ and } x\omega y \\ x \in Y \text{ and } y \in Y \text{ and } x\omega' y \end{cases}$$

$$x(\omega<\omega')y \iff \begin{cases} x\in X \text{ and } y \in Y \\ x\in X \text{ and } y \in X \text{ and } x\omega y \\ x \in Y \text{ and } y \in Y \text{ and } x\omega' y \end{cases}$$

Glueing. — If ω and ω' are two partial orders on disjoint sets X and

Y, then the following equality holds:

$$\overline{\omega < \omega'} \;=\; \overline{\omega|\omega'} \;=\; \overline{\omega' < \omega}$$

This enables to *glue* two partial orders ω on X and ω' on Y, and obtain an order variety $\omega * \omega' = \overline{\omega|\omega'}$ on $X + Y$. The two main properties of glueing are:

$$(\alpha_x) * x = \alpha \qquad\qquad (\omega * x)_x = \omega$$

for α an order variety on X, x an element of X, and ω a partial order on $X - \{x\}$.

Next and tensor. — Given two order varieties α on X and β on Y, and two elements $x \in X$ and $y \in Y$, one glues α and β together on x and y, in a *series* or *parallel* fashion, to obtain an order variety on $(X - \{x\}) + (Y - \{y\}) + \{z\}$:

$$\alpha \odot^z_{x,y} \beta \;=\; \overline{\alpha_x < z < \beta_y} \;=\; (\beta_y < \alpha_x) * z$$

$$\alpha \otimes^z_{x,y} \beta \;=\; \overline{\alpha_x|z|\beta_y} \;=\; (\beta_y|\alpha_x) * z$$

Interior. — Every cyclic order α on X contains a *largest* order variety $\natural\alpha$. The order variety $\natural\alpha$ is called the *interior* of the cyclic order α, and defined as

$$\natural\alpha \;=\; \bigcap_{x \in X} \alpha_x * x$$

Notation. — Consider an order variety α on X, and a subset Y of X. We write $\alpha \restriction_Y$ the order variety obtained by restricting the ternary relation α to the subset Y of X. Given an element x of X, the order variety $\alpha[z/x]$ is the order variety on $(X - \{x\}) + \{z\}$ obtained by replacing x by z in X.

Par. — Given an order variety α on X and two different elements $x, y \in X$, one defines the order variety $\alpha[z/x, y]$ on $(X - \{x, y\}) + \{z\}$ as

$$\alpha[z/x, y] \;=\; \natural\,(\alpha \restriction_{X-\{y\}} [z/x] \cap \alpha \restriction_{X-\{x\}} [z/y])$$

We write $\alpha[x, y]$ when x and y are two different elements of X.

MNL. — The multiplicative fragment (without units) of non commutative logic (MNL) extends both MLL and McyLL. Its formulas are

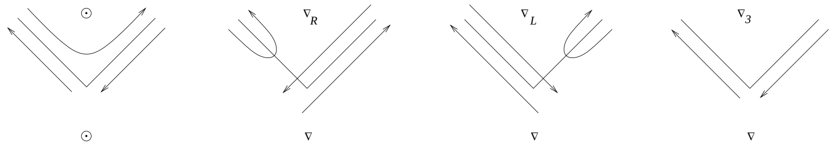


Fig. 1.3. Abrusci-Ruet switching positions for next and sequential

constructed using the connectives $\otimes$, $\parr$ (from MLL) and $\odot$, $\triangledown$ (from McyLL). Negation in MNL simply extends negation in MLL and McyLL. An MNL sequent $\vdash \omega$ is an order variety on a finite set of (occurrences of) MNL formulas. An MNL derivation tree is a tree of MNL sequents constructed according to the following rules.

$$(\textsc{Ax})\ \frac{}{\vdash A^{\perp} * A} \qquad (\textsc{Cut})\ \frac{\vdash \omega * A \qquad \vdash A^{\perp} * \omega'}{\vdash \omega * \omega'}$$

$$(\otimes)\ \frac{\vdash \omega * A \qquad \vdash \omega' * B}{\vdash (\omega|\omega')\ *\ A \otimes B} \qquad (\parr)\ \frac{\vdash \alpha[A, B]}{\vdash \alpha[A \parr B / A, B]}$$

$$(\odot)\ \frac{\vdash \omega * A \qquad \vdash \omega' * B}{\vdash (\omega < \omega')\ *\ A \odot B} \qquad (\triangledown)\ \frac{\vdash \omega * (A < B)}{\vdash \omega\ *\ A \triangledown B}$$

1.7 Abrusci and Ruet's long trip criterion for MNL

In this section, we recall the correctness criterion for McyLL and MNL developped by V. Michele Abrusci and then Paul Ruet in [2, 3]. The criterion adapts Girard long trip condition for MLL, by:

- keeping the switching positions of MLL for $\otimes$ and $\parr$ links,
- considering $\odot$-links as $\otimes$-links limited to the unique switching position $\odot = \otimes_R$,
- considering $\triangledown$-links as $\parr$-links with the usual switching positions $\triangledown_L = \parr_L$ and $\triangledown_R = \parr_R$, and an additional switching position $\triangledown_3$.

Abrusci-Ruet switching positions appear in figures 1.1 and 1.3. Contrarily to the other switching positions, the position $\triangledown_3$ is not *total*: a $\triangledown$-link in position $\triangledown_3$ does not necessarily reemit a particle which enters it! Accordingly, Abrusci and Ruet weaken Girard's long trip condition in definition 1.7.2, and require only that, for a given proof-net Θ and switching s, there exists a unique cycle in $\mathbf{trip}(\Theta, s)$ which visits all the conclusions, *but not necessarily all the proof-net* Θ.

MNL switching. — A switching of an MNL proof-structure Θ is the data of

- a switching position in $\{\otimes_L, \otimes_R\}$ for every $\otimes$-link of Θ,
- a switching position in $\{⅋_L, ⅋_R\}$ for every $⅋$-link of Θ,
- a switching position in $\{\nabla_L, \nabla_R, \nabla_3\}$ for every ∇-link of Θ.

Every MNL switching s defines a switched proof-structure $\mathbf{trip}(\Theta, s)$ as in section 1.2.

Bilaterality. — An additional (and technical) condition of "bilaterality" is required on the cycle. The condition ensures for instance that the proof-structure illustrated in (1.1) with $\otimes$-links replaced by $\odot$-links, is not a proof-net.

Definition 1.7.1 (bilateral) *Let Θ be an MNL proof-structure, and s an MNL switching of Θ. A trip σ in $\mathbf{trip}(\Theta, s)$ is bilateral if σ is not of the form*

$$A^x, ..., B^y, ..., A^{\overline{x}}, ..., B^{\overline{y}}$$

where A and B are occurrences of formulas in Θ, and $\overline{\uparrow} = \downarrow$, $\overline{\downarrow} = \uparrow$.

Abrusci-Ruet long trip criterion. —

Definition 1.7.2 (Abrusci-Ruet proof-net) *An* Abrusci-Ruet proof-net *is an MNL proof-structure Θ such that, for every MNL switching s:*

(i) *there is exactly one cycle σ in $\mathbf{trip}(\Theta, s)$, called the* long trip*,*
(ii) *σ contains all the conclusions,*
(iii) *σ is bilateral.*

Three important properties are established in [3].

(i) *soundness:* every MNL derivation tree of conclusion $\vdash \alpha$ translates as an Abrusci-Ruet proof-net Θ, in such a way that α is the largest order variety contained in each α_s, where α_s denotes the total cyclic order (or oriented cycle) on the conclusions of Θ defined by the long trip of $\mathbf{trip}(\Theta, s)$, for s an MNL switching. It is worth noting for section 1.8 that the characterization of α still works when the switchings s are restricted to the $\{\nabla_L, \nabla_R\}$-free ones,
(ii) *sequentialization:* every *cut-free* Abrusci-Ruet proof-net sequentializes as an MNL derivation tree,

(iii) *cut-elimination:* MNL enjoys cut-elimination.

In fact, points 2. and 3. are proved using an alternative characterization of Abrusci-Ruet proof-nets, rather than the original definition 1.7.2. — see theorem 2.20 in [3], or the discussion in section 1.9.

Remark. — Virgil Mogbil and Quintijn Puite observe in [15] that the bilaterality condition of definition 1.7.2 (point (iii)) may be replaced by the condition that the MNL proof-structure Θ translates as a MLL proof-net Θ^*. Obviously, this condition also rejects the proof-structure illustrated in (1.1).

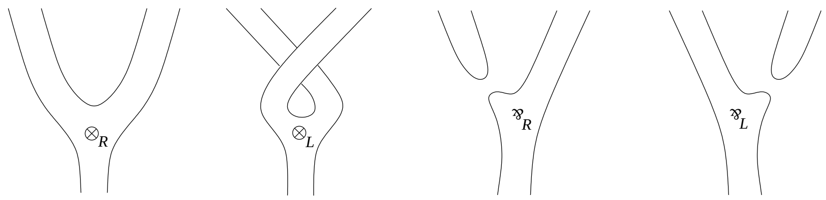
1.8 A planarity correctness criterion for MNL

In this section, we extend to MNL the well-known planarity criterion for McyLL, discussed at length in section 1.5. We will see in section 1.9 that the resulting planarity criterion for MNL reformulates topologically Abrusci-Ruet long trip criterion. Thus, just as in the commutative case of MLL, the topological point of view federates seemingly different correctness criteria (eg. planarity vs. long trip).

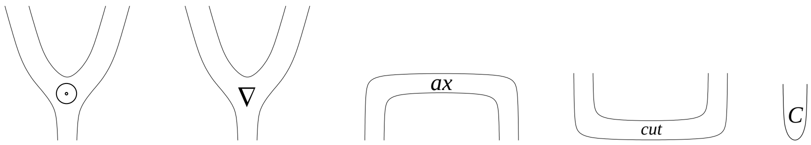
Topological switching. — A topological switching of an MNL proof-structure Θ is simply defined as a $\{\nabla_L, \nabla_R\}$-free MNL switching of Θ. Alternatively, it is the data of

- a switching position in $\{\otimes_L, \otimes_R\}$ for every $\otimes$-link of Θ,
- a switching position in $\{⅋_L, ⅋_R\}$ for every $⅋$-link of Θ.

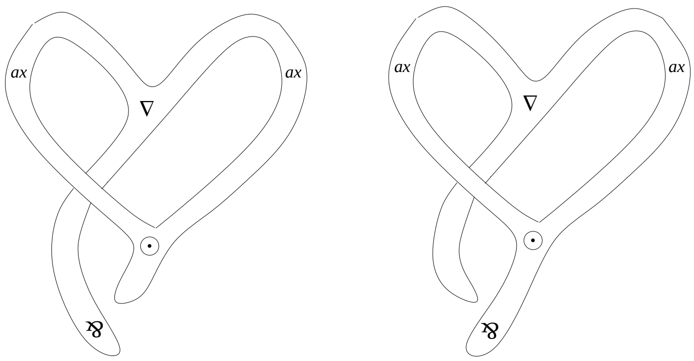
Switched surface. — To every MNL proof-structure Θ and topological switching s, we associate the surface $\mathbf{ribbon}(\Theta, s)$ by replacing every $\otimes$ and $⅋$-link by the ribbon diagram corresponding to its MNL switching



and every $\odot$ or ∇ or axiom or cut-link and conclusion by the ribbon diagram



Planarity criterion for MNL. — Just as for McyLL in section 1.5, requiring planarity of $\mathbf{ribbon}(\Theta, s)$ for every switching s is not sufficient to characterize MNL proofs. We have seen that requiring in addition that all conclusions lie on the same border of $\mathbf{ribbon}(\Theta)$ is sufficient to characterize *cut-free* McyLL proofs. Note that this is not even the case in MNL. For instance, the *cut-free* proof-structure of conclusion $\vdash (B \odot A) \multimap (A \odot B)$ which is not sequentializable in MNL, has its two switched surfaces planar, with all conclusions (= one conclusion in each case) on the same border.



(1.7)

Fortunately, proof-structures like (1.7) may be rejected in the same way as in McyLL: by considering *external* and *internal* borders. These notions are adapted to MNL in the obvious way: given an MNL proof-structure Θ and a topological switching s, the *index* of a border b of the surface $\mathbf{ribbon}(\Theta, s)$, is the number of *internal sides* of ∇-link of Θ the border b visits; A border of $\mathbf{ribbon}(\Theta, s)$ is *external* or *internal* when it is of index 0, and of index 1 or more, respectively. The criterion below is a "conservative" extension to MNL of definition 1.5.2 for McyLL.

Definition 1.8.1 (topological MNL proof-net) *A* topological MNL proof-net *is an MNL proof-structure* Θ

1. *whose commutative translation* Θ^* *is an MLL proof-net,*

and such that, for every topological switching s:

2. *the switched surface* $\mathbf{ribbon}(\Theta, s)$ *is planar and has a unique external border* σ*,*
3. σ *contains all the conclusions.*

Obviously, the proof-structure (1.7) is rejected by the criterion: its unique conclusion lies on an internal border when $⅋$ is switched in position $⅋_R$.

Remark. — For the same reasons as in section 1.5, definition 1.5.2, it follows from definition 1.8.1 that every internal border of $\mathbf{ribbon}(\Theta, s)$ is of index 1, when Θ is an MNL proof-net, and s is a topological switching.

Soundness. — Given a proof derivation π, its associated proof-structure Θ in MNL, and a topological switching s, one proves by structural induction on π that the long trip in the proof-structure $\mathbf{trip}(\Theta, s)$ *is* precisely the external border of the switched surface $\mathbf{ribbon}(\Theta, s)$. It follows that the long trip of $\mathbf{trip}(\Theta, s)$ visits the conclusions of Θ in the same order as the external border of $\mathbf{ribbon}(\Theta, s)$. By property of soundness, in section 1.7, the order variety $\vdash \alpha$ is the maximal order variety on the conclusions of Θ included in all oriented cycles induced by the external border of $\mathbf{ribbon}(\Theta, s)$, for s a topological switching of Θ. Soundness follows easily.

Sequentialization. — Just as in [3, 12] we limit our sequentialization theorem to *cut-free* MNL proof-nets.

Theorem 1.8.2 (MNL sequentialization) *Every* cut-free *MNL proof-net is the translation of an MNL derivation tree.*

Proof The proof proceeds as in theorem 1.5.4 for $\odot$ and $\triangledown$-links. $\otimes$-links can be treated as $\odot$-link, and $⅋$-links are treated as follows. Suppose that l is a terminal $⅋$-link of conclusion $A ⅋ B$ in a cut-free MNL proof-net Θ. Let Λ be the proof-structure obtained by removing l from Θ. Its MLL translation is a proof-net. There remains to check on Λ conditions 2 and 3 of definition 1.8.1. Let s be a topological switching of Λ, and $s_L = s + \{l \mapsto ⅋_L\}$ and $s_R = s + \{l \mapsto ⅋_R\}$ the two associated topological switchings on Θ. Obviously, $\mathbf{ribbon}(\Lambda, s)$, $\mathbf{ribbon}(\Theta, s_L)$ and $\mathbf{ribbon}(\Theta, s_R)$ denote the same surface S. Planarity of $\mathbf{ribbon}(\Lambda, s)$ follows. Moreover, the unique external border of $\mathbf{ribbon}(\Theta, s_L)$ (on which A lies in $\mathbf{ribbon}(\Lambda, s)$) and the unique external border of $\mathbf{ribbon}(\Theta, s_R)$

(on which B lies in $\mathbf{ribbon}(\Lambda, s)$) are necessarily the same border of S. It follows that $\mathbf{ribbon}(\Lambda, s)$ has a unique external border, on which A and B lie. We conclude that Λ is an MNL proof-net. □

Cut-elimination. — The proof of cut-elimination for MNL follows a purely topological argument, instead of the algebraic one presented by Abrusci and Ruet in [3].

Corollary 1.8.3 *MNL enjoys cut-elimination.*

Proof Follows from soundness and sequentialization of MNL proof-nets in the same way as corollary 1.5.5 follows from soundness and sequentialization of McyLL proof-nets. The only difficulty is to establish that MNL proof-nets are preserved by cut-elimination.

Consider a topological MNL proof-net Θ containing a cut-elimination pattern, and the MNL proof-structure Λ obtained after cut-elimination of the pattern. We prove that Λ is a proof-net. Two cases may occur: either the cut-elimination pattern is "non-commutative", that is, involves a $\odot$ and a $\triangledown$ link, in which case we proceed as in corollary 1.5.5, with an obvious adaptation regarding preservation of uniqueness of the external border; or the cut-elimination pattern is "commutative", that is, involves a $\otimes$ link $l_{\otimes}$ and a $⅋$ link $l_{⅋}$, with respective conclusions $A \otimes B$ and $B^{\perp} ⅋ A^{\perp}$, in which case we proceed as follows. We fix a topological switching s of Λ, and consider the four topological switchings of Θ

$$s_{XY} = s + (l_{⅋} \mapsto ⅋_X) + (l_{\otimes} \mapsto \otimes_Y)$$

for $X, Y \in \{L, R\}$. From now on, we call S the surface obtained by cutting (with scisors) the branch A of the $\otimes$-link $l_{\otimes}$ in $\mathbf{ribbon}(\Theta, s_{LR})$. Like $\mathbf{ribbon}(\Theta, s_{LR})$, the surface S is planar. The cut-link connecting $l_{\otimes}$ and $l_{⅋}$ has two borders σ and τ in S, which may be called informally the respective borders of A and B. To avoid ambiguities, we will only indicate that the surfaces $\mathbf{ribbon}(\Theta, s_{LR})$ and $\mathbf{ribbon}(\Theta, s_{LL})$ are obtained by glueing in S the branch A to the borders σ and τ, respectively. We prove by case analysis that $\mathbf{ribbon}(\Lambda, s)$ is planar, and has a unique external border, visiting all the conclusions of Λ.

[**A**] When the two borders σ and τ are different, planarity of both $\mathbf{ribbon}(\Theta, s_{LR})$ and $\mathbf{ribbon}(\Theta, s_{LL})$ implies that the surface S is not connected. More, S has two disconnected components S_1 and S_2, with the branch A in one component, say S_1, and the borders σ and τ in the other component S_2. The surface $\mathbf{ribbon}(\Lambda, s)$ is the result of glueing A in S_1 and $A^\perp$ in S_2. This shows that $\mathbf{ribbon}(\Lambda, s)$ is planar. By our correctness criterion, each border of the surface $\mathbf{ribbon}(\Theta, s_{LR})$ or $\mathbf{ribbon}(\Theta, s_{LL})$ visits either all the conclusions of Θ, or the internal border of exactly one ∇-link. Call ν the border of A in S_1. We claim that ν does not visit any conclusion, nor any internal border of a ∇-link. We proceed by contradiction. Suppose that the border ν visits a conclusion of S_1; then, by the last remark on $\mathbf{ribbon}(\Theta, s_{LR})$ or $\mathbf{ribbon}(\Theta, s_{LL})$, neither σ nor τ visits the internal border of a ∇-link in S_2; thus, two borders of $\mathbf{ribbon}(\Theta, s_{LR})$ are external; this contradicts the hypothesis that Θ is a proof-net. Suppose now that ν visits the internal border of a ∇-link; then, for the same reason as above, neither σ nor τ visits the internal border of a ∇-link in S_2, or a conclusion of S_2; it follows that the external border of $\mathbf{ribbon}(\Theta, s_{LL})$ is the residual of the border σ after glueing τ and ν together; the border visits no conclusion of Θ; according to the correctness criterion, the proof-net Θ does not have any conclusion; this contradicts the fact that Θ translates as an MLL proof-net. This proves our claim that ν visits no internal border of a ∇-link, and no conclusion of S_1. From this, we conclude easily that just like $\mathbf{ribbon}(\Theta, s_{LR})$, the surface $\mathbf{ribbon}(\Lambda, s)$ has a unique external border, visiting all the conclusions of Λ.

[**B**] When $\sigma = \tau$, and the surface S has two connected components, we call S_1 the component containing the branch A, and S_{23} the component containing the border $\sigma = \tau$. The surface $\mathbf{ribbon}(\Lambda, s)$ is connected because the proof-structure Λ translates as a MLL proof-net. This ensures that the branch $A^\perp$ appears in S_{23}, not in S_1; and implies that the proof-structure $\mathbf{ribbon}(\Lambda, s)$ is planar. There remains to show that $\mathbf{ribbon}(\Lambda, s)$ contains a unique external border, visiting all the conclusions of Λ. We proceed by case analysis. Either σ visits, or does not visit, the branch with conclusion $A^\perp$ in S. When σ visits $A^\perp$, the surface $\mathbf{ribbon}(\Theta, s_{LR})$ may be deformed into $\mathbf{ribbon}(\Lambda, s)$ by letting the component S_1 "slide" along the border σ of S_{23}, until S_1 reaches the branch $A^\perp$. It follows that, like $\mathbf{ribbon}(\Theta, s_{LR})$, the surface $\mathbf{ribbon}(\Lambda, s)$ has a unique external border visiting all conclusions of Λ.

Now, we treat the case when the border σ does not visit the branch with conclusion $A^\perp$ in S. Let S' denote the surface obtained by cutting (with scisors) the branch $B^\perp$ in the surface S_{23}. By planarity of S_{23} and equality of borders $\sigma = \tau$, the surface S' has two connected components: one component, called S_3, contains the branch with conclusion $B^\perp$; the other component, called S_2, contains the cut-elimination pattern $l_{⅋}, l_\otimes$. The three components S_1, S_2 and S_3 are also the result of cutting (with scisors) the branch $B^\perp$ in $\mathbf{ribbon}(\Theta, s_{LR})$, this resulting in two components S_{12} and S_3; then of cutting (with scisors) the branch A in S_{12}, this resulting in the two components S_1 and S_2. Let σ_1 denote the border of A in S_1, and σ_2 and σ_{12} denote the border of the cut-elimination pattern $l_\otimes, l_{⅋}$ in S_2 and S_{12} respectively. The surface $\mathbf{ribbon}(\Theta, s_{RR})$ is obtained by glueing the border σ_{12} with the border of $A^\perp$ in $S_{12} + S_3$. Connectedness of $\mathbf{ribbon}(\Theta, s_{RR})$ implies that $A^\perp$ appears in the component S_3, not in the component S_{12}. Now, let τ_A and τ_B denote the borders of $A^\perp$ and $B^\perp$ in the component S_3, respectively. We claim that τ_A and τ_B are different, and prove it as follows: the surface S is the result of glueing σ_2 in S_2 with τ_B in S_3; if τ_A and τ_B were equal in S_3, the border σ would visit $A^\perp$, contradicting our hypothesis. Now, the surfaces $\mathbf{ribbon}(\Theta, s_{LR})$ and $\mathbf{ribbon}(\Theta, s_{RR})$ are obtained by pasting (with glue) the borders σ_{12} in S_{12} with the borders τ_B and τ_A in S_3, respectively. From this, the inequality $\tau_A \neq \tau_B$, and an argument similar to case [A], it follows that the border σ_{12} visits no internal border of a ∇-link, and no conclusion of Θ. It follows that the border σ_1 of A in S_1, which is (in a sense) a segment of the border σ_{12}, visits no internal border of a ∇-link, and no conclusion of Θ. We conclude easily that the surface $\mathbf{ribbon}(\Lambda, s)$ obtained by pasting the border σ_1 in S_1 with the border of the conclusion $A^\perp$ in S_{23}, has a unique external border, visiting all the conclusions of Λ.

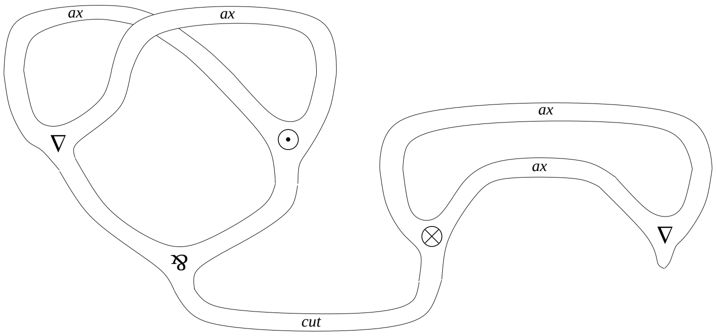
[C] When $\sigma = \tau$, and the surface S is connected, we may suppose by symmetry, wlog. that the surface S' obtained by cutting (with scisors) the branch B of the $\otimes$-link $l_\otimes$ in $\mathbf{ribbon}(\Theta, s_{RR})$ is also connected, and that the two borders σ' and τ' of the cut-elimination pattern $l_{⅋}, l_\otimes$ are equal in S'. Removing the cut-link connecting B and $B^\perp$ in S and removing the cut-link connecting A and $A^\perp$ in S' induce the same surface T, modulo minor topological alterations. The equality $\sigma = \tau$, alternatively the equality $\sigma' = \tau'$, implies that the surface T has two connected components. We call T_1 the component of B and T_2 the component of $B^\perp$, and claim that T_1 is also the component of A and T_2 the component of $A^\perp$ in the surface T. Indeed, consider the $\{\nabla_3\}$-free MNL switching s' of Θ obtained by replacing in s every switching position ∇_3 by the switching position ∇_L; let T' be the surface obtained from $\mathbf{ribbon}(\Theta, s'_{LR})$ or $\mathbf{ribbon}(\Theta, s'_{RR})$ by cutting (with scisors) the two branches A and B of the $\otimes$-link $l_\otimes$. Because Θ translates as an

MLL proof-net Θ^*, the surface T' has three components: one component contains A, the other component contains B, and the last component contains both $A^\perp$ and $B^\perp$. The surface T is obtained by replacing some of the positions ∇_L in s' by the position ∇_3 in s. Thus, the two formulas $A^\perp$ and $B^\perp$ which appear in the same component of T', appear a fortiori in the same component of T. The component T_2 is this component of $A^\perp$ and $B^\perp$; while T_1 is the component of A and B. Let σ_A and σ_B denote the respective borders of $A^\perp$ and $B^\perp$ in T_2.

Now, glue in S the conclusion A to the surface σ, to obtain the surface $\mathbf{ribbon}(\Theta, s_{LR})$. The surface S is connected, and the surface $\mathbf{ribbon}(\Theta, s_{LR})$ is planar. It follows that the border σ visits the conclusion A. On the other hand, after cutting (with scisors) the branch B in S, the border σ becomes the border σ' of B in T_1. We conclude that the conclusions A and B lie on the same border σ' of the component T_1. Glue these two conclusions A and B together in T_1, and call T' the resulting surface. The operation divides the border σ' of T_1 into two borders of T', denoted σ'_1 and σ'_2, in such a way that (1) the surface $\mathbf{ribbon}(\Theta, s_{LR})$ is obtained by glueing $B^\perp$ in T_2 to σ'_1 in T', and (2) the surface $\mathbf{ribbon}(\Theta, s_{LL})$ is obtained by glueing $B^\perp$ in T_2 to σ'_2 in T'. Our correctness criterion, and an argument similar to case [A] show that each border σ'_1 and σ'_2 visits exactly one internal border of a ∇-link in T'; and that the border σ_B visits no internal border of a ∇-link, and no conclusion of Θ. Now, the surface $\mathbf{ribbon}(\Theta, s_{RR})$ is obtained by glueing A in T_2 with σ'_1 in T'. It follows from our correctness criterion that the border σ_A visits no internal border of a ∇-link, and no conclusion of Θ.

We claim that the two border σ_A and σ_B coincide in T_2. Suppose not: $\sigma_A \neq \sigma_B$. In that case, the external border of $\mathbf{ribbon}(\Theta, s_{LR})$ is the residual of σ_A after glueing $B^\perp$ and σ'_2. It follows that the external border of $\mathbf{ribbon}(\Theta, s_{LR})$ which visits all conclusions of Θ by the correctness criterion, visits also no conclusion of Θ by our previous result. We concluded that Θ has no conclusion, which contradicts the fact that Θ translates as a MLL proof-net Θ^*. This establishes the claim: $\sigma_A = \sigma_B$. We are nearly done. Recall that σ_A does not visit any internal border of a ∇-link, and no conclusion of Θ. We deduce that $\mathbf{ribbon}(\Theta, s_{LR})$ may be deformed into $\mathbf{ribbon}(\Lambda, s)$ by "sliding" A along σ_A in T_2, until it reaches $A^\perp$. This proves that $\mathbf{ribbon}(\Lambda, s)$ is planar, has a unique external border, which visits all its conclusions. $\square$

Remark. — Try this alternative (but wrong!) definition of MNL proof-net: relax the condition on internal and external borders, and consider the class of MNL proof-structures Θ translating as an MLL proof-net Θ^*, and whose surface $\mathbf{ribbon}(\Theta, s)$ is planar for every topological switching s. It happens that the class is not closed under cut-elimination, as the proof-structure below of conclusion $\vdash (B \odot A) \multimap\!\!\bullet (A \odot B)$ illustrates.



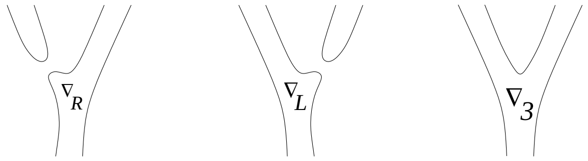
It should be noted that in the figure above, we use a topological notation for proof-structures, adapted from our notation for switchings. This is discussed in the appendix, section 1.10.

1.9 The planarity vs. the long trip criterion for MNL

Here, we reformulate our definition of MNL topological proof-nets in three different ways. The first formulation is topological, but *emancipated* of all reference to MLL, in V. Michele Abrusci's style. We call the second formulation *intermediate* because it prepares the third formulation, in which any reference to the topology disappears. Planarity is replaced by a *well-bracketing* condition on the ∇-links of the switched proof-structures. We benefit from the fact that this third formulation appears already in [3] and characterizes Abrusci-Ruet proof-nets, to conclude that our planarity criterion coincides with the long trip criterion for MNL.

Switched surface (2). — Here, we want to extend the definition of section 1.8, and define a surface $\mathbf{ribbon}(\Theta, s)$ for every MNL proof-structure Θ and MNL switching s, instead of $\{\nabla_L, \nabla_R\}$-free switchings. This is easy. The surface $\mathbf{ribbon}(\Theta, s)$ is defined as before, except that every ∇-link l is replaced by the ribbon diagram of its switching position

∇_L, ∇_R or ∇_3:



The previous definitions of *index*, of *external* and *internal* borders are extended in the obvious way: only ∇-links in position ∇_3 increase the index of a border of $\mathbf{ribbon}(\Theta, s)$.

The emancipated criterion. — An alternative characterization of topological MNL proof-nets follows, which does not mention commutative linear logic.

Lemma 1.9.1 *An MNL proof-structure* Θ *is a* topological MNL proof-net *iff for every MNL switching s:*

(i) *the surface* $\mathbf{ribbon}(\Theta, s)$ *is planar and has a unique external border* σ,

(ii) σ *contains all the conclusions.*

Note that the formulation is very close to Abrusci and Ruet definition 1.7.2 of an MNL proof-net, except that bilaterality is replaced here by planarity.

The intermediate criterion. — The next criterion makes the first step towards a non topological reformulation of our topological criterion, definition 1.8.1. Consider an MNL proof-structure Θ whose MLL-translation is an MLL proof-net Θ^*. Obviously, every ∇_3-free switching s of Θ defines a surface $\mathbf{ribbon}(\Theta, s)$ homeomorphic to the disk. The positions of each ∇-link l of Θ may be indicated on the unique border σ of $\mathbf{ribbon}(\Theta, s)$:

- by an opening bracket $(_l$
- by a closing bracket $)_l$.

in such a way that the segment of σ put inside brackets $(_l \ldots)_l$ coincides with the internal border of the surface $\mathbf{ribbon}(\Theta, s + (l \mapsto \nabla_3))$. Then, a necessary and sufficient condition for Θ to be a topological proof-net is that, for every ∇_3-free switching s of Θ:

(i) the brackets $(_l$ and $)_l$ may be pasted together in $\mathbf{ribbon}(\Theta, s)$ in such a way that the surface remains planar,

(ii) no conclusion of Θ appears inside brackets.

The well-bracketing criterion. — Now, we make topology disappear entirely from the intermediate criterion, by reformulating the planarity condition of point (i) as a well-bracketing condition on $(_l ...)_l$.

Lemma 1.9.2 *A MNL proof-structure Θ is a topological MNL proof-net iff:*

1. *its MLL translation Θ^* is an MLL proof-net,*

and for every ∇_3-free switching s of Θ:

2. *the brackets $(_l$ and $)_l$ are well-bracketed on the border* $\mathbf{trip}(\Theta, s)$ *of* $\mathbf{ribbon}(\Theta, s)$*,*
3. *no conclusion of Θ appears inside brackets.*

The planarity and the long trip criteria coincide. — The series of conditions in lemma 1.9.2 is already mentioned in [3], theorem 2.20, where it characterizes Abrusci-Ruet MNL proof-nets. We conclude that

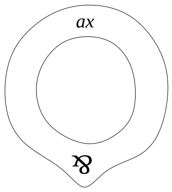
Theorem 1.9.3
The topological MNL proof-nets coincide with the Abrusci-Ruet MNL proof-nets.

Remark. — The remark by Mogbil and Puite about bilaterality (see the end of section 1.7) adapted to our topological setting, indicates that the planarity condition of lemma 1.9.1 may be replaced by the hypothesis that the proof-structure Θ translates as a MLL proof-net Θ^*. Indeed, a topological argument shows that in that case, the surface $\mathbf{ribbon}(\Theta, s)$ is planar for every MNL switching s. Suppose not: there exists a MNL switching s making $\mathbf{ribbon}(\Theta, s)$ non planar. Let s' denote the ∇_3-free switching obtained by switching as ∇_L (or ∇_R) all ∇-links switched ∇_3 in the switching s. The surface $\mathbf{ribbon}(\Theta, s')$ is homeomorphic to the disk because Θ^* is a MLL proof-net, and the MNL switching s' is ∇_3-free. Lemma 1.9.2 indicates that there exist two ∇-links l_1 and l_2 switched as ∇_3 in the MNL switching s such that the surface $\mathbf{ribbon}(\Theta, s'')$ is already non planar, when one alters s' into $s'' = s' + (l_1, l_2 \mapsto \nabla_3)$. We leave the reader check that the surface $\mathbf{ribbon}(\Theta, s'')$ has a unique border σ, of index 2, which visits all the conclusions of Θ. This contradicts the other hypothesis of lemma 1.9.1 (there exists a unique external border, which visits all conclusions) and we conclude that $\mathbf{ribbon}(\Theta, s)$ is planar, for every MNL switching s.

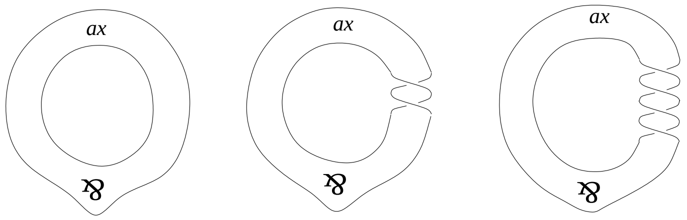
We also leave the reader check (hint: by a counter-example in McyLL) that the planarity condition is necessary in our definition 1.8.1 of topological proof-net, despite the fact that we assume that Θ translates as MLL proof-net Θ^*. The difference with lemma 1.9.1 is that MNL switchings are restricted here to topological (that is: $\{\nabla_L, \nabla_R\}$-free) switchings.

1.10 Appendix: is MNL an embedded logic?

In this article, we advocate that *switchings* are better expressed as topological objects, than as graphs. One may go further, and declare boldly that proofs *themselves* are topological objects, from which switched surfaces are deduced by topological surgery. From that perspective, the MLL proof π of $\vdash A^{\perp} \parr A$ defines a surface homeomorphic to the annulus.

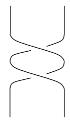


Each of the switching positions $\parr_L$ and $\parr_R$ of the $\parr$-link indicates to cut (with scisors) the annulus π from one border σ_1 to the other border σ_2. In each case, one obtains a surface homeomorphic to the disk. Except for inessential details in the presentation of proofs (ribbon diagrams vs. simplicial complexes) this topological presentation may be found in [14]. It may be worth stressing that the topology of proofs is understood *internally*. In particular, neither the proof theory, nor the topology, reflects the fact that the annulus π may be embedded in several ways in the ambient space, forming all kinds of "twisted knots" like:



etc...

The idea of representing a proof as a surface *embedded* in an ambient space may be found in [6] where Arnaud Fleury interprets the exchange rule as a “braided” permutation, and introduces a “twist” operation on formulas, inspired by similar operations in tortile tensor categories [11].

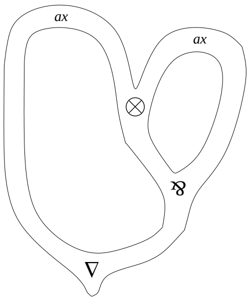


In the resulting “embedded logic” $\overline{\mathrm{MLL}}$, every embedding of the annulus π in the ambient space happens to be a particular proof of the formula $\vdash A^{\perp} \parr A$. More generally, a $\overline{\mathrm{MLL}}$ proof is either constructed sequentially, or characterized geometrically (this is the correctness criterion) as a proof-structure embedded in space, whose switchings are all homeomorphic to the disk. Similarly, one defines an embedded version $\overline{\mathrm{McyLL}}$ of McyLL, whose proofs π are the proofs of $\overline{\mathrm{MLL}}$ verifying the extra condition that π is planar, and has a unique external border visiting all conclusions.

In contrast, there does not seem to exist any satisfactory embedded version of MNL, for the following reason. Consider the MNL proof

$$\dfrac{\dfrac{\dfrac{\vdash A^{\perp}, A \qquad \vdash B, B^{\perp}}{\vdash A^{\perp}, A \otimes B, B^{\perp}}\otimes}{\vdash A^{\perp}, (A \otimes B) \parr B^{\perp}}\parr}{\vdash A^{\perp} \triangledown ((A \otimes B) \parr B^{\perp})}\triangledown \tag{1.8}$$

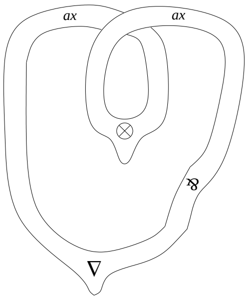
As in the case of the annulus, there may be several way to embed the proof in ambient space. We choose one of them, which we draw below.



In this particular embedding of the MNL proof (1.8), the switching position s

$$\nabla \mapsto \nabla_3 \qquad \invamp \mapsto \invamp_R \qquad \otimes \mapsto \otimes_L$$

induces a surface admitting a "twist" between the formulas $A^\perp$ and $(A \otimes B) \invamp B^\perp$.



(1.9)

So, the switched surface, seen as embedded in ambient space, is not planar. More generally, there exists no embedding of (1.8) able to induce only planar MNL switching surfaces. The phenomenon is a consequence of the *see-saw* rule of non-commutative logic, which says that every proof of $\vdash A \invamp B$ is also a proof of $\vdash A \nabla B$. This principle is fine when the topology of proofs is understood internally, but becomes problematic when the topology of proofs is embedded in an ambient space — at least in our ribbon presentation. Typically, the see-saw rule justifies the last ∇-introduction rule of the derivation tree (1.8) which implies in turn that the surface (1.9) is not planar.

1.11 Conclusion

In their correctness criteria [2, 3] Abrusci and Ruet characterize McyLL and MNL proof-nets without mentioning commutative MLL. This conveyed the hope for a theory of McyLL and MNL "emancipated" from any reference to MLL. In this article, we choose to step back, and understand McyLL and MNL as commutative MLL + *a planarity condition*:

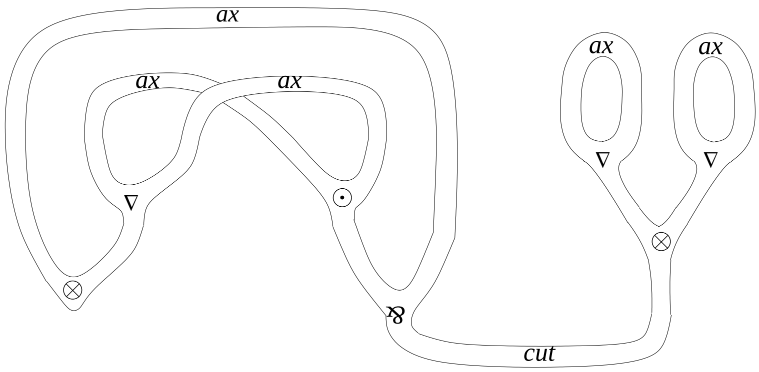
- MLL + planarity of proof-nets, for McyLL,
- MLL + planarity of *switched* proof-nets, for MNL.

One reason is that cut-elimination of McyLL and MNL follows essentially from planarity — and its preservation by cut-elimination in MLL. Another reason is that the switching positions $⅋_L$ and $⅋_R$ are internalized in MLL by the "linear" distributivity formulas below, see [5, 1]:

$$A \otimes (B ⅋ C) \multimap (A \otimes B) ⅋ C, \qquad\qquad A \otimes (B ⅋ C) \multimap B ⅋ (A \otimes C).$$

In contrast, there exists (today) no such internal justification in McyLL or MNL for the "emancipated" criteria formulated in [2, 3] and recalled in sections 1.7 and 1.9.

To conclude, we will mention the open problem of designing a correctness criterion for MNL proof-structures *with cuts.* Abrusci and Ruet illustrate this problem in [3] by exhibiting the MNL proof-net (1.10) which cannot be sequentialized in MNL. (Here again, we use a topological notation to draw the proof-net (1.10), as discussed in the appendix.)



(1.10)

Finding a satisfactory solution may require to alter MNL — as cyclic linear logic was altered into planar logic in section 1.5. For what matters is not the details of the logic, but its relationship to a geometric (or computational) property of proofs, preserved by cut-elimination.

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