Five Basic Concepts of

Axiomatic Rewriting Theory

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Rewriting paths modulo homotopy

An algebraic and topological notion of confluence

The λ-calculus with explicit substitutions

Terms	M	$::=$	$\mathbf{1} \quad \mid \quad MN \quad \mid \quad \lambda M \quad \mid \quad M[s]$
Substitutions	s	$::=$	$id \quad \mid \quad \uparrow \quad \mid \quad M \cdot s \quad \mid \quad s \circ t$

Key idea: replace the β-rule of the λ-calculus

$$(\lambda x.M)\, N \quad \longrightarrow \quad M\, [x := N]$$

by the Beta-rule of the $\lambda\sigma$-calculus

$$(\lambda\, M)\, N \quad \longrightarrow \quad M\, [N \cdot id]$$

where the substitution is explicit – and thus similar to a closure.

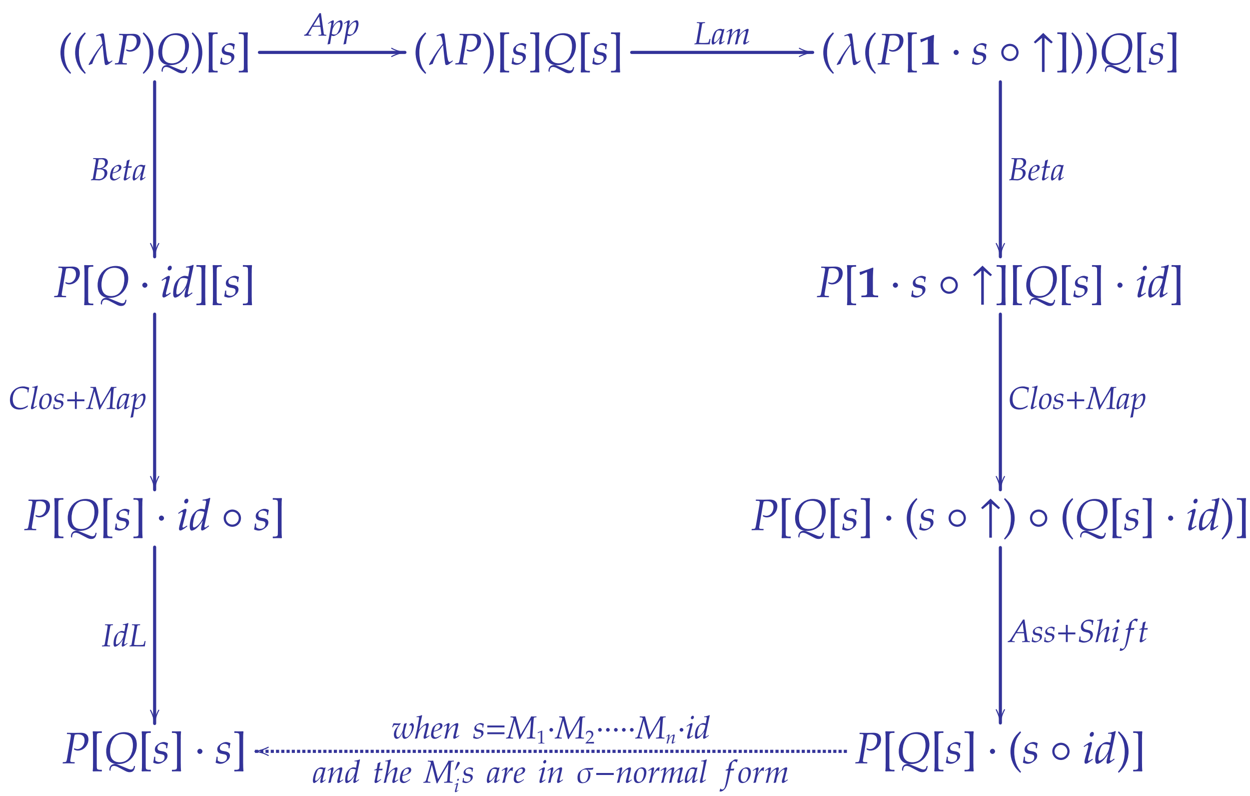
The eleven rewriting rules of the $\lambda\sigma$-calculus

Beta	$(\lambda M)N$	$\longrightarrow$	$M[N \cdot id]$
App	$(MN)[s]$	$\longrightarrow$	$M[s]N[s]$
Abs	$(\lambda M)[s]$	$\longrightarrow$	$\lambda(M[\mathbf{1} \cdot (s \circ \uparrow)])$
Clos	$M[s][t]$	$\longrightarrow$	$M[s \circ t]$
VarCons	$\mathbf{1}[M \cdot s]$	$\longrightarrow$	M
VarId	$\mathbf{1}[id]$	$\longrightarrow$	$\mathbf{1}$
Map	$(M \cdot s) \circ t$	$\longrightarrow$	$M[t] \cdot (s \circ t)$
IdL	$id \circ s$	$\longrightarrow$	s
Ass	$(s_1 \circ s_2) \circ s_3$	$\longrightarrow$	$s_1 \circ (s_2 \circ s_3)$
ShiftCons	$\uparrow \circ (M \cdot s)$	$\longrightarrow$	s
ShiftId	$\uparrow \circ id$	$\longrightarrow$	$\uparrow$

The eleven critical pairs of the $\lambda\sigma$-calculus

$App + Beta$	$(\lambda M)[s](N[s])$	$\overset{App}{\longleftarrow}$	$((\lambda M)N)[s]$	$\overset{Beta}{\longrightarrow}$	$M[N \cdot id][s]$
$Clos + App$	$(MN)[s \circ t]$	$\overset{Clos}{\longleftarrow}$	$(MN)[s][t]$	$\overset{App}{\longrightarrow}$	$(M[s](N[s]))[t]$
$Clos + Abs$	$(\lambda M)[s \circ t]$	$\overset{Clos}{\longleftarrow}$	$(\lambda M)[s][t]$	$\overset{Abs}{\longrightarrow}$	$(\lambda(M[\mathbf{1} \cdot s \circ \uparrow]))[t]$
$Clos + VarId$	$\mathbf{1}[id \circ s]$	$\overset{Clos}{\longleftarrow}$	$\mathbf{1}[id][s]$	$\overset{VarId}{\longrightarrow}$	$\mathbf{1}[s]$
$Clos + VarCons$	$\mathbf{1}[(M \cdot s) \circ t]$	$\overset{Clos}{\longleftarrow}$	$\mathbf{1}[M \cdot s][t]$	$\overset{VarCons}{\longrightarrow}$	$M[t]$
$Clos + Clos$	$M[s][t \circ t']$	$\overset{Clos}{\longleftarrow}$	$M[s][t][t']$	$\overset{Clos}{\longrightarrow}$	$M[s \circ t][t']$
$Ass + Map$	$(M \cdot s) \circ (t \circ t')$	$\overset{Ass}{\longleftarrow}$	$((M \cdot s) \circ t) \circ t'$	$\overset{Map}{\longrightarrow}$	$(M[t] \cdot s \circ t) \circ t'$
$Ass + IdL$	$id \circ (s \circ t)$	$\overset{Ass}{\longleftarrow}$	$(id \circ s) \circ t$	$\overset{IdL}{\longrightarrow}$	$s \circ t$
$Ass + ShiftId$	$\uparrow \circ (id \circ s)$	$\overset{Ass}{\longleftarrow}$	$(\uparrow \circ id) \circ s$	$\overset{ShiftId}{\longrightarrow}$	$\uparrow \circ s$
$Ass + ShiftCons$	$\uparrow \circ ((M \cdot s) \circ t)$	$\overset{Ass}{\longleftarrow}$	$(\uparrow \circ (M \cdot s)) \circ t$	$\overset{ShiftCons}{\longrightarrow}$	$s \circ t$
$Ass + Ass$	$(s \circ s') \circ (t \circ t')$	$\overset{Ass}{\longleftarrow}$	$((s \circ s') \circ t) \circ t'$	$\overset{Ass}{\longrightarrow}$	$(s \circ (s' \circ t)) \circ t'$

A very dangerous critical pair

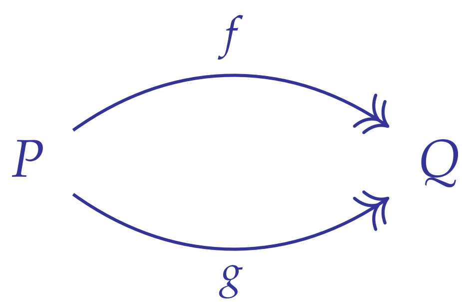


This critical pair leads to a counter-example to **strong normalization** in the simply-typed $\lambda\sigma$-calculus (TLCA 1995).

A fundamental problem

Hence, one main challenge of Rewriting Theory:

Classify the rewriting paths from a term P to its normal form Q



Very complicated in the case of the $\lambda\sigma$-calculus...

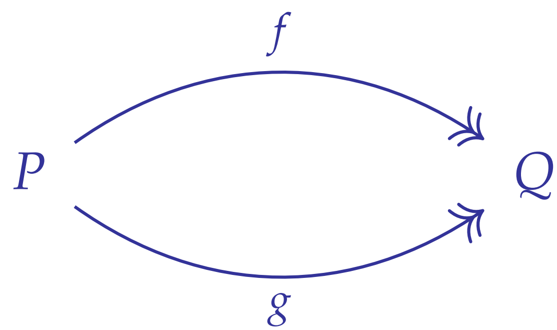
I. Permutation tiles

A geometric account of redex permutations

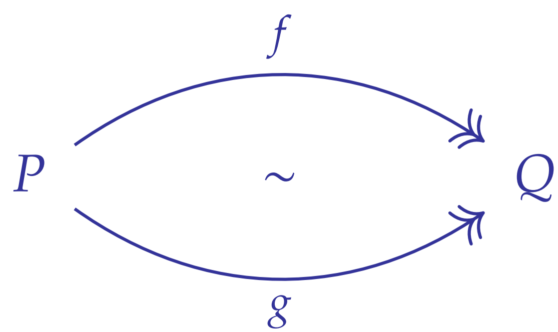
A key observation

Theorem (Lévy 1978)

In the λ-calculus, every two paths to the normal form



are equal modulo a series of β-redex permutations:



A geometric intuition

It is nice and clarifying to think of

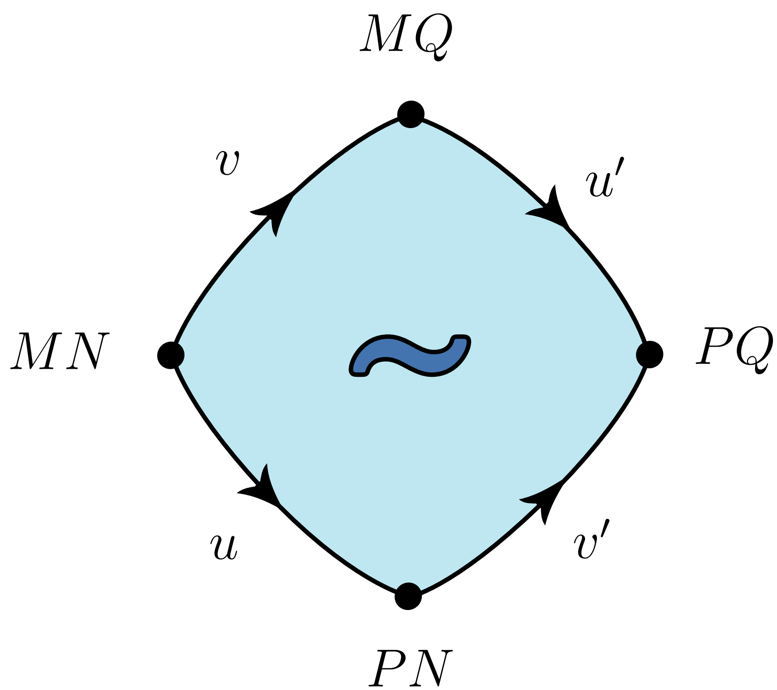
the redex permutation equivalence

between f and g in a geometric way as

a homotopy relation between rewriting paths

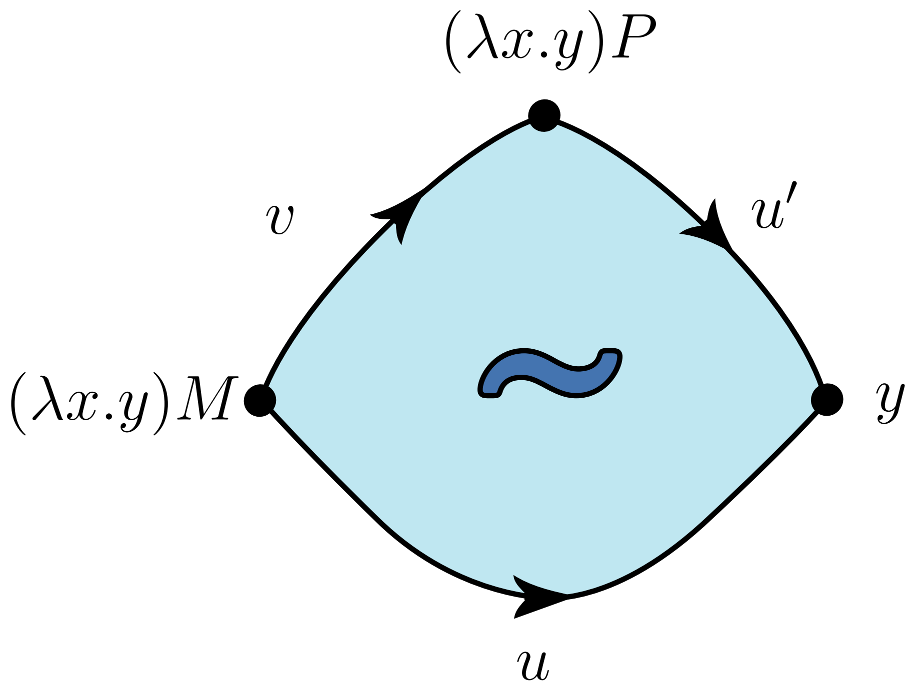
This intuition can be made rigorous mathematically using Albert Burroni's notion of **polygraph.**

Permutation tiles (1)



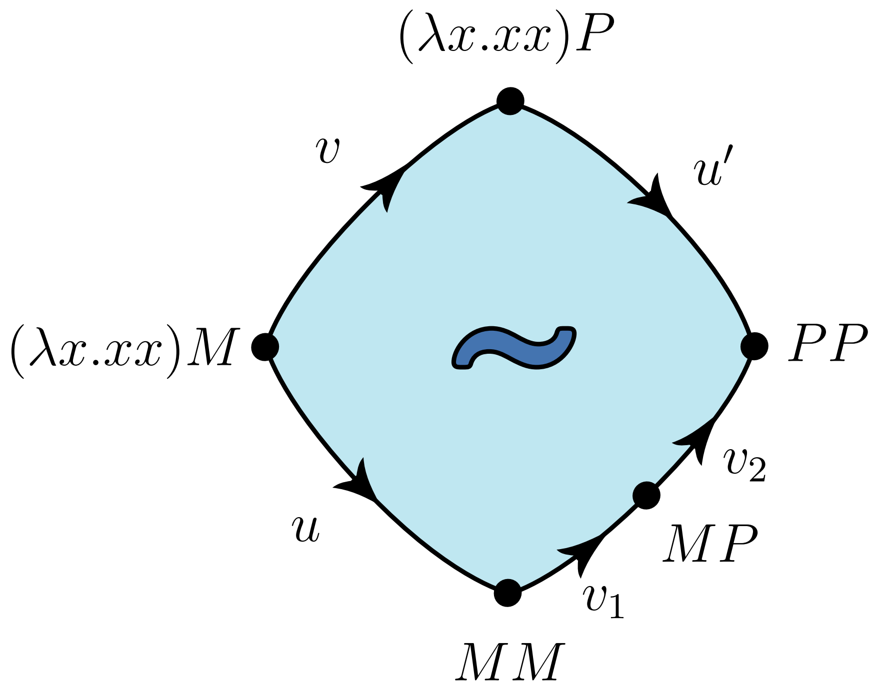
the redex
$u : M \to P$
and the redex
$v : N \to Q$
are **independent**

Permutation tiles (2)



the outer redex
$u : (\lambda x.y)M \to y$
erases
the inner redex
$v : M \to P$

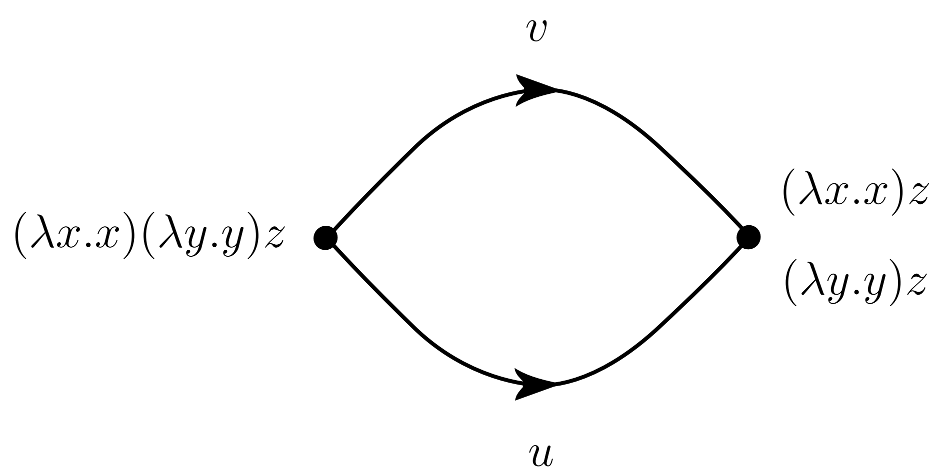
Permutation tiles (3)



the outer redex
$u : (\lambda x.xx)\,M \to MM$
duplicates
the inner redex
$v : M \to P$

Illustration of the theorem

There is a 2-dimensional hole in the λ-calculus



because the outer redex

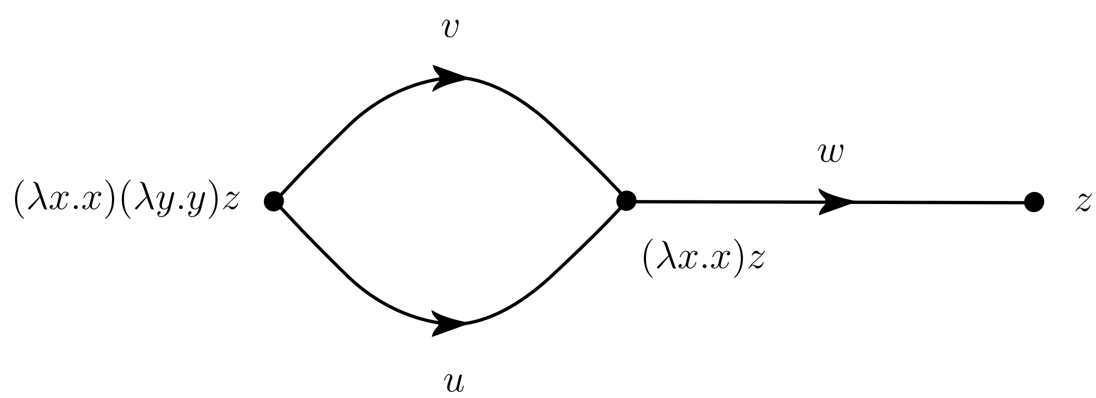
$$u \quad : \quad \underline{(\lambda x.x)\,(\lambda y.y)\,z} \quad \longrightarrow \quad (\lambda y.y)\,z$$

is not equivalent modulo homotopy to the inner redex

$$v \quad : \quad (\lambda x.x)\,\underline{(\lambda y.y)\,z} \quad \longrightarrow \quad (\lambda x.x)\,z$$

Illustration of the theorem

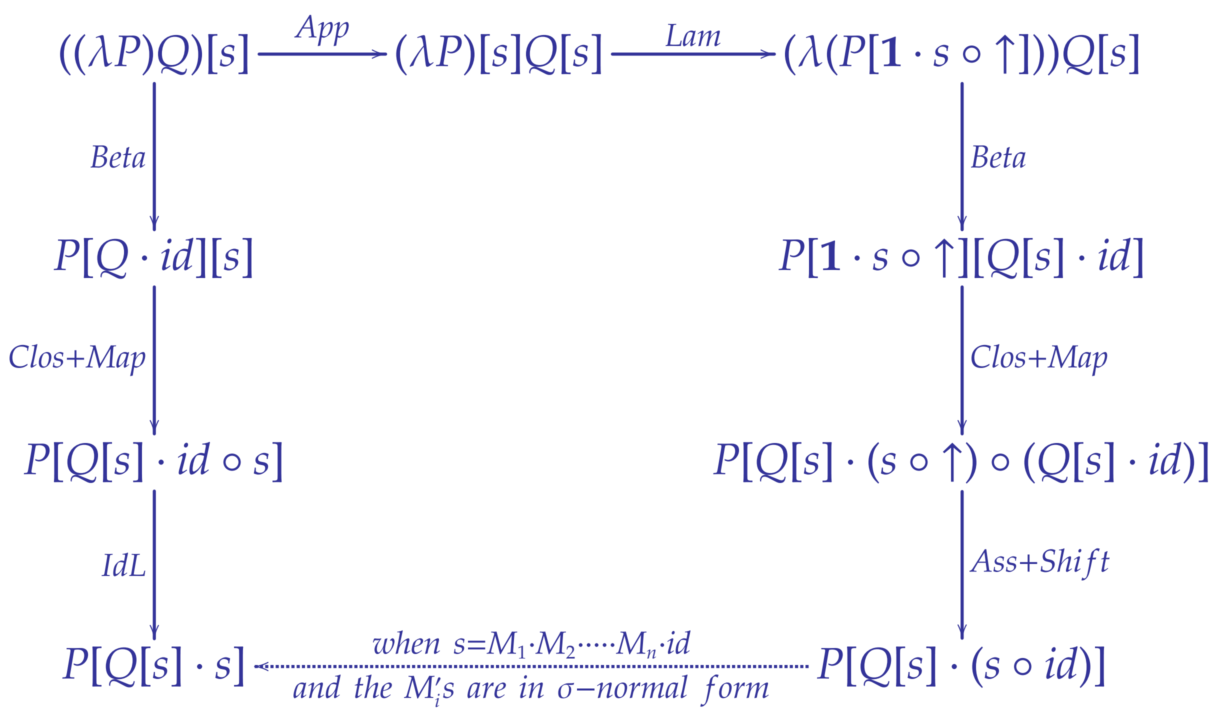
When one extends the two redexes u and v with w



the resulting rewriting paths $u \cdot w$ and $v \cdot w$ are normalizing and thus equivalent modulo homotopy!

In the $\lambda\sigma$-calculus...

Critical pairs like



generate 2-dimensional holes in the rewriting geometry and thus obstructions to homotopy equivalence...

A bridge between $\lambda\sigma$ and λ

Key theorem (Abadi-Cardelli-Curien-Lévy 1990)

Every rewriting path between $\lambda\sigma$-terms

$$f \quad : \quad P \longrightarrow\!\!\!\!\rightarrow Q$$

induces a rewriting path (modulo homotopy)

$$\sigma(f) \quad : \quad \sigma(P) \longrightarrow\!\!\!\!\rightarrow \sigma(Q)$$

between the underlying λ-terms.

Moreover, the translation preserves homotopy equivalence:

$$f \sim_{\lambda\sigma} g \quad \Rightarrow \quad \sigma(f) \sim_{\lambda} \sigma(g)$$

A remarkable consequence

Fact. Two rewriting paths in the $\lambda\sigma$-calculus

$$P \underset{g}{\overset{f}{\rightrightarrows}} Q$$

are transported to the **same homotopy class** of rewriting paths

$$\sigma(P) \underset{\sigma(g)}{\overset{\sigma(f)}{\rightrightarrows}} \sigma(Q) \qquad \sim_{\lambda}$$

when the $\lambda\sigma$-term Q is in normal form.

What about head-normal forms?

Can we classify the **head-rewriting paths** of the $\lambda\sigma$-calculus?

This requires to resolve two very serious difficulties:

- ▷ define a general notion of **head-rewriting path** for a term rewriting system admitting critical pairs

- ▷ establish that every head-rewriting path in the $\lambda\sigma$-calculus

$$f \quad : \quad P \longrightarrow\!\!\!\!\!\!\rightarrow V$$

is transported to a head-rewriting path

$$\sigma(f) \quad : \quad \sigma(P) \longrightarrow\!\!\!\!\!\!\rightarrow \sigma(V)$$

of the λ-calculus.

Axiomatic Rewriting Theory

Main claim.

This problem is arguably too difficult to resolve
by working directly on the syntax of the $\lambda\sigma$-calculus.

One should move to a purely **diagrammatic** approach
based on the 2-dimensional notion of permutation tile.

The purpose of **Axiomatic Rewriting Theory** is to establish
a number of important structural properties:

- ▷ standardisation theorem
- ▷ factorisation theorem
- ▷ stability theorem

from the generic properties of permutation tiles in rewriting.

Axiomatic rewriting system

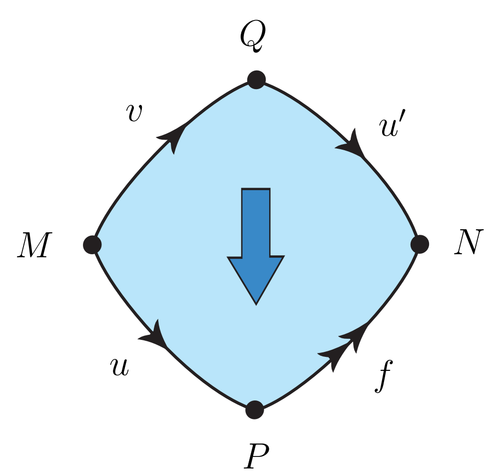
Definition. A graph

$$G \quad = \quad (V, E, \partial_0, \partial_1)$$

defined by its source and target functions

$$\partial_0 \, , \, \partial_1 \quad : \quad E \quad \longrightarrow \quad V$$

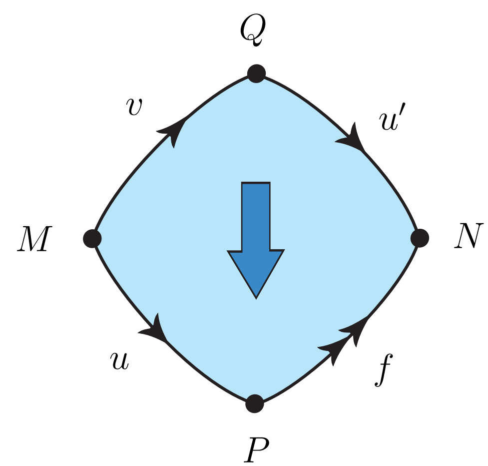
together with a set of 2-dimensional tiles of the form



where the rewriting path f is of arbitrary length.

Reversible permutations

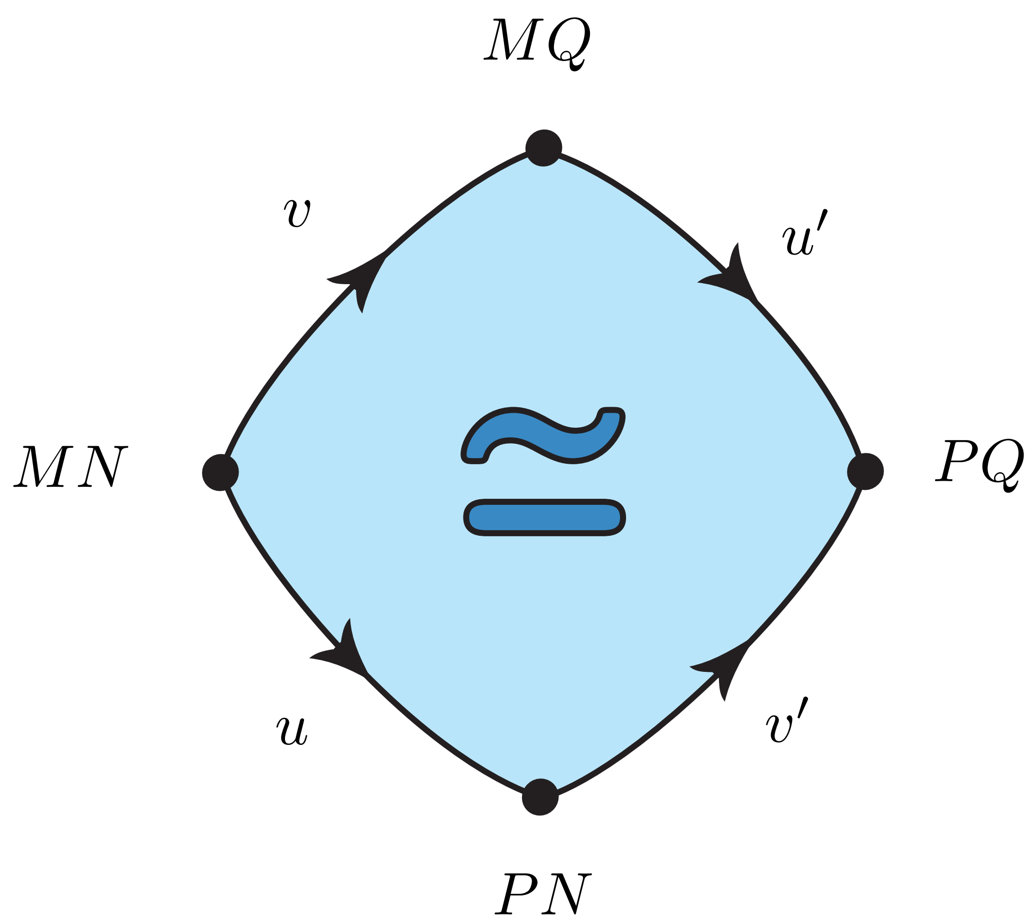
Definition: a permutation tile



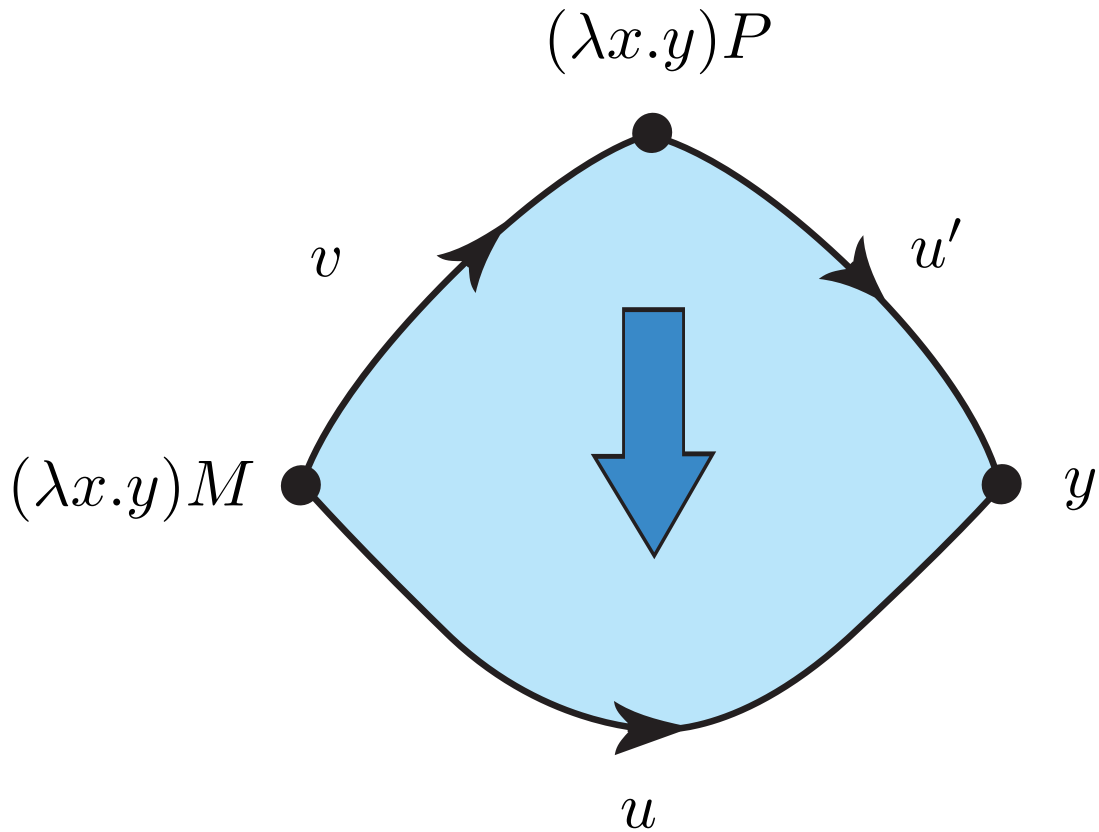
is called **reversible** when it has an inverse.

Note that f is of length 1 in that specific case.

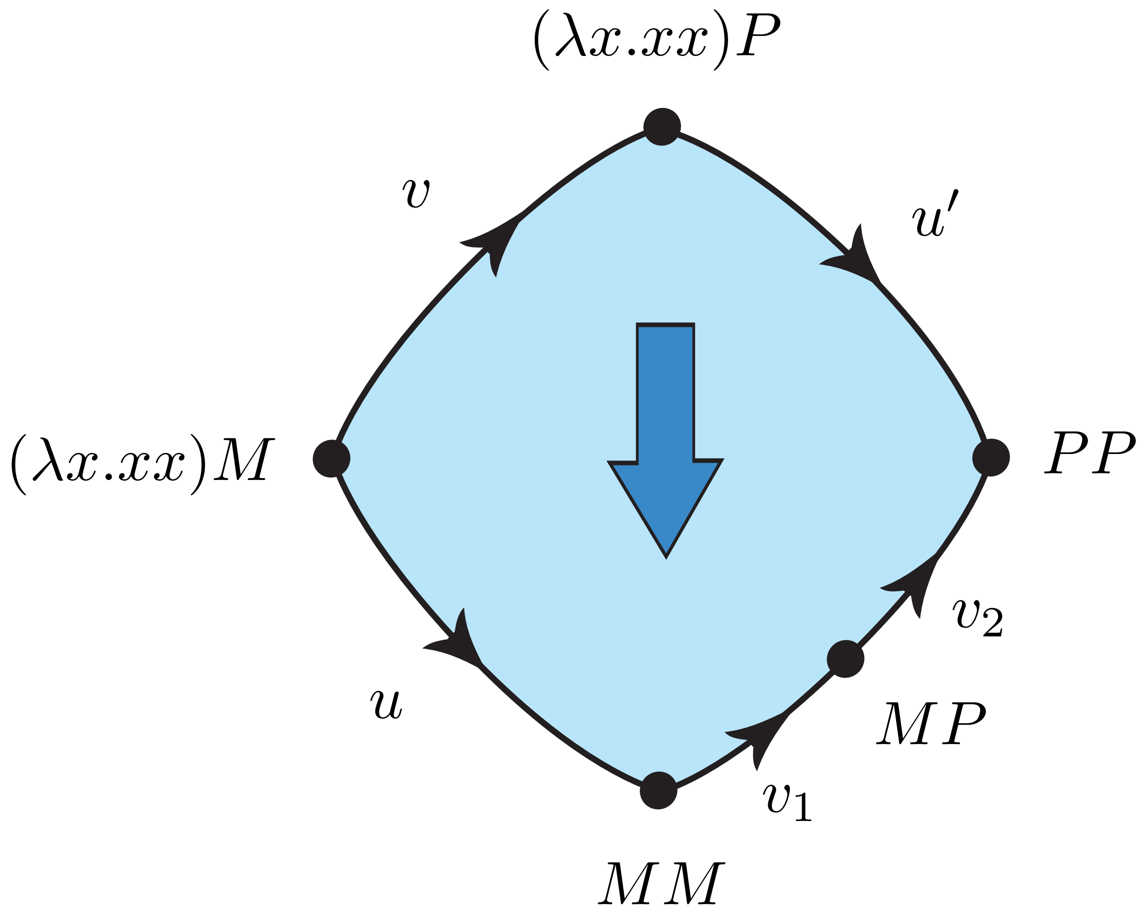
Reversible permutation tiles



Irreversible permutation tiles



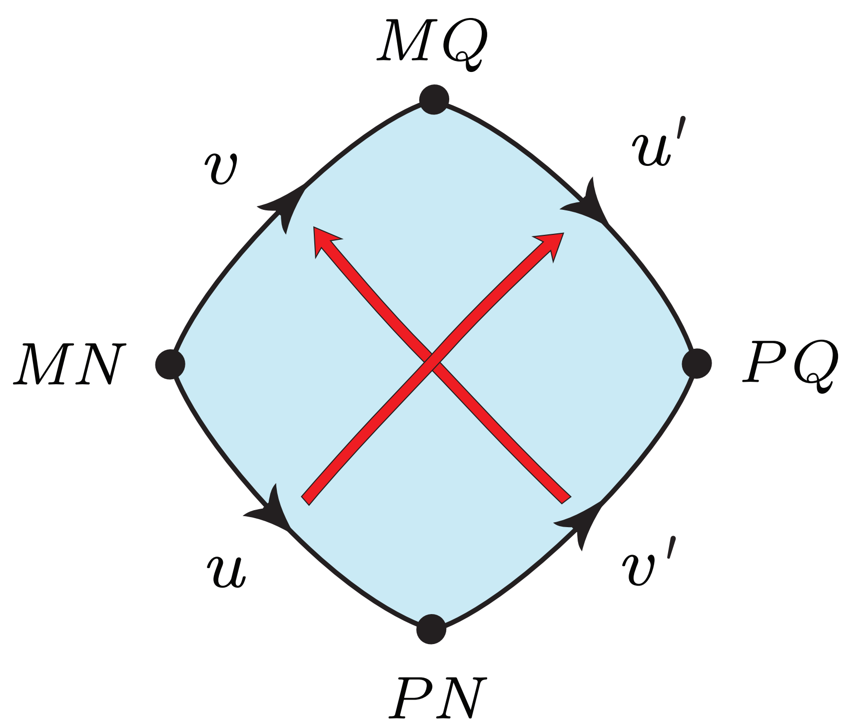
Irreversible permutation tiles



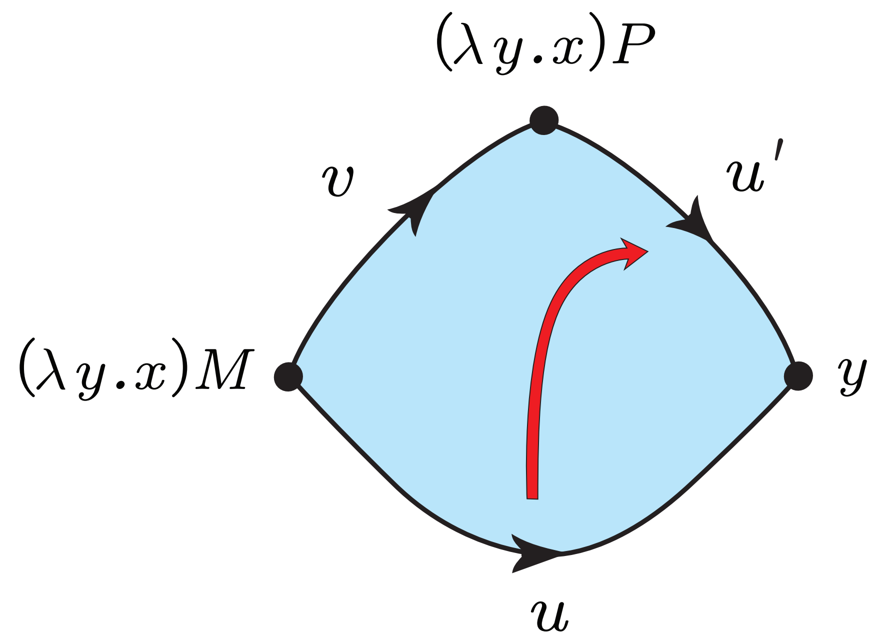
II. Standardisation cells

Rewriting surfaces between rewriting paths

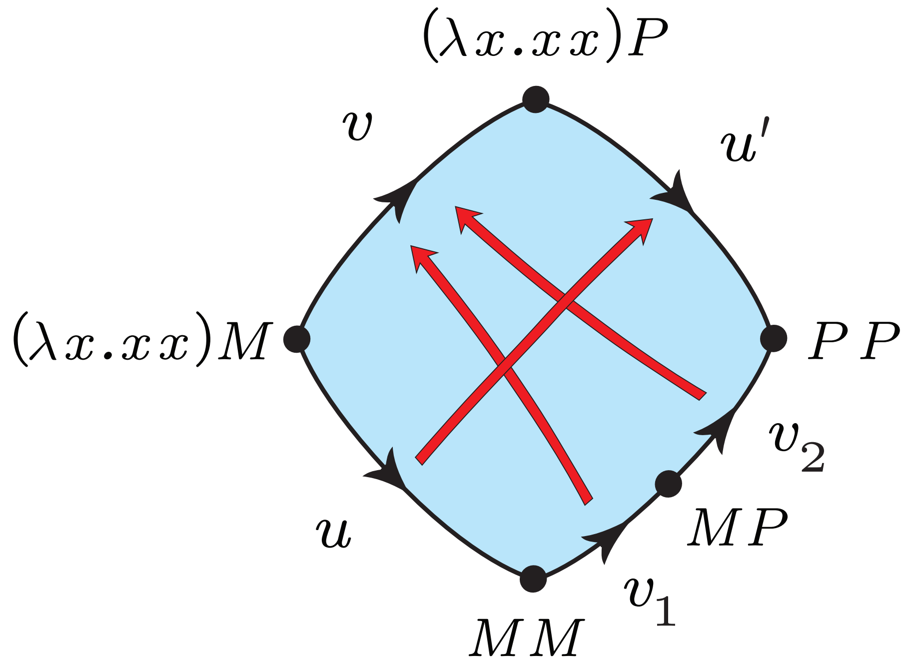
Key idea: let us track ancestors!



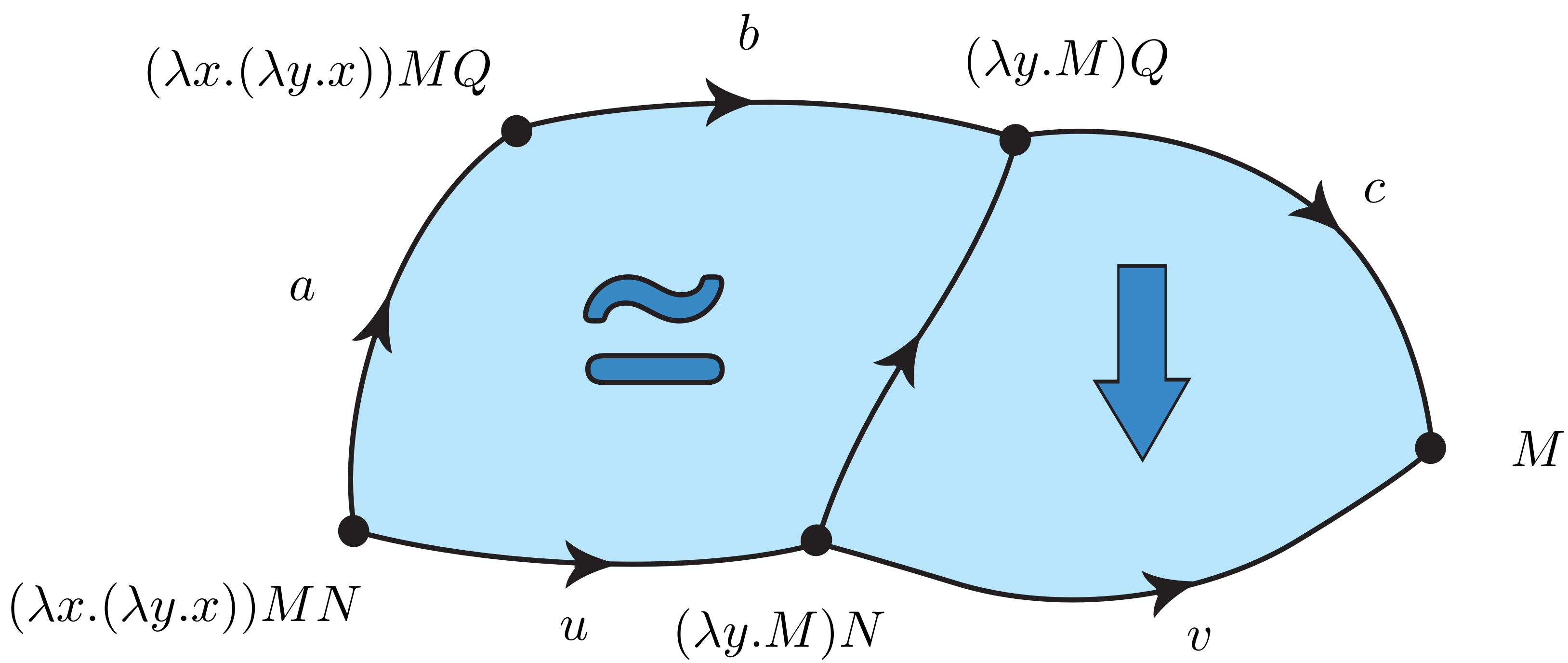
Key idea: let us track ancestors!



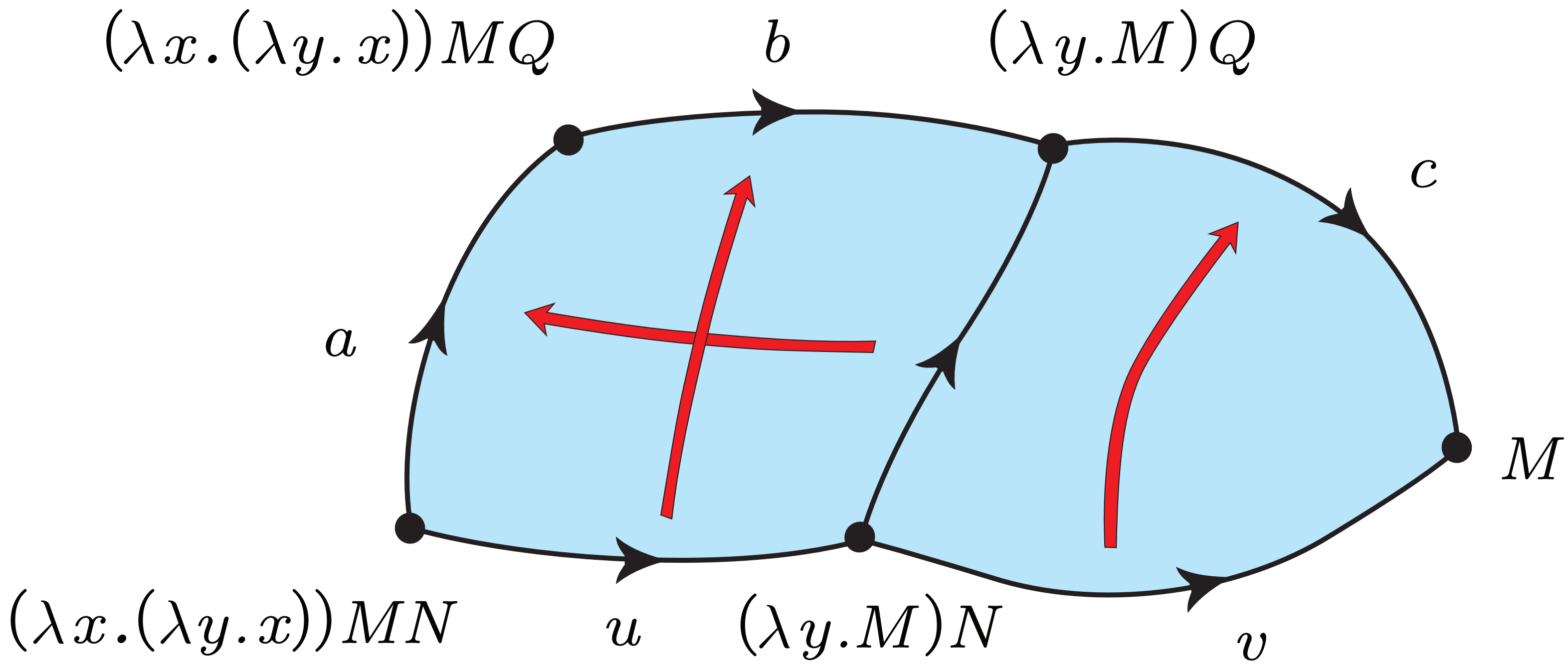
Key idea: let us track ancestors!



Standardisation cells



Illustration



Standardisation cells

Definition.

A standardisation cell

$$\theta \quad : \quad f \Longrightarrow g \quad : \quad M \longrightarrow\!\!\!\!\twoheadrightarrow N$$

is a triple (f, g, φ) consisting of two coinitial and cofinal paths

$$f = \; M \xrightarrow{u_1} \xrightarrow{u_2} \cdots \xrightarrow{u_p} N \qquad g = \; M \xrightarrow{v_1} \xrightarrow{v_2} \cdots \xrightarrow{v_q} N$$

and of a function

$$\varphi \quad : \quad \{1, \ldots, q\} \longrightarrow \{1, \ldots, p\}$$

called the **ancestor function** of the standardisation cell.

A 2-category of rewriting and standardisation

Theorem.

Every axiomatic rewriting system G induces a 2-category

- ▷ its objects are the terms,
- ▷ its morphisms are the rewriting paths,
- ▷ its cells are the standardisation cells.

III. Standard rewriting paths

The normal forms of the 2-dimensional rewriting

Standard rewriting paths

Definition.

A rewriting path

$$f \quad : \quad M \longrightarrow\!\!\!\!\twoheadrightarrow N$$

is called **standard** when every standardisation cell

$$f \quad \Rightarrow \quad g$$

is reversible.

A diagrammatic standardisation theorem

Theorem.

For every rewriting path f, there exists a 2-dimensional cell

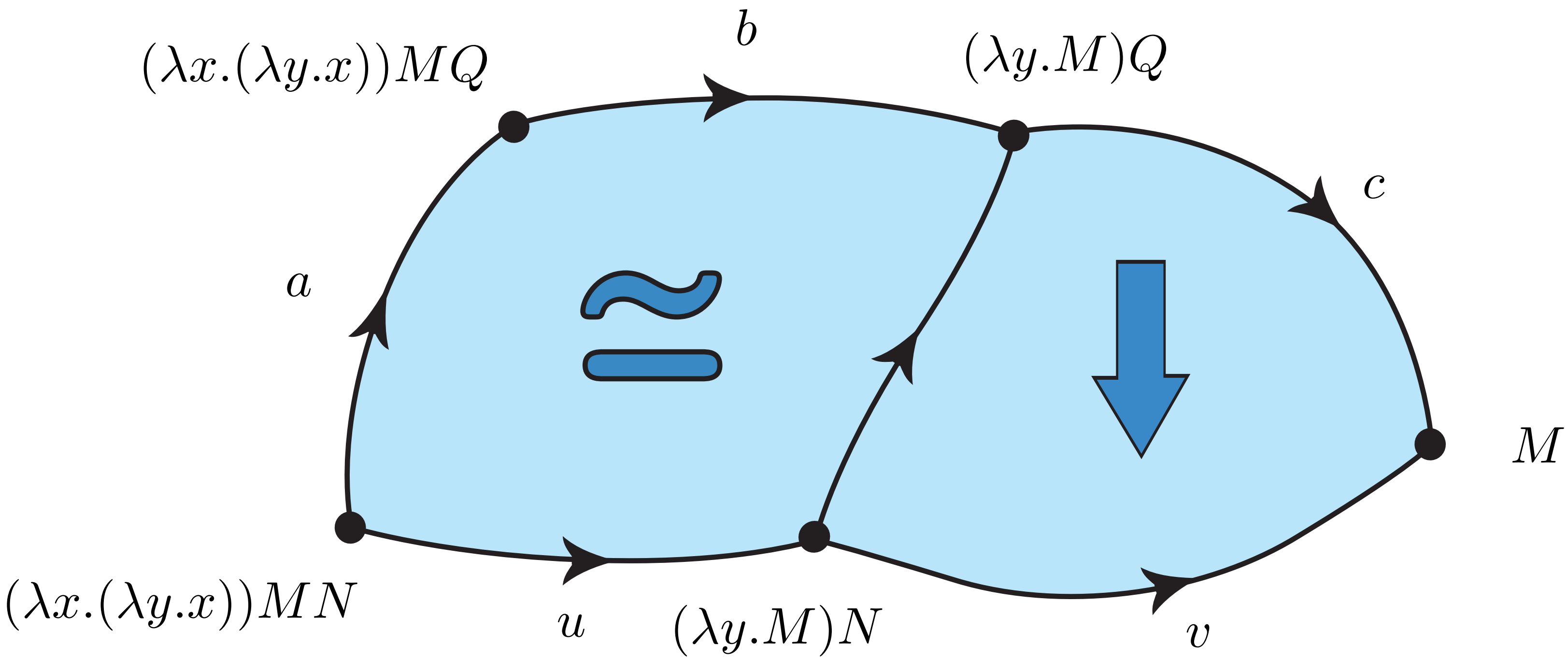
$$f \Rightarrow g$$

transforming f into a standard path g.

Moreover, the standard path g associated to the path f is unique, modulo reversible permutations.

The 2-dimensional cell $f \Rightarrow g$ itself is unique,
up to canonical 3-dimensional deformations.

Standardisation



A two-dimensional process revealing the causal dependencies

IV. External rewriting paths

The external-internal factorisation theorem

External rewriting paths

Definition. A rewriting path

$$M \xrightarrow{\ e\ } \!\!\!\!\rightarrow P$$

is called **external** when it satisfies the following property:

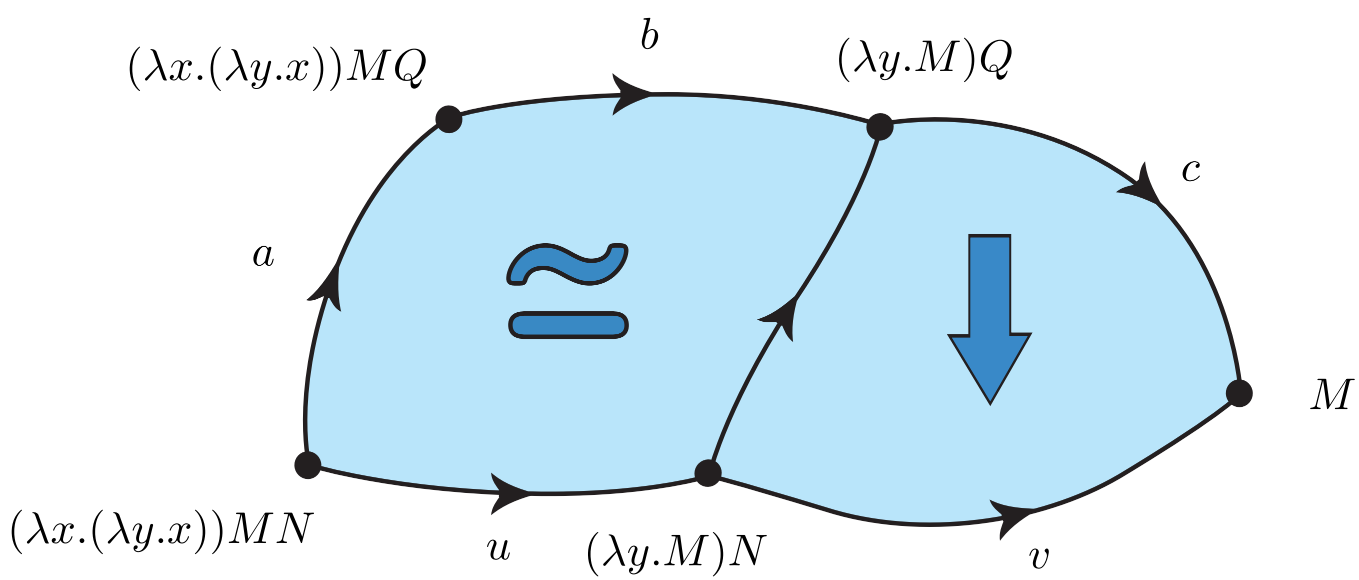
$$P \xrightarrow{f} \!\!\!\!\rightarrow Q \text{ standard} \quad \Longrightarrow \quad M \xrightarrow{e} \!\!\!\!\rightarrow P \xrightarrow{f} \!\!\!\!\rightarrow Q \text{ standard.}$$

External rewriting paths

The β-redex

$$(\lambda x.(\lambda y.x))MN \xrightarrow{a} (\lambda x.(\lambda y.x))MP$$

is standard but not external in the diagram below:

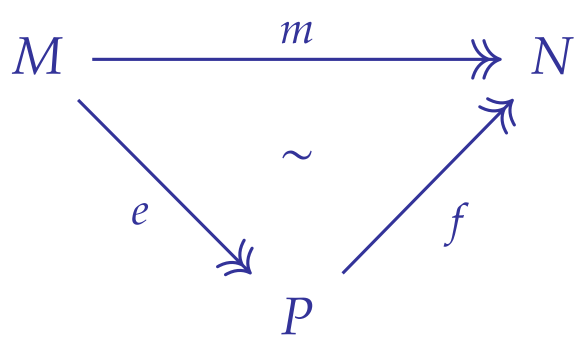


Internal rewriting paths

Definition. A rewriting path

$$M \overset{m}{\twoheadrightarrow} N$$

is called **internal** when every factorization up to homotopy



satisfies the following property:

$$e \text{ is external} \quad \Longrightarrow \quad e \text{ is equal to the identity.}$$

Factorization theorem (Existence)

Suppose given an axiomatic rewriting system.

Theorem. Every rewriting path

$$M \xrightarrow{\quad f \quad} N$$

factors as

$$M \xrightarrow{e} P \xrightarrow{m} N$$

where

- the rewriting path e is external
- the rewriting path m is internal

Factorization theorem (uniqueness)

For every commutative diagram

$$\begin{array}{ccccc} M_1 & \xrightarrow{e_1} & P_1 & \xrightarrow{m_1} & N_1 \\ {\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle v} \\ M_2 & \xrightarrow[e_2]{} & P_2 & \xrightarrow[m_2]{} & N_2 \end{array}$$

e_1 and e_2 external
m_1 and m_2 internal

there exists a unique path $h : P_1 \twoheadrightarrow P_2$ such that

$$\begin{array}{ccccc} M_1 & \xrightarrow{e_1} & P_1 & \xrightarrow{m_1} & N_1 \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle v} \\ M_2 & \xrightarrow{e_2} & P_2 & \xrightarrow{m_2} & N_2 \end{array}$$

commutes.

Factorization theorem (uniqueness)

For every commutative diagram (up to homotopy)

$$\begin{array}{ccccc} M_1 & \xrightarrow{e_1} & P_1 & \xrightarrow{m_1} & N_1 \\ {\scriptstyle u}\downarrow & & \sim & & \downarrow{\scriptstyle v} \\ M_2 & \xrightarrow[e_2]{} & P_2 & \xrightarrow[m_2]{} & N_2 \end{array}$$

e_1 and e_2 external
m_1 and m_2 internal

there exists a unique path $h : P_1 \twoheadrightarrow P_2$ (up to homotopy) such that

$$\begin{array}{ccccc} M_1 & \xrightarrow{e_1} & P_1 & \xrightarrow{m_1} & N_1 \\ {\scriptstyle u}\downarrow & \sim & \downarrow{\scriptstyle h} & \sim & \downarrow{\scriptstyle v} \\ M_2 & \xrightarrow{e_2} & P_2 & \xrightarrow{m_2} & N_2 \end{array}$$

commutes (up to homotopy.)

V. Head rewriting paths

A universal cone of head-rewriting paths

Axiomatic set of values

Definition.

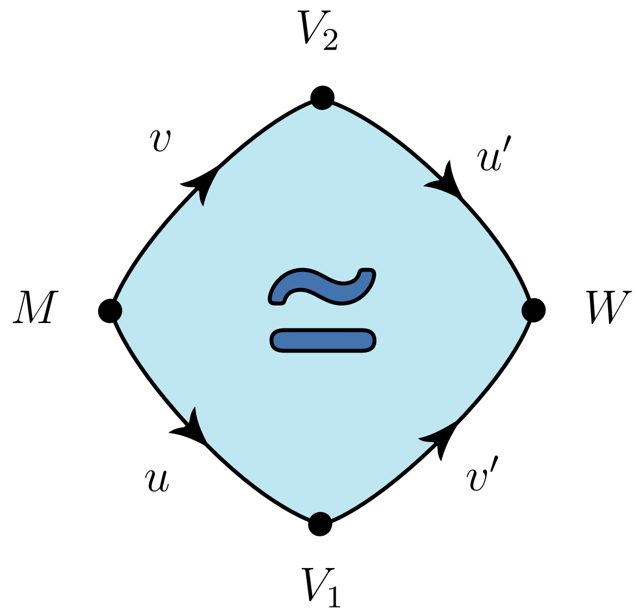
An axiomatic set $\mathcal{H}$ of values is a set of terms satisfying three properties:

(1) the set $\mathcal{H}$ is closed under reduction:

$$V \in \mathcal{H} \quad \text{and} \quad V \longrightarrow W \quad \Longrightarrow \quad W \in \mathcal{H}$$

Axiomatic set of values

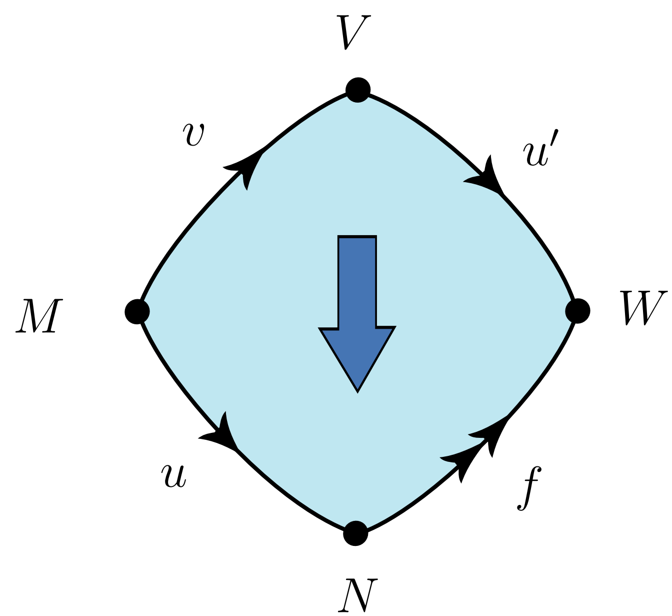
(2) In every reversible tile



$V_1 \in \mathcal{H}$ and $V_2 \in \mathcal{H}$ $\Longrightarrow$ $M \in \mathcal{H}$

Axiomatic set of values

(3) In every irreversible tile



$$V \in \mathcal{H} \quad \Longrightarrow \quad M \in \mathcal{H}$$

Stability theorem

Suppose given an axiomatic set $\mathcal{H}$ of values.

Theorem. For every term M, there exists a cone of paths

$$\left(\quad M \xrightarrow{e_i} V_i \quad \right)_{i \in I}$$

satisfying the following property: for every rewriting path

$$f \; : \; M \longrightarrow W \qquad \text{where } W \in \mathcal{H}$$

there exists a **unique** index $i \in I$ and a unique path

$$h \; : \; V_i \longrightarrow W \qquad \text{up to homotopy}$$

such that

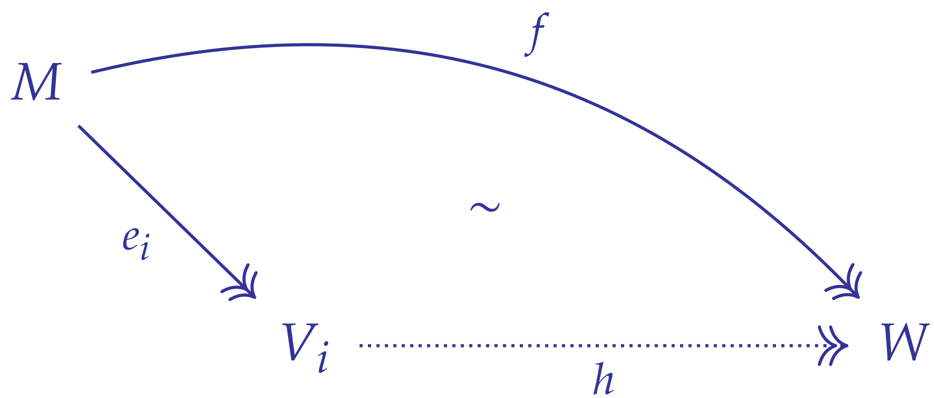
$$M \xrightarrow{f} W \quad \sim \quad M \xrightarrow{e_i} V_i \xrightarrow{h} W$$

A cone of head-rewriting paths

This means that there is a unique head-rewriting path

$$e_i \; : \; M \twoheadrightarrow V_i$$

in the cone such that f factors as



up to homotopy.

Application to the $\lambda\sigma$-calculus

Suppose that M is a λ-term seen as a $\lambda\sigma$-term.

Theorem. Every head rewriting path

$$e_i \; : \; M \longrightarrow\!\!\!\!\!\rightarrow V_i \qquad\qquad V_i \in \mathcal{H}_{\lambda\sigma}$$

to the set $\mathcal{H}_{\lambda\sigma}$ of $\lambda\sigma$-head-normal forms is transported to

$$e \; : \; M \longrightarrow\!\!\!\!\!\rightarrow \sigma(V_i)$$

the **unique** head-rewriting path from M to the set

$$\mathcal{H}_{\lambda}$$

of head-normal forms in the λ-calculus.

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