# Categorical combinatorics
# of scheduling and synchronization

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CNRS & Université Paris Diderot

ACM Symposium on Principles of Programming Languages
Cascais ⚓ POPL'19 ⚓ 16 $\longrightarrow$ 19 January 2019

# Categorical combinatorics of scheduling and synchronization in game semantics

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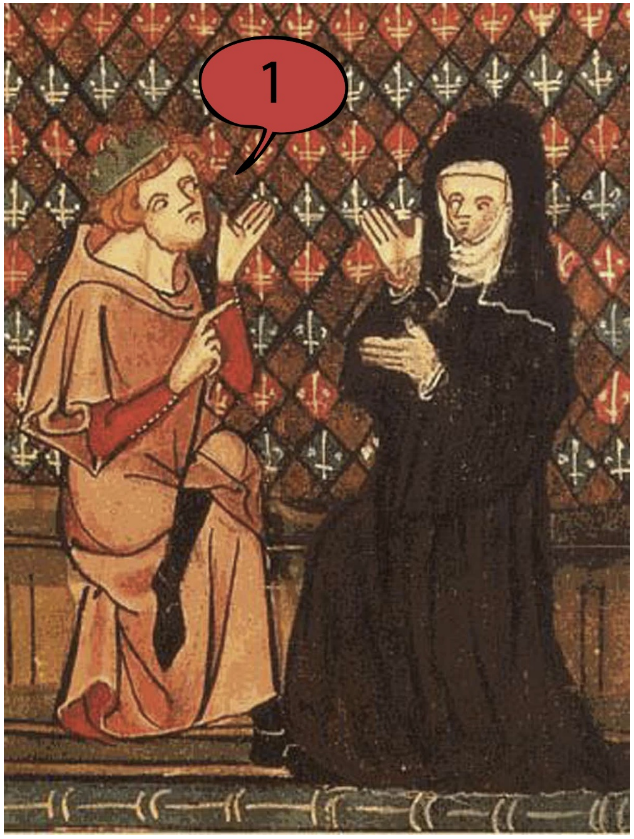
## Understanding logic in space and time




What are the principles at work in a dialogue?

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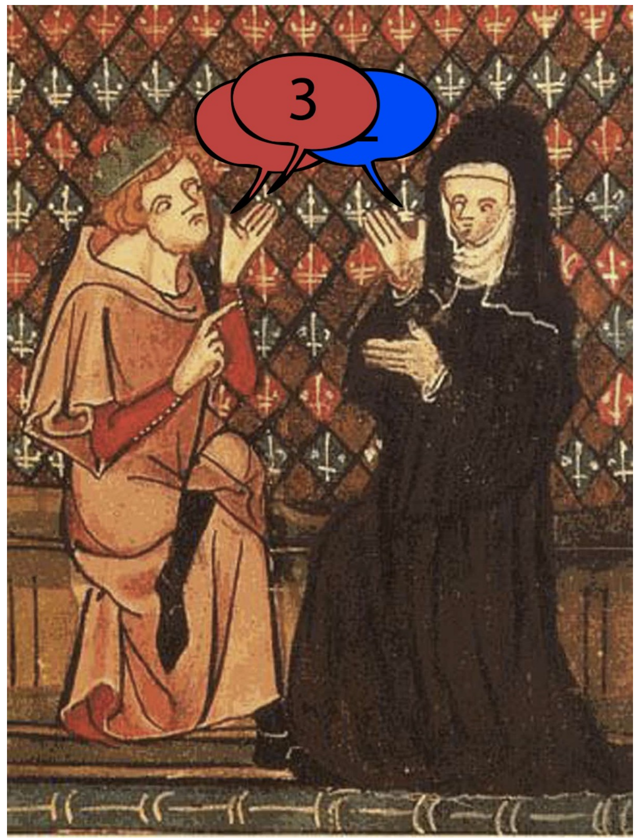
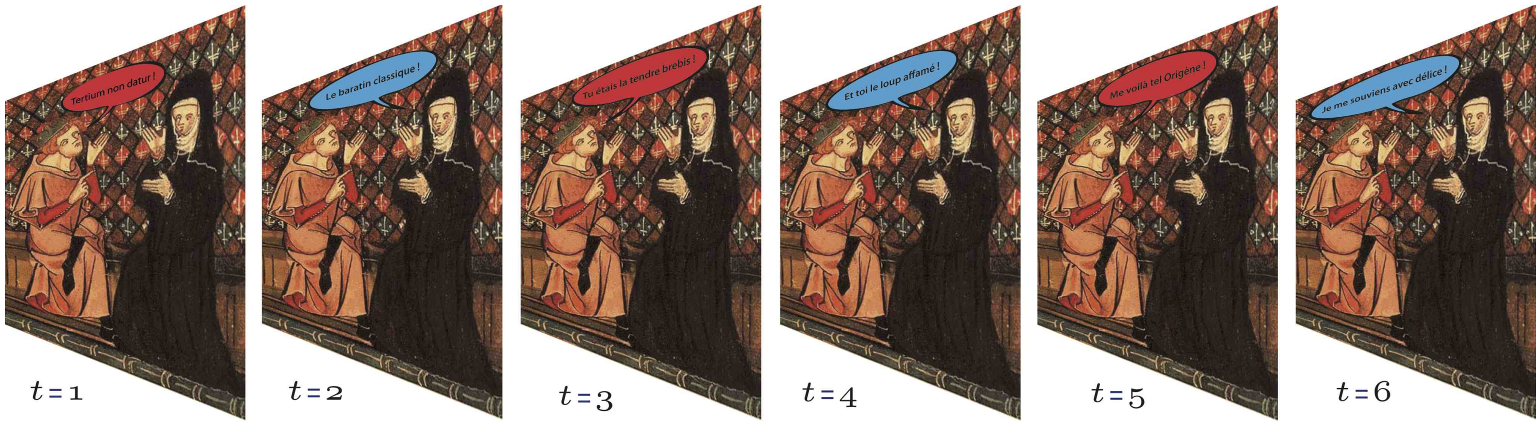



What are the principles at work in a dialogue?

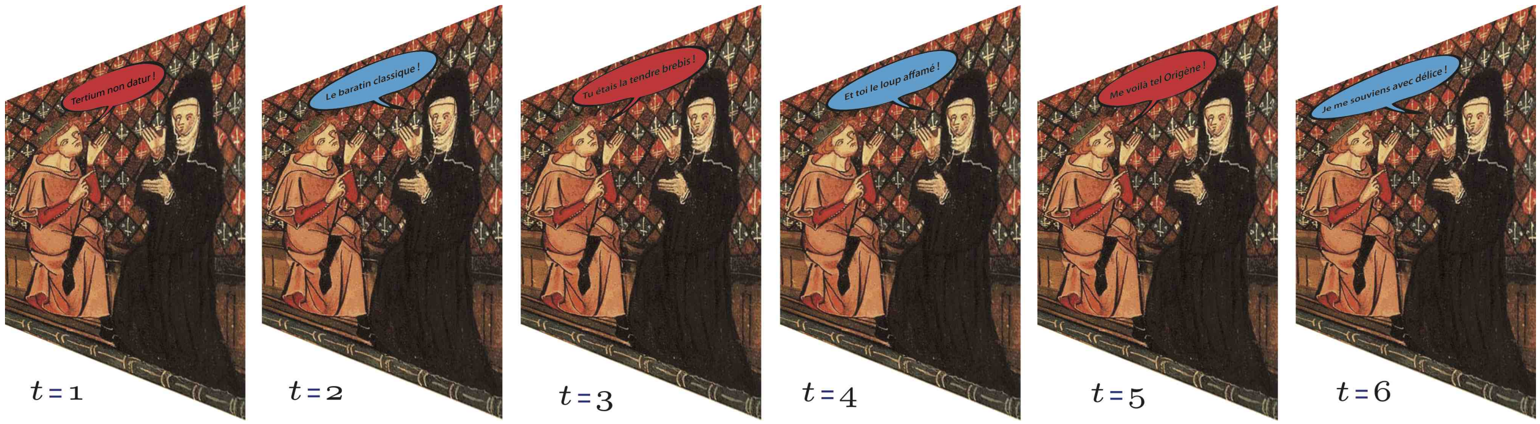



**Purpose of this talk:**

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by linear logic.




**Purpose of this talk:**

Understand how different proofs and programs may be

- combined together in space
- synchronized together in time

in the rich and modular ecosystem provided by game semantics.

# Template games

Categorical combinatorics of synchronization

# The category of polarities

We introduce the category

$$\pm_{\mathrm{game}}$$

freely generated by the graph

$$\langle \ominus \rangle \; \underset{P}{\overset{O}{\leftrightarrows}} \; \langle \oplus \rangle$$

the category $\pm_{\mathrm{game}}$ will play a fundamental role in the talk

## Template games

**First idea:**

Define a **game** as a category $A$ equipped with a functor

$$\begin{array}{c} A \\ \Big\downarrow \lambda_A \\ \pm_{\mathrm{game}} \end{array}$$

to the category $\pm_{\mathrm{game}}$ freely generated by the graph

$$\langle \ominus \rangle \underset{P}{\overset{O}{\rightleftarrows}} \langle \oplus \rangle$$

Inspired by the notion of **coloring** in graph theory

## Positions and trajectories

It is convenient to use the following terminology

$$\begin{array}{ccc} \text{objects} & \leftrightarrow & \text{positions} \\ \text{morphisms} & \leftrightarrow & \text{trajectories} \end{array}$$

and to see the category $A$ as an **unlabelled** transition system.

# The polarity functor

The polarity functor

$$\lambda_A \quad : \quad A \longrightarrow \pm_{\text{game}}$$

assigns a polarity $\oplus$ or $\ominus$ to every position of the game $A$.

**Definition.** A position $a \in A$ is called

| | | | |
|---|---|---|---|
| **Player** | when its polarity | $\lambda_A(a) = \oplus$ | is positive |
| **Opponent** | when its polarity | $\lambda_A(a) = \ominus$ | is negative |

# Opponent moves

**Definition.** An **Opponent move**

$$m \quad : \quad a^{\oplus} \longrightarrow b^{\ominus}$$

is a trajectory of the game $A$ transported to the edge

$$O \quad : \quad \langle \oplus \rangle \longrightarrow \langle \ominus \rangle$$

of the template category $\pm_{\mathrm{game}}$.

# Player moves

**Definition.** A **Player move**

$$m \quad : \quad a^{\ominus} \longrightarrow b^{\oplus}$$

is a trajectory of the game $A$ transported to the edge

$$P \quad : \quad \langle \ominus \rangle \longrightarrow \langle \oplus \rangle$$

of the template category $\pm_{\text{game}}$.

# Silent trajectories

**Definition.** A **silent move**

$$m \quad : \quad a \longrightarrow b$$

is a trajectory of the game $A$ transported to an identity morphism

$$id_{\langle \oplus \rangle} \quad : \quad \langle \oplus \rangle \longrightarrow \langle \oplus \rangle$$

$$id_{\langle \ominus \rangle} \quad : \quad \langle \ominus \rangle \longrightarrow \langle \ominus \rangle$$

of the template category $\pm_{\mathrm{game}}$.

# The template of strategies

Categorical combinatorics of synchronization

# The template of strategies

In order to describe the strategies between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!\mid\;\;\;\;\;\; B$$

we introduce the **template of strategies**

$$\pitchfork_{\mathrm{strat}}$$

defined as the category freely generated by the graph

$$\langle \ominus, \ominus \rangle \underset{O_s}{\overset{P_s}{\rightleftarrows}} \langle \oplus, \ominus \rangle \underset{P_t}{\overset{O_t}{\rightleftarrows}} \langle \oplus, \oplus \rangle$$

# The template of strategies

Each of the four labels

$$O_s \qquad P_s \qquad O_t \qquad P_t$$

describes a specific kind of Opponent and Player move

| | | | | |
|---|---|---|---|---|
| $O_s$ | : | Opponent move | played at | the source game |
| $P_s$ | : | Player move | played at | the source game |
| $O_t$ | : | Opponent move | played at | the target game |
| $P_t$ | : | Player move | played at | the target game |

which may appear on the interactive trajectory played by a strategy

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; B.$$

# The template of strategies

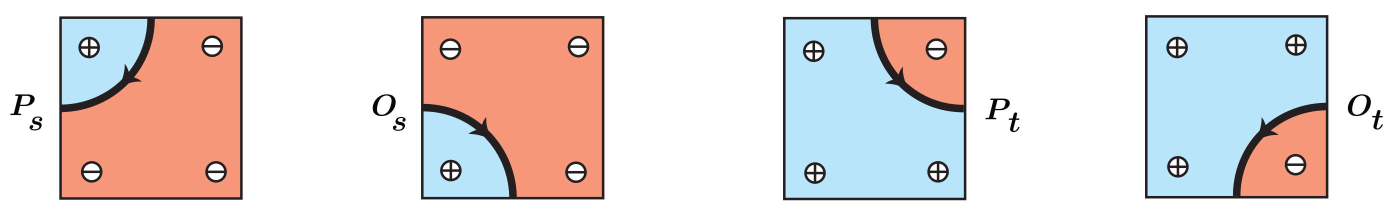
The four generators

$$\langle \ominus, \ominus \rangle \underset{O_s}{\overset{P_s}{\leftrightarrows}} \langle \oplus, \ominus \rangle \underset{P_t}{\overset{O_t}{\leftrightarrows}} \langle \oplus, \oplus \rangle$$

of the category

$$⚓_{\text{strat}}$$

may be depicted as follows:

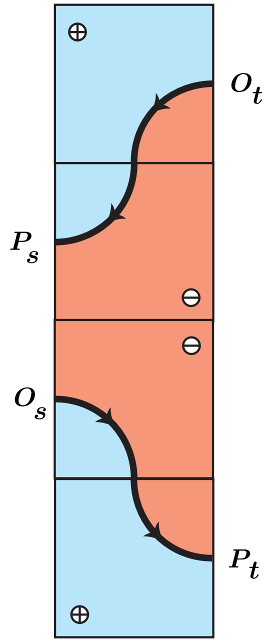


# The template of strategies

In that graphical notation, the sequence

$$O_t \cdot P_s \cdot O_s \cdot P_t$$

is depicted as



# The template of strategies

The category $\pitchfork_{\mathrm{strat}}$ comes equipped with a span of functors

$$\pitchfork_{\mathrm{game}} \xleftarrow{s=(1)} \pitchfork_{\mathrm{strat}} \xrightarrow{t=(2)} \pitchfork_{\mathrm{game}}$$

defined as the projection $s = (1)$ on the first component:

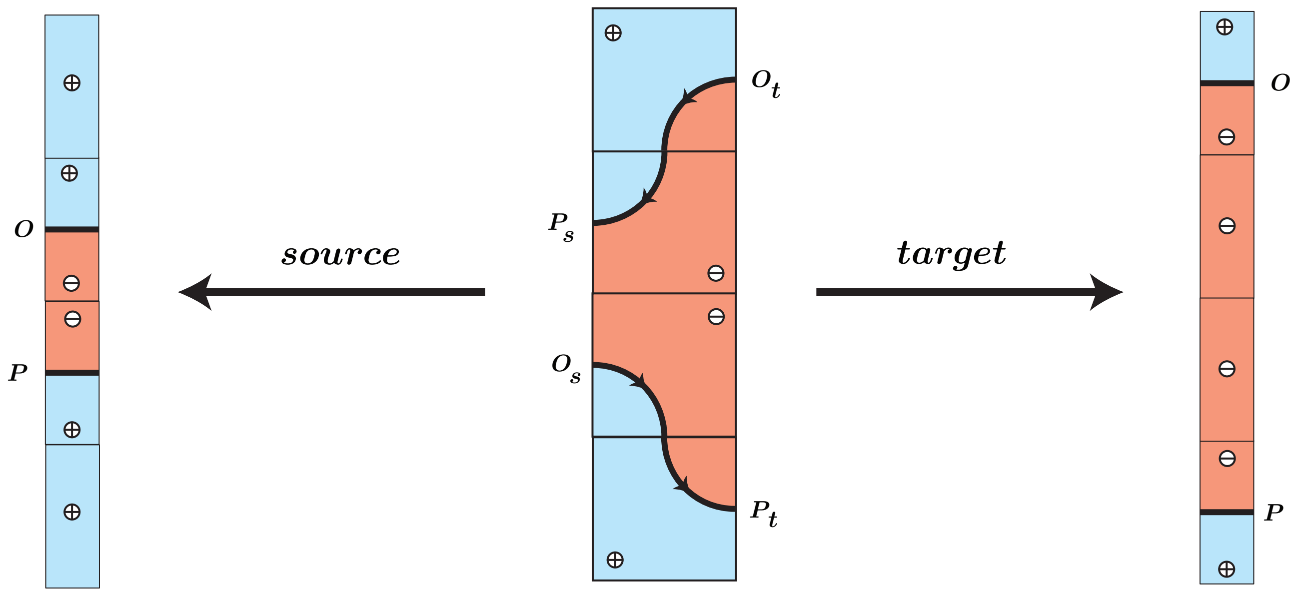
$$\begin{array}{rcl} \langle \ominus, \ominus \rangle & \mapsto & \langle \ominus \rangle \\ \langle \oplus, \ominus \rangle \, , \, \langle \oplus, \oplus \rangle & \mapsto & \langle \oplus \rangle \end{array} \qquad \begin{array}{c} O_s \mapsto P \qquad P_s \mapsto O \\ O_t \, , \, P_t \mapsto id_{\langle \oplus \rangle} \end{array}$$

and as the projection $t = (2)$ on the second component:

$$\begin{array}{rcl} \langle \oplus, \oplus \rangle & \mapsto & \langle \oplus \rangle \\ \langle \ominus, \ominus \rangle \, , \, \langle \oplus, \ominus \rangle & \mapsto & \langle \ominus \rangle \end{array} \qquad \begin{array}{c} O_t \mapsto O \qquad P_t \mapsto P \\ O_s \, , \, P_s \mapsto id_{\langle \ominus \rangle} \end{array}$$

# The template of strategies

The two functors $s$ and $t$ are illustrated below:



# Strategies between games

**Second idea:**

Define a **strategy** between two games

$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!\!\!+\quad B$$

as a **span of functors**

$$A \xleftarrow{\quad s \quad} S \xrightarrow{\quad t \quad} B$$

together with a **scheduling functor**

$$S \xrightarrow{\quad \lambda_\sigma \quad} \pm_{\text{strat}}$$

# Strategies between games

making the diagram below commute

$$
\begin{array}{ccccc}
A & \xleftarrow{\;s\;} & S & \xrightarrow{\;t\;} & B \\
\downarrow{\scriptstyle \lambda_A} & & \downarrow{\scriptstyle \lambda_\sigma} & & \downarrow{\scriptstyle \lambda_B} \\
\pitchfork_{\mathrm{game}} & \xleftarrow{\;s\;} & \pitchfork_{\mathrm{strat}} & \xrightarrow{\;t\;} & \pitchfork_{\mathrm{game}}
\end{array}
$$

**Key idea:**

Every trajectory $s \in S$ induces a pair of trajectories $s_A \in A$ and $s_B \in B$.

The functor $\lambda_\sigma$ describes how $s_A$ and $s_B$ are scheduled together by $\sigma$.

# Support of a strategy

**Terminology.** The category $S$ defining the span

$$A \xleftarrow{\ s\ } S \xrightarrow{\ t\ } B$$

is called the **support** of the strategy

$$\sigma \quad : \quad A \longrightarrow B$$

**Basic intuition:**

« the support $S$ contains the trajectories played by $\sigma$ »

# A typical scheduling $B \cdot A \cdot A \cdot B$

A trajectory $s \in S$ of the strategy $\sigma$ with schedule

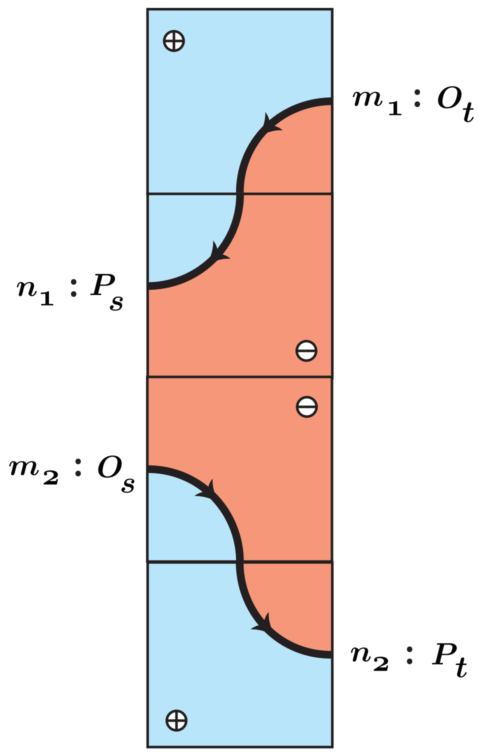
$$\langle \oplus, \oplus \rangle \xrightarrow{O_t} \langle \oplus, \ominus \rangle \xrightarrow{P_s} \langle \ominus, \ominus \rangle \xrightarrow{O_s} \langle \ominus, \oplus \rangle \xrightarrow{P_t} \langle \oplus, \oplus \rangle$$

is traditionally depicted as

| | $A$ | $\xrightarrow{\sigma}$ | $B$ |
|---|---|---|---|
| first move $m_1$ of polarity $O_t$ | | | $m_1$ |
| second move $n_1$ of polarity $P_s$ | $n_1$ | | |
| third move $m_2$ of polarity $O_s$ | $m_2$ | | |
| fourth move $n_2$ of polarity $P_t$ | | | $n_2$ |

# A typical scheduling $B \cdot A \cdot A \cdot B$

Thanks to the approach, one gets the more informative picture:



# Simulations

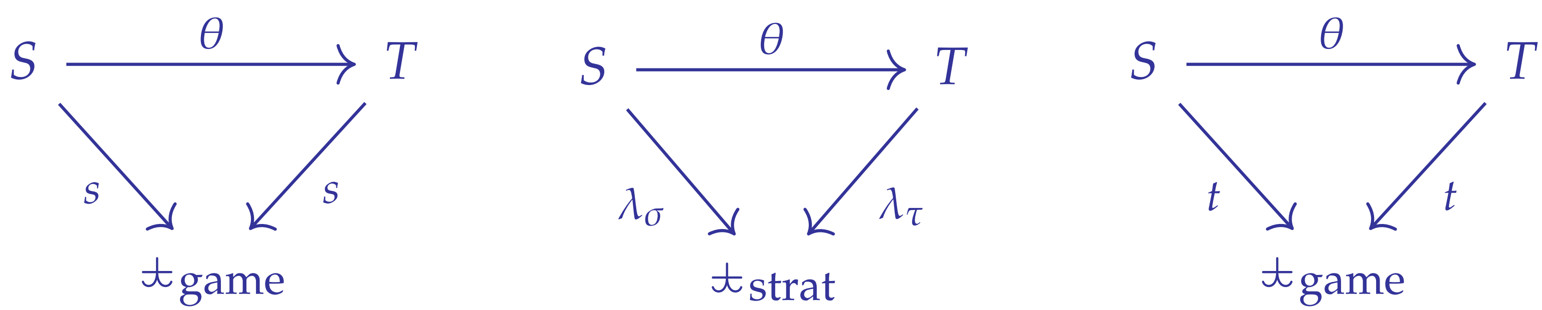
**Definition:** A **simulation** between strategies

$$\theta \;:\; \sigma \Longrightarrow \tau \;:\; A \;\mathrel{\longrightarrow\!\!\!\!\!\!\!+\;\;}\; B$$

is a **functor** from the support of $\sigma$ to the support of $\tau$

$$\theta \;:\; S \longrightarrow T$$

making the three triangles commute



## The category of strategies and simulations

Suppose given two games $A$ and $B$.

The category **Games** $(A, B)$ has **strategies** between $A$ and $B$

$$\sigma, \tau \quad : \quad A \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; B$$

as objects and **simulations** between strategies

$$\theta \;:\; \sigma \Longrightarrow \tau \;:\; A \longrightarrow\!\!\!\!\!\!\!\!+\;\;\; B$$

as morphisms.

# The bicategory **Games**

A bicategory of games, strategies and simulations

## The bicategory **Games** of games and strategies

At this stage, we want to turn the family of categories

$$\textbf{Games}\,(A, B)$$

into a **bicategory**

$$\textbf{Games}$$

of games and strategies.

## The bicategory **Games** of games and strategies

To that purpose, we need to define a composition functor

$$\circ_{A,B,C} \; : \; \textbf{Games}\,(B,C) \times \textbf{Games}\,(A,B) \longrightarrow \textbf{Games}\,(A,C)$$

which composes a pair of strategies

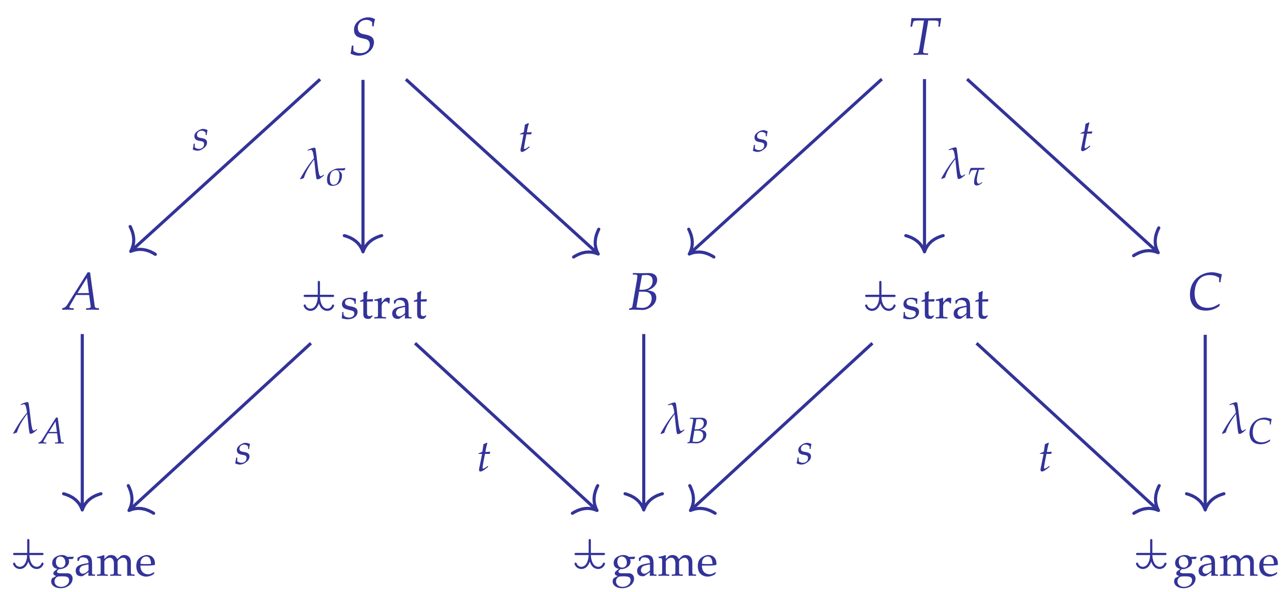
$$\sigma \quad : \quad A \longrightarrow\!\!\!\!\!\!\!+\;\; B \qquad\qquad \tau \quad : \quad B \longrightarrow\!\!\!\!\!\!\!+\;\; C$$

into a strategy

$$\sigma \circ_{A,B,C} \tau \quad : \quad A \longrightarrow\!\!\!\!\!\!\!+\;\; C$$

# Composition of strategies

The construction starts by putting the pair of functorial spans side by side:




Fine, but how shall one carry on and perform the composition?

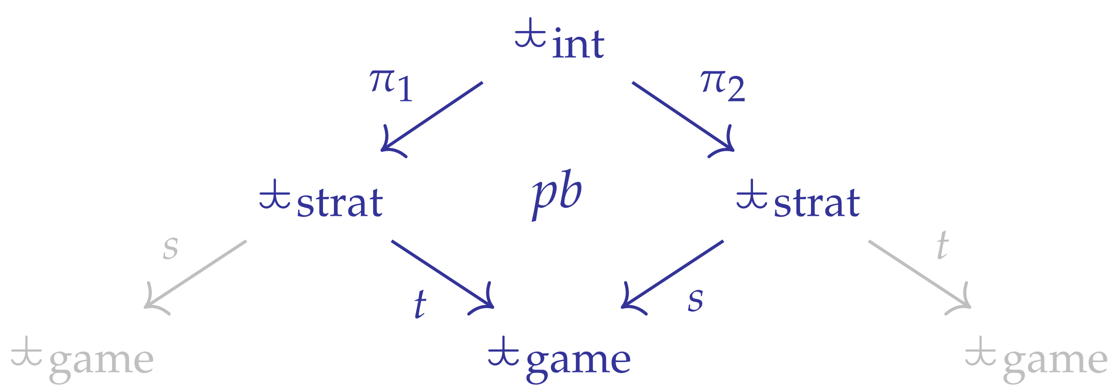
# The template of interactions

**Third idea:**

We define the **template of interactions**

$$ホ_{\mathrm{int}}$$

as the category obtained by the pullback diagram below



# The template of interactions

Somewhat surprisingly, the category

$$\pitchfork_{\mathrm{int}}$$

is simple to describe, as the **free category** generated by the graph

$$\langle\ominus,\ominus,\ominus\rangle \underset{O_s}{\overset{P_s}{\leftrightarrows}} \langle\oplus,\ominus,\ominus\rangle \underset{P|O}{\overset{O|P}{\leftrightarrows}} \langle\oplus,\oplus,\ominus\rangle \underset{P_t}{\overset{O_t}{\leftrightarrows}} \langle\oplus,\oplus,\oplus\rangle$$

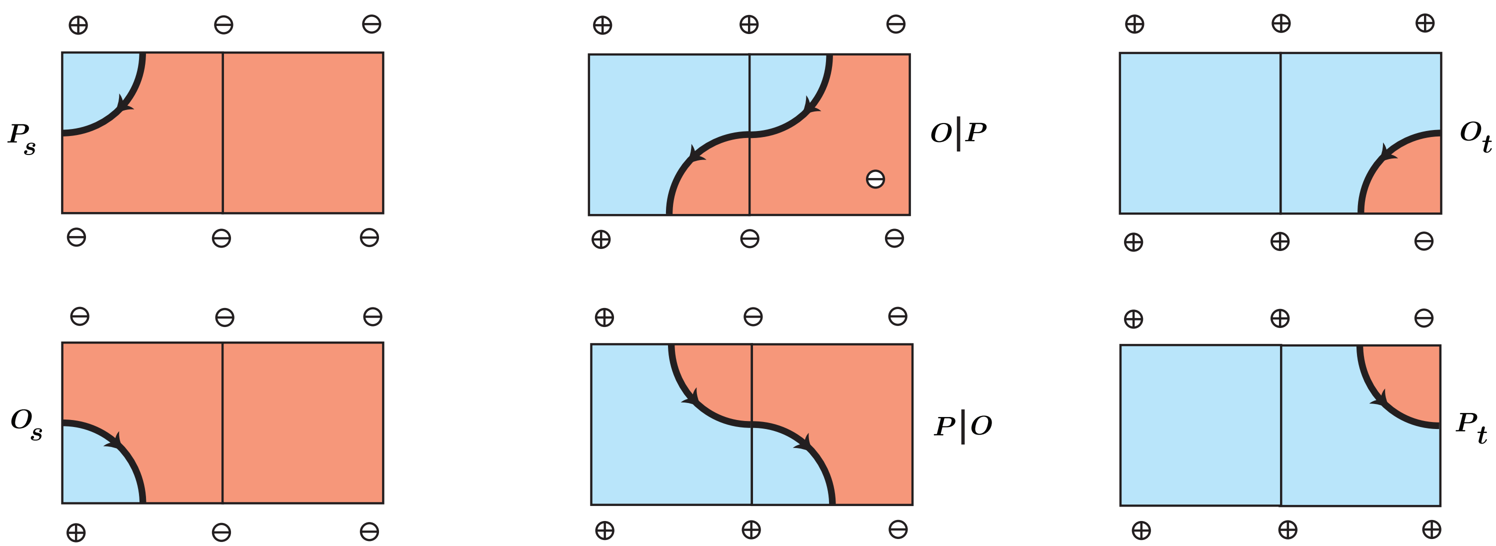
with four states or positions.

# The template of interactions

The six generators

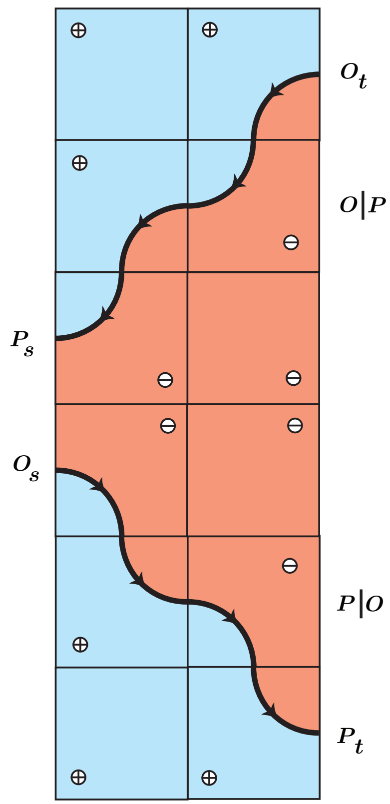
$$\langle\ominus,\ominus,\ominus\rangle \underset{O_s}{\overset{P_s}{\leftrightarrows}} \langle\oplus,\ominus,\ominus\rangle \underset{P|O}{\overset{O|P}{\leftrightarrows}} \langle\oplus,\oplus,\ominus\rangle \underset{P_t}{\overset{O_t}{\leftrightarrows}} \langle\oplus,\oplus,\oplus\rangle$$

may be depicted as follows:



# The template of interactions

A typical sequence of interactions is thus depicted as follows:



# Key observation

The template $\pitchfork_{\mathrm{int}}$ of interactions comes equipped with a functor

$$hide \quad : \quad \pitchfork_{\mathrm{int}} \longrightarrow \pitchfork_{\mathrm{strat}}$$

which makes the diagram below commute:

$$\begin{array}{ccccc}
\pitchfork_{\mathrm{strat}} & \xleftarrow{(12)} & \pitchfork_{\mathrm{int}} & \xrightarrow{(23)} & \pitchfork_{\mathrm{strat}} \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle hide} & & \downarrow{\scriptstyle (2)} \\
\pitchfork_{\mathrm{game}} & \xleftarrow{s=(1)} & \pitchfork_{\mathrm{strat}} & \xrightarrow{t=(2)} & \pitchfork_{\mathrm{game}}
\end{array}$$

and thus defines a map of span.

# Key observation

The functor

$$hide \quad : \quad \pm_{\mathrm{int}} \longrightarrow \pm_{\mathrm{strat}}$$

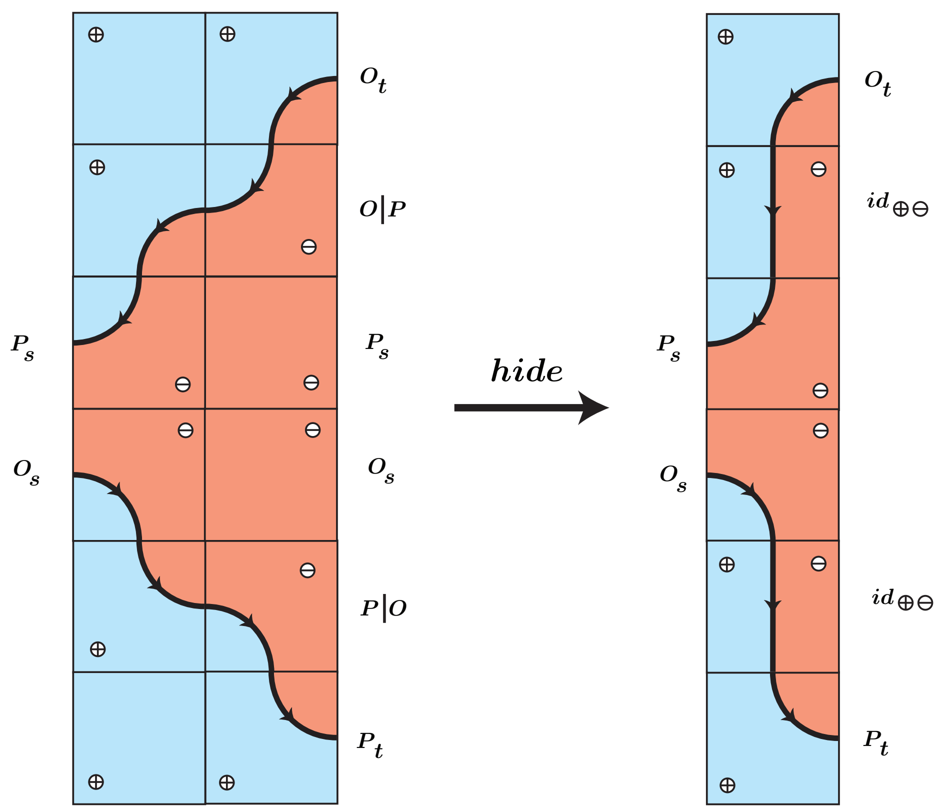
is defined by **projecting** the positions of the interaction category

$$\langle\, \varepsilon_1 \,,\, \varepsilon_2 \,,\, \varepsilon_3 \,\rangle$$

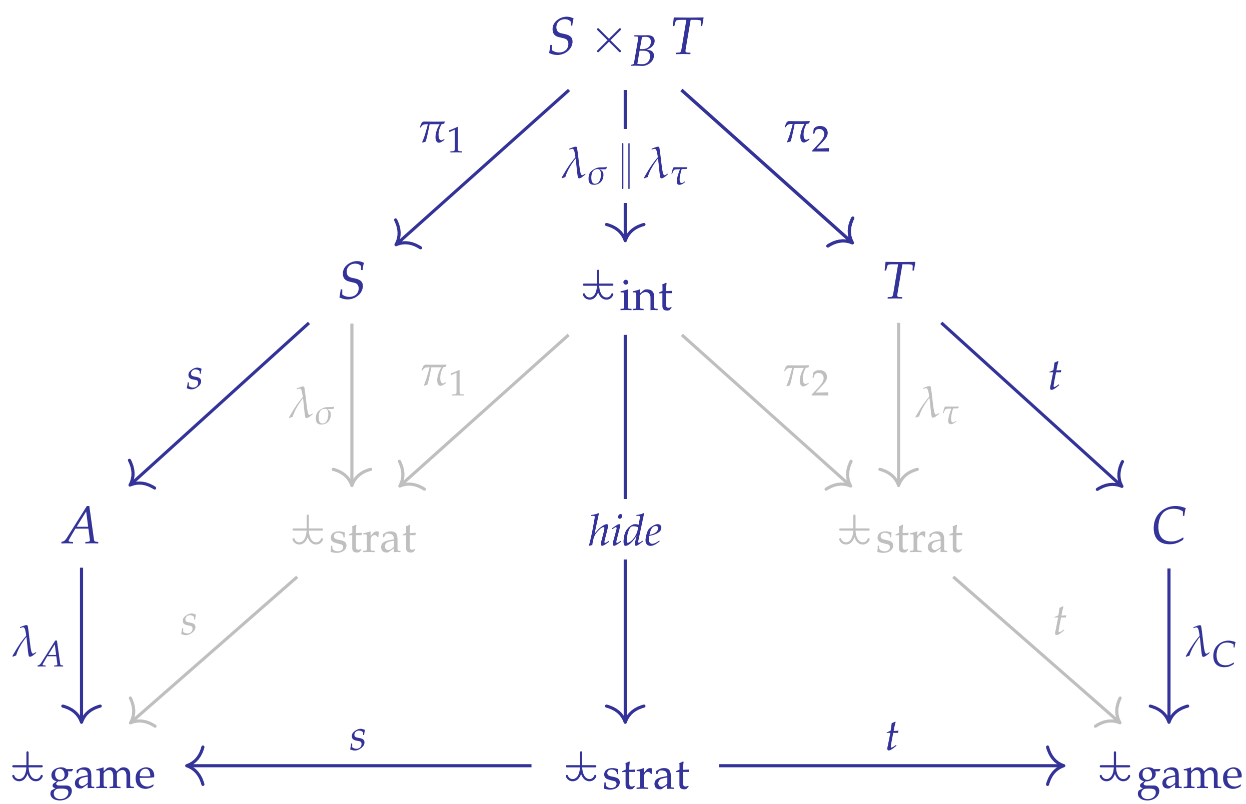
on their first and third components:

$$\begin{array}{rcl@{\qquad}rcl}
\langle \ominus, \ominus, \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle & O_s \mapsto O_s & & P_s \mapsto P_s \\
\langle \oplus, \ominus, \ominus \rangle \,,\, \langle \oplus, \oplus, \ominus \rangle & \mapsto & \langle \oplus, \ominus \rangle & O|P \,,\, P|O & \mapsto & id_{\langle \oplus, \ominus \rangle} \\
\langle \oplus, \oplus, \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle & O_s \mapsto O_s & & P_s \mapsto P_s
\end{array}$$

# Illustration



# Composition of strategies



## Composition of strategies

This definition of composition implements the slogan that

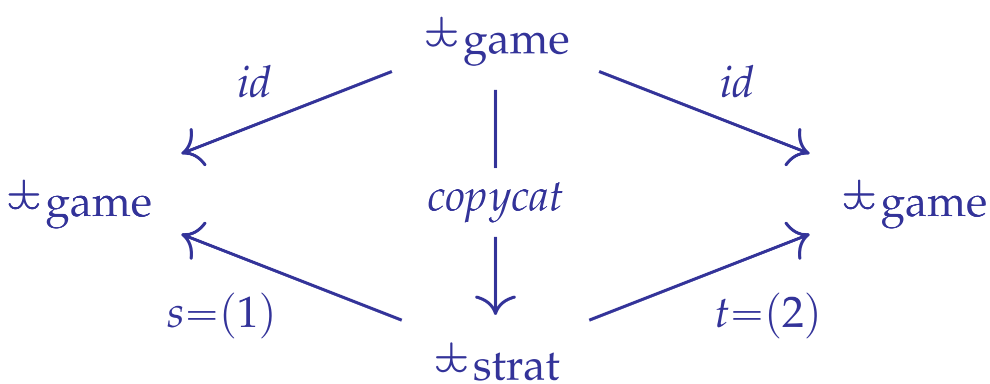
| |
|---|
| composition = synchronization + hiding |

# What about identities?

There exists a functor

$$copycat \quad : \quad \pitchfork\mathrm{game} \longrightarrow \pitchfork\mathrm{strat}$$

which makes the diagram commute:




and thus defines a morphism of spans.

# What about identities?

The functor

$$\textit{copycat} \quad : \quad \pm_{\mathrm{game}} \longrightarrow \pm_{\mathrm{strat}}$$

is defined by **duplicating** the positions of the polarity category

$$\langle \varepsilon \rangle$$

in the following way:

$$\begin{array}{rcl} \langle \ominus \rangle & \mapsto & \langle \ominus, \ominus \rangle \\ \langle \oplus \rangle & \mapsto & \langle \oplus, \oplus \rangle \end{array} \qquad \begin{array}{rcl} O & \mapsto & O_t \cdot P_s \\ P & \mapsto & O_s \cdot P_t \end{array}$$

# A synchronous copycat strategy

The functor

$$copycat \quad : \quad \pm_{\mathrm{game}} \longrightarrow \pm_{\mathrm{strat}}$$

transports the edge

$$\langle \ominus \rangle \xleftarrow{\;O\;} \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xleftarrow{\;P_s\;} \langle \oplus, \ominus \rangle \xleftarrow{\;O_t\;} \langle \oplus, \oplus \rangle$$

# A synchronous copycat strategy

The functor

$$copycat \quad : \quad \pm_{\text{game}} \longrightarrow \pm_{\text{strat}}$$

transports the edge

$$\langle \ominus \rangle \xrightarrow{\quad P \quad} \langle \oplus \rangle$$

to the trajectory consisting of two moves

$$\langle \ominus, \ominus \rangle \xrightarrow{\quad O_s \quad} \langle \oplus, \ominus \rangle \xrightarrow{\quad P_t \quad} \langle \oplus, \oplus \rangle$$

# The identity strategy

Given a game $A$, the copycat strategy

$$cc_A \quad : \quad A \longrightarrow A$$

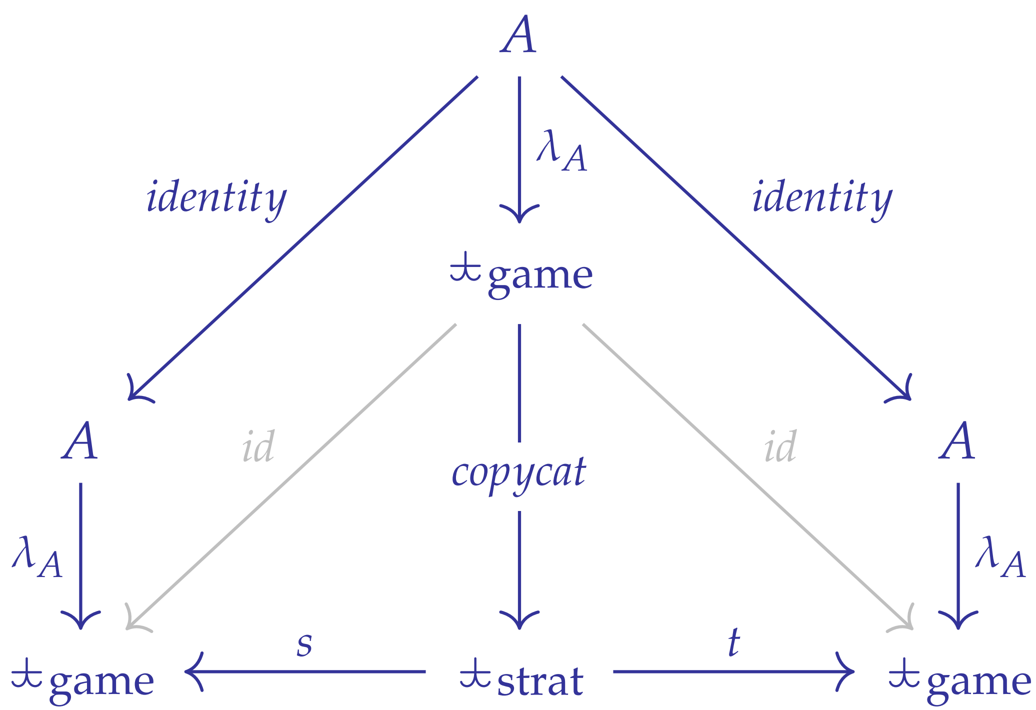
is defined as the functorial span

$$A \xleftarrow{\quad identity \quad} A \xrightarrow{\quad identity \quad} A$$

together with the scheduling functor

$$\lambda_{cc_A} \;=\; A \xrightarrow{\quad \lambda_A \quad} \pitchfork_{\text{game}} \xrightarrow{\quad copycat \quad} \pitchfork_{\text{strat}}$$

# Identity strategy



# Discovery of an unexpected principle

**Key observation:** the categories

$$\text{ナ}[0] = \text{ナ}_{\text{game}} \qquad \text{ナ}[1] = \text{ナ}_{\text{strat}} \qquad \text{ナ}[2] = \text{ナ}_{\text{int}}$$

and the span of functors

$$\text{ナ}[0] \xleftarrow{\;s\;} \text{ナ}[1] \xrightarrow{\;t\;} \text{ナ}[0]$$

define an **internal category** in $Cat$ with composition and identity

$$\text{ナ}[2] \xrightarrow{\;hide\;} \text{ナ}[1] \qquad\qquad \text{ナ}[0] \xrightarrow{\;copycat\;} \text{ナ}[1]$$

## As an immediate consequence...

**Theorem A.** The construction just given defines a **bicategory**

**Games**

of games, strategies and simulations.

## Main technical result of the paper

**Theorem B.** The bicategory

**Games**

of games, strategies and simulations is **symmetric monoidal.**

## Main technical result of the paper

**Theorem C.** The bicategory

**Games**

of games, strategies and simulations is **star-autonomous.**

## All these results are based on the same recipe!

One constructs an **internal category** of tensorial schedules

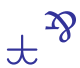
$$\text{ш}^{\otimes}$$

together with a pair of **internal functors**

$$\text{ш} \times \text{ш} \xleftarrow{\;pick\;} \text{ш}^{\otimes} \xrightarrow{\;pince\;} \text{ш}$$

## All these results are based on the same recipe!

One constructs an **internal category** of cotensorial schedules



together with a pair of **internal functors**

## All these results are based on the same recipe!

One constructs an **internal functor**

$$reverse \quad : \quad \pitchfork^{op} \longrightarrow \pitchfork$$

which reverses the polarity of every position and move

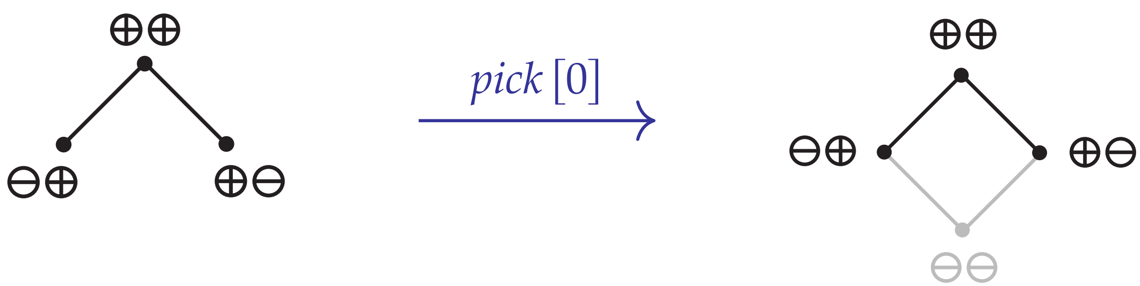
$$\begin{array}{ccc} \oplus & \mapsto & \ominus \\ \ominus & \mapsto & \oplus \end{array} \qquad\qquad \begin{array}{ccc} O & \mapsto & P \\ P & \mapsto & O \end{array}$$

# The pick functor

The internal functor

$$pick \quad : \quad \pitchfork^{\otimes} \longrightarrow \pitchfork \times \pitchfork$$

is defined at dimension 0 by the functor:

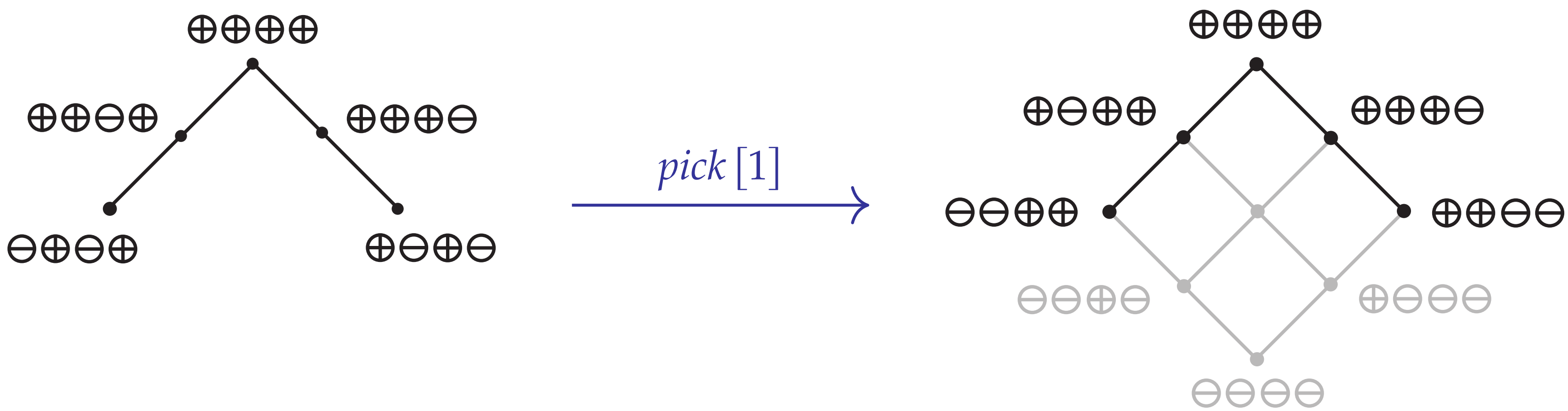


# The pick functor

The internal functor

$$pick \quad : \quad \pm^{\otimes} \longrightarrow \pm \times \pm$$

is defined at dimension 1 by the functor:



# The pince functor

The internal functor

$$pince \quad : \quad \pitchfork^{\otimes} \longrightarrow \pitchfork$$

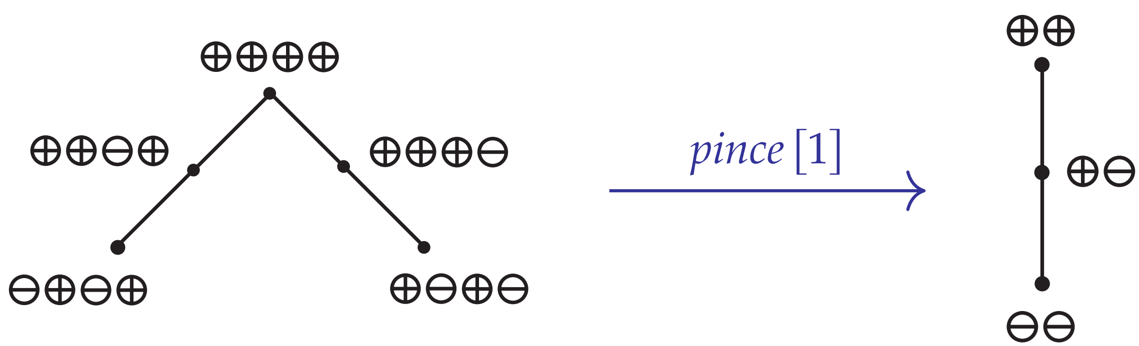
is defined at dimension 0 by the functor:

# The pince functor

The internal functor

$$pince \quad : \quad \pitchfork^{\otimes} \longrightarrow \pitchfork$$

is defined at dimension 1 by the functor:



# Conclusion and future work

- games played on **categories** with **synchronous copycats**
- an easy recipe to construct **new game semantics**
- **three templates** considered in the paper:

| | |
|---|---|
| $⅄_{\text{alt}}$ | alternating games and strategies |
| $⅄_{\text{conc}}$ | concurrent games and strategies |
| $⅄_{\text{span}}$ | functorial spans with no scheduling |

- same basic principles in **concurrent separation logic**
- a model of **differential linear logic** based on **homotopy theory**

# Selected bibliography

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Two Sides of the Same Coin: Session Types and Game Semantics.
POPL'19 – **Capabilities and Session Types session this afternoon!**

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Concurrent Strategies.
LICS 2011

# The distributivity law of linear logic

A game semantics of linear logic

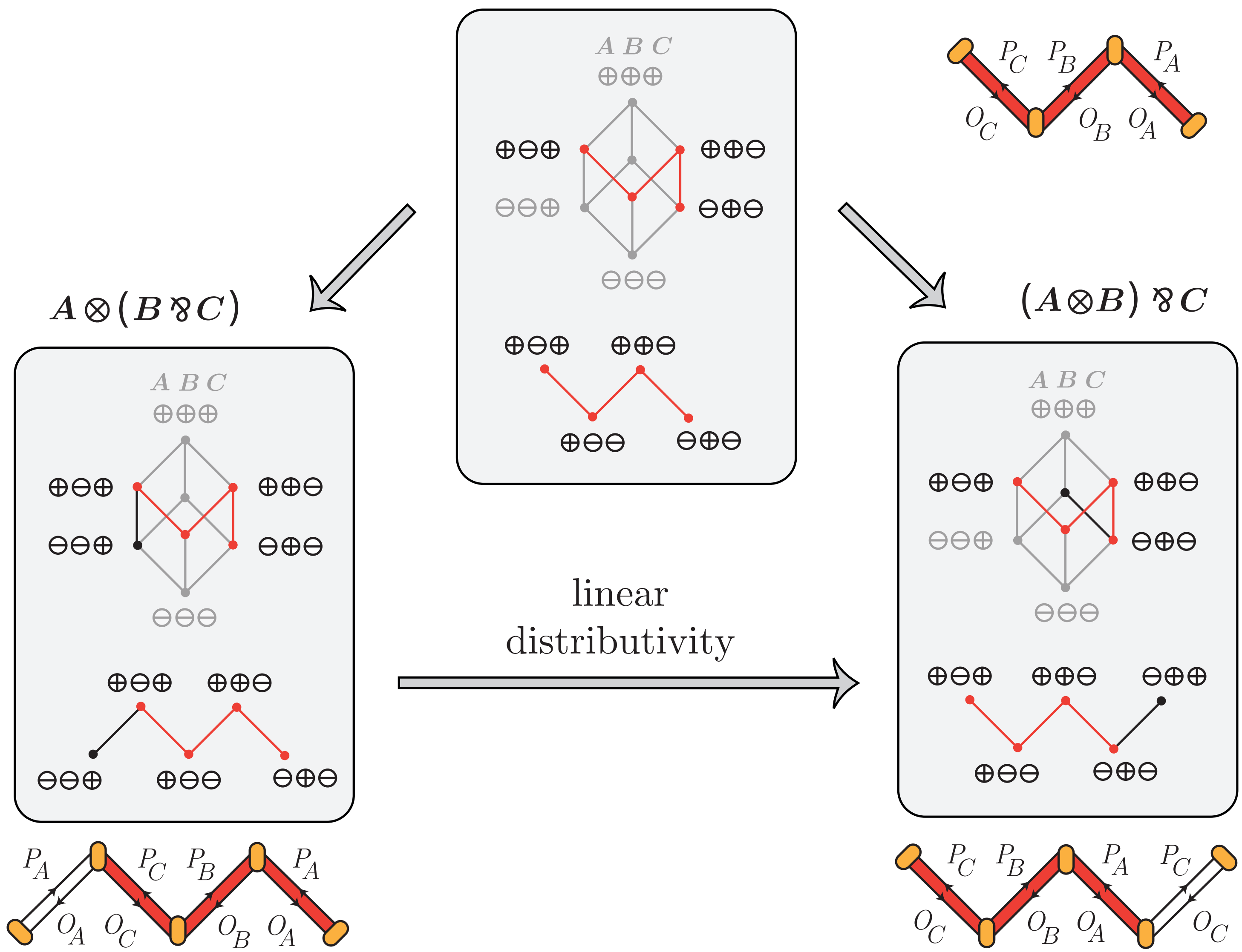
## The distributivity law of linear logic

The main ingredient of linear logic

$$\kappa_{A,B,C} \quad : \quad A \otimes (B \parr C) \longrightarrow (A \otimes B) \parr C$$

cannot be interpreted in traditional game semantics.

When one interprets it in template games, here is what one gets...


A ⊗ (B ⅋ C)
(A ⊗ B) ⅋ C
linear
distributivity
A B C
⊕⊕⊕
⊕⊖⊕
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$P_A$
$P_B$
$P_C$
$O_A$
$O_B$
$O_C$

# The template of interactions

How the category $大_{\mathrm{int}}$ is computed as a pullback

# The template of interactions

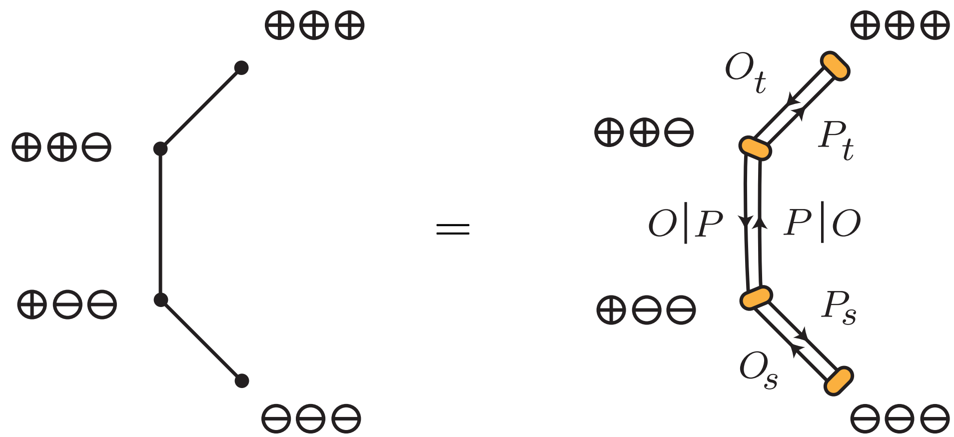
We find illuminating to depict the canonical functor

$$\pitchfork_{\mathrm{int}} \xrightarrow{(1223)} \pitchfork_{\mathrm{strat}} \times \pitchfork_{\mathrm{strat}}$$

induced by the pullback diagram in the following way:

# The template of interactions

In order to fully appreciate the diagram, one needs to “fatten” it




in such a way as to recover the template of interactions

$$\langle\ominus,\ominus,\ominus\rangle \;\underset{O_s}{\overset{P_s}{\leftrightarrows}}\; \langle\oplus,\ominus,\ominus\rangle \;\underset{P|O}{\overset{O|P}{\leftrightarrows}}\; \langle\oplus,\oplus,\ominus\rangle \;\underset{P_t}{\overset{O_t}{\leftrightarrows}}\; \langle\oplus,\oplus,\oplus\rangle$$

## The template of concurrent games

Templates of concurrent games as commutative monoids

## The template of games

The category

$$\pitchfork_{\text{conc}}[0]$$

is generated by the graph

$$P \circlearrowright \langle * \rangle \circlearrowleft O$$

together with the additional equation

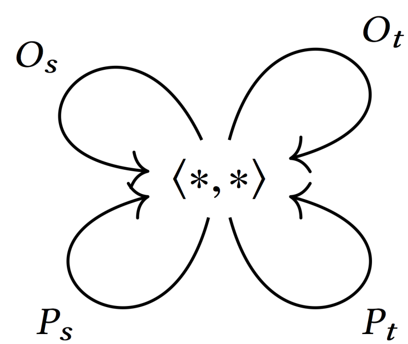
$$O \cdot P = P \cdot O$$

# The template of strategies

The category

$$\text{⹀}_{\text{conc}}[1]$$

is generated by the graph




together with the six elementary equations

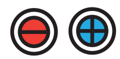
$$O_s \cdot P_s = P_s \cdot O_s \qquad O_s \cdot P_t = P_t \cdot O_s \qquad O_s \cdot O_t = O_t \cdot O_s$$
$$O_t \cdot P_s = P_s \cdot O_t \qquad O_t \cdot P_t = P_t \cdot O_t \qquad P_s \cdot P_t = P_t \cdot O_s$$

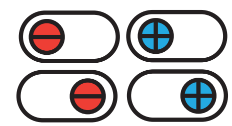
# The templates of games and strategies

The two templates

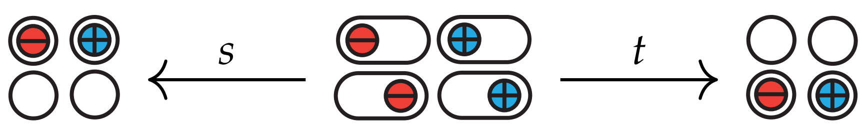
$$\pm_{\mathrm{conc}}[0] \qquad \pm_{\mathrm{conc}}[1]$$

are **commutative monoids** generated by the sets of moves:

$$\pm_{\mathrm{conc}}[0] \quad = $$ 

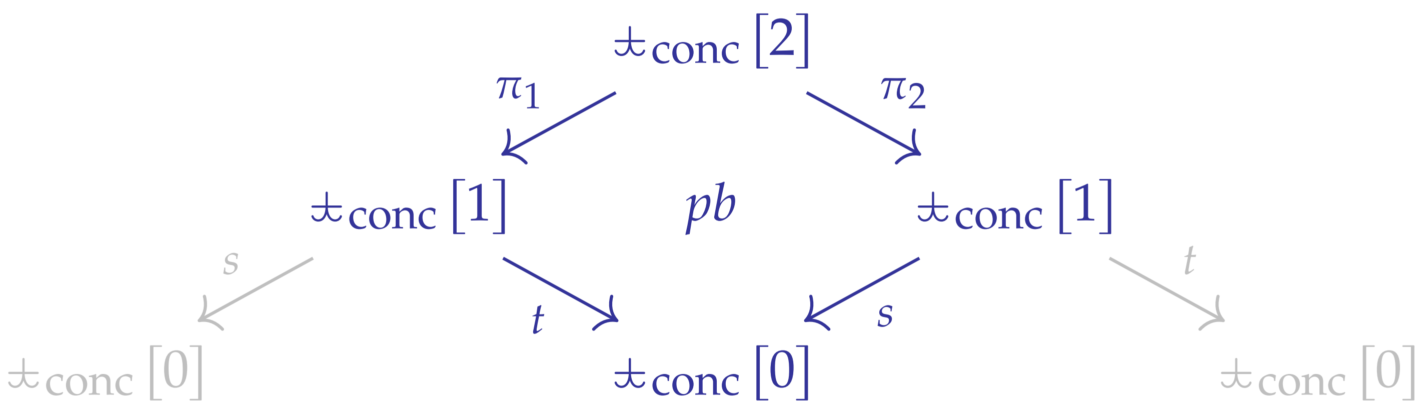
$$\pm_{\mathrm{conc}}[1] \quad = $$ 

The representation is nice to describe the source and target functors:



# The template of interactions

When one computes the pullback




one obtains the commutative monoid:

$$\pitchfork_{\mathrm{conc}}[2] \quad = $$