The rabbit calculus: convolution products on double categories and categorification of rule algebra

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The quest for causality in rewriting theory

An important insight coming from Huet and Lévy:

In order to track the **causality structure** relating different β -redexes, one needs to consider rewriting paths modulo **permutations** of the form



Consider the term rewriting system on the signature with two letters:

a binary letter m:2 a constant letter e:0

together with the unique rewrite rule

 $r : m(e, x) \longrightarrow x$

which we depict as follows:



The term



admits exactly two redexes

$$m(e, m(e, e)) \xleftarrow{red} m(m(e, e), m(e, e)) \xrightarrow{blue} m(m(e, e), e)$$

which are **independent** and can be computed in parallel.

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$$m(e, m(e, e)) \xleftarrow{red} m(m(e, e), m(e, e)) \xrightarrow{blue} m(m(e, e), e)$$

which are **independent** and can be computed in parallel.

One obtains a local confluence or permutation diagram



expressing that the blue redex and the red redex are independent.

Independence and permutation



At the same time, rewriting the redex

$$m(m(e,e),m(e,e)) \xrightarrow{red} m(e,m(e,e))$$

in the same term



rewrites to the term



which admits the redex

$$m(e, m(e, e)) \xrightarrow{green} m(e, e)$$

rewrites to the term



which admits the redex

$$m(e, m(e, e)) \xrightarrow{green} m(e, e)$$

Here, the red redex creates the green redex



because the green redex cannot be permuted before the red redex.

Creation



Composing redexes in term rewriting



Composing redexes in term rewriting

The term



admits one **composite redex** obtained by composing red and green.

$$m(m(e,e),m(e,e)) \xrightarrow{red} m(e,m(e,e)) \xrightarrow{green} m(e,e)$$

Composing redexes in term rewriting

The term



admits one composite redex obtained by composing red and green.

 $m(m(e,e),m(e,e)) \xrightarrow{red} m(e,m(e,e)) \xrightarrow{green} m(e,e)$

The quest for causality in rewriting theory

In the λ -calculus and term rewriting systems

A well-established tradition based on **optimality** and **residual theory**

- ▶ the notion of **Lévy families** in the λ -calculus (Lévy 1980)
- ▶ their generalisation to any CRS (Asperti, Laneve 1995)
- ▶ a residual theory based on **the notion of trek** (PAM, 2002)

More recently, in categorical graph rewriting

▶ the notion of **tracelet** emerging in the work by **Nicolas Behr**.

Our ambition in this work is to initiate a convergence between these lines by revisiting/categorifying the work on tracelets using **double categories**.

Double categories

A convenient framework for term and graph rewriting

Double categories

Definition. A (weak) **double category** \mathbb{D} consists of

- \triangleright a category \mathbb{D}_0 of objects,
- \triangleright a category \mathbb{D}_1 of horizontal maps,
- a pair of source and target functors

$$\mathbb{D}_0 \xleftarrow{T} \mathbb{D}_1 \xrightarrow{S} \mathbb{D}_0$$

a horizontal composition functor

 $\diamond_h \quad : \quad \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$

a horizontal identity functor

 $idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$

satisfying a number of **associativity** and **neutrality** properties.

The category \mathbb{D}_0 of vertical maps

A morphism in the category \mathbb{D}_0 is represented as a vertical map



which may be **composed vertically** with other vertical maps.

The category \mathbb{D}_1 of horizontal maps

An object in the category \mathbb{D}_1 is represented as a **horizontal map**

$$B \xleftarrow{r} A$$

A morphism in the category \mathbb{D}_1 is represented as a **double cell**



which may be **composed vertically** with other double cells.

The category \mathbb{D}_1 of horizontal maps

We often find convenient to use the pictorial notation



for the double cell usually noted



The category \mathbb{D}_2 of paths of length 2

Every double category D comes with

a category $\mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ of horizontal paths of length 2

defined as the limit of the diagram of functors



in the category Cat of categories and functors.

The category \mathbb{D}_2 of paths of length 2

A typical morphism of \mathbb{D}_2 has the shape



which we also like to depict as



The category \mathbb{D}_3 of paths of length 3

Every double category \mathbb{D} comes with

a category \mathbb{D}_3 of horizontal paths of length 3

defined as the limit of the diagram of functors



in the category Cat of categories and functors.

The category \mathbb{D}_3 of paths of length 3

A typical morphism of \mathbb{D}_3 has the shape



which we also like to depict as



The category \mathbb{D}_4 of paths of length 4

Every double category D comes with

a category \mathbb{D}_4 of horizontal paths of length 4

defined as the limit of the diagram of functors



in the category Cat of categories and functors.

The category \mathbb{D}_4 of paths of length 4

A typical morphism of \mathbb{D}_4 has the shape



which we also like to depict as



Unbiased presentation of a double category

Every double category \mathbb{D} comes equipped with a family of functors

 $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$

called the **horizontal composition** functors, and satisfying a number of **associativity** and **neutrality** properties.

This leads to an alternative (unbiased) definition of (weak) double category.

Note that the functors h_2 and h_0 coincide with the functors \diamond_h and *idh*

$$h_2 = \diamond_h \quad : \quad \mathbb{D}_2 \longrightarrow \mathbb{D}_1$$
$$h_0 = idh \quad : \quad \mathbb{D}_0 \longrightarrow \mathbb{D}_1$$

The double category **DPO** of double pushouts

The double category $\mathbb{D} = \mathbb{D}PO$ on an adhesive category G

- \triangleright whose objects are objects A, B, C of the adhesive category G,
- ▷ whose horizontal maps M = (S, s, t) are spans in **G**,
- ▷ whose vertical maps $\lambda_A : A \to A'$ are monos in **G**,
- ▶ whose double cells $\theta : M \Rightarrow M'$ are monos $\lambda_{\theta} : S \rightarrow S'$ making the pushout diagram commute:



The double category LTRS of linear term rewriting

The double category $\mathbb{D} = LTRS$ on a first-order signature

$$\Sigma = \prod_{n \in \mathbb{N}} \Sigma_n$$

is defined as follows:

▷ its objects are sequences of **closed linear** λ **-terms**

 $t_1: A_1 \otimes \ldots \otimes t_n: A_n$

whose types are generated by the grammar

A,B ::= $\circ \mid A \multimap B$

extended with the rule for each letter $a \in \Sigma_n$ of the signature:

The double category LTRS of linear term rewriting

its vertical maps

 $u_1: A_1 \otimes \ldots \otimes u_p: A_p \xrightarrow{f_1 \otimes \ldots \otimes f_q} v_1: B_1 \otimes \ldots \otimes v_q: B_q$

are sequences of **linear** λ **-terms**

 $\Gamma_1 \vdash f_1 : B_1 \qquad \dots \qquad \Gamma_q \vdash f_q : B_q$

separating the context linearly

 $A_1, \ldots, A_p \cong \Gamma_1, \ldots, \Gamma_q$

and satisfying the series of expected equations

 $v_1 = f_1[u_1, ..., u_p]$... $v_q = f_q[u_1, ..., u_p]$

The double category LTRS of linear term rewriting

whose double cell are of the form



where the horizontal morphism

$$t_1 \otimes \ldots \otimes t_p \longrightarrow u_1 \otimes \ldots \otimes u_p$$

is a **pair of sequences** of closed linear λ -terms with same types.

Rewriting rules as covariant presheaves





which **implement** the transformation u as an instance of the rule r.

The rewrite rule

 $r : m(e, x) \longrightarrow x$

implements the red redex using the double cell:



The rewrite rule

 $r \quad : \quad m(e,x) \longrightarrow x$

implements the blue redex using the double cell:



The rewrite rule

$$r \quad : \quad m(e,x) \longrightarrow x$$

implements the green redex using the double cell:


Goal: composing rules using convolution



Category of elements of a presheaf

The Grothendieck construction

Elements of a covariant presheaf

Recall that an element

 $(a, x) \in \operatorname{Elts}(F)$ of a covariant presheaf $F : \mathbb{C} \longrightarrow \operatorname{Set}$ is defined as a pair $\left(\begin{array}{cc} a \in \mathbb{C} & , & x \in F(a) \end{array}\right)$ consisting of

- \triangleright an object *a* of the underlying category **C**,
- \triangleright an element x of the set F(a).

Elements of a covariant presheaf

We find enlightening to draw such a pair

$$\left(a \in \mathbf{C} , x \in F(a) \right) \in \mathbf{Elts}(F)$$

F

а

in the following way

with the intuition that the element

 $x \in F(a)$

provides a witness of the covariant presheaf F at instance $a \in C$.

Covariant action of a presheaf

By definition of a covariant presheaf

$$F : \mathbf{C} \longrightarrow \mathbf{Set}$$

every element

$$\left(\begin{array}{cc} a \in \mathbf{C} & , & x \in F(a) \end{array}\right) \in \mathbf{Elts}(F)$$

and morphism of the category C

$$\gamma \quad : \quad a \longrightarrow a'$$

induces an element

$$\left(\begin{array}{cc}a' \in \mathbf{C} , \quad \gamma \cdot x = F(\gamma)(x) \in F(a')\end{array}\right) \in \mathbf{Elts}\left(F\right)$$

Covariant action of a presheaf

This means that every diagram



can be completed into the diagram



The category of elements

The category Elts(F) of elements of a covariant presheaf

$$F \quad : \quad \mathbf{C} \longrightarrow \mathbf{Set}$$

is defined in the following way:

- \triangleright its objects are the elements (a, x) of the covariant presheaf F
- its morphisms

$$(f,x) \quad : \quad (a,x) \longrightarrow (a',x')$$

are the pairs consisting of a morphism

$$f : a \longrightarrow a'$$

of the category C and an element $x \in F(a)$ such that

$$f \cdot x = F(f)(x) = x'$$

The category of elements

The category of elements

Elts (F)

associated to a covariant presheaf

 $F : \mathbf{C} \longrightarrow \mathbf{Set}$

comes equipped with a projection functor

 π_F : **Elts** (F) \longrightarrow **C**

which transports every element

 $(a, x) \in \mathbf{Elts}(F)$

to the object $a \in C$ of the underlying category C.

Fact. The functor π_F defines a **discrete opfibration**.

Grothendieck opfibrations

Definition. A functor

 $p : \mathbf{E} \longrightarrow \mathbf{C}$

is an opfibration when there exists an opcartesian morphism



for every object $R \in p^{-1}(A)$ and every morphism $u : A \to B$.

Opcartesian morphisms

A morphism $f : R \to S$ in **E** is opcartesian above $u : A \to B$ in **C** when the following property holds:



The Grothendieck correspondence

The projection functor

 π_F : **Elts** (F) \longrightarrow **C**

is a discrete opfibration. Indeed, every diagram



can be completed with the opcartesian morphism (f, x) as follows:



The Grothendieck correspondence

Moreover, every natural transformation



induces a commutative diagram of discrete opfibrations:



The Grothendieck correspondence

- Fact. This induces a categorical equivalence between
- ▷ The category [C, Ens] of covariant presheaves

F,G : $\mathbf{C} \longrightarrow \mathbf{Set}$

and natural transformations between them.

 \triangleright The slice category **DiscOpFib**/**C** of **discrete opfibrations** above **C**.

Moreover, there is an adjunction



A construction on monoidal categories

Given two covariant presheaves

F,G : $\mathbf{C} \longrightarrow \mathbf{Set}$

on a monoidal category **C** with tensor product

 $\otimes \quad : \quad \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$

the **Day convolution product** of F and G is the covariant presheaf

 $G \otimes F : \mathbf{C} \longrightarrow \mathbf{Set}$

defined by the coend formula

$$G \otimes F = c \mapsto \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

Equivalently, the convolution product

 $G \otimes F : \mathbf{C} \longrightarrow \mathbf{Set}$

may be defined as the left Kan extension of the functor

$$\mathbf{C} \times \mathbf{C} \xrightarrow{G \times F} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set}$$

along the tensor product functor:



An element of the coend

$$G \otimes F(c) = \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

consists of a morphism

$$b \otimes a \xrightarrow{\gamma} c$$

together with a pair of elements

 $y \in G(b)$ $x \in F(a)$

considered modulo an equivalence relation \sim .

As we did before, we find enlightening to draw the two elements

 $y \in G(b)$ $x \in F(a)$

in the following way:



Accordingly, we like to draw the triple

$$(b \otimes a \xrightarrow{\gamma} c , x \in F(a) , y \in G(b))$$

in the following way:



Suppose given a pair of elements

 $x \in F(a) \qquad \qquad y \in G(b)$

a pair of morphisms

 $\alpha : a \longrightarrow a' \qquad \beta : b \longrightarrow b'$

and a morphism

 $\gamma : a' \otimes b' \longrightarrow c$

The situation may be depicted as follows:



The diagram may be completed as follows:



This equivalence relation \sim defined by the coend

$$G \otimes F(c) = \int^{(b,a) \in \mathbf{C} \times \mathbf{C}} \mathbf{C}(b \otimes a, c) \times G(b) \times F(a)$$

identifies every triple of the form

$$\left(b \otimes a \xrightarrow{\beta \otimes \alpha} b' \otimes a' \xrightarrow{\gamma} c \quad , \quad x \in F(a) \quad , \quad y \in G(b) \right)$$

with the corresponding triple

$$\left(\begin{array}{ccc} b' \otimes a' \xrightarrow{\gamma} c & , & \alpha \cdot x \in F(a') & , & \beta \cdot y \in G(b') \end{array} \right)$$

Diagrammatically, the equivalence relation \sim identifies the two triples:



Theorem [Day 1970] The convolution product

 $G, F \mapsto G \otimes F$

on a monoidal category C with tensor product \otimes defines a functor

 $\hat{\otimes}$: $[C, Set] \times [C, Set] \longrightarrow [C, Set]$

which equips the category of covariant presheaves

[C, Set]

with the structure of a monoidal closed category.

In particular, the convolution product is associative:

 $H \,\hat{\otimes}\, (G \,\hat{\otimes}\, F) \quad \cong \quad (H \,\hat{\otimes}\, G) \,\hat{\otimes}\, F$

Step 0. We start from the functor

Elts (G) × Elts (F)
$$\xrightarrow{\pi_G \times \pi_F} \mathbf{C} \times \mathbf{C} \xrightarrow{\otimes} \mathbf{C}$$

whose objects in the source category are pairs

$$\begin{pmatrix} x \in F(a) & , y \in G(b) \end{pmatrix}$$

may be depicted in the following way:



Step 1. We replace the functor by its free split opfibration

Elts (*G*, *F*)
$$\longrightarrow$$
 C

where the source category Elts (G, F) has objects defined as triples

$$b \otimes a \xrightarrow{\gamma} c$$
 , $x \in F(a)$, $y \in G(b)$

which may be depicted in the following way:



Step 1. We replace the functor by its free split opfibration

Elts (*G*, *F*)
$$\longrightarrow$$
 C

whose morphisms in each fiber above $c \in \mathbb{C}$ are of the form:



Step 2. Replace each fiber category of the opfibration



by its set of **connected components**, using the equivalence relation:



A key observation

From this follows that there exists a cofinal functor

Elts (G) × **Elts** (F) \longrightarrow **Elts** (G \otimes F)

making the diagram commute:



in the category Cat of categories and functors.

A key observation

The category Cat/C inherits a tensor product

 $\tilde{\otimes}$: Cat/C × Cat/C \longrightarrow Cat/C

from the monoidal structure of the category C.

The Day tensor product

 $\hat{\otimes}$: DiscOpFib/C \times DiscOpFib/C \longrightarrow DiscOpFib/C

is the monoidal structure obtained by transporting $\tilde{\otimes}$ along the adjunction



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whose morphisms in each fiber above $c \in \mathbb{C}$ are of the form:



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in the category Cat of categories and functors.
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The category Cat/C inherits a tensor product

 $\tilde{\otimes}$: Cat/C × Cat/C \longrightarrow Cat/C

from the monoidal structure of the category C.

The Day tensor product

 $\hat{\otimes}$: DiscOpFib/C \times DiscOpFib/C \longrightarrow DiscOpFib/C

is the monoidal structure obtained by transporting $\tilde{\otimes}$ along the adjunction



The convolution product on double categories

Extending the Day construction

The convolution product on double categories

Given two covariant presheaves

F,G : $\mathbb{D}_1 \longrightarrow \mathbf{Set}$

on a double category D with horizontal composition

 \diamond_h : $\mathbb{D}_2 = \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \longrightarrow \mathbb{D}_1$

the **convolution product** of F and G is the covariant presheaf

 $G * F : \mathbb{D}_1 \longrightarrow \mathbf{Set}$

defined by the coend formula:

$$G * F = t \mapsto \int^{(s,r) \in \mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

The convolution product

Equivalently, the convolution product

 $G * F : \mathbb{D}_1 \longrightarrow \mathbf{Set}$

may be defined as the left Kan extension of the functor

$$\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \xrightarrow{\text{proj}} \mathbb{D}_1 \times \mathbb{D}_1 \xrightarrow{G \times F} \mathbf{Set} \times \mathbf{Set} \xrightarrow{\times} \mathbf{Set}$$

along the tensor product functor:



An element of the coend

$$G * F(t) = \int^{(s,r) \in \mathbb{D}_2} \mathbb{D}_1(s \diamond_h r, t) \times G(s) \times F(r)$$

consists of a double cell of the form



together with a pair of elements

 $y \in G(s)$ $x \in F(r)$

considered modulo an equivalence relation noted \sim .

We find enlightening to draw the triple

$$\left(s \diamond_h r \xrightarrow{\gamma} t , x \in F(r) , y \in G(s) \right)$$

in the following way:



Suppose given a pair of elements

 $x \in F(r) \qquad \qquad y \in G(s)$

a pair of double cells

$$\alpha : r \Longrightarrow r' \qquad \beta : s \Longrightarrow s'$$

and a double cell

$$\gamma : s' \diamond_h r' \Longrightarrow t$$

The five components may be depicted as follows:



The equivalence relation \sim defined by the coend

$$G * F(t) = \int_{0}^{(s,r) \in \mathbb{D}_{2}} \mathbb{D}_{1}(s \diamond_{h} r, t) \times G(s) \times F(r)$$

identifies every triple of the form



Key observation

Theorem [Behr, PAM, Zeilberger]

The convolution product

 $G, F \mapsto G * F$

on a double category ${\rm I\!D}$ defines a functor

 $* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$

which equips the category of covariant presheaves

 $\widehat{\mathbb{D}}$:= $[\mathbb{D}_1, \mathbf{Set}]$

with the structure of an oplax monoidal closed category.

The category of covariant presheaves

 $\widehat{\mathbb{D}}$:= $[\mathbb{D}_1, \mathbf{Set}]$

comes equipped with a family of convolution products

$$*_n : \widehat{\mathbb{D}} \times \cdots \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$$

where we use the notation

 $(F_n \ast \cdots \ast F_1) \quad := \quad \ast_n \ (F_n, \dots, F_1)$

for the n-ary product of n covariant presheaves

 F_n,\ldots,F_1 : $\mathbb{D}_1 \longrightarrow \mathbf{Set}$.

The ternary convolution product

Typically, the ternary convolution product

 $H * G * F : \mathbf{C} \longrightarrow \mathbf{Set}$

of three covariant presheaves H, G, F is defined by the coend formula

$$H * G * F = u \mapsto \int^{(t,s,r) \in \mathbb{D}_3} \mathbb{D}_1(t \diamond_h s \diamond_h r, u) \times H(t) \times G(s) \times F(r)$$

where \mathbb{D}_3 is the category of horizontal paths of length 3.

The ternary convolution product

The elements of the ternary convolution product are quadruples

$$\left(\begin{array}{cccc}t\diamond_hs\diamond_hr \xrightarrow{\delta}u & , & x\in F(r) & , & y\in G(s) & , & z\in G(t)\end{array}\right)$$

which may be depicted in the following way:



The ternary convolution product

The elements of the convolution product

$$\left(\begin{array}{cccc}t\diamond_hs\diamond_hr\xrightarrow{\delta}u&,&x\in F(r)&,&y\in G(s)&,&z\in G(t)\end{array}\right)$$

are identified modulo the equivalence relation:



The convolution products are related by associativity maps such as

$$H * (G * F) \xleftarrow{assoc} (H * G * F) \xrightarrow{assoc} (H * G) * F$$

which are **not reversible** in general, for the following reason:



In a general double category D, not every composite shape of the form



defining an element of the presheaf H * (G * F) at instance $u : A \longrightarrow A'$

is equivalent modulo \sim in \mathbb{D} to a ternary shape of the form



defining an element of H * G * F at the same instance $u : A \longrightarrow A'$.

Cylindrical decomposition property

A sufficient condition to ensure strong associativity

We want to find a **sufficient condition** on a double category

 $(\mathbb{D}, h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1)$

ensuring that the **associativity maps** of the convolution product

$$H * (G * F) \xleftarrow{assoc} (H * G * F) \xrightarrow{assoc} (H * G) * F$$

are reversible.

In particular, this requires to show that every **composite shape**



defining an element of the presheaf H * (G * F) at instance $u : A \longrightarrow A'$

is equivalent modulo \sim in \mathbb{D} to a ternary shape of the form



defining an element of H * G * F at the same instance $u : A \longrightarrow A'$.

Suppose that every double cell of the form



factors in the following way:



In that case, one can rewrite the original composite shape



We then into the shape where the cell γ has been factored:



then into the equivalent shape using the equivalence relation \sim



then into the equal shape by vertical composition:



and finally in the ternary shape we were looking for:



Every double category \mathbb{D} comes equipped with a family of categories

$\operatorname{Cyl}_{\mathbb{D}}[n]$

called cylinder categories and defined in the following way:

 \triangleright the objects of $\mathbf{Cyl}_{\mathbb{D}}[n]$ are the tuples

 $\sigma = (s_n, \dots, s_1, s, \sigma : s_n \diamond_h \dots \diamond_h s_1 \Rightarrow s)$

defining a globular cell of the form



▷ given globular cells

$$\sigma = (s_n, \dots, s_1, s, \sigma : s_n \diamond_h \dots \diamond_h s_1 \Rightarrow s)$$

$$\tau = (t_n, \dots, t_1, t, \tau : t_n \diamond_h \dots \diamond_h t_1 \Rightarrow t)$$

the morphisms of $\mathbf{Cyl}_{\mathbb{D}}[n]$ of the form

$$(\varphi_n, \cdots, \varphi_1, \varphi) \quad : \quad \sigma \longrightarrow \tau$$

are tuples consisting of a map in \mathbb{D}_n

$$(\varphi_n,\ldots,\varphi_1)$$
 : $(s_n,\ldots,s_1) \Rightarrow (t_n,\ldots,t_1)$

and of a double cell

$$\varphi : s \Rightarrow t$$

such that the double cell $\varphi \circ \sigma$ depicted below



is equal to the double cell $\tau \circ (\varphi_n \diamond_h \dots \diamond_h \varphi_1)$ depicted below



Typically, a map of the cylinder category $Cyl_{\mathbb{D}}[2]$ of the form



is defined as a tuple of double cells $\varphi, \varphi_1, \varphi_2$ of double cells of the form





satisfying the equation:



This justifies to see every $Cyl_{\mathbb{D}}[n]$ as a cylinder category of \mathbb{D} .

The cylindrical decomposition property

Key observation: each composition functor

 $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$

of the double category \mathbb{D} factors as

 $\mathbb{D}_n \longrightarrow \operatorname{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$

Definition. A double category **D** satisfies

the *n*-cylindrical decomposition property (*n*-CDP)

when the functor

 $\operatorname{Cyl}_{\mathbb{D}}[n] \xrightarrow{\pi_n} \mathbb{D}_1$

is an opfibration (not necessarily discrete).

Main theorem

Theorem [Behr, PAM, Zeilberger in this FSCD]

Suppose that a double category \mathbb{D} satisfies

the *n*-cylindrical decomposition property (*n*-CDP)

for all $n \in \mathbb{N}$.

In that case, the convolution product defines a functor

 $* \quad : \quad \widehat{\mathbb{D}} \times \widehat{\mathbb{D}} \longrightarrow \widehat{\mathbb{D}}$

which equips the category of covariant presheaves

 $\widehat{\mathbb{D}}$:= $[\mathbb{D}_1, \mathbf{Set}]$

with the structure of an strong monoidal closed category.
Main theorem

In particular, the associativity maps are **reversible** in that case:



Reversibility comes from the cylindrical decomposition property of D.

Illustrations

The theorem applies to the following situations:

- ▷ every **bicategory** $\mathbb{D} = \mathcal{W}$ satisfies *n*-CDP,
- ▷ every framed bicategory $\mathbb{D} = \mathcal{W}$ satisfies *n*-CDP for $n \ge 1$,
- ▷ the double category $\mathbb{D} = \mathbb{DPO}$ satisfies *n*-CDP for $n \ge 1$.
- ▷ the double category $\mathbb{D} = LTRS$ of term rewriting satisfies *n*-CDP.

More generally, the theorem enables us to use the convolution product for a number of categorical graph and term rewriting frameworks.

Categorifying rule algebras

Composing representable presheaves by convolution

Categorification of rule algebras

One main ingredient of rule algebras is the following equation

$$\delta(r) \star \delta(s) = \sum_{\mu \in \mathcal{M}_r(s)} \delta(r_{\mu}s)$$

where

- \triangleright $\mathcal{M}_r(s)$ is the set of **admissible matches** of rule *r* into rule *s*
- $ightarrow r_{\mu}s$ denotes one possible way to get a **composite rule** from *r* and *s*.

Similarly, we want to find sufficient conditions on \mathbb{D} such that

$$\hat{\Delta}_{r} * \hat{\Delta}_{s} = \sum_{\mu \in \mathcal{M}_{r}(s)} \hat{\Delta}_{r_{\mu}s}$$

where the sum is now set-theoretic union.

Multi-sums

Suppose that *A* and *B* are objects in a category **C**.

Definition. A **multi-sum** of *A* and *B* is a family of cospans

$$(A \xrightarrow{a_i} U_i \xleftarrow{b_i} B)_{i \in I}$$

such that for any cospan

$$A \xrightarrow{f} X \xleftarrow{g} B$$

there exists a unique $i \in I$ and a unique morphism

$$[f,g] \quad : \quad U_i \stackrel{f}{\longrightarrow} X$$

such that

$$f = [f,g] \circ a_i$$
 and $g = [f,g] \circ b_i$

Categorification of rule algebras

Theorem. Assume \mathbb{D} is a small double category satisfying

- \triangleright the vertical category \mathbb{D}_0 has multi-sums,
- ▷ the source and target functors $S, T : \mathbb{D}_1 \to \mathbb{D}_0$ are opfibrations.

In that case, the convolution product of two representable presheaves is isomorphic to the sum of representables

$$\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1} \cong \sum_{i \in I} \hat{\Delta}_{r_2 \langle c_i \rangle \diamond_h \langle b_i \rangle r_1}$$

where the multi-sum of B and C is given by a family of cospans

$$(B \longrightarrow U_i \xleftarrow{c_i} C)_{i \in I}$$

and where $r_2 \langle c_i \rangle$ denotes the *S*-pushforward of r_2 along c_i and $\langle b_i \rangle r_1$ denotes the *T*-pushforward of r_1 along b_i .

Sketch of the proof



Sketch of the proof



Illustration

From this, we obtain that the convolution product with itself

$$\hat{\Delta}_r * \hat{\Delta}_r : \mathbb{D}_1 \longrightarrow \mathbf{Set}$$

of the representable presheaf

 $\hat{\Delta}_r : \mathbb{D}_1 \longrightarrow \mathbf{Set}$

is isomorphic to the sum of two representable presheaves

 $\hat{\Delta}_r * \hat{\Delta}_r \cong \hat{\Delta}_{r_1} + \hat{\Delta}_{r_2}$

associated to the rewrite rules r_1 and r_2

$$\lambda x.m(e,x): \mathfrak{o} \multimap \mathfrak{o} \ , \ \lambda y.m(e,y): \mathfrak{o} \multimap \mathfrak{o} \ \xrightarrow{rule r_1} \ \lambda x.x: \mathfrak{o} \multimap \mathfrak{o} \ , \ \lambda y.y: \mathfrak{o} \multimap \mathfrak{o} \ \lambda x.m(m(e,e),x): \mathfrak{o} \multimap \mathfrak{o} \ \xrightarrow{rule r_2} \ \lambda x.x: \mathfrak{o} \multimap \mathfrak{o}$$

in the double category $\mathbb{D} = \mathbf{LTRS}$.

Illustration



Conclusion and future works

What we have done in the FSCD paper:

- a categorification of tracelets and rule algebras
- > an **axiomatic and unified framework** for term and graph rewriting
- ▷ a convolution product $G, F \mapsto G * F$ for double categories
- > a cylindrical decomposition property for strong associativity

A few topics we like to think about:

- make sure the framework works for higher-order rewrite systems
- categorify the more quantitative and stochastic aspects of tracelets
- ▷ improve our understanding of **causality** in rewriting systems

Thank you!

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Probability generating functions

Key idea: given a chemical reaction described as a transition

 $\kappa_{i,o} : iX \longrightarrow oX$

where

 \triangleright $i \in \mathbb{N}$ denotes the number of particles X entering the transition

 \triangleright $o \in \mathbb{N}$ denotes the number of particles X exiting the transition

▷ $\kappa_{i,o} \in \mathbb{R}_{>0}$ denotes the **base rate** of the transition

the dynamics may be encoded using a probability generating function

$$P(t;x) = \sum_{n \ge 0} p_n(t) x^n$$

where $p_n(t) \ge 0$ is the probability at time *t* that the system is in a state with *n* particles.

Probability generating functions

Delbruck's formulation of the evolution of the system

 $\frac{\partial}{\partial t}P(t;x) = \mathcal{H}P(t;x)$

$$P(0;x) = P_0(x)$$
$$\mathcal{H} = \sum_{i,o} \kappa_{i,o} (\hat{x}^o - \hat{x}^i) \left(\frac{\partial}{\partial x}\right)^i$$

The Heisenberg-Weyl algebra