Title

Axiomatic Rewriting Theory VI

Residual theory revisited

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Original motivation: the $\lambda\sigma$-calculus and its critical pairs

One critical pair plays a fundamental role in our counter-exam
to preservation of strong normalization in the $\lambda\sigma$-calculus (199

$$
\begin{tikzcd}
((\lambda P)Q)[s] \arrow[r, "App"] \arrow[d, "Beta"'] & (\lambda P)[s]Q[s] \arrow[r, "Lam"] & (\lambda(P[\mathbf{1}\cdot s \circ \arrow[d] \\
P[Q\cdot id][s] \arrow[d, "Clos+Map"'] & \boxed{\text{Always turn left!}} & P[\mathbf{1}\cdot s\circ\uparrow][\arrow[d] \\
P[Q[s]\cdot id\circ s] \arrow[d, "IdL"'] & & P[Q[s]\cdot(s\circ\uparrow) \arrow[d] \\
P[Q[s]\cdot s] & & P[Q[s]\cdot(\arrow[ll, dotted, "when\ \ s=M_1\cdot M_2\cdot\ldots\cdot M_n\cdot id"', "and\ the\ M_i's\ are\ in\ \sigma-normal\ form"]
\end{tikzcd}
$$

Ten other critical pairs!

$App + Beta$	$(\lambda a)[s](b[s])$	$\overset{App}{\longleftarrow}$	$((\lambda a)b)[s]$	$\overset{Beta}{\longrightarrow}$
$Clos + App$	$(ab)[s \circ t]$	$\overset{Clos}{\longleftarrow}$	$(ab)[s][t]$	$\overset{App}{\longrightarrow}$
$Clos + Abs$	$(\lambda a)[s \circ t]$	$\overset{Clos}{\longleftarrow}$	$(\lambda a)[s][t]$	$\overset{Abs}{\longrightarrow}$
$Clos + VarId$	$\mathbf{1}[id \circ s]$	$\overset{Clos}{\longleftarrow}$	$\mathbf{1}[id][s]$	$\overset{VarId}{\longrightarrow}$
$Clos + VarCons$	$\mathbf{1}[(a \cdot s) \circ t]$	$\overset{Clos}{\longleftarrow}$	$\mathbf{1}[a \cdot s][t]$	$\overset{VarCon}{\longrightarrow}$
$Clos + Clos$	$a[s][t \circ t']$	$\overset{Clos}{\longleftarrow}$	$a[s][t][t']$	$\overset{Clos}{\longrightarrow}$
$Ass + Map$	$(a \cdot s) \circ (t \circ t')$	$\overset{Ass}{\longleftarrow}$	$((a \cdot s) \circ t) \circ t'$	$\overset{Map}{\longrightarrow}$
$Ass + IdL$	$id \circ (s \circ t)$	$\overset{Ass}{\longleftarrow}$	$(id \circ s) \circ t$	$\overset{IdL}{\longrightarrow}$
$Ass + ShiftId$	$\uparrow \circ (id \circ s)$	$\overset{Ass}{\longleftarrow}$	$(\uparrow \circ id) \circ s$	$\overset{ShiftId}{\longrightarrow}$
$Ass + ShiftCons$	$\uparrow \circ ((a \cdot s) \circ t)$	$\overset{Ass}{\longleftarrow}$	$(\uparrow \circ (a \cdot s)) \circ t$	$\overset{ShiftCon}{\longrightarrow}$
$Ass + Ass$	$(s \circ s') \circ (t \circ t')$	$\overset{Ass}{\longleftarrow}$	$((s \circ s') \circ t) \circ t'$	$\overset{Ass}{\longrightarrow}$

The associativity rule = a simplified $\lambda\sigma$-calculus

The $\lambda\sigma$-calculus originates from Curien's CCL, a categorical
on the associativity rule

$$x \otimes (y \otimes z) \xrightarrow{\alpha} (x \otimes y) \otimes z$$

For instance, the distinction between terms and substitution
ad'hoc in the $\lambda\sigma$-calculus.

$$M[t][u] \xrightarrow{Clos} M[t \circ u] \qquad (s \circ t) \circ u \xrightarrow{Ass} s \circ (t \circ u$$

$$MN[t] \xrightarrow{App} M[t]N[t] \qquad (M \cdot s)[t] \xrightarrow{Map} M[t] \cdot s \circ$$

$$\mathbf{1}[M \cdot s] \xrightarrow{VarCons} M \qquad \uparrow \circ (M \cdot s) \xrightarrow{Shift} s$$

What are the critical pairs of the associativity rule?

The associativity rule generates one fake critical pair:

$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{w \otimes \alpha} w \otimes ((x \otimes y) \otimes z)$$
$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{\alpha} (w \otimes x) \otimes (y \otimes z)$$

which is convergent by the diagram below:

$$\begin{tikzcd}
w \otimes (x \otimes (y \otimes z)) \arrow[r, "w \otimes \alpha"] \arrow[dd, "\alpha"'] & w \otimes ((x \otimes y) \otimes z) \arrow[d, "\alpha"] \\
 & (w \otimes (x \otimes y)) \otimes z \arrow[d, "\alpha \otimes z"] \\
(w \otimes x) \otimes (y \otimes z) \arrow[r, "\alpha"] & ((w \otimes x) \otimes y) \otimes z
\end{tikzcd}$$

This diagram is called Mac Lane's pentagon.

A coherence theorem (Mac Lane)

A classical problem of category theory: compare the several r
which rewrite the left comb

$$x_1 \otimes (x_2 \otimes \ldots \otimes (x_{n-1} \otimes x_n))))))$$

into the right comb:

$$((((((x_1 \otimes x_2) \otimes \ldots) \otimes x_{n-1}) \otimes x_n$$

★ Theorem [Mac Lane 1970]

Every two such rewriting paths f and g are equal, modulo a se

— square permutations (to permute non-overlapping redexes)

— pentagonal permutations (to permute overlapping redexes)

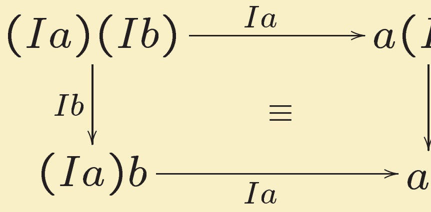
A theory of residuals (Lévy)

A classical problem of rewriting theory: compare the severa
rewrite a λ-term P into its normal form Q.

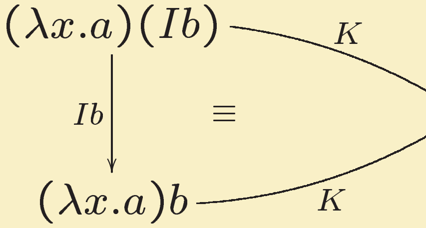
★ Theorem [Lévy 1978]

Every two such paths f, g are equal, modulo a series of:

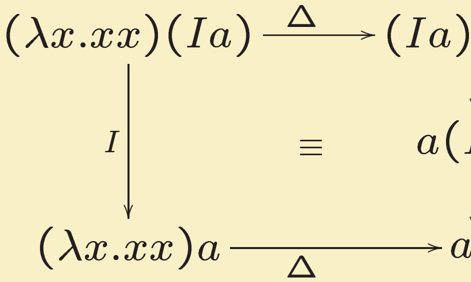
— disjoint permutations:



— erasing permutations:



— duplicating permutations:



Towards a general theory of residuals

J-J. Lévy deduces his theorem from a theory of redexes and re

Idea: in the permutation diagram:

$$\begin{tikzcd}
(\lambda x.xx)(Ia) \arrow[rr, "\Delta"] \arrow[dd, "I"'] & & (Ia)(Ia) \arrow[d, "I_1"] \\
& & a(Ia) \arrow[d, "I_2"] \\
(\lambda x.xx)a \arrow[rr, "\Delta"] & & aa
\end{tikzcd}$$

the "residual" path of I after Δ:

$$(Ia)(Ia) \xrightarrow{I_1} a(Ia) \xrightarrow{I_2} aa$$

is the parallel reduction of the two β-redexes residual of I afte

What would be a theory of residuals for the associativi

Treks

In the permutation diagram

$$\begin{tikzcd}
w \otimes (x \otimes (y \otimes z)) \arrow[r, "w \otimes \alpha"] \arrow[dd, "\alpha"'] & w \otimes ((x \otimes y) \otimes z) \arrow[d, "\alpha"] \\
& (w \otimes (x \otimes y)) \otimes z \arrow[d, "\alpha \otimes z"] \\
(w \otimes x) \otimes (y \otimes z) \arrow[r, "\alpha"] & ((w \otimes x) \otimes y) \otimes z
\end{tikzcd}$$

the "residual" path of α after $w \otimes \alpha$:

$$w \otimes ((x \otimes y) \otimes z) \xrightarrow{\alpha} (w \otimes (x \otimes y)) \otimes z \xrightarrow{\alpha \otimes z} ((w \otimes x) \otimes y$$

is not the parallel reduction of two redexes.

$\boxed{\text{Introduce an intermediate atomicity of computation =}}$

$$\begin{tikzcd}
\text{Redexes} \arrow[rr, leftrightarrow, bend left, "\text{Lévy}"] \arrow[dr, leftrightarrow, bend right] & & \text{Paths} \arrow[dl, leftrightarrow, bend left] \\
& \text{Treks} &
\end{tikzcd}$$

I. Lévy's theory of residuals (1978

Rewriting graph and residual relation

★ A graph $\mathcal{G}$ is a quadruple $(V, E, \mathbf{source}, \mathbf{target})$:

— a set V of terms,

— a set E of redexes,

— two functions $\mathbf{source}, \mathbf{target} : E \longrightarrow V$.

★ $R(M)$ denotes the set of redexes outgoing from the term M

★ A residual relation $[\![-]\!]$ over a graph $\mathcal{G}$ is a relation

$$[\![u]\!] \subset R(M) \times R(N)$$

for every redex $u : M \longrightarrow N$.

Axioms 1 and 2

1. **self-destruction:** for every redex u, the set $u[\![u]\!]$ is empty

2. **finiteness:** for every two coinitial redexes u and v, the set

$\boxed{u[\![v]\!] \text{ denotes the set of residuals of } u \text{ after } v}$

Multi-treks and developments

★ A multi-redex is a pair (M, U) consisting of

- a term M,
- a finite subset U of $R(M)$.

★ A path $f = u_1; u_2; \cdots; u_n$ develops a multi-redex (M, U) part

— u_1 is element of U

— u_2 is residual (after u_1) of an element $u \in U$,

...

— u_n is residual (after $u_1 ... u_n$) of an element $u \in U$.

The path f develops the multi-redex (M, U) when $U[\![u_1]\!][\![u_2]\!] \cdots$

Axioms 3. and 4.

3. **finite developments:** for every multi-redex (M, U), there
   any infinite sequence

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \dots M_n \xrightarrow{u_n} M_{n+1} \dots$$

   of redexes such that, for every index i, the path

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} \dots M_{i-1} \xrightarrow{u_{i-1}} M_i$$

   develops the multi-redex (M, U) partially.

4. **permutation:** for every two coinitial redexes

$$u : M \longrightarrow P \qquad v : M \longrightarrow Q$$

   there exists two cofinal paths

$$h_u : Q \longrightarrow N \qquad h_v : P \longrightarrow N$$

   such that:

   - h_u develops $(Q, u[\![v]\!])$ and h_v develops $(P, v[\![u]\!])$,
   - h_u and h_v induce the same residual relation $[\![u; h_v]\!] = [\![$

Homotopy

This defines a homotopy relation $\equiv$ on the rewriting paths.

$$\begin{tikzcd} M \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, "h_v"] \\ Q \arrow[r, "h_u"] & N \end{tikzcd}$$

h_u is a development of $(Q, u$

h_v is a development of $(P, v$

Universal confluence = pushout

For every two coinitial rewriting paths $f : M \longrightarrow P$ and $g : M$
exists two paths $f' : Q \longrightarrow N$ and $g' : P \longrightarrow N$ such that:

(1) the paths $f; g' : M \longrightarrow N$ and $g; f' : M \longrightarrow N$ are equal mod

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, dotted, "g'"] \\
Q \arrow[r, dotted, "f'"'] & N
\end{tikzcd}
$$

(2) for every two paths $f'' : Q \longrightarrow O$ and $g'' : P \longrightarrow O$ such tha
there exists a path h (unique modulo homotopy) such that
$g'' \equiv g'; h$

$$
\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "g"'] & P \arrow[d, "g'"] \arrow[ddr, bend left, "g''"] & \\
Q \arrow[r, "f'"] \arrow[drr, bend right, "f''"'] & N \arrow[dr, dotted, "h"] & \\
 & & O
\end{tikzcd}
$$

Theorem [Lévy 1978] Every residual system is universally conf

Corollary Two paths from a term P to its normal form Q are

II. The theory revisited: treks and resi

Trek graph

★ A trek graph is a graph $\mathcal{G}$ equipped with a partially ordered

$$(T(M), \preceq_M)$$

for every term M, such that

$\boxed{\text{the minimal elements of } T(M) \text{ are the redexes } R(M) = \text{m}}$

★ A residual relation over a trek graph $\mathcal{G}$ is a relation

$$[\![u]\!] \subset T(P) \times T(Q)$$

for every redex $u : P \longrightarrow Q$, such that:

▷ ▷ ▷ For every treks $\mathfrak{t}_1, \mathfrak{t}_2 \in T(P)$ and $\mathfrak{t}'_1 \in T(Q)$ and redex u :

$$\mathfrak{t}_1 \preceq \mathfrak{t}_2 \quad \text{and} \quad \mathfrak{t}_1 [\![u]\!] \mathfrak{t}'_1$$

implies that there exists a trek $\mathfrak{t}'_2 \in T(Q)$, such that:

$$\mathfrak{t}_2 [\![u]\!] \mathfrak{t}'_2 \quad \text{and} \quad \mathfrak{t}'_1 \preceq \mathfrak{t}'_2$$

Axioms 1. and 2.

1. **self-destruction:** for every redex u, the set $u[\![u]\!]$ is empty

2. **finiteness:** for every $\mathfrak{t} \in T(M)$ and $u \in R(M)$, the set $\mathfrak{t}[\![u]\!]$

$\boxed{\mathfrak{t}[\![u]\!] = \{\mathfrak{t}' \in T(Q) \mid \mathfrak{t}[\![u]\!]\mathfrak{t}'\} \text{ denotes the set of treks residual}}$

Multi-treks and developments

★ A multi-trek is a pair $(M, \mathfrak{S})$ consisting of

- a term M,
- a finite subset $\mathfrak{S}$ of $T(M)$.

★ A path $f = u_1; u_2; \cdots; u_n$ develops a multi-trek $(M, \mathfrak{S})$ partia

— $u_1 \preceq \mathfrak{t}_1$ for some element $\mathfrak{t}_1 \in \mathfrak{S}$,

— $u_2 \preceq \mathfrak{t}_2$ for some element $\mathfrak{t}_2 \in \mathfrak{S}[\![u_1]\!]$,

...

— $u_n \preceq \mathfrak{t}_n$ for some element $\mathfrak{t}_n \in \mathfrak{S}[\![u_1]\!] \cdots [\![u_n]\!]$.

The path f develops the multi-trek $(M, \mathfrak{S})$ when $\mathfrak{S}[\![u_1]\!] \cdots [\![u_n]\!]$

Axioms 3. and 4.

3. **finite developments:** for every multi-trek $(M, \mathfrak{S})$, there
any infinite sequence

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} ... M_n \xrightarrow{u_n} M_{n+1} ...$$

of redexes such that, for every index i, the path

$$M_1 \xrightarrow{u_1} M_2 \xrightarrow{u_2} ... M_{i-1} \xrightarrow{u_{i-1}} M_i$$

develops the multi-trek $(M, \mathfrak{S})$ partially.

4. **permutation:** for every two coinitial redexes

$$u : M \longrightarrow P \qquad v : M \longrightarrow Q$$

there exists two cofinal paths

$$h_u : Q \longrightarrow N \qquad h_v : P \longrightarrow N$$

such that:

- h_u develops $(Q, u[\![v]\!])$ and h_v develops $(P, v[\![u]\!])$,
- h_u and h_v induce the same residual relation $[\![u; h_v]\!] = [\![$

Homotopy (extended)

Define a homotopy relation $\equiv$ on the rewriting paths.

$$
\begin{tikzcd}
M \arrow[r, "u"] \arrow[d, "v"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, "h_v"] \\
Q \arrow[r, "h_u"] & N
\end{tikzcd}
$$

h_u is a development of $(Q, u$

h_v is a development of $(P, v$

Universal confluence (extended)

For every two coinitial rewriting paths $f : M \longrightarrow P$ and $g : M$
exists two paths $f' : Q \longrightarrow N$ and $g' : P \longrightarrow N$ such that:

(1) the paths $f ; g' : M \longrightarrow N$ and $g ; f' : M \longrightarrow N$ are equal mod

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\equiv"] & P \arrow[d, dotted, "g'"] \\ Q \arrow[r, dotted, "f'"'] & N \end{tikzcd}$$

(2) for every two paths $f'' : Q \longrightarrow O$ and $g'' : P \longrightarrow O$ such tha
there exists a path h (unique modulo homotopy) such that
$g'' \equiv g' ; h$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & P \arrow[d, "g'"] \arrow[ddr, bend left, "g''"] & \\ Q \arrow[r, "f'"] \arrow[drr, bend right, "f''"'] & N \arrow[dr, dotted, "h"] & \\ & & O \end{tikzcd}$$

Theorem Every trek system is universally confluent.

Corollary Two paths from a term P to its normal form Q are

III. The associativity treks and residu

Definition of treks (the associativity rule)

Given $n \geq 1$ variables $y_1, ..., y_n$, let $[y_1, ..., y_n]$ denote the right co

$$[y_1] = y_1 \qquad [y_1, ..., y_n] = [y_1, ..., y_{n-1}] \otimes y_n$$

A trek $\mathfrak{t}$ of $T(M)$ is defined as a pattern

$$x \otimes [y_1, ..., y_n]$$

for $n \geq 1$, and its occurrence o in the binary tree M.

So, a trek $\mathfrak{t} \in T(M)$ is a pair (o, n) where the occurrence o is a se of L's and R's (for left and right) identifying the position of the product of the trek $\mathfrak{t}$.

Notation: We say that the trek $\mathfrak{t}$ has a head: the tensor produc and a spine: the $n - 1$ tensor products inside $[y_1, ..., y_n]$.

Treks and residuals (the associativity rule)

Suppose that $\mathfrak{t}, \mathfrak{t}' \in T(M)$ are two treks of the binary tree M.

We write $\mathfrak{t} \preceq \mathfrak{t}'$ when $\mathfrak{t}$ and $\mathfrak{t}'$ occur at the same occurrence, ar

$$\mathfrak{t} = x \otimes [[y_1, \ldots, y_m], y_{m+1}, \ldots, y_n] \xrightarrow{n-m+1} [x \otimes [y_1, \ldots, y_m], y_{m+}$$

$$\mathfrak{t}' = x \otimes [y_1, \ldots, y_m, y_{m+1}, \ldots, y_n] \xrightarrow{n} [x, y_1, \ldots, y_m, y_{m+1}, \cdot$$

The two treks $\mathfrak{t}$ and $\mathfrak{t}'$ verify $\mathfrak{t}[\![u]\!]\mathfrak{t}'$ precisely when:

- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (\phi_u(o), n)$ when the redex u and the overlap,
- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (o \cdot L, n-1)$ when $n \geq 3$ and the redex u both the head and the spine of $\mathfrak{t}$. Observe that $u \preceq \mathfrak{t}$ in th
- $\mathfrak{t} = (o \cdot R, n)$ and $\mathfrak{t}' = (o, n)$ when the redex u overlap with not the spine) of $\mathfrak{t}$,
- $\mathfrak{t} = (o, n)$ and $\mathfrak{t}' = (o, n+1)$ when the redex u overlap with not the head) of $\mathfrak{t}$.

Reunderstanding Mac Lane's pentagon

$$
\begin{tikzcd}
w \otimes (x \otimes (y \otimes z)) \arrow[r, "w\otimes\alpha"] \arrow[dd, "\alpha"'] & w \otimes ((x \otimes y) \otimes z) \arrow[d, "\alpha"] \\
 & (w \otimes (x \otimes y)) \otimes z \arrow[d, "\alpha\otimes z"] \\
(w \otimes x) \otimes (y \otimes z) \arrow[r, "\alpha"] & ((w \otimes x) \otimes y) \otimes z
\end{tikzcd}
$$

The trek "w wants to visit x"

$$w \otimes ((x \otimes y) \otimes z) \xrightarrow{\alpha} (w \otimes (x \otimes y)) \otimes z \xrightarrow{\alpha\otimes z} ((w \otimes x) \otimes y$$

is the residual of the trek "w wants to visit x"

$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{\alpha} (w \otimes x) \otimes (y \otimes z)$$

after the redex "x wants to visit y"

$$w \otimes (x \otimes (y \otimes z)) \xrightarrow{w\otimes\alpha} w \otimes ((x \otimes y) \otimes z)$$

Conclusion

Examples of the article:

— the associativity rule ($\Rightarrow$ a standardization theorem),

— the positive braids permutations,

— optimality theory in the λ-calculus.

More generally, another atomicity for

$\boxed{\text{Non orthogonal and universally confluent rewriting sys}}$

Perspectives:

— a $\lambda\sigma$-calculus without critical pairs... with a unique head-re

— Automatic/semi-automatic criterion in order to detect fake

— Connections with coherence theorems in n-dimensional cat