A micrological study of negation

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Abstract

Tensorial logic is a primitive logic of tensor and negation which refines linear logic by relaxing the hypothesis that linear negation is involutive. Thanks to this mild modification, tensorial logic provides a type-theoretic account of game semantics where innocent strategies are portrayed as temporal refinements of traditional proof-nets in linear logic. In this paper, we study the algebraic and combinatorial structure of negation in a non-commutative variant of tensorial logic. The analysis is based on a 2-categorical account of dialogue categories, which unifies tensorial logic with linear logic, and discloses a primitive symmetry between proofs and anti-proofs. The micrological analysis of tensorial negation reveals that it can be decomposed into a series of more elementary components: an adjunction $L \dashv R$ between the left and right negation functors L and R ; a pair of linear distributivity laws $\kappa^{\mathbb{O}}$ and $\kappa^{\mathbb{O}}$ which refines the linear distributivity law between $\otimes$ and $\parr$ in linear logic, and generates the Opponent and Proponent views of innocent strategies between dialogue games ; a pair of axiom and cut combinators adapted from linear logic ; an involutive change of frame $(-)^*$ reversing the point of view of Prover and of Denier on the logical dispute, and reversing the polarity of moves in the dialogue game associated to the tensorial formula.

1 Introduction

The general ambition of our work in tensorial logic is to lift the ideas and constructions of linear logic to this more primitive logic where tensorial negation $A \mapsto \neg A$ is not required to be involutive. This research programme bumps against the apparent objection that tensorial logic is an "intuitionistic" logic and thus lacks the so-called "classical" symmetries of linear logic [6, 7]. We show how to circumvent this objection by establishing that the "classical" symmetries of linear logic are derived from a more primitive symmetry of logic which governs both tensorial logic and linear logic. The action of this symmetry

$$\textit{Prover} \leftrightarrow \textit{Denier}$$

is to interchange the two sides *Prover* and *Denier* of a logical dispute. Seen from a 2-categorical point of view, this symmetry internal to logic amounts to the 2-dimensional duality internal to category theory

$$\mathscr{C} \leftrightarrow \mathscr{C}^{op}$$

between a category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. As such, the symmetry is "proto-logical" and works in the same way for classical logic and for intuitionistic logic.

In this introduction, we explain how the symmetry *Prover* $\leftrightarrow$ *Denier* emerged from our attempts to extend the "classical" symmetries of linear logic to an intuitionistic logic like tensorial logic. Since our approach to proof theory is mainly algebraic, we find convenient to start from the notion of $*$-autonomous category which provides the categorical counterpart of

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linear logic. Moreover, in order to work on the most precise and general notion of negation, we focus in this paper on *non commutative* variants of linear logic and tensorial logic. We leave the study of pivotal, braided or symmetric versions of tensorial logic to subsequent papers.

Recall that a non-symmetric $*$-autonomous category is traditionally defined as a biclosed monoidal category $\mathscr{C}$ equipped with a dualizing object $\bot$, see [2] for details. An object $\bot$ of a biclosed monoidal category $\mathscr{C}$ is called *dualizing* when the two canonical morphisms

$$x \longrightarrow \bot \circ\!\!- (x \multimap \bot) \qquad\qquad x \longrightarrow (\bot \circ\!\!- x) \multimap \bot$$

transporting x into its double negation are isomorphisms, for every object x of the dialogue category. The terminology of "dualizing object" comes from the fact that the two negation functors

$$\begin{array}{lccccccc} (-)^\bot & : & x & \mapsto & x \multimap \bot & : & \mathscr{C} & \longrightarrow & \mathscr{C}^{op} \\ {}^\bot(-) & : & x & \mapsto & \bot \circ\!\!- x & : & \mathscr{C}^{op} & \longrightarrow & \mathscr{C} \end{array}$$

define in that case an equivalence

$$\mathscr{C} \;\underset{{}^\bot(-)}{\overset{(-)^\bot}{\rightleftarrows}}\; \mathscr{C}^{op} \qquad \text{(equivalence)} \tag{1}$$

between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. This establishes that every $*$-autonomous category $\mathscr{C}$ is self-dual, in the technical sense that it is equivalent to its opposite category.

This specific definition of $*$-autonomous category is prevailing today. However, one is entitled to complain about the fact that this accepted definition is not sufficiently "symmetric" for the following reason: it starts from the conjunction $\otimes$ and from the two implications $\multimap$ and $\circ\!\!-$ provided by the biclosed monoidal category, rather than from the conjunction $\otimes$ and from the disjunction $\parr$ provided by the traditional presentation of linear logic. From a purely aesthetic point of view, this non-symmetric presentation of a perfectly self-dual logic like linear logic looks awkward. One thus wonders whether the formulation may be replaced by a properly symmetric one. A preliminary step in that direction is to observe is that there are two canonical ways to define the disjunction in a biclosed monoidal category:

$$\begin{array}{lcll} x \parr_1 y & := & \bot \circ\!\!- ((y \multimap \bot) \otimes (x \multimap \bot)) & := \quad {}^\bot(y^\bot \otimes x^\bot) \\ x \parr_2 y & := & ((\bot \circ\!\!- y) \otimes (\bot \circ\!\!- x)) \multimap \bot & := \quad ({}^\bot y \otimes {}^\bot x)^\bot \end{array}$$

and that the two disjunctions $\parr_1$ and $\parr_2$ are isomorphic when the object $\bot$ is dualizing. The reason is that the two canonical morphisms

$$\begin{array}{l} x \circ\!\!- y \longrightarrow (\bot \circ\!\!- (x \multimap \bot)) \circ\!\!- y \xrightarrow{iso} \bot \circ\!\!- (y \otimes (x \multimap \bot)) \\ x \multimap y \longrightarrow x \multimap ((\bot \circ\!\!- y) \multimap \bot) \xrightarrow{iso} ((\bot \circ\!\!- y) \otimes x) \multimap \bot \end{array}$$

are isomorphisms in that case. This enables one to reformulate the disjunctions as:

$$x \parr_1 y \;\cong\; x \circ\!\!- (y \multimap \bot) \qquad\qquad x \parr_2 y \;\cong\; (\bot \circ\!\!- x) \multimap y$$

or even more boldly, as:

$$x \parr_1 y \;\cong\; ((\bot \circ\!\!- x) \multimap \bot) \circ\!\!- (y \multimap \bot) \qquad\qquad x \parr_2 y \;\cong\; (\bot \circ\!\!- x) \multimap (\bot \circ\!\!- (y \multimap \bot))$$

One concludes by applying the isomorphism below (which is valid in any biclosed monoidal category) on the specific instances $u = \bot \circ\!\!- x$ and $v = y \multimap \bot$:

$$(u \multimap \bot) \circ\!\!- v \quad \cong \quad u \multimap (\bot \circ\!\!- v).$$

Since the left and the right implication functors coincide up to isomorphism, the disjunction $\parr$ may be thus defined in a non equivocal way as follows:

$$x \parr y \quad := \quad (\bot \circ\!\!- x) \multimap \bot \circ\!\!- (y \multimap \bot). \tag{2}$$

This discussion leads us to the question of presenting the notion of $*$-autonomous category starting from the symmetric pair of connectives $\otimes$ and $\parr$ instead of the asymmetric pair $\otimes$ and $\multimap$. This question was studied and resolved in a precise and elegant way by Cockett and Seely with their notion of *linearly distributive category*. Recall from [4, 3] that a linearly distributive category is defined as a category $\mathscr{C}$ equipped with two monoidal structures $(\mathscr{C}, \otimes, 1)$ and $(\mathscr{C}, \parr, \bot)$ together with a pair of distributivity laws

$$\begin{array}{lcccc} \kappa^R_{x,y,z} & : & (x \parr y) \otimes z & \longrightarrow & x \parr (y \otimes z) \\ \kappa^L_{x,y,z} & : & x \otimes (y \parr z) & \longrightarrow & (x \otimes y) \parr z \end{array} \tag{3}$$

between the two tensor products, which satisfy a series of coherence diagrams recalled in the paper. The main result established in [4] states (in particular) that a symmetric $*$-autonomous category is the same thing as a symmetric linearly distributive category where every object m comes equipped with a dual object m^* together with two maps

$$\begin{array}{lcccc} \mathbf{AX}[m] & : & \mathbf{1} & \longrightarrow & m^* \parr m \\ \mathbf{CUT}[m] & : & m \otimes m^* & \longrightarrow & \bot \end{array} \tag{4}$$

satisfying a series of well-chosen coherence diagrams, see §2 for details. This alternative presentation of $*$-autonomous categories provides the desired symmetric formulation based on the conjunction $\otimes$ and on the disjunction $\parr$ of linear logic.

A nice aspect of the approach developed by Cockett and Seely is that the notion of linearly distributive category does not require the category $\mathscr{C}$ to be self-dual. Indeed, in their presentation of $*$-autonomous categories, the self-duality of linear logic is entirely decoupled from the connectives $\otimes$ and $\parr$ and comes only at the very last stage, when one equips every object x of the linearly distributive category $\mathscr{C}$ with a dual object x^*. This very last step enforces the category $\mathscr{C}$ to be $*$-autonomous, and thus self-dual, in the technical sense that $\mathscr{C}$ is equivalent to its opposite category $\mathscr{C}^{op}$.

Now, we are ready to turn ourselves to tensorial logic. Looking at it from the functorial point of view, the notion of *dialogue category* provides a categorical counterpart to tensorial logic in exactly the same way as the notion of $*$-autonomous category does for linear logic. So, if one believes in the apparently extravagant idea that tensorial logic is a *refinement* of linear logic — rather than simply an intuitionistic *fragment* of it — then one should be able to present dialogue categories in a similarly symmetric fashion. As we have already pointed out, the project does not seem to make a lot of sense when one looks at it with the spectacles of linear logic. Recall that a dialogue category is a monoidal category equipped with an object $\bot$ and two natural isomorphisms

$$\begin{array}{lcccc} \varphi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(y, x \multimap \bot) \\ \psi_{x,y} & : & \mathscr{C}(x \otimes y, \bot) & \cong & \mathscr{C}(x, \bot \circ\!\!- y) \end{array}$$

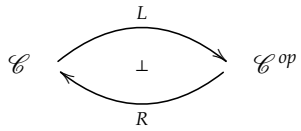
So, if one wants to adapt the notion of linearly distributive category to the case of a dialogue category, one needs

a. to understand how and where the conjunction $\otimes$ and the disjunction $\parr$ should be interpreted in a dialogue category,

b. to decorrelate very carefully the self-duality which will emerge from step a. from the conjunction $\otimes$ and the disjunction $\parr$ of the dialogue category, in the same way as the self-duality $m \mapsto m^*$ is decoupled from the connectives $\otimes$ and $\parr$ in order to get a linearly distributive category from a $*$-autonomous category.

The reader will agree with us that this discussion does not make much sense in the way it is currently formulated... In particular, one does not see (1) how to interpret the disjunction $\parr$ in a dialogue category and (2) what self-duality means since a dialogue category $\mathscr{C}$ has no reason to be equivalent to its opposite category $\mathscr{C}^{op}$.

In order to resolve these difficulties, one needs to look at the situation in a radically different way, following the guideline offered by game semantics and 2-dimensional algebra. The first observation is that every dialogue category comes with an adjunction



(5)

between the two negation functors

$$L \ : \ x \ \mapsto \ \perp \multimap x \qquad\qquad R \ : \ x \ \mapsto \ x \multimap \perp$$

This adjunction induces a monad in the category $\mathscr{C}$

$$T \quad : \quad x \quad \mapsto \quad (\perp \multimap x) \multimap \perp \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

and a comonad in the opposite category $\mathscr{C}^{op}$

$$T' \quad : \quad x \quad \mapsto \quad \perp \multimap (x \multimap \perp) \quad : \quad \mathscr{C}^{op} \quad \longrightarrow \quad \mathscr{C}^{op}$$

which may be also seen as a monad in the original category $\mathscr{C}$. At this stage, it is worth mentioning that a $*$-autonomous category may be alternatively defined as a dialogue category where the adjunction (5) is an equivalence. By equivalence, we mean that the units η and η' of the two double negation monads T and T'

$$\eta_x \ : \ x \ \longrightarrow \ (\perp \multimap x) \multimap \perp \qquad\qquad \eta'_x \ : \ x \ \longrightarrow \ \perp \multimap (x \multimap \perp)$$

are invertible for every object x of the category, and thus, that the object $\perp$ is dualizing. This basic observation shows that the primitive notion of dialogue category is sufficient to define the notion of $*$-autonomous category, and that the more sophisticated notion of biclosed monoidal category may be avoided for that purpose.

At this stage, a tentative solution for problem a. emerges from the contemplation of the adjunction (5) between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. The idea is to interpret the conjunction $\otimes$ as the tensor product taken in the dialogue category $\mathscr{C}$ and the disjunction $\parr$ as the tensor product taken in the opposite dialogue category $\mathscr{C}^{op(0,1)}$. Here, we write $\mathscr{C}^{op(0,1)}$ for the monoidal category $\mathscr{C}$ where the orientation of tensors (dimension 0) and of morphisms (dimension 1) have been reversed. After this properly symmetric but partly speculative resolution of problem a. we are facing two additional and quite serious difficulties:

c. on the one hand, in contrast to what happened in the case of linear logic, the conjunction $\otimes$ and the disjunction $\parr$ do not live in the same category, since the conjunction $\otimes$ lives in the category $\mathscr{C}$ and the disjunction $\parr$ lives in its opposite category,

d. on the other hand, it is difficult to understand in what sense the conjunction $\otimes$ is really different from the disjunction $\parr$ since they are both defined as the tensor product of the category $\mathscr{C}$, and consequently, they only differ modulo the apparently conventional difference between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$.

We will resolve the three difficulties b. c. and d. in three combined steps. The first step is probably the most important one, although it appears at first sight as purely notational. It consists in decoupling the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op(0,1)}$ by writing

$$\mathscr{A} \quad = \quad (\mathscr{A}, \with, \mathit{true})$$

for the monoidal category $\mathscr{C}$ and

$$\mathscr{B} \quad = \quad (\mathscr{B}, \parr, \mathit{false})$$

for its opposite category $\mathscr{C}^{op(0,1)}$. The recipe may not seem very convincing at this stage, but it does resolve problem d. or at least overcomes it, by declaring that the two categories $\mathscr{A}$ and $\mathscr{B}$ are considered *from now on* as intrinsically different. However, the two categories are still related by the identity

$$\mathscr{B}^{op(0,1)} \quad = \quad \mathscr{A} \tag{6}$$

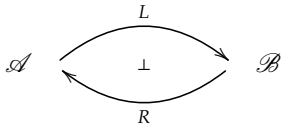
which we will choose to forget (in the case of linearly distributive chiralities) or to remember (in the case of dialogue chiralities) depending on the situation. Accordingly, and for clarity's sake, we choose to write conjunction as $\with$ in the category $\mathscr{A}$ and disjunction as $\parr$ in the category $\mathscr{B}$.

A guiding intuition is that $\mathscr{A}$ denotes a category of proofs or strategies implemented by one side of the logical dispute (typically called Prover) whereas $\mathscr{B}$ denotes a category of anti-proofs or counter-strategies implemented by the other side (typically called Denier). We find convenient to call "polarization" this decoupling in two independent components $\mathscr{A}$ and $\mathscr{B}$ of the original dialogue category $\mathscr{C}$ and of its opposite category $\mathscr{C}^{op(0,1)}$. The terminology pays tribute to what this decorrelation owes to the notion of polarity introduced by Jean-Yves Girard in his seminal work on classical logic [8]. Girard's key idea was that the formulas of classical logic should be polarized in order to interpret every positive formula as a commutative $\otimes$-comonoid and every negative formula as a commutative $\parr$-monoid in the category of coherence spaces. Our own decision of decoupling the category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op(0,1)}$ is inspired by this construction. The novelty of our work compared to the traditional understanding of polarities is (*i*) to reverse the point of view and to take polarities as more primitive than linear logic or tensorial logic themselves (*ii*) to reveal the 2-categorical rather than 1-categorical nature of these polarities and thanks to this new insight (*iii*) to generalize polarities to a large variety of proof systems besides the original system devised by Girard for classical logic. Typically, decoupling a cartesian closed category $\mathscr{C}$ from its opposite category $\mathscr{C}^{op}$ enables one to write the same "classical" decomposition of implication as in linear logic:

$$A \Rightarrow B \quad = \quad A^* \text{ or } B \tag{7}$$

with a notion of polarity different from Girard's table of polarities for classical logic, see [12] for an elaboration of the table and of the formula. The fact that the formula (7) holds in an "intuitionistic" framework like cartesian closed categories establishes it as a fundamental principle of logic itself, not circumscribed anymore to classical logic or to linear logic.

Coming back to our discussion on dialogue categories, the decorrelation of $\mathscr{C}$ and $\mathscr{C}^{op(0,1)}$ enables us to resolve problem c. by recasting the original adjunction (5) as the adjunction



(8)

between the categories $\mathscr{A}$ and $\mathscr{B}$. In that way, the conjunction $\parr$ and the disjunction $\otimes$ live in *different* categories $\mathscr{A}$ and $\mathscr{B}$ related by the functors L and R of the adjunction (8).

The analysis is sufficiently advanced at this stage to provide the ground for a polarized or two-sided refinement of "linearly distributive categories". The key idea is that the linearly distributive category $\mathscr{C}$ together with its two tensor products $\otimes$ and $\parr$ should be replaced by *a pair* of monoidal categories

$$(\mathscr{A}, \parr, \mathit{true}) \qquad\qquad (\mathscr{B}, \otimes, \mathit{false}) \tag{9}$$

equipped with an adjunction (8) relating them. In addition, and there lies a key contribution of the paper, the original pair of distributivity laws (3) should be replaced by the following pair of distributivity laws

$$\begin{array}{lcccc} \kappa^{\parr} & : & R(b \otimes L(a)) \parr m & \longrightarrow & R(b \otimes L(a \parr m)) \\ \kappa^{\otimes} & : & L(a \parr R(b \otimes n)) & \longrightarrow & L(a \parr R(b)) \otimes n \end{array} \tag{10}$$

Although these distributivity laws $\kappa^{\parr}$ and $\kappa^{\otimes}$ may appear intimidating at first sight, we will see that they are the building block of tensorial logic.

Putting (8-9-10) together leads us to the notion of *linearly distributive chirality* introduced in the core of the paper, see §5 for details. The two-sided notion of chirality refines the one-sided notion of linearly distributive category in just the same way as the notion of dialogue category refines the notion of $*$-autonomous category. It is worth mentioning that the notion of chirality we have in mind here is *mixed* rather than *pure* in the sense of [12]. A mixed chirality is a pair of categories in which the side $\mathscr{B}$ is not required to be equivalent to the opposite of the side $\mathscr{A}$ in contrast to the case of a pure chirality where $\mathscr{B} \cong \mathscr{A}^{op}$. The resulting notion of linearly distributive chirality satisfies two cardinal properties:

- every dialogue category $\mathscr{C}$ defines a linearly distributive chirality whose two sides $\mathscr{A}$ and $\mathscr{B}$ are defined as $\mathscr{A} = \mathscr{C}$ and $\mathscr{B} = \mathscr{C}^{op(0,1)}$ and whose functors L and R are defined as negation functors,
- a linearly distributive category is the same thing as a linearly distributive chirality where the two sides $\mathscr{A}$ and $\mathscr{B}$ happen to coincide with the same category $\mathscr{C}$ and where the two functors L and R are defined as the identity functor between the category $\mathscr{C}$ and itself.

In the same way as we like to call "polarization" the process of decoupling $\mathscr{C}$ and its opposite category $\mathscr{C}^{op(0,1)}$, we find clarifying to call "depolarization" the operation of restricting the definition of a linearly distributive chirality to the specific case when:

- the two sides $\mathscr{A}$ and $\mathscr{B}$ coincide,
- and the two functors L and R are defined as identity functors.

The philosophy of tensorial logic is that every concept and construction of linear logic should have a counterpart which refines it in tensorial logic — and from which the original concept may be recovered by depolarizing tensorial logic into linear logic. The notion of *linearly distributive category* offers a typical illustration of this principle, since the notion may be recovered by depolarization from the notion of *linearly distributive chirality* introduced in the paper.

At this stage, it is worth stressing the perfect symmetry between the two distributivity laws $\kappa^{\parr}$ and $\kappa^{\otimes}$. This symmetry is an instance of a general *chirality principle* between proofs and anti-proofs observed in [11]. This symmetry is witnessed in this specific case by the fact that each of the two laws may be obtained from the other one by applying the involution

$$R \;\leftrightarrow\; L \qquad x \parr y \;\leftrightarrow\; x \otimes y \qquad a \;\leftrightarrow\; b \qquad m \;\leftrightarrow\; n$$

and by reversing the orientation of the map. After depolarization, this fundamental symmetry between the two combinators $\kappa^{⅋}$ and $\kappa^{⊗}$ boils down to the *self-symmetry* between the two sides κ^R and κ^L of the original distributivity law (3) for linearly distributive categories. The discovery of this unexpected degeneracy of linear logic was the starting point of tensorial logic, about ten years ago. Indeed, the distinction between $\kappa^{⅋}$ and $\kappa^{⊗}$ in tensorial logic reflects the distinction between Opponent views (generated by the combinator $\kappa^{⅋}$) and Proponent views (generated by the combinator $\kappa^{⊗}$) in innocent strategies, see [11] for details. The discovery that the two combinators $\kappa^{⅋}$ and $\kappa^{⊗}$ of tensorial logic are identified as $\kappa^R = \kappa^L$ after depolarization into linear logic was a clear challenge to the belief that linear logic should be taken as the ultimate horizon of proof-theory.

Now that we have designed a tensorial counterpart to the notion of linearly distributive category, we are very close to the resolution of problem b. Our original problem was to understand how the self-duality of $*$-autonomous categories may be adapted to dialogue categories. The key insight is to take the identity (6) very seriously, and to relax it to a monoidal equivalence

$$\mathscr{A} \;\underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}}\; \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)} \tag{11}$$

between the monoidal categories $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$. Our decision of relaxing the identity (6) into a monoidal equivalence (11) is motivated by the observation that the negation functor of a $*$-autonomous category is not strictly involutive in general, but only involutive up to equivalence. Shifting from an equality to an equivalence also enable us to encapsulate the original notion of *duality* introduced by Cockett and Seely [4, 3] for linearly distributive categories, see §2.3 for details. In this way, we ensure that the notion of duality on a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ captures the two important cases below:

- the case of a dialogue category $\mathscr{C}$ where the equivalence (11) is defined by taking the identity functor $\mathscr{C} \to \mathscr{C}$ between the category $\mathscr{C}$ and itself,
- the case of a $*$-autonomous category $\mathscr{C}$ where the equivalence (11) is defined by taking the negation functors $\mathscr{C} \to \mathscr{C}^{op(0,1)}$ and $\mathscr{C}^{op(0,1)} \to \mathscr{C}$ between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op(0,1)}$.

We have resolved the four obstructions a. b. c. and d. However, in order to complete our program, there still remains a last step to perform: we need to refine (4) and to formulate a variant of duality between an object m in the category $\mathscr{A}$ and an object m^* in the category $\mathscr{B}$. This will lead us to introduce in the course of the paper (§5.3) the notion of *left-handed duality* in linearly distributive dialogue chiralities, which consists of the two combinators

$$\begin{array}{rcl} \mathbf{AX}[m] & : & \mathit{true} \longrightarrow R(m^* ⅋ L(m)) \\ \mathbf{CUT}[m] & : & L(m ⊗ R(m^*)) \longrightarrow \mathit{false} \end{array}$$

together with a series of $2 + 2 + 2 + 2 = 8$ coherence diagrams. As explained in §5.5, one recovers after depolarization the two combinators (4) of a duality in a linearly distributive category $\mathscr{C}$.

After all these efforts, we are ready to state the main technical result of the paper (Prop. 17 in §5.4) which connects the notion of "linearly distributive chirality equipped with a left-handed duality" to the notion of "dialogue category" we started from. In order to make the comparison simpler, the notion of dialogue category is formulated in a two-sided way using the notion of *left-handed dialogue chirality* formulated in a companion paper [12] and recalled in §3.

Main proposition. *Every linearly distributive chirality* $(\mathscr{A}, \mathscr{B})$ *equipped with a left-handed duality defines a left-handed dialogue chirality* $(\mathscr{A}, \mathscr{B})$.

We have shown in [12] that the notion of left-handed dialogue chirality $(\mathscr{A}, \mathscr{B})$ is equivalent to the notion of dialogue category $\mathscr{C}$ in an appropriate 2-categorical sense. Since the face $\mathscr{A}$ of a left-handed dialogue chirality is a dialogue category, the proposition establishes that every linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ equipped with a left-handed duality defines a dialogue category $\mathscr{A}$. We also explain in our conclusion that in the case of *symmetric* dialogue categories and chiralities, the statement can be strengthened (see Prop. 26) into a one-to-one correspondence:

Main proposition (symmetric case). *A symmetric dialogue chirality* $(\mathscr{A}, \mathscr{B})$ *is the same thing as a symmetric linearly distributive chirality* $(\mathscr{A}, \mathscr{B})$ *equipped with a duality.*

This one-to-one correspondence justifies the claim that the notion of "linearly distributive chirality" refines the notion of "linearly distributive category" in the same way as tensorial logic refines linear logic. Although we do not develop it here, a similar one-to-one correspondence theorem holds in the non-commutative case of *pivotal* or *braided* dialogue categories and chiralities, see the companion paper [14] for details.

A brief summary. We advocate that the resulting scenography of logic is more harmonious than the traditional picture with its misleading separation between classical logic and intuitionistic logic. In our picture,

- every category $\mathscr{A} = \mathscr{C}$ of proofs comes with an opposite category $\mathscr{B} = \mathscr{C}^{op(0,1)}$ of anti-proofs (or refutations) which may be entirely decoupled from the original category $\mathscr{C}$,
- every formula A of the logic may be either seen from the point of view of Prover or from the point of view of Denier ; these two points of view are related by a change of reference frame $A \mapsto A^*$ between Prover in $\mathscr{A}$ and Denier in $\mathscr{B}$.

One main benefit of the understanding of logic inspired by the dialogical interpretation of proofs and programs is that the change of reference frame $A \mapsto A^*$ between Prover and Denier remains involutive in the case of an intuitionistic system like tensorial logic.

Beside our debt to Girard's notion of polarity, we want to stress what this alternative picture of logic owes to the technical shift from 1-dimensional to 2-dimensional categories. The higher-dimensional point of view was indeed necessary in order to understand that the primitive symmetry of logic between Prover and Denier lives at the 2-dimensional level, rather than at the more traditional 1-dimensional level. This idea is supported by a microcosm principle adapted in [12] from a similar microcosm principle formulated by Baez and Dolan [1] for higher dimensional algebra. In the case of monoidal categories, the microcosm principle tells that any contravariant operation like negation or implication relies on the existence of a symmetry

$$op(0,1) \quad : \quad \mathscr{C} \mapsto \mathscr{C}^{op(0,1)} \quad : \quad LaxMonCat \quad \longrightarrow \quad OpLaxMonCat^{op(0,2)}$$

between the cartesian 2-category *LaxMonCat* of monoidal categories and lax monoidal functors and the cartesian 2-category *OpLaxMonCat* constructed with oplax monoidal functors. In other words, the primary duality of logic *Prover* $\leftrightarrow$ *Denier* is implemented by the 2-dimensional operation of "reversing" a category $\mathscr{C}$ of proofs into its opposite category $\mathscr{C}^{op(0,1)}$ of anti-proofs. We like to think that this purely formal and algebraic observation reflects the phylogenetic principle that *symmetry comes before logic* [1].

[1] An intellectual challenge for logicians today is to develop a sufficiently fine-grained understanding (or anatomy) of formal logic in order to articulate it with the investigations on the evolutionary mechanisms which lead to the emergence of logical or prelogical reasoning in our human societies.

Plan of the paper. We start by recalling in §2 the notion of linearly distributive category together with the notion of duality we will be specifically interested in. We recall in §3 the notion of left-handed dialogue chirality introduced in [12] as a two-sided and symmetric notion of dialogue category. The section includes (see §3.5) a discussion on the integration of "time" in the traditional proof-nets of linear logic. The notion of bimodulation between modulations (also called lax actions or parametric monads) is then recalled in §4. We reach in §5 the technical core of the paper, where we introduce the notion of linearly distributive chirality, and formulate the corresponding notion of left-handed duality. We establish in §5 the main result of the paper (Prop. 17) which is stated there in the framework of non-commutative tensoria logic. We conclude in §6 by observing that a mismatch remains between the two notions of dialogue chirality and of linearly distributive chirality in the case of non-symmetric monoidal categories. We leave the task of resolving this matter to the companion paper devoted to pivotal negation [14]. However, in order to settle the matter in the symmetric case, we establish at the end of the section that the notion of "dialogue chirality" coincides with the notion of "linearly distributive chirality with a duality" when the underlying monoidal categories $\mathscr{A}$ and $\mathscr{B}$ happen to be symmetric. This correspondence also holds in the pivotal or braided cases.

2 Linearly distributive categories

The purpose of the section is to recall the main definitions and results appearing in the original work by Cockett and Seely [4]. We start by recalling the notion of linearly distributive category in §2.1 and then introduce the notion of exact pairing in §2.2. After recalling in §2.3 the definition of "duality" in that setting, we state in §2.4 the main result of Cockett and Seely: a linearly distributive category equipped with a duality is the same thing as a (non symmetric) $*$-autonomous category. In order to prepare the later comparison in §5 with linearly distributive chiralities and left-handed dualities, we introduce in §2.5 the category $\mathscr{C}^{\mathbf{ep}}$ of exact pairings of a linearly distributive category $\mathscr{C}$, and deliver in §2.6 a number of alternative formulations of the notion of "duality" in a linearly distributive category.

2.1 Definition

A linearly distributive category $\mathscr{C}$ is defined in [4, 3] as a category equipped with two monoidal structures, the first one called "tensor product" given by the bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit $\mathbf{1}$ and natural isomorphisms

$$\alpha^{\otimes}_{x,y,z} : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$$

$$\lambda^{\otimes}_{x} : \mathbf{1} \otimes x \longrightarrow x \qquad \rho^{\otimes}_{x} : x \otimes \mathbf{1} \longrightarrow x$$

and the second one called "cotensor product" given by the bifunctor $⅋ : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ with unit $\bot$ and natural isomorphisms

$$\alpha^{⅋}_{x,y,z} : (x ⅋ y) ⅋ z \longrightarrow x ⅋ (y ⅋ z)$$

$$\lambda^{⅋}_{x} : \bot ⅋ x \longrightarrow x \qquad \rho^{⅋}_{x} : x ⅋ \bot \longrightarrow x.$$

Linearly distributive categories are moreover equipped with two natural morphisms

$$\kappa^{L}_{x,y,z} : x \otimes (y ⅋ z) \longrightarrow (x \otimes y) ⅋ z$$

$$\kappa^{R}_{x,y,z} : (x ⅋ y) \otimes z \longrightarrow x ⅋ (y \otimes z)$$

required to satisfy a series of commutativity axioms, consisting of six pentagons and four triangles. The six pentagons may be separated into three series of two pentagons. The first series describes how the distributive law κ^R interacts with the associativity laws:

$$\begin{array}{ccccc}
((w \parr x) \otimes y) \otimes z & \xrightarrow{\kappa^R} & (w \parr (x \otimes y)) \otimes z & \xrightarrow{\kappa^R} & w \parr ((x \otimes y) \otimes z) \\
\downarrow{\scriptstyle \alpha^\otimes} & & & & \downarrow{\scriptstyle \alpha^\otimes} \\
(w \parr x) \otimes (y \otimes z) & & \xrightarrow{\kappa^R} & & w \parr (x \otimes (y \otimes z))
\end{array}$$

$$\begin{array}{ccccc}
((w \parr x) \parr y) \otimes z & & \xrightarrow{\kappa^R} & & (w \parr x) \parr (y \otimes z) \\
\downarrow{\scriptstyle \alpha^\parr} & & & & \downarrow{\scriptstyle \alpha^\parr} \\
(w \parr (x \parr y)) \otimes z & \xrightarrow{\kappa^R} & w \parr ((x \parr y) \otimes z) & \xrightarrow{\kappa^R} & w \parr (x \parr (y \otimes z))
\end{array}$$

the second series describes how the distributive law κ^L interacts with the associativity laws:

$$\begin{array}{ccccc}
(w \otimes x) \otimes (y \parr z) & & \xrightarrow{\kappa^L} & & ((w \otimes x) \otimes y) \parr z \\
\downarrow{\scriptstyle \alpha^\otimes} & & & & \downarrow{\scriptstyle \alpha^\otimes} \\
w \otimes (x \otimes (y \parr z)) & \xrightarrow{\kappa^L} & w \otimes ((x \otimes y) \parr z) & \xrightarrow{\kappa^L} & (w \otimes (x \otimes y)) \parr z
\end{array}$$

$$\begin{array}{ccccc}
w \otimes ((x \parr y) \parr z) & \xrightarrow{\kappa^L} & (w \otimes (x \parr y)) \parr z & \xrightarrow{\kappa^L} & ((w \otimes x) \parr y) \parr z \\
\downarrow{\scriptstyle \alpha^\parr} & & & & \downarrow{\scriptstyle \alpha^\parr} \\
w \otimes (x \parr (y \parr z)) & & \xrightarrow{\kappa^L} & & (w \otimes x) \parr (y \parr z)
\end{array}$$

and the last series describes how κ^L and κ^R interact together with the associativity laws:

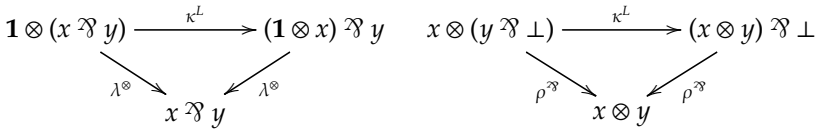
$$\begin{array}{ccccc}
(w \otimes (x \parr y)) \otimes z & & \xrightarrow{\kappa^L} & & ((w \otimes x) \parr y) \otimes z \\
\downarrow{\scriptstyle \alpha^\otimes} & & & & \downarrow{\scriptstyle \kappa^R} \\
w \otimes ((x \parr y) \otimes z) & \xrightarrow{\kappa^R} & w \otimes (x \parr (y \otimes z)) & \xrightarrow{\kappa^L} & (w \otimes x) \parr (y \otimes z)
\end{array}$$

$$\begin{array}{ccccc}
(w \parr x) \otimes (y \parr z) & \xrightarrow{\kappa^L} & ((w \parr x) \otimes y) \parr z & \xrightarrow{\kappa^R} & (w \parr (x \otimes y)) \parr z \\
\downarrow{\scriptstyle \kappa^R} & & & & \downarrow{\scriptstyle \alpha^\parr} \\
w \parr (x \otimes (y \parr z)) & & \xrightarrow{\kappa^L} & & w \parr ((x \otimes y) \parr z)
\end{array}$$

Similarly, the four coherence triangles may be separated in two series of two triangles. The first series describes how the distributive law κ^R interacts with the units:

$$\begin{array}{ccc}
(\bot \parr x) \otimes y & \xrightarrow{\kappa^R} & \bot \parr (x \otimes y) \\
& {\scriptstyle \lambda^\parr} \searrow \quad \swarrow {\scriptstyle \lambda^\parr} & \\
& x \otimes y &
\end{array}
\qquad
\begin{array}{ccc}
(x \parr y) \otimes \mathbf{1} & \xrightarrow{\kappa^R} & x \parr (y \otimes \mathbf{1}) \\
& {\scriptstyle \rho^\otimes} \searrow \quad \swarrow {\scriptstyle \rho^\otimes} & \\
& x \parr y &
\end{array}$$

and the second series describes how the distributive law κ^L interacts with the units:



One requires as expected that these diagrams commute for all objects w, x, y, z of the linearly distributive category $\mathscr{C}$.

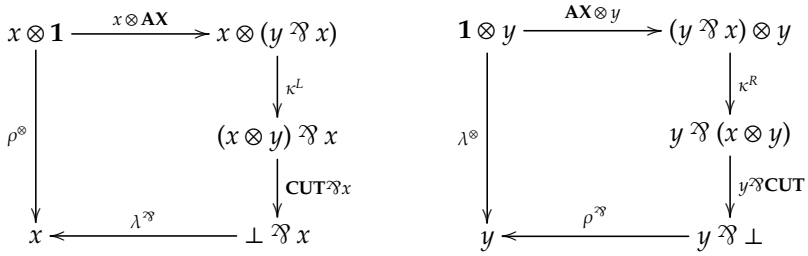
2.2 Exact pairings in linearly distributive categories

Before defining the notion of duality in a linearly distributive category, we find useful to recall the notion of *exact pairing* originally formulated by Egger [5] under the name of *linear adjunction*.

Definition 1 (exact pairing) *An exact pairing $x \dashv y$ in a linearly distributive category $\mathscr{C}$ consists of a pair (x, y) of objects of the category $\mathscr{C}$ together with two morphisms*

$$\mathbf{AX} : \mathbf{1} \longrightarrow y \parr x \qquad \mathbf{CUT} : x \otimes y \longrightarrow \bot$$

making the following diagrams commute:



(12)

In that case, one says that the object y is a left dual to the object x, and conversely, that the object x is a right dual to the object y.

This definition is motivated by the following observation:

Proposition 1 *In a linearly distributive category $\mathscr{C}$, every exact pairing $x \dashv y$ induces an adjunction*

$$(x \otimes -) \quad \dashv \quad (y \parr -) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

where the functor $(x \otimes -)$ is left adjoint to the functor $(y \parr -)$, and an adjunction

$$(- \otimes y) \quad \dashv \quad (- \parr x) \quad : \quad \mathscr{C} \quad \longrightarrow \quad \mathscr{C}$$

where the functor $(- \parr x)$ is right adjoint to the functor $(- \otimes y)$.

The proof of the first statement is essentially immediate. One associates to every morphism

$$f \quad : \quad x \otimes w \quad \longrightarrow \quad z$$

the morphism $\varphi_{x,w,z}(f)$ defined as the composite

$$w \xrightarrow{(\lambda^{\otimes})^{-1}} \mathbf{1} \otimes y \xrightarrow{\mathbf{AX}} (y \parr x) \otimes w \xrightarrow{\kappa^R} y \parr (x \otimes w) \xrightarrow{f} y \parr z$$

Conversely, one associates to every morphism

$$g \quad : \quad w \quad \longrightarrow \quad y \parr z$$

the morphism $\psi_{x,w,z}(g)$ defined as the composite

$$x \otimes w \xrightarrow{\ g\ } x \otimes (y \parr z) \xrightarrow{\ \kappa^L\ } (x \otimes y) \parr z \xrightarrow{\ \mathbf{CUT}\ } \bot \parr z \xrightarrow{\ \rho^{\parr}\ } z$$

The coherence properties of the linearly distributive category $\mathscr{C}$ and of the exact pairing $x \dashv y$ imply together that the correspondence is one-to-one. This establishes the property. Note that the unit and counit of the adjunction $(x \otimes -) \dashv (y \parr -)$ are natural transformations defined at instance z as the unique morphisms **axiom** and **cut** making the diagrams commute:

$$\begin{array}{ccc} \mathbf{1} \otimes z & \xrightarrow{\ \mathbf{AX} \otimes z\ } & (y \parr x) \otimes z \\ {\scriptstyle \lambda^{\otimes}}\downarrow & & \downarrow{\scriptstyle \kappa^R} \\ z & \xrightarrow{\ \mathbf{axiom}\ } & y \parr (x \otimes z) \end{array} \qquad \begin{array}{ccc} (x \otimes y) \parr z & \xrightarrow{\ \mathbf{CUT} \parr z\ } & \bot \parr z \\ {\scriptstyle \kappa^L}\uparrow & & \downarrow{\scriptstyle \rho^{\parr}} \\ x \otimes (y \parr z) & \xrightarrow{\ \mathbf{cut}\ } & z \end{array}$$

The proof of the second statement is similar. The unit and counit of the adjunction $(- \otimes y) \dashv (- \parr x)$ are in that case defined at instance z as the unique morphisms making the diagrams commute:

$$\begin{array}{ccc} z \otimes \mathbf{1} & \xrightarrow{\ z \otimes \mathbf{AX}\ } & z \otimes (y \parr x) \\ {\scriptstyle \rho^{\otimes}}\downarrow & & \downarrow{\scriptstyle \kappa^L} \\ z & \longrightarrow & (z \otimes y) \parr x \end{array} \qquad \begin{array}{ccc} z \otimes (x \otimes y) & \xrightarrow{\ z \otimes \mathbf{CUT}\ } & z \otimes \mathbf{1} \\ {\scriptstyle \kappa^R}\uparrow & & \downarrow{\scriptstyle \lambda^{\otimes}} \\ (z \parr x) \otimes y & \longrightarrow & z \end{array}$$

It is worth noting that the definitions require both distributivity laws κ^L and κ^R. A similar phenomenon will reappear in §5 when we lift the notion of exact pairing $x \dashv y$ to linearly distributive chiralities $(\mathscr{A}, \mathscr{B})$.

2.3 Dualities in linearly distributive categories

A general notion of duality in a linearly distributive category $\mathscr{C}$ is introduced in [4]. We find convenient to recall this notion for a later comparison with our notion of duality in a linearly distributive chirality.

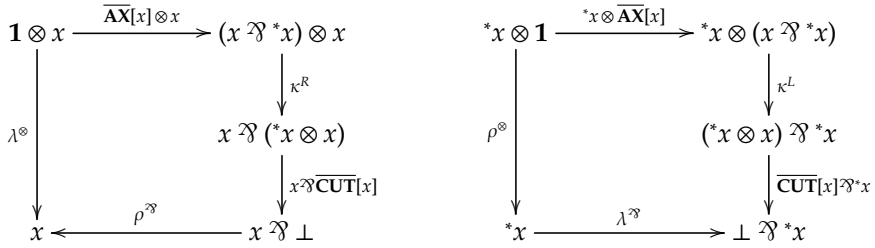
Definition 2 (duality) *A duality in a linearly distributive category $\mathscr{C}$ consists of the following data:*

- *an object x^* and an object ${}^*x$,*
- *four morphisms*

$$\begin{array}{ll} \mathbf{AX}[x] : \mathbf{1} \longrightarrow x^* \parr x & \mathbf{CUT}[x] : x \otimes x^* \longrightarrow \bot \\ \overline{\mathbf{AX}}[x] : \mathbf{1} \longrightarrow x \parr {}^*x & \overline{\mathbf{CUT}}[x] : {}^*x \otimes x \longrightarrow \bot \end{array}$$

for every object x of the category $\mathscr{C}$. The morphisms are moreover required to make the diagrams below commute:

$$\begin{array}{ccc} x \otimes \mathbf{1} & \xrightarrow{\ x \otimes \mathbf{AX}[x]\ } & x \otimes (x^* \parr x) \\ & & \downarrow{\scriptstyle \kappa^L} \\ {\scriptstyle \rho^{\otimes}}\downarrow & & (x \otimes x^*) \parr x \\ & & \downarrow{\scriptstyle \mathbf{CUT}[x] \parr x} \\ x & \xrightarrow{\ \lambda^{\parr}\ } & \bot \parr x \end{array} \qquad \begin{array}{ccc} \mathbf{1} \otimes x^* & \xrightarrow{\ \mathbf{AX}[x] \otimes x^*\ } & (x^* \parr x) \otimes x^* \\ & & \downarrow{\scriptstyle \kappa^R} \\ {\scriptstyle \lambda^{\otimes}}\downarrow & & x^* \parr (x \otimes x^*) \\ & & \downarrow{\scriptstyle x^* \parr \mathbf{CUT}[x]} \\ x^* & \xleftarrow{\ \rho^{\parr}\ } & x^* \parr \bot \end{array} \qquad (13)$$



(14)

In other words, a duality in a linearly distributive category $\mathscr{C}$ is the same thing as the data of an exact pairing $x \dashv x^*$ and of an exact pairing ${}^*x \dashv x$, for every object x of the category $\mathscr{C}$.

2.4 Main statement of the section

As preparation to the next statement, we first show that

Proposition 2 *In any linearly distributive category $\mathscr{C}$ with a duality,*

- *the functor $(x \otimes -)$ is left adjoint to the functor $(x^* \parr -)$*
- *the functor $(- \parr y)$ is right adjoint to the functor $(- \otimes y^*)$*
- *the functor $(- \otimes x)$ is left adjoint to the functor $(- \parr {}^*x)$*
- *the functor $(y \parr -)$ is right adjoint to the functor $({}^*y \otimes -)$*

for all objects x, y of the category $\mathscr{C}$. In particular, every such linearly distributive category $\mathscr{C}$ is biclosed monoidal, with left and right implications defined as

$$x \multimap y = x^* \parr y \qquad\qquad x \multimap y = y \parr {}^*x.$$

The property is a direct consequence of Proposition 1 and of Definition 2. At this stage, one is ready to state the main result established by Cockett and Seely [4] which justifies the notion of duality introduced in §2.3

Proposition 3 (Cockett-Seely) *The notion of non-symmetric $*$-autonomous category coincides with the notion of linearly distributive category equipped with a duality.*

In particular, every linearly distributive category $\mathscr{C}$ equipped with a duality is a $*$-autonomous category with left and right negation functors defined using the duality, what one writes:

$$(-)^\perp = x \multimap \perp := x^* \qquad\qquad {}^\perp(-) = \perp \multimap x := {}^*x \tag{15}$$

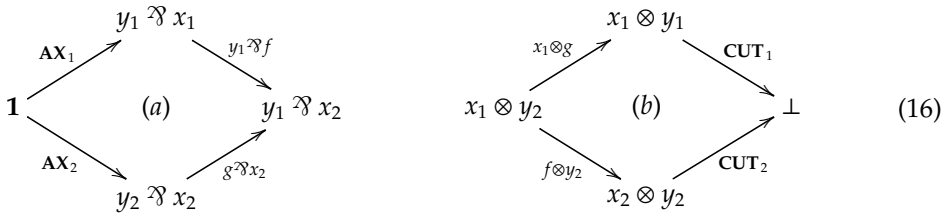
using the notations of the introduction, see Diagram (1).

2.5 The category of exact pairings

This section and the next one (§2.6) are essentially technical. Their main purpose is to prepare the later comparison with the notion of "left-handed duality" in linearly distributive chiralities. Suppose given a linearly distributive category $\mathscr{C}$ and two exact pairings $x_1 \dashv y_1$ and $x_2 \dashv y_2$. We start by studying when a pair of morphisms

$$f : x_1 \longrightarrow x_2 \qquad\qquad g : y_2 \longrightarrow y_1$$

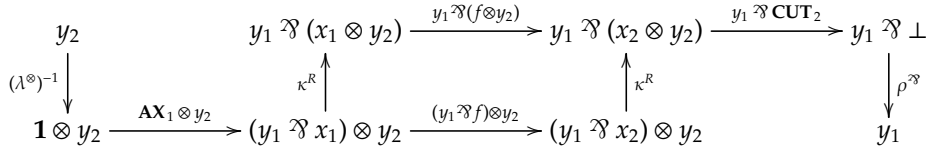
make the two diagrams below commute:



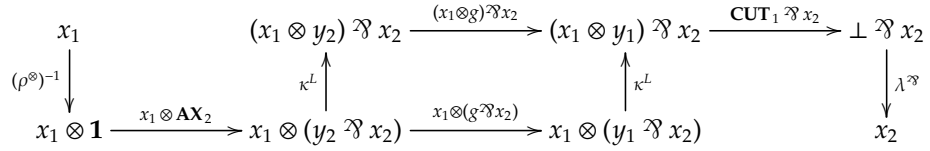
(16)

Proposition 4 *Suppose given two exact pairings $x_1 \dashv y_1$ and $x_2 \dashv y_2$ and two morphisms $f : x_1 \to x_2$ and $g : y_2 \to y_1$ in a linearly distributive category $\mathscr{C}$. The following statements are equivalent:*

- *the diagram* (16 *a*) *commutes,*
- *the diagram* (16 *b*) *commutes,*
- *the morphism $g : y_2 \to y_1$ is equal to the composite morphism*



- *the morphism $f : x_1 \to y_2$ is equal to the composite morphism*



This observation leads us to introduce the category $\mathscr{C}^{\mathbf{ep}}$ whose objects are the exact pairings $x \dashv y$ of the linearly distributive category $\mathscr{C}$, and whose morphisms

$$(f, g) \quad : \quad (x_1 \dashv y_1) \quad \longrightarrow \quad (x_2 \dashv y_2)$$

are the pairs of morphisms $f : x_1 \to x_2$ and $g : y_2 \to y_1$ satisfying the equivalent statements of Prop. 4. The category $\mathscr{C}^{\mathbf{ep}}$ comes together with projection functors

$$\begin{array}{rccc} \pi_{\text{left}} & : & \mathscr{C}^{\mathbf{ep}} & \longrightarrow & \mathscr{C} \\ & & x \dashv y & \mapsto & x \end{array} \qquad\qquad \begin{array}{rccc} \pi_{\text{right}} & : & \mathscr{C}^{\mathbf{ep}} & \longrightarrow & \mathscr{C}^{op} \\ & & x \dashv y & \mapsto & y \end{array} \tag{17}$$

which transport every exact pairing $x \dashv y$ to its left dual x and to its right dual y, respectively. It follows from Proposition 4 that the two functors are fully faithful. Indeed, given two exact pairings $x_1 \dashv y_1$ and $x_2 \dashv y_2$, every morphism $f : x_1 \to x_2$ determines a unique morphism $g : y_2 \to y_1$ such that (f, g) is a morphism of the category $\mathscr{C}^{\mathbf{ep}}$; and symmetrically, every morphism $g : y_2 \to y_1$ determines a unique morphism $f : x_1 \to x_2$ such that (f, g) is a morphism of the category $\mathscr{C}^{\mathbf{ep}}$.

We make the two complementary observations in any linearly distributive category:

Proposition 5 *Every pair of exact pairings $x_1 \dashv y_1$ and $x_2 \dashv y_2$ induces an exact pairing*

$$x_1 \otimes x_2 \;\dashv\; y_2 \parr y_1$$

with structure morphisms

$$\mathbf{AX} : \mathbf{1} \longrightarrow (y_2 \parr y_1) \parr (x_1 \otimes x_2) \qquad\qquad \mathbf{CUT} : (x_1 \otimes x_2) \otimes (y_2 \parr y_1) \longrightarrow \bot$$

defined as the composite morphisms

$$\mathbf{1} \xrightarrow{\mathbf{AX}_2} y_2 \parr x_2 \xrightarrow{(\lambda^{\otimes})^{-1}} y_2 \parr (\mathbf{1} \otimes x_2) \xrightarrow{\mathbf{AX}_1} y_2 \parr ((y_1 \parr x_1) \otimes x_2) \xrightarrow{(a^{\parr})^{-1} \circ \kappa^R} (y_2 \parr y_1) \parr (x_1 \otimes x_2)$$

$$(x_1 \otimes x_2) \otimes (y_2 \parr y_1) \xrightarrow{\kappa^L \circ a^{\otimes}} x_1 \otimes ((x_2 \otimes y_2) \parr y_1) \xrightarrow{\mathbf{CUT}_2} x_1 \otimes (\bot \parr y_1) \xrightarrow{\lambda^{\parr}} x_1 \otimes y_1 \xrightarrow{\mathbf{CUT}_1} \bot.$$

Proposition 6 *Every pair of exact pairings $x_1 \dashv y_1$ and $x_2 \dashv y_2$ induces an exact pairing*

$$x_1 \parr x_2 \;\dashv\; y_2 \otimes y_1$$

with structure morphisms

$$\mathbf{AX} : \mathbf{1} \longrightarrow (y_2 \otimes y_1) \parr (x_1 \parr x_2) \qquad\qquad \mathbf{CUT} : (x_1 \parr x_2) \otimes (y_2 \otimes y_1) \longrightarrow \bot$$

defined as the composite morphisms

$$\mathbf{1} \xrightarrow{\mathbf{AX}_2} y_2 \parr x_2 \xrightarrow{(\rho^{\otimes})^{-1}} (y_2 \otimes \mathbf{1}) \parr x_2 \xrightarrow{\mathbf{AX}_1} (y_2 \otimes (y_1 \parr x_1)) \parr x_2 \xrightarrow{a^{\parr} \circ \kappa^L} (y_2 \otimes y_1) \parr (x_1 \parr x_2)$$

$$(x_1 \parr x_2) \otimes (y_2 \otimes y_1) \xrightarrow{\kappa^L \circ (a^{\otimes})^{-1}} (x_1 \parr (x_2 \otimes y_2)) \otimes y_1 \xrightarrow{\mathbf{CUT}_2} (x_1 \parr \bot) \parr y_1 \xrightarrow{\rho^{\parr}} x_1 \otimes y_1 \xrightarrow{\mathbf{CUT}_1} \bot.$$

Starting from these preliminary observations, one establishes that

Proposition 7 *For every linearly distributive category $\mathscr{C}$, the category $\mathscr{C}^{\mathbf{ep}}$ is linearly distributive.*

The tensor product ($\otimes$) and cotensor product ($\parr$) of the category $\mathscr{C}^{\mathbf{ep}}$ are defined using Propositions 5 and 6, and its tensor unit ($\mathbf{1}$) and cotensor unit ($\bot$) are respectively defined as the obvious exact pairings $\mathbf{1} \dashv \bot$ and $\bot \dashv \mathbf{1}$ in the category $\mathscr{C}$.

2.6 Alternative definitions of duality

Here, we take advantage of the technical results established in §2.5 to reformulate the notion of "duality" defined in Definition 2 in a more conceptual and functorial way:

Proposition 8 *A duality in a linearly distributive category $\mathscr{C}$ is alternatively defined as a pair of functors*

$$\mathit{rightdual} \;:\; \mathscr{C} \longrightarrow \mathscr{C}^{\mathbf{ep}} \qquad\qquad \mathit{leftdual} \;:\; \mathscr{C}^{op} \longrightarrow \mathscr{C}^{\mathbf{ep}}$$

defining sections of the projection functors (17) in the sense that

$$\pi_{\mathrm{left}} \circ \mathit{rightdual} : \mathscr{C} \to \mathscr{C} \qquad\qquad \pi_{\mathrm{right}} \circ \mathit{leftdual} : \mathscr{C}^{op} \to \mathscr{C}^{op}$$

are the identity functors.

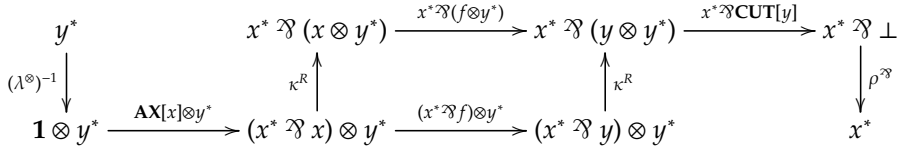
Every duality in a linearly distributive category thus defines a contravariant functor

$$(-)^* \;=\; x \mapsto x^* \;:\; \mathscr{C} \longrightarrow \mathscr{C}^{op}$$

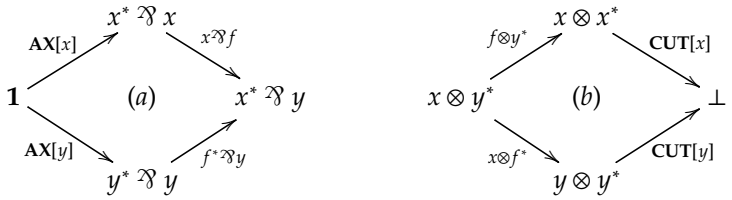
defined as the composite

$$(-)^* \;=\; \pi_{\mathrm{right}} \circ \mathit{rightdual} \;:\; \mathscr{C} \longrightarrow \mathscr{C}^{\mathbf{ep}} \longrightarrow \mathscr{C}^{op}. \tag{18}$$

Expounding the definition, the functor $(-)^*$ transports every morphism $f : x \to y$ to the morphism $f^* : y^* \to x^*$ defined as the composite morphism:



Note that the definition of $(-)^*$ and the fact that it factors through $\mathscr{C}^{\mathbf{ep}}$ ensures that the diagrams below commute



(19)

for every morphism $f : x \to y$ of the category $\mathscr{C}$. The two families $\mathbf{AX}[-]$ and $\mathbf{CUT}[-]$ are thus natural (or dinatural). In the same way, every duality defines a contravariant functor

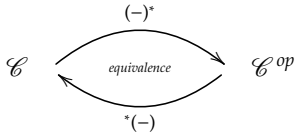
$$ {}^*(-) \quad = \quad x \mapsto {}^*x \quad : \quad \mathscr{C}^{op} \longrightarrow \mathscr{C} $$

defined as the composite

$$ {}^*(-) \quad = \quad \pi_{\text{left}} \circ \mathit{leftdual} \quad : \quad \mathscr{C}^{op} \longrightarrow \mathscr{C}^{\mathbf{ep}} \longrightarrow \mathscr{C}. \tag{20} $$

Moreover, the two resulting functors $(-)^*$ and ${}^*(-)$ define an equivalence between the category $\mathscr{C}$ and its opposite category $\mathscr{C}^{op}$. This observation leads us to formulate the notion of "duality" in a linearly distributive category $\mathscr{C}$ in the following equivalent way:

Proposition 9 *A duality in a linearly distributive category $\mathscr{C}$ is alternatively defined as an equivalence*



(21)

together with two families of morphisms

$$ \mathbf{AX}[x] : \mathbf{1} \longrightarrow x^* \parr x \qquad\qquad \mathbf{CUT}[x] : x \otimes x^* \longrightarrow \bot $$

making the diagrams (13) and (19) commute for all object x, y and all morphisms $f : x \to y$ of the category $\mathscr{C}$.

The first requirement that the diagrams (13) commute ensures that the family of morphisms $\mathbf{AX}[x]$ and $\mathbf{CUT}[x]$ define a section $\mathit{rightdual} : \mathscr{C} \to \mathscr{C}^{\mathbf{ep}}$ of the functor π_{left}. The second requirement that the diagrams (19) commute is then equivalent to requiring that the functor $(-)^*$ factors as described in (18). So, given a duality formulated in this way (Prop. 9), one recovers the original notion of duality (Def. 2) in the following way: for every object $x \in \mathscr{C}$, one defines the exact pairing ${}^*x \dashv x$ as the exact pairing ${}^*x \dashv ({}^*x)^*$ associated to the object ${}^*x$, combined with the isomorphism in the category $\mathscr{C}$ between x and $({}^*x)^*$ induced by the equivalence (21) between $\mathscr{C}$ and $\mathscr{C}^{op}$. This provides a section $\mathit{leftdual} : \mathscr{C}^{op} \to \mathscr{C}^{\mathbf{ep}}$ of the functor π_{right} such that the functor ${}^*(-)$ factors as described in (20).

We conclude the section by formulating the notion of duality in yet another way. To that purpose, we start from the observation that the projection functors

$$\pi_{\text{left}} \;:\; (\mathscr{C}^{\,\mathbf{ep}}, \otimes, \mathbf{1}) \longrightarrow (\mathscr{C}, \otimes, \mathbf{1}) \qquad\qquad \pi_{\text{right}} \;:\; (\mathscr{C}^{\,\mathbf{ep}}, \otimes, \mathbf{1}) \longrightarrow (\mathscr{C}, \parr, \bot)^{op(0,1)}$$

become monoidal functors, in the strict sense, when one considers $\mathscr{C}^{\,\mathbf{ep}}$ as a monoidal category equipped with the tensor product $\otimes$ defined in Propositions 5 and 7. This leads us to reformulate Proposition 8 in this monoidal spirit:

Proposition 10 *A duality in a linearly distributive category $\mathscr{C}$ is alternatively defined as a pair of monoidal functors*

$$\mathit{rightdual} \;:\; (\mathscr{C}, \otimes, \mathbf{1}) \longrightarrow (\mathscr{C}^{\,\mathbf{ep}}, \otimes, \mathbf{1}) \qquad\qquad \mathit{leftdual} \;:\; (\mathscr{C}, \parr, \bot)^{op(0,1)} \longrightarrow (\mathscr{C}^{\,\mathbf{ep}}, \otimes, \mathbf{1})$$

defining sections (in the monoidal sense) of the monoidal projection functors (17).

In this definition, the functors *rightdual* and *leftdual* are not expected to be monoidal in the strict sense, but only up to isomorphic coercion maps

$$\mathit{rightdual}(x) \otimes \mathit{rightdual}(y) \;\longrightarrow\; \mathit{rightdual}(x \otimes y) \qquad\qquad \mathbf{1} \;\longrightarrow\; \mathit{rightdual}(\mathbf{1})$$

natural in $x, y \in \mathscr{C}$. Moreover, because *rightdual* and *leftdual* are required to be sections in the monoidal sense, these coercion maps are inverse to morphisms of the form $(id_{x\otimes y}, m_{x,y})$ and $(id_{\mathbf{1}}, m_{\mathbf{1}})$ for a family of isomorphisms

$$m_{x,y} \;:\; y^* \parr x^* \;\longrightarrow\; (x \otimes y)^* \qquad\qquad m_{\mathbf{1}} \;:\; \bot \;\longrightarrow\; \mathbf{1} \tag{22}$$

natural in $x, y \in \mathscr{C}$. The coercion maps $m_{x,y}$ and $m_{\mathbf{1}}$ result from the uniqueness (up to isomorphism) of a right dual to the objects $x \otimes y$ and $\mathbf{1}$, for all objects $x, y \in \mathscr{C}$. They are characterized by the fact that they make the diagrams below commute:

$$\begin{array}{ccccc}
y^* \parr (\mathbf{1} \otimes y) & \xrightarrow{\mathbf{AX}[x]} & y^* \parr ((x^* \parr x) \otimes y) & \xrightarrow{\kappa^R} & y^* \parr (x^* \parr (x \otimes y)) \\
{\scriptstyle (\lambda^{\otimes})^{-1}}\uparrow & & & & \downarrow{\scriptstyle (\alpha^{\parr})^{-1}} \\
y^* \parr y & & & & (y^* \parr x^*) \parr (x \otimes y) \\
{\scriptstyle \mathbf{AX}[y]}\uparrow & & & & \downarrow{\scriptstyle m_{x,y}} \\
\mathbf{1} & & \xrightarrow{\qquad\mathbf{AX}[x\otimes y]\qquad} & & (x \otimes y)^* \parr (x \otimes y)
\end{array} \tag{23}$$

$$\begin{array}{ccccc}
x \otimes (y \otimes (y^* \parr x^*)) & \xrightarrow{\kappa^L} & x \otimes ((y \otimes y^*) \parr x^*) & \xrightarrow{\mathbf{CUT}[y]} & x \otimes (\bot \parr x^*) \\
{\scriptstyle \alpha^{\otimes}}\uparrow & & & & \downarrow{\scriptstyle \lambda^{\parr}} \\
(x \otimes y) \otimes (y^* \parr x^*) & & & & x \otimes x^* \\
{\scriptstyle m_{x,y}^{-1}}\uparrow & & & & \downarrow{\scriptstyle \mathbf{CUT}[x]} \\
(x \otimes y) \otimes (x \otimes y)^* & & \xrightarrow{\qquad\mathbf{CUT}[x\otimes y]\qquad} & & \bot
\end{array} \tag{24}$$

$$\begin{array}{ccc}
\mathbf{1}^* \parr \mathbf{1} & \xrightarrow{m_{\mathbf{1}}} & \bot \parr \mathbf{1} \\
{\scriptstyle \mathbf{AX}[\mathbf{1}]}\uparrow & & \downarrow{\scriptstyle \lambda^{\parr}} \\
\mathbf{1} & \xrightarrow{id} & \mathbf{1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathbf{1} \otimes \bot & \xrightarrow{m_{\mathbf{1}}} & \mathbf{1} \otimes \mathbf{1}^* \\
{\scriptstyle \lambda^{\otimes}}\downarrow & & \downarrow{\scriptstyle \mathbf{CUT}[\mathbf{1}]} \\
\bot & \xrightarrow{id} & \bot
\end{array} \tag{25}$$

for all $x, y \in \mathscr{C}$. The coercion maps $m_{x,y}$ and $m_{\mathbf{1}}$ equip the functor $(-)^*$ defined in (18) with the structure of a monoidal functor

$$(-)^* \quad : \quad (\mathscr{C}, \otimes, \mathbf{1}) \longrightarrow (\mathscr{C}, \parr, \bot)^{op(0,1)}$$

For just the same reasons, the functor ${}^*(-)$ described in (20) defines a monoidal functor

$$ {}^*(-) \quad : \quad (\mathscr{C}, \parr, \bot)^{op(0,1)} \longrightarrow (\mathscr{C}, \otimes, \mathbf{1}). $$

Put together, the two monoidal functors $(-)^*$ and ${}^*(-)$ define a monoidal equivalence

$$ (\mathscr{C}, \otimes, \mathbf{1}) \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} (\mathscr{C}, \parr, \bot)^{op(0,1)} \qquad \text{(monoidal equivalence)} \tag{26} $$

Starting from these observations, one shows that a duality in a linearly distributive category $\mathscr{C}$ may be alternatively defined in the following monoidal way:

Proposition 11 *A duality in a linearly distributive category $\mathscr{C}$ is alternatively defined as a monoidal equivalence (26) together with two families of morphisms*

$$ \mathbf{AX}[x] : \mathbf{1} \longrightarrow x^* \parr x \qquad\qquad \mathbf{CUT}[x] : x \otimes x^* \longrightarrow \bot $$

making the diagrams (13), (19) as well as (23), (24), (25) commute for all object x, y and all morphisms $f : x \to y$ of the category $\mathscr{C}$.

Remark: We will see the monoidal equivalence (26) as well as the exact pairing (13), the naturality (19) and the monoidality conditions (23), (24), (25) reappear in §5.5 when we compare the notion of "duality" in a linearly distributive category to the corresponding notion of "left-handed duality" in a linearly distributive chirality.

3 Dialogue chiralities

The purpose of the section is to introduce the notion of dialogue chirality [12] which we choose to formulate here in its left-handed variant. As explained in the introduction, the notion of dialogue chirality was designed in order to provide a two-sided formulation of the familiar but one-sided notion of dialogue category. The concept is particularly important in the present paper since we establish at a later stage (Prop. 17 in §5) that every right-handed linearly distributive chirality (§5.1, Def. 9) equipped with a left-handed duality (§5.3) defines a left-handed dialogue chirality. We start the section by recalling in §3.1 the original definition of dialogue chirality in its left-handed variant. In this variant, currification occurs on the left, instead on the right. We then investigate a series of equivalent formulations of this notion of dialogue chirality, either based on a family of adjunctions in §3.2 or on a family of transjunctions in §3.3. We complete the section by recalling in §3.4 the notion of transjunction introduced in [13] together with its pictorial representation in string diagrams.

3.1 Left-handed dialogue chiralities

We recall the definition of dialogue chirality [12] which we choose to formulate here in its left-handed variant.

Definition 3 (dialogue chirality) *A left-handed dialogue chirality is a pair of monoidal categories*

$$ (\mathscr{A}, \oslash, \mathit{true}) \qquad\qquad (\mathscr{B}, \otimes, \mathit{false}) $$

equipped with an adjunction

$$ \mathscr{A} \underset{R}{\overset{L}{\underset{\bot}{\rightleftarrows}}} \mathscr{B} \tag{27} $$

whose unit and counit are denoted as

$$\eta \;:\; Id \longrightarrow R \circ L \qquad\qquad \varepsilon \;:\; L \circ R \longrightarrow Id$$

together with a monoidal equivalence

$$\mathscr{A} \;\underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}}\; \mathscr{B}^{op(0,1)} \qquad \text{monoidal equivalence} \tag{28}$$

and a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, m \parr a \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, m^* \otimes b \,\rangle$$

natural in m, a, b, which we call currification *isomorphisms. Here, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:*

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b)\,).$$

The family χ is moreover required to make the diagram

$$\begin{array}{ccccc}
\langle\, (m \parr n) \parr a \,|\, b \,\rangle & & \xrightarrow{\quad\chi_{m \parr n}\quad} & & \langle\, a \,|\, (m \parr n)^* \otimes b \,\rangle \\
\Big\downarrow{\scriptstyle associativity} & & & & \Big\uparrow{\scriptstyle \begin{array}{c} associativity \\ monoidality\ of\ negation \end{array}} \\
\langle\, m \parr (n \parr a) \,|\, b \,\rangle & \xrightarrow{\chi_m} & \langle\, n \parr a \,|\, m^* \otimes b \,\rangle & \xrightarrow{\chi_n} & \langle\, a \,|\, n^* \otimes (m^* \otimes b) \,\rangle
\end{array} \tag{29}$$

commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$.

3.2 A formulation based on adjunctions

A preliminary step towards the algebraic presentation of dialogue chiralities performed later in the paper (see §5) is to replace the currification isomorphism χ by a family of adjunctions, in the following way.

Proposition 12 *A left-handed dialogue chirality is the same thing as a pair of monoidal categories $(\mathscr{A}, \parr, true)$ and $(\mathscr{B}, \otimes, false)$ equipped with an adjunction (27) and a monoidal equivalence (28) together with an adjunction*

$$L(m \parr -) \quad \dashv \quad R(m^* \otimes -) \tag{30}$$

for every object m of the category $\mathscr{A}$, whose unit and counit are denoted

$$\eta[m] \;:\; a \longrightarrow R(m^* \otimes L(m \parr a)) \qquad\qquad \varepsilon[m] \;:\; L(m \parr R(m^* \otimes b)) \longrightarrow b.$$

The family $\eta[-]$ is moreover required to be natural and monoidal, in the sense that the diagrams below

$$\begin{array}{ccccc}
 & \overset{\eta[m]}{\nearrow} & R(m^* \otimes L(m \parr a)) & \overset{f}{\searrow} & \\
a & & & & R(m^* \otimes L(n \parr a)) \\
 & \underset{\eta[n]}{\searrow} & R(n^* \otimes L(n \parr a)) & \underset{f^*}{\nearrow} &
\end{array} \tag{31}$$

$$\begin{array}{ccc} a & \xrightarrow{\eta[m \parr n]} & R((m \parr n)^* \otimes L((m \parr n) \parr a)) \\ {\scriptstyle \eta[n]}\downarrow & & \downarrow{\scriptstyle \text{associativity \& monoidality of negation}} \\ R(n^* \otimes L(n \parr a)) & & \\ {\scriptstyle \eta[m]}\downarrow & & \\ R(n^* \otimes LR(m^* \otimes L(m \parr (n \parr a)))) & \xrightarrow{\varepsilon} & R(n^* \otimes (m^* \otimes L(m \parr (n \parr a)))) \end{array} \tag{32}$$

should commute for all objects a, m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

Remark. The family of adjunctions (30) instantiated at the unit *true* induces an adjunction $L \dashv R$ between the functors L and R, whose unit and counit are defined as the expected families of morphisms

$$\eta'_a \;:\; a \xrightarrow{\eta[true]} R(true^* \otimes L(true \parr a)) \xrightarrow{\text{monoidality \& units}} RL(a)$$

$$\varepsilon'_b \;:\; LR(b) \xrightarrow{\text{monoidality \& units}} L(true \parr R(true^* \otimes b)) \xrightarrow{\varepsilon[true]} b$$

An important question is thus to understand whether this adjunction necessarily coincides with the original adjunction (5) between L and R. The answer is positive. The fact that $\eta = \eta'$ and $\varepsilon = \varepsilon'$ and thus, that the adjunctions are equal, is deduced from the coherence diagram (32). The main idea of the proof is to instantiate the diagram (32) with $m = n = e$, and to apply the coherence laws of the monoidal categories $\mathscr{A}$ and $\mathscr{B}$ in order to show that the diagram

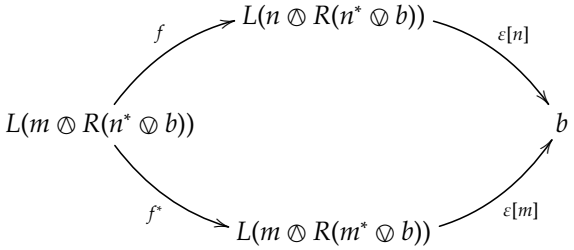
$$\begin{array}{ccc} a & \xrightarrow{\eta'} & RLa \\ {\scriptstyle \eta'}\downarrow & & \| \\ RLa & & \| \\ {\scriptstyle \eta'}\downarrow & & \| \\ RLRLa & \xrightarrow{\varepsilon} & RLa \end{array}$$

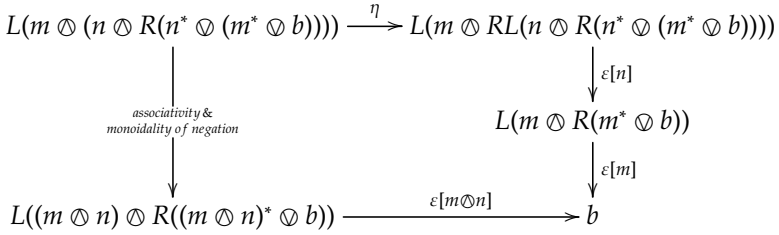
commutes. The two equalities $\eta = \eta'$ and $\varepsilon' = \varepsilon$ easily follow from this and from the properties of adjunctions. This means in particular that the coherence diagram

$$\begin{array}{ccc} R(true^* \otimes L(true \parr a)) & \xrightarrow{\text{monoidality}} & R(false \otimes L(true \parr a)) \\ {\scriptstyle \eta[true]}\uparrow & & \downarrow{\scriptstyle units} \\ a & \xrightarrow{\eta} & RL(a) \end{array} \tag{33}$$

is a consequence of the two coherence diagrams (31) and (32) formulated in the statement of Proposition 12.

Remark. Once established that the adjunction $(L, R, \eta', \varepsilon')$ coincides with the adjunction $(L, R, \eta, \varepsilon)$, it is not difficult to deduce from their companion diagrams (31) and (32) that the two coherence diagrams





commute for all object b of the category $\mathscr{B}$ and for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$. At this point, it is worth mentioning that there is an element of choice in the definition of dialogue chiralities formulated in Proposition 12 since the two coherence diagrams for the counit $\varepsilon[-]$ of the adjunction $L(m \parr -) \dashv R(m^* \otimes -)$ may very well replace the two corresponding coherence diagrams (31) and (32) for the unit $\eta[-]$.

3.3 A formulation based on transjunctions

The alternative formulation of dialogue chiralities exposed in §3.2 can be marginally improved and oriented in the direction of proof theory. The idea is to replace the original pair of combinators $\eta[-]$ and $\varepsilon[-]$ presenting the adjunctions $L(m \parr -) \dashv R(m^* \otimes -)$ by a pair of arguably simpler combinators:

$$\begin{array}{rcccc} \mathbf{axiom}[m] & : & L(a) & \longrightarrow & m^* \otimes L(m \parr a) \\ \mathbf{cut}[m] & : & m \parr R(m^* \otimes b) & \longrightarrow & R(b) \end{array}$$

The two combinators are defined from $\eta[-]$ and $\varepsilon[-]$ in the following way:

$$\begin{array}{rcccccc} \mathbf{axiom}[m] & : & L(a) & \xrightarrow{\eta[m]} & LR(m^* \otimes L(m \parr a)) & \xrightarrow{\varepsilon} & m^* \otimes L(m \parr a) \\ \mathbf{cut}[m] & : & m \parr R(m^* \otimes b) & \xrightarrow{\eta} & RL(m \parr R(m^* \otimes b)) & \xrightarrow{\varepsilon[m]} & R(b) \end{array}$$

Conversely, the combinators $\eta[-]$ and $\varepsilon[-]$ are recovered from the combinators $\mathbf{axiom}[-]$ and $\mathbf{cut}[-]$ in the following way:

$$\begin{array}{rcccccc} \eta[m] & : & a & \xrightarrow{\eta} & RL(a) & \xrightarrow{\mathbf{axiom}[m]} & R(m^* \otimes L(m \parr a)) \\ \varepsilon[m] & : & L(m \parr R(m^* \otimes b)) & \xrightarrow{\mathbf{cut}[m]} & LR(b) & \xrightarrow{\varepsilon} & b \end{array}$$

It is easy to show that this back-and-forth translation between the pair of combinators $(\eta[-], \varepsilon[-])$ and the pair of combinators $(\mathbf{axiom}[-], \mathbf{cut}[-])$ defines a one-to-one relationship. This observation leads us to an alternative formulation of dialogue chiralities, based this time on a family of *transjunctions*

$$(m \parr -) \quad \dashv\!\!\!\vdash \quad (m^* \otimes -) \tag{34}$$

between the functors

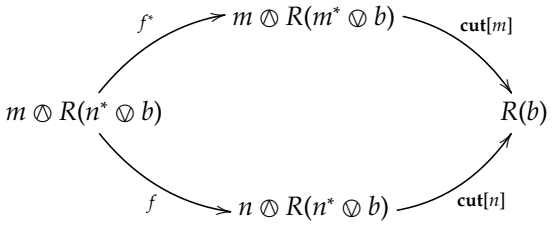
$$m \parr - \; : \; \mathscr{A} \longrightarrow \mathscr{A} \qquad\qquad m^* \otimes - \; : \; \mathscr{B} \longrightarrow \mathscr{B}$$

accross the adjunction $L \dashv R$ between $\mathscr{A}$ and $\mathscr{B}$. The reader unaware of the notion of transjunction will find it recalled in §3.4. As just explained, the notion enables us to replace the original combinators $\eta[-]$ and $\varepsilon[-]$ by the logically flavoured combinators $\mathbf{axiom}[-]$ and $\mathbf{cut}[-]$.

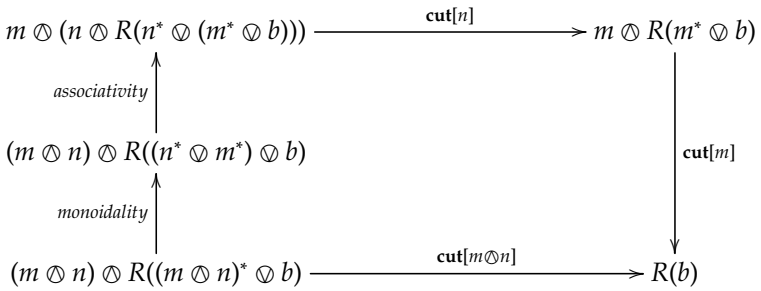
Proposition 13 *A left-handed dialogue chirality may be alternatively defined as a pair of categories* $(\mathscr{A}, \parr, true)$ *and* $(\mathscr{B}, \otimes, false)$ *equipped with an adjunction (27) and a monoidal equivalence (28) together with a family of transjunctions*

$$\mathbf{axiom}[m] \,:\, L(-) \longrightarrow m^* \otimes L(m \parr -) \qquad \mathbf{cut}[m] \,:\, m \parr R(m^* \otimes -) \longrightarrow R(-)$$

The family $\mathbf{cut}[-]$ *is moreover required to be natural and monoidal, in the sense that the two diagrams*



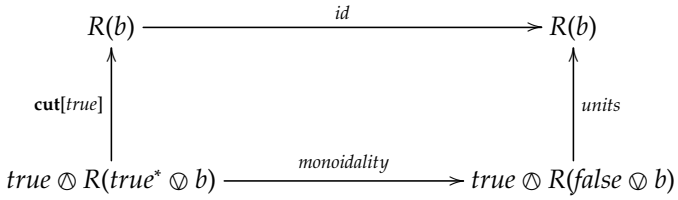
(35)



(36)

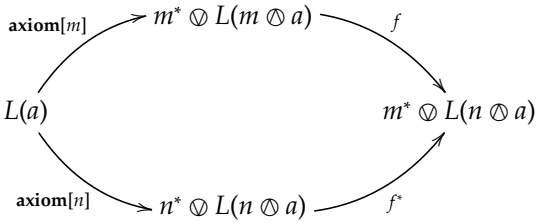
commute for all objects a, m, n *and morphisms* $f : m \to n$ *of the category* $\mathscr{A}$.

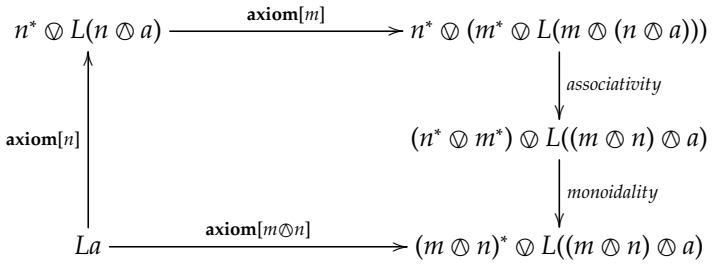
Remark. In the same way as in §3.2, the coherence diagram



(37)

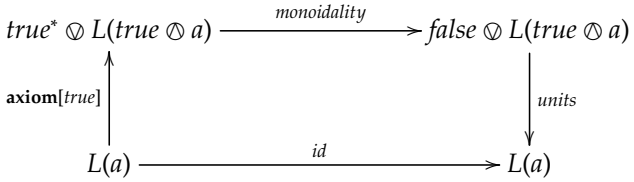
follows from the coherence diagrams (35) and (36). Similarly, one can easily check that the two coherence diagrams for the combinator $\mathbf{cut}[-]$ are equivalent to the corresponding coherence diagrams for $\mathbf{axiom}[-]$ below:





(38)

and the coherence diagram



(39)

follows from the coherence diagrams (35) and (36).

3.4 Transjunctions

We recall below the notion of transjunction introduced in [13].

Definition 4 (transjunction) *Suppose given a pair of adjunctions*

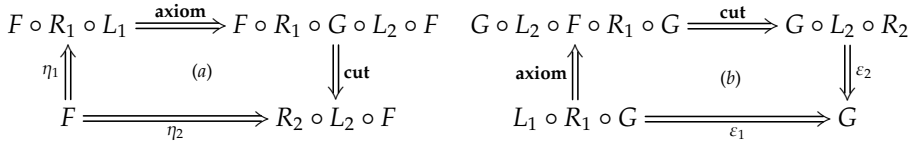
whose units and counits are denoted η_1, η_2 and $\varepsilon_1, \varepsilon_2$ respectively. A transjunction $F \dashv\!\!\!\vdash G$ between a pair of functors

$$F : \mathscr{A}_1 \to \mathscr{A}_2 \qquad G : \mathscr{B}_2 \to \mathscr{B}_1$$

accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations

$$\textbf{axiom} : L_1 \Rightarrow G \circ L_2 \circ F \qquad \textbf{cut} : F \circ R_1 \circ G \Rightarrow R_2$$

making the two diagrams



(40)

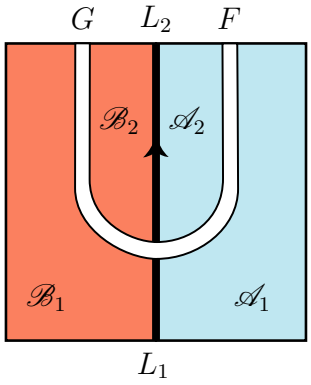
commute.

Although the notion of transjunction may appear complicated at first sight, it is ultimately justified by the following observation.

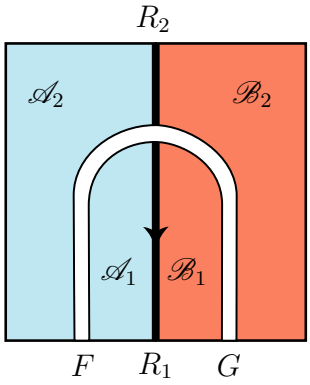
Proposition 14 *A transjunction $F \dashv\!\!\!\vdash G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is the same thing as an adjunction $F \circ L_2 \dashv R_1 \circ G$.*

One benefit of shifting from the notion of adjunction $F \circ L_1 \dashv R_2 \circ G$ to the notion of transjunction $F \dashv\!\!\!\circ\, G$ is that transjunctions are much easier to depict in the language of string diagrams. Remember that we work here in the 2-category of categories, functors and natural transformations. The generators **axiom** and **cut** of a transjunction $F \dashv\!\!\!\circ\, G$ accross the adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ are then depicted in the following way:

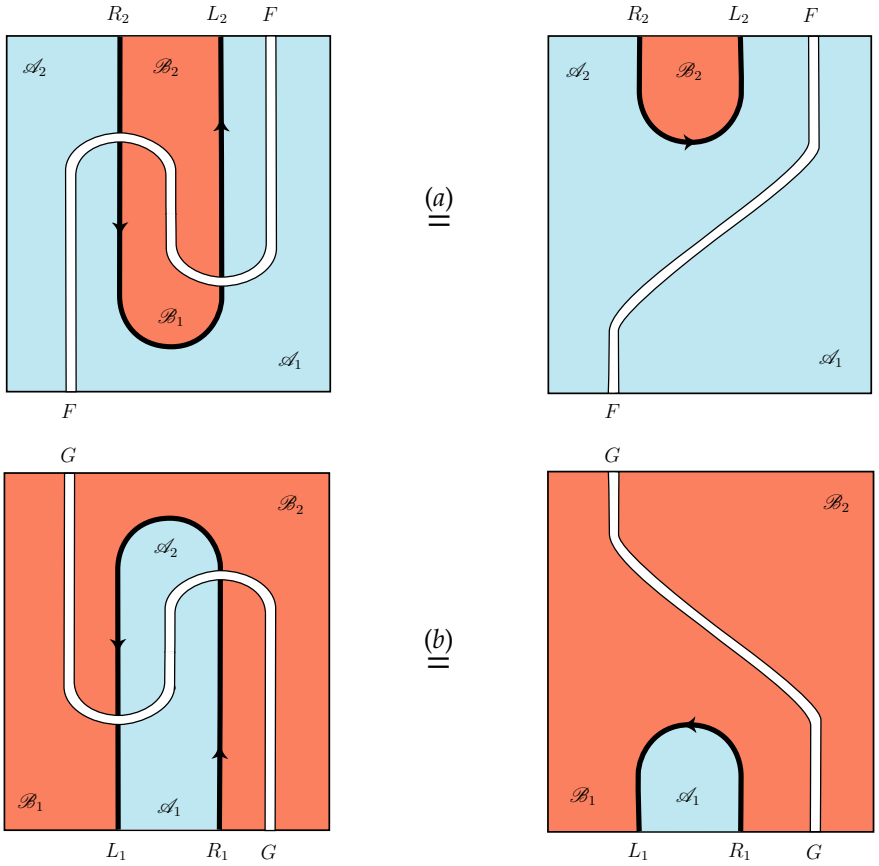
axiom =



cut =



Accordingly, the two commutative diagrams (*a*) and (*b*) of Definition 4 are depicted as:

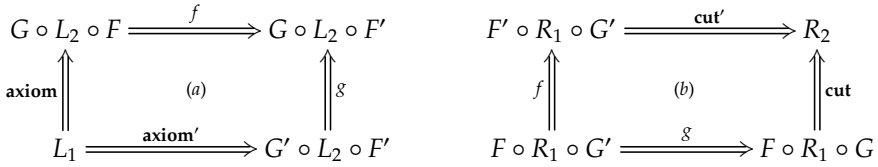


A notion of homorphism between transjunctions may be also introduced, thus giving rise to a category of transjunctions.

Definition 5 (homomorphism) *A homomorphism between two transjunctions $F \dashv\!\!\!\circ\, G$ and $F' \dashv\!\!\!\circ\, G'$ accross the same pair of adjunctions $L_1 \dashv R_1$ and $L_2 \dashv R_2$ is defined as a pair of natural transformations*

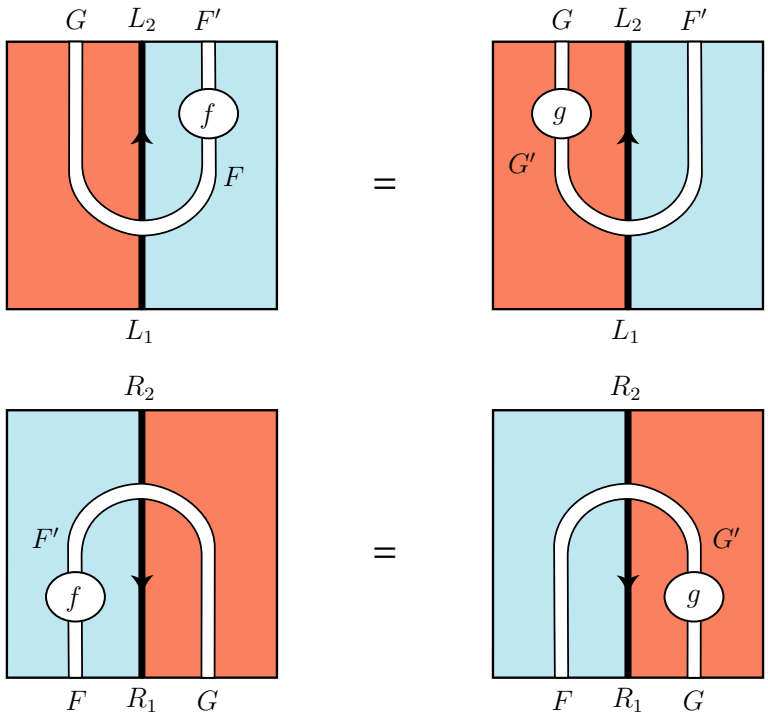
$$f : F \Rightarrow F' \qquad g : G' \Rightarrow G$$

making the two diagrams



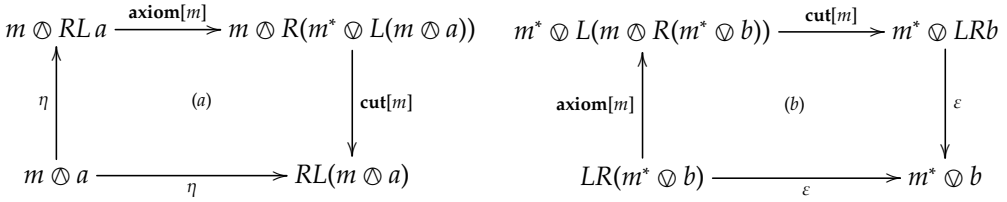
commute.

Pictorially, such a homomorphism (f, g) is defined as a pair of natural transformations $f : F \Rightarrow F'$ and $g : G' \Rightarrow G$ satisfying the diagrammatic equalities below:



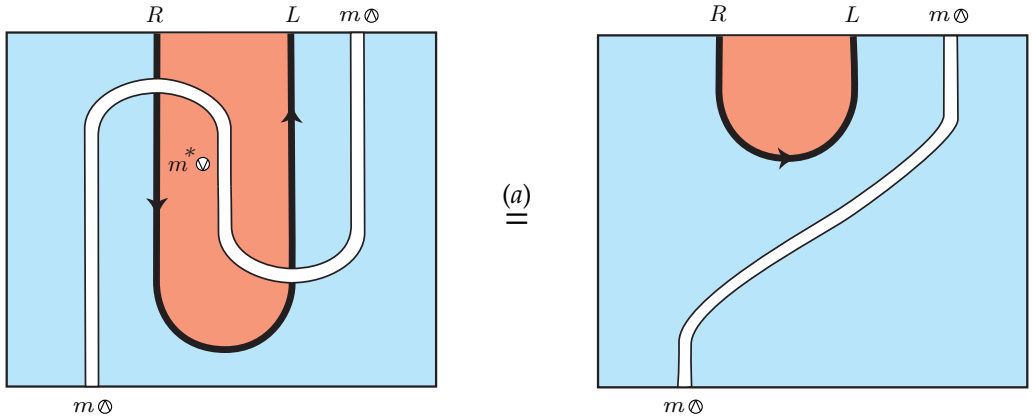
3.5 Tensorial proof-nets vs. multiplicative proof-nets

As mentioned in the introduction, our work on tensorial logic is guided by the observation that many constructions of linear logic can be lifted to a situation where linear negation is not required to be involutive anymore. This leads us to raise the following question: is it possible to refine the axiom-cut annihilation of linear logic — as it appears in multiplicative proof-nets — in order to make sense in tensorial logic? One benefit of our algebraic approach to this question is that we have just resolved it, although somewhat secretly. Indeed, recall from §3.3 and §3.4 (Def. 4 and Prop. 13) that the two coherence diagrams (40) required of a transjunction should be instantiated in the following way:

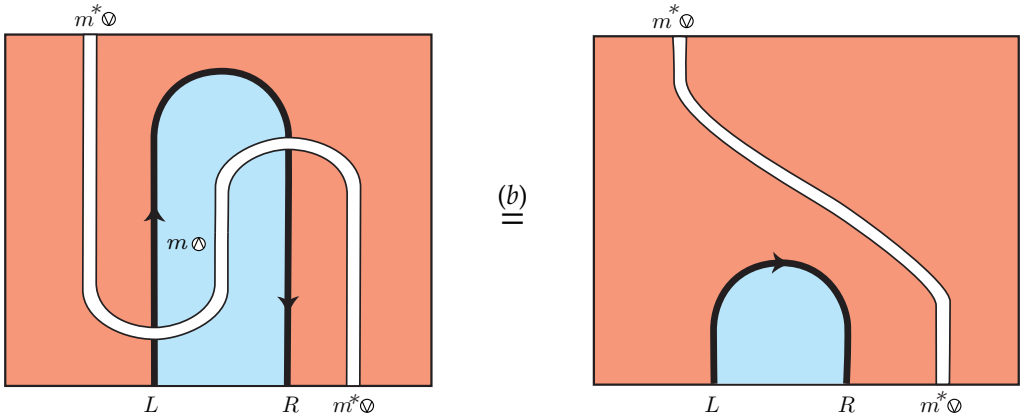


(41)

in the definition of a left-handed dialogue chirality. These diagrams are required to commute for all objects m, a of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. Once translated in string diagrams, the coherence diagram (a) is depicted as



while its companion diagram (b) is depicted as its mirror image:



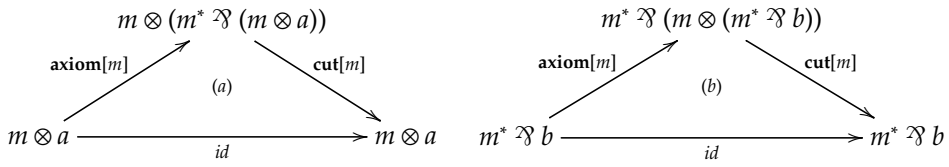
An interesting observation is that the two equations (a) and (b) refine the traditional axiom-cut annihilation of linear logic, and integrate the non involutive negation functors L and R in it. This general situation of a dialogue chirality $(\mathscr{A}, \mathscr{B})$ may be specialized to the "depolarized" situation $(\mathscr{A}, \mathscr{B}) = (\mathscr{C}, \mathscr{C})$ where the two sides $\mathscr{A}$ and $\mathscr{B}$ coincide with a $*$-autonomous category $\mathscr{C}$, and where the two functors R and L are the identity functor $\mathscr{C} \to \mathscr{C}$, see §5.5 for a discussion on depolarisation. In that degenerate situation, the family of transjunctions defining the dialogue chirality $(\mathscr{C}, \mathscr{C})$ in Proposition 13 boils down to a family of adjunctions

$$m \otimes - \quad \dashv \quad m^* \parr - \qquad : \qquad \mathscr{C} \longrightarrow \mathscr{C} \tag{42}$$

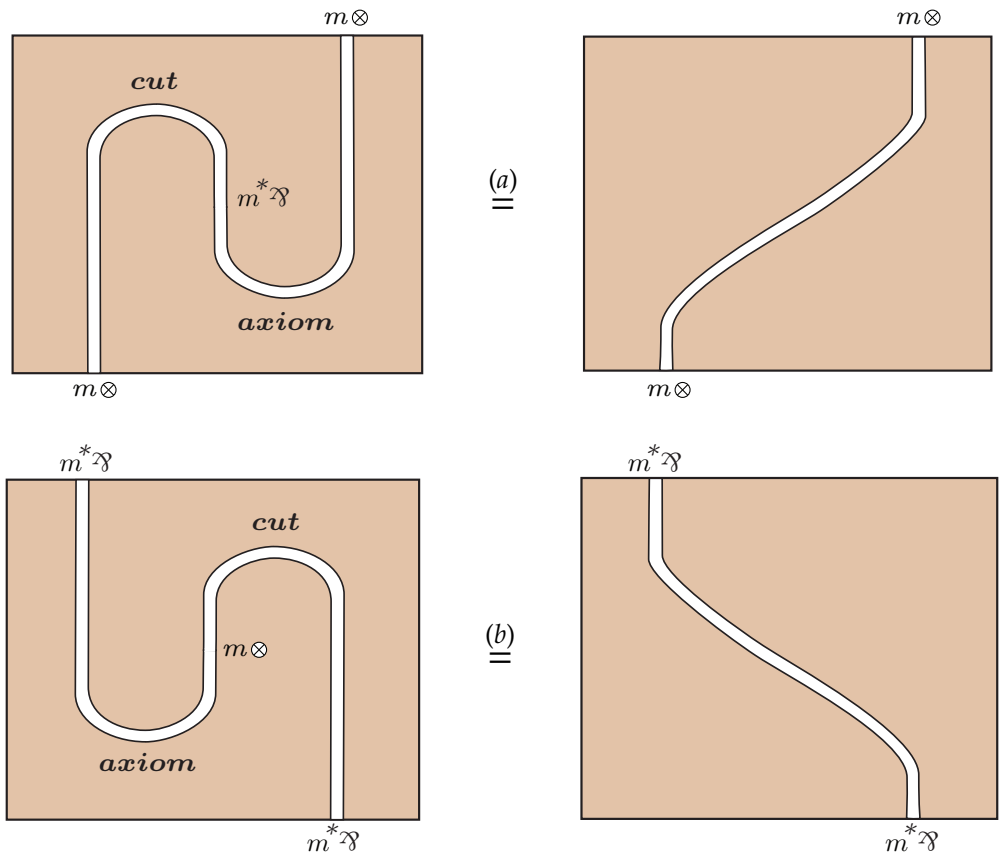
with unit and counit defined as

$$\mathbf{axiom}[m] : a \longrightarrow m^* \parr (m \otimes a) \qquad\qquad \mathbf{cut}[m] : m \otimes (m^* \parr b) \longrightarrow b.$$

Accordingly, the original equations (a) and (b) are replaced by the familiar triangular laws of the adjunction (42):



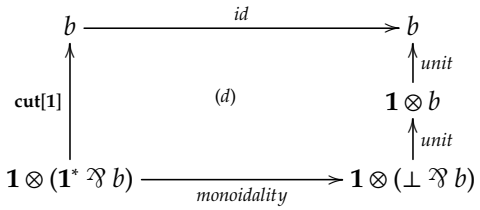
which are required to commute for all objects m, a of the category $\mathscr{A}$ and all objects b of the category $\mathscr{B}$. Once formulated in the graphical language of string diagrams, one recovers the familiar axiom-cut annihilation of multiplicative proof-nets in linear logic, turned upside-down:



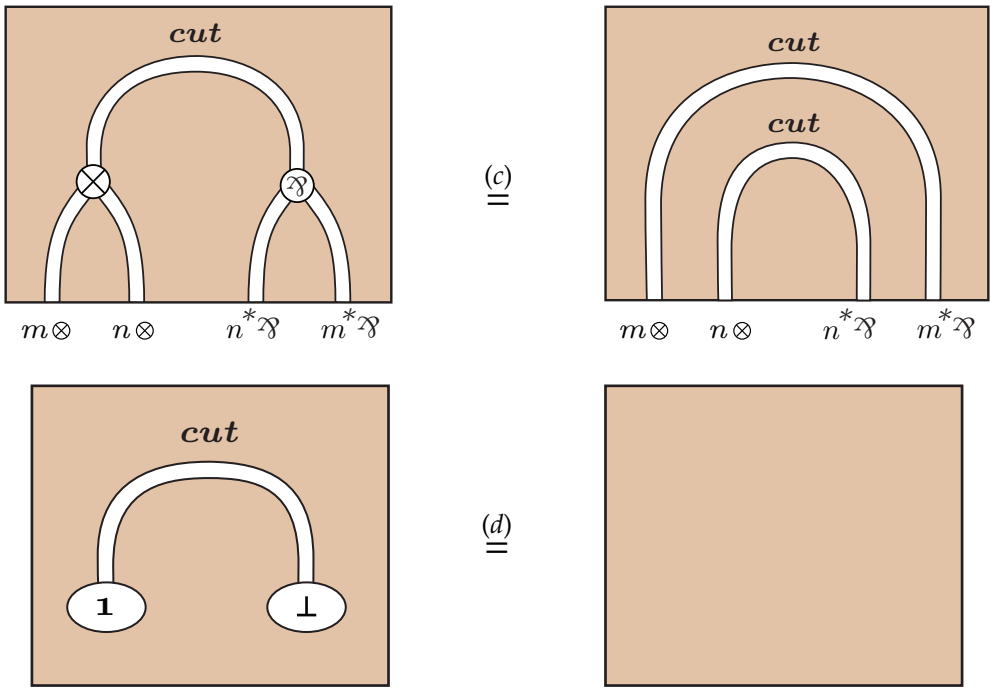
At this stage, it appears that the purpose of each transjunction (34) mentioned in Proposition 13 is to refine the corresponding adjunction (42) of $*$-autonomous categories by taking care of the explicit negation functors L and R of dialogue chiralities.

As we have just seen, the refinement of linear logic into tensorial logic works surprisingly well in the case of the axiom-cut annihilation. One thus wonders whether the two other fundamental equations of multiplicative proof nets — namely the tensor-par and the **1**-$\bot$ annihilation rules — may be similarly upgraded to tensorial logic. The two equations of linear logic are conveniently formulated as the coherence diagrams (c) and (d) which do indeed commute in any $*$-autonomous category. Note that the diagrams express that the family of adjunctions (42) is monoidal in its parameter m.

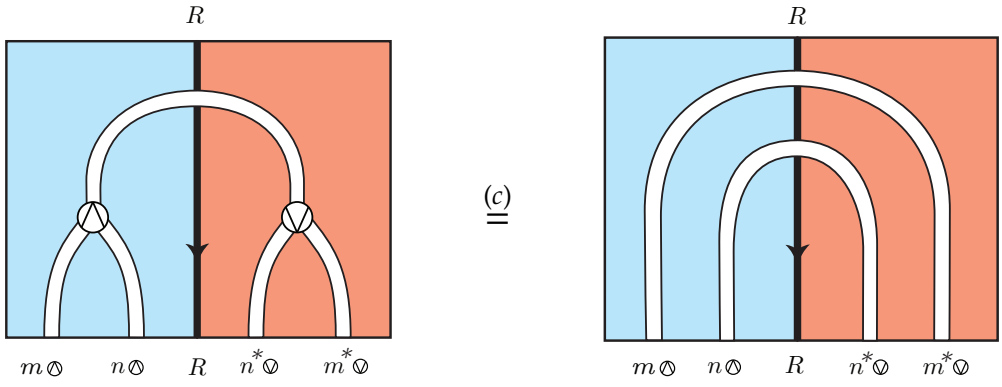
$$\begin{array}{ccc}
m \otimes (n \otimes (n^* \parr (m^* \parr b))) & \xrightarrow{\textbf{cut}[n]} & m \otimes (m^* \parr b) \\
\uparrow{\scriptstyle \textit{associativity}} & & \\
(m \otimes n) \otimes ((n^* \parr m^*) \parr b) & (c) & \downarrow{\scriptstyle \textbf{cut}[m]} \\
\uparrow{\scriptstyle \textit{monoidality}} & & \\
(m \otimes n) \otimes ((m \otimes n)^* \parr b) & \xrightarrow[\textbf{cut}[m \otimes n]]{} & b
\end{array}$$

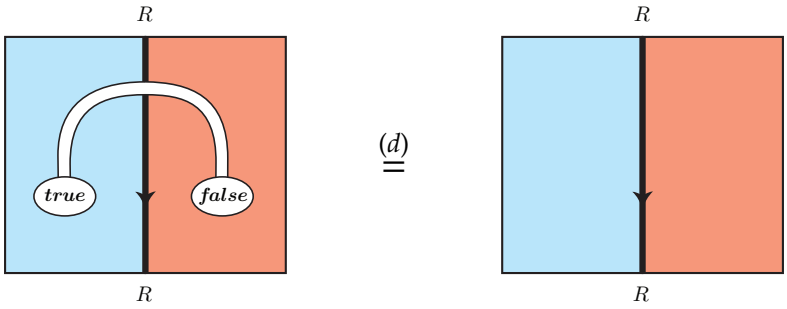


When one reformulates the coherence diagrams (*c*) and (*d*) in the language of string diagrams, one realizes that they correspond to the usual equations satisfied by multiplicative proof-nets of linear logic, turned upside-down.



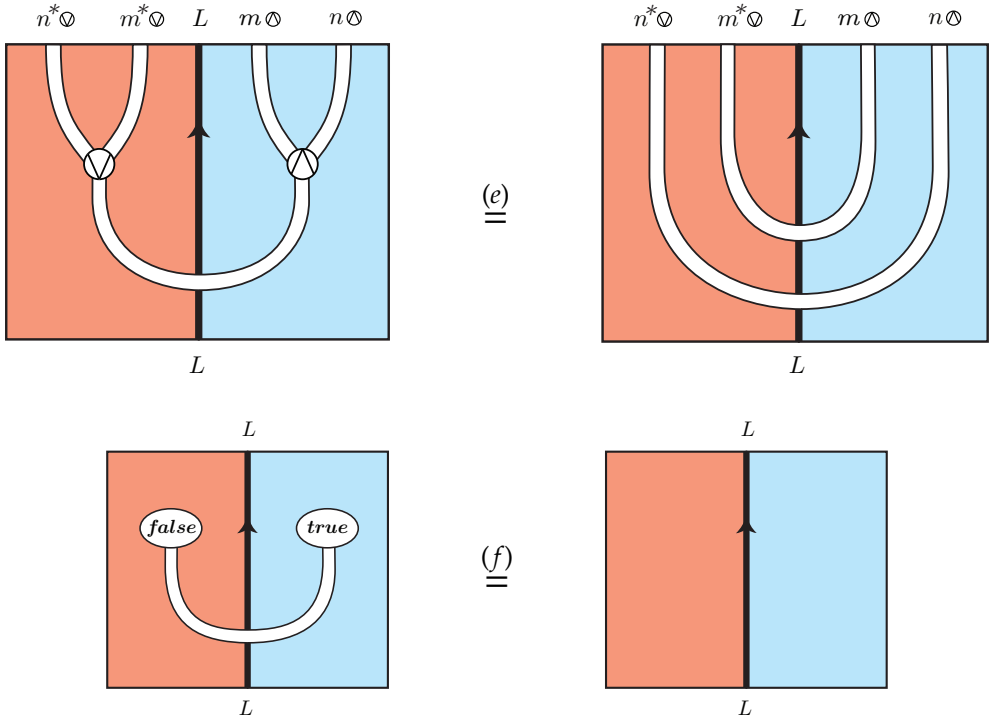
Once again, our purely algebraic approach to proof-theory (combined with string diagrams) provides a simple answer to the question of upgrading equations (*c*) and (*d*) to tensorial logic. We have seen how the first part of Proposition 13 upgrades the axiom-cut annihilation equations (*a*) and (*b*). We claim that the second part of Proposition 13 does the same for the two equations (*c*) and (*d*). This claim becomes apparent when one depicts the coherence diagrams (36) and (37) as string diagrams:



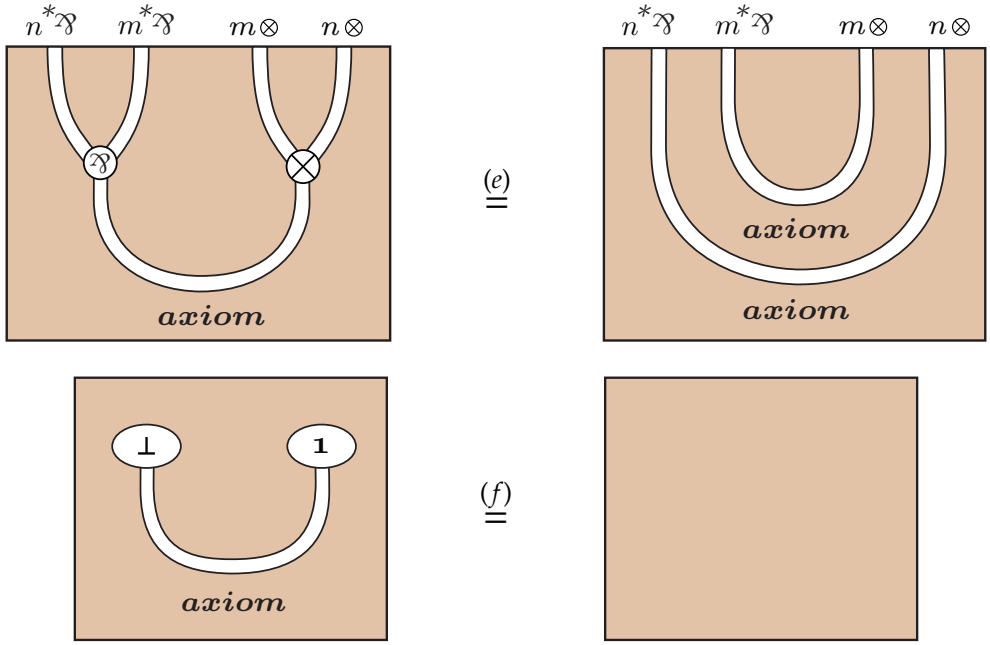


In the same way as for the two equations (a) and (b), one recovers the equations (c) and (d) of linear logic by "depolarizing" the corresponding equations of tensorial logic. Diagrammatically speaking, depolarization amounts to replacing the light blue area for $\mathscr{A}$ and the dark red area for $\mathscr{B}$ by a unique cream area for $\mathscr{C}$, and by removing accordingly the separating frontiers L and R between $\mathscr{A}$ and $\mathscr{B}$.

As already mentioned, a basic principle of dialogue chiralities is that every coherence diagram comes with a mirror image. A typical illustration is provided by diagram (a) whose mirror image is diagram (b). Similarly, the mirror images of the two diagrams (c) and (d) are provided by the coherence diagrams (38) and (39) depicted below as two equations (e) and (f) between string diagrams:



These diagrams refine the familiar η-expansion rule of multiplicative proof-nets in linear logic, which can be deduced by "depolarizing" their tensorial counterpart (e) and (f) in the expected way:



This series of elementary observations leads us to the idea that the theory of proof-nets in linear logic can be refined to a similarly rich theory of proof-nets in tensorial logic. A preliminary step in this research program has been performed in [11] where a notion of tensorial proof-net at the crossroad of linear logic and of game semantics has been introduced. Seen from that point of view, the process of depolarizing a tensorial proof-net into a multiplicative proof-net of linear logic deserves the name of "detemporalization" or "desequentialization" because it removes the flow of execution (and the flow of time) performed by the negation functors R and L inside the proof. Note that this flow of negations inside the tensorial proof-net is precisely what reflects the innocent strategy associated to the proof, with the functor R standing for the Opponent moves, and the functor L standing for the Proponent moves.

4 Bimodulations

The notion of *bimodulation* between two *modulations* was introduced in our companion paper [13] in order to describe the algebraic structure of dialogue categories. By modulation, we mean a *lax action* of a monoidal category $\mathscr{I}$ on a category $\mathscr{A}$, what we also like to call a *parametric monad* depending on the context of application. The purpose of a bimodulation is then to relate two such modulations. One main reason for introducing the notion of bimodulation is that it unifies the familiar notions of *monadic strength* on the one hand, and of *distributivity law* between two monads on the other hand. The notion of bimodulation will play a central rôle in §5 when we formulate the notion of *linearly distributive chirality*. We start the section by recalling the two notions of modulation in §4.1 and of bimodulation in §4.2. The precise definition of a right-handed (rather than left-handed) dialogue chirality is then formulated in §4.3. This leads us to §4.5 where we illustrate the notion of bimodulation with the example of the continuation bimodulation associated to any right-handed dialogue chirality.

4.1 Modulations

Instead of directly defining a modulation as a lax action of a monoidal category $\mathscr{I}$ on a category $\mathscr{A}$, we prefer to slightly extend our span of application and thus to work in a general 2-category $\mathscr{W}$ in the spirit of formal monad theory [15].

Definition 6 (modulation) *A modulation in a 2-category $\mathscr{W}$ is defined as a 0-cell $\mathscr{A}$ equipped with a lax monoidal functor*

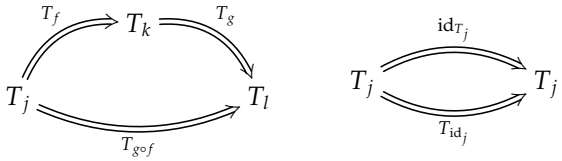
$$(T, m) \;:\; \mathscr{I} \longrightarrow \mathrm{End}(\mathscr{A})$$

where $\mathrm{End}(\mathscr{A}) = \mathscr{W}(\mathscr{A}, \mathscr{A})$ *denotes the monoidal category of 0-cells (or endomorphisms) from $\mathscr{A}$ to $\mathscr{A}$. The monoidal category $(\mathscr{I}, \otimes, e)$ is called the parameter category of the $\mathscr{I}$-modulation. Accordingly, an object j of the category $\mathscr{I}$ is called a parameter of the modulation.*

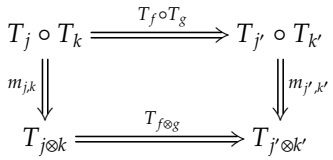
Hence, a $\mathscr{I}$-modulation (T, m) consists of

- a 1-cell $T_j : \mathscr{A} \longrightarrow \mathscr{A}$ for every parameter j and a 2-cell $T_f : T_j \Rightarrow T_k$ for every morphism $f : j \longrightarrow k$ between such parameters,
- a 2-cell $m_e : 1_A \Rightarrow T_e$ called the unit of the modulation,
- a 2-cell $m_{j,k} : T_j \circ T_k \Rightarrow T_{j \otimes k}$ called the (j,k)-component of the multiplication of the modulation, for every pair of parameters j and k.

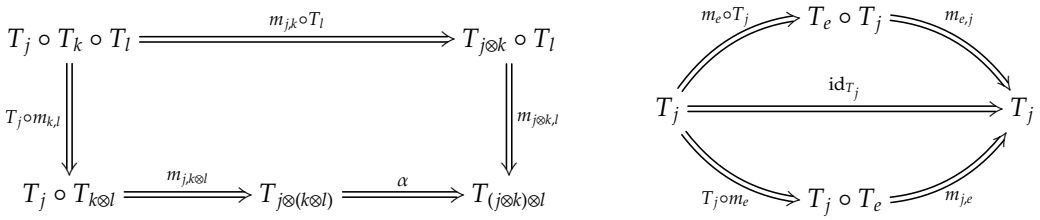
These data are moreover required to make a series of coherence diagrams commute in the category $\mathrm{End}(\mathscr{A})$. First, the diagrams



express the functoriality of T. Then, the diagrams



express the naturality of m. Finally, the two diagrams



express the monoidality of m, for all indices j, j', k, k', l and morphisms f, g, h in the parameter category $\mathscr{I}$. Note that a modulation T parametrized by the trivial monoidal category $\mathbf{1}$ with exactly one object $*$ and exactly one morphism (thus equal to the identity morphism of $*$) is the same thing as a formal monad in the 2-category $\mathscr{W}$, in the sense of Street [15].

4.2 Bimodulations between modulations

From now on, we suppose given a 0-cell $\mathscr{C}$ in a 2-category $\mathscr{W}$ equipped with a $\mathscr{I}$-modulation

$$T = \bullet \;:\; \mathscr{I} \longrightarrow \mathrm{End}(\mathscr{C})$$

and a $\mathscr{M}^{\,op(0)}$-modulation

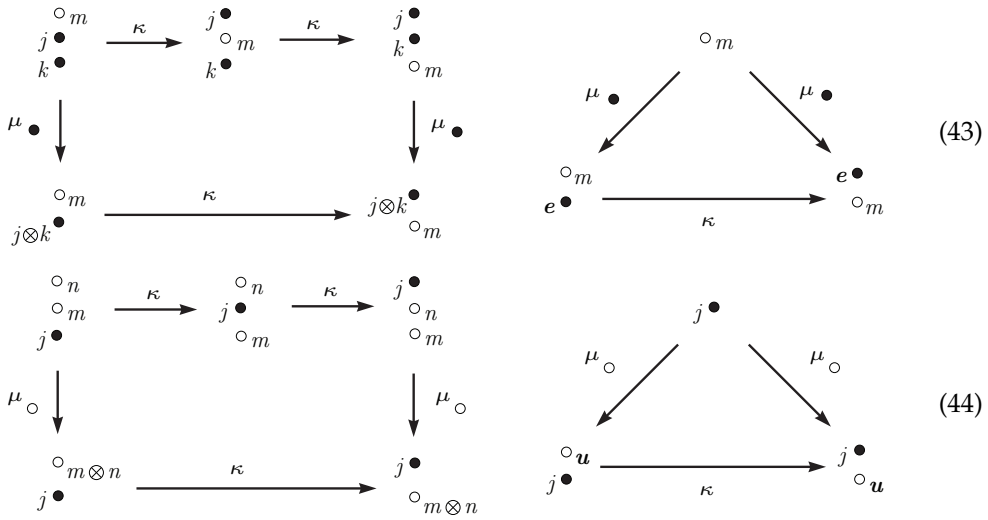
$$S = \circ \;:\; \mathscr{M}^{\,op(0)} \longrightarrow \mathrm{End}(\mathscr{C})$$

with parameters taken in the monoidal categories $(\mathscr{I}, \otimes, \boldsymbol{e})$ and $(\mathscr{M}, \otimes, \boldsymbol{u})$.

Definition 7 (bimodulation) *A bimodulation (S, T, κ) is a pair of modulations $T = \bullet$ and $S = \circ$ together with a natural transformation*

$$\kappa \quad : \quad ST \quad \Rightarrow \quad TS \quad : \quad \mathscr{I} \times \mathscr{M}^{\,op(0)} \quad \longrightarrow \quad \mathrm{End}(\mathscr{C})$$

called the commutator κ of the bimodulation, making the diagrams below commute



(43)

(44)

for all objects i, j of the category $\mathscr{I}$ and all objects m, n of the category $\mathscr{M}$.

In the particular case when $\mathscr{W} = \boldsymbol{Cat}$, the commutator κ may be alternatively formulated as a natural transformation

$$\kappa \quad : \quad (- \bullet -) \circ - \quad \Rightarrow \quad - \bullet (- \circ -) \quad : \quad \mathscr{I} \times \mathscr{C} \times \mathscr{M} \quad \longrightarrow \quad \mathscr{C}$$

with components

$$\kappa_{j,m,A} \quad : \quad (j \bullet A) \circ m \quad \longrightarrow \quad j \bullet (A \circ m)$$

parametrized by the objects j of the category $\mathscr{I}$, m of the category $\mathscr{M}$ and A of the category $\mathscr{C}$, and making a number of coherence diagrams commute. We like to think of a bimodulation κ as a lax version of a $(\mathscr{I}, \mathscr{M})$-bimodule. This justifies our terminology. The interested reader will find in [13] the coherence diagrams (43) and (44) depicted in string diagrams, with the bimodulation κ represented as a braiding commuting the actions of $S = \circ$ and $T = \bullet$.

Remark. A *comodulation* parametrized by the monoidal category $(\mathscr{I}, \otimes, e)$ in a 2-category $\mathscr{W}$ is defined as a $\mathscr{I}$-modulation in the 2-category $\mathscr{W}^{\,op(2)}$. Here, the notation $\mathscr{W}^{\,op(2)}$ stands for the 2-category obtained from $\mathscr{W}$ by reversing the orientation of its 2-cells. Similarly, a *bimodulation* between two comodulations is defined as a bimodulation between the corresponding modulations in the 2-category $\mathscr{W}^{\,op(2)}$. It is also possible to express a notion of bimodulation between a modulation and a comodulation which generalizes the notion of distributivity law between a monad and a comonad.

4.3 Right-handed dialogue chiralities

There is an element of choice in our definition of a dialogue chirality in §3 and it thus makes sense to call it a "left-handed" dialogue chirality. As we explained, every dialogue category $\mathscr{C}$ induces such a left-handed dialogue chirality, obtained by defining L and R as the negation functors

$$L \; : \; x \; \mapsto \; \bot \multimap x \qquad\qquad R \; : \; x \; \mapsto \; x \multimap \bot.$$

The other choice of negation functors

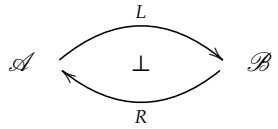
$$L \,:\, x \;\mapsto\; x \multimap \bot \qquad\qquad R \,:\, x \;\mapsto\; \bot \multimap x$$

also induces a dialogue chirality, which is *right-handed* this time, instead of left-handed. As expected, the notion of "right-handed" dialogue chirality defined below is the mirror image of the notion of left-handed dialogue chirality.

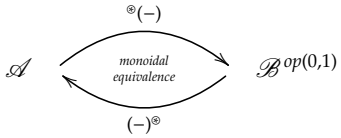
Definition 8 (dialogue chirality) *A right-handed dialogue chirality is defined as a pair of monoidal categories*

$$(\mathscr{A}, \parr, \mathit{true}) \qquad\qquad (\mathscr{B}, \otimes, \mathit{false})$$

equipped with an adjunction



with a monoidal equivalence



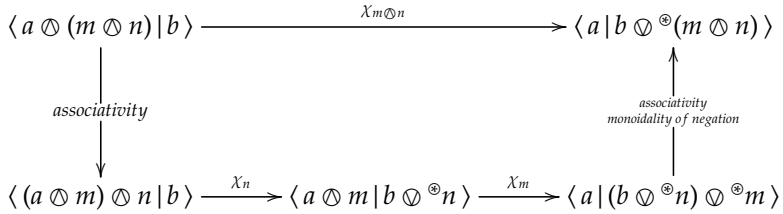
and with a family of bijections

$$\chi_{m,a,b} \quad : \quad \langle\, a \parr m \,|\, b \,\rangle \quad \longrightarrow \quad \langle\, a \,|\, b \otimes {}^{\circledast} m \,\rangle$$

natural in m, a, b. In the same way as in the case of a left-handed dialogue chirality, the bracket $\langle\, a \,|\, b \,\rangle$ denotes the set of morphisms from a to $R(b)$ in the category $\mathscr{A}$:

$$\langle\, a \,|\, b \,\rangle \quad = \quad \mathscr{A}(\, a \,,\, R(b) \,).$$

The family χ is moreover required to make the diagram



commute for all objects a, m, n of the category $\mathscr{A}$, and all objects b of the category $\mathscr{B}$.

4.4 Internal symmetries of dialogue chiralities

One should think of the symmetry *Prover* $\leftrightarrow$ *Denier* as an internal symmetry of logic, which comes with a *principle of invariance*. This principles states that the symmetry *Prover* $\leftrightarrow$ *Denier* preserves the laws of logic. Seen from an algebraic point of view, and applied to the particular case of tensorial logic, this principle implies that the notion of dialogue chirality is invariant under a number of 2-categorical symmetries. First of all, every left-handed (resp. right-handed) dialogue chirality becomes a left-handed (resp. right-handed) dialogue chirality by applying the following symmetry

$$\mathscr{A} \mapsto \mathscr{B}^{\,op(1)} \quad \mathscr{B} \mapsto \mathscr{A}^{\,op(1)} \quad \parr \mapsto \otimes^{\,op(1)} \quad \otimes \mapsto \parr^{\,op(1)} \quad L \mapsto R^{\,op(1)} \quad R \mapsto L^{\,op(1)} \qquad (45)$$

which inverts the directions of the morphisms (dimension 1) of $\mathscr{A}$ and $\mathscr{B}$, and thus transports the dialogue chirality $(\mathscr{A}, \mathscr{B})$ into the dialogue chirality $(\mathscr{B}^{op(1)}, \mathscr{A}^{op(1)})$. This symmetry reflects the principle of invariance just mentioned. Since it will reappear on several occasions in the paper, we will call it for conciseness the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(1)}$. Somewhat less surprisingly, every left-handed (resp. right-handed) dialogue chirality becomes a right-handed (resp. left-handed) dialogue chirality by applying the symmetry

$$\mathscr{A} \mapsto \mathscr{A}^{op(0)} \quad \mathscr{B} \mapsto \mathscr{B}^{op(0)} \quad \parr \mapsto \parr^{op(0)} \quad \otimes \mapsto \otimes^{op(0)} \quad L \mapsto L^{op(0)} \quad R \mapsto R^{op(0)} \tag{46}$$

which inverts the direction of the conjunction and of the disjunction (dimension 0) and which we call the symmetry $\mathscr{A} \leftrightarrow \mathscr{A}^{op(0)}$. Finally, by composing the two symmetries, one obtains another illustration of the principle of invariance, which tells that every left-handed (resp. right-handed) dialogue chirality becomes a right-handed (resp. left-handed) dialogue chirality by applying the symmetry

$$\mathscr{A} \mapsto \mathscr{B}^{op(0,1)} \quad \mathscr{B} \mapsto \mathscr{A}^{op(0,1)} \quad \parr \mapsto \otimes^{op(0,1)} \quad \otimes \mapsto \parr^{op(0,1)} \quad L \mapsto R^{op(0,1)} \quad R \mapsto L^{op(0,1)} \tag{47}$$

which we will call the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(0,1)}$. The interested reader is advised to look at the companion paper [12] for a more detailed discussion on this point.

4.5 The continuation bimodulation

An important reason for introducing the notion of bimodulation in the companion paper [13] is that every right-handed dialogue chirality $(\mathscr{A}, \mathscr{B})$ comes together with a bimodulation

$$\kappa^{\parr} \quad : \quad S\,T \quad \longrightarrow \quad T\,S$$

between the two modulations

$$\begin{array}{lllllllll} T & : & (b,a) & \mapsto & R(b \otimes L(a)) & : & \mathscr{B} \times \mathscr{A} & \longrightarrow & \mathscr{A} \\ S & : & (a,m) & \mapsto & a \parr m & : & \mathscr{A} \times \mathscr{A} & \longrightarrow & \mathscr{A} \end{array}$$

The same construction may be applied to the right-handed dialogue chirality $(\mathscr{B}^{op(1)}, \mathscr{A}^{op(1)})$ obtained by applying the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(1)}$ discussed in §4.4. This establishes that every right-handed dialogue chirality $(\mathscr{A}, \mathscr{B})$ comes equipped with a bimodulation

$$\kappa^{\otimes} \quad : \quad K\,L \quad \longrightarrow \quad L\,K$$

between the two comodulations

$$\begin{array}{lllllllll} L & : & (b,n) & \mapsto & b \otimes n & : & \mathscr{B} \times \mathscr{B} & \longrightarrow & \mathscr{B} \\ K & : & (a,b) & \mapsto & L(a \parr R(b)) & : & \mathscr{A} \times \mathscr{B} & \longrightarrow & \mathscr{B} \end{array}$$

We will see in the following section §5 that this pair of bimodulations $\kappa^{\parr}$ and $\kappa^{\otimes}$ plays a central role in the definition of a linearly distributive chirality.

5 Linearly distributive chiralities

At this point, we are ready to formulate the notion of linearly distributive chirality discussed in the introduction. The reader will find it fully exposed in §5.1. We then proceed by analogy with linearly distributive categories, and introduce in §5.3 a notion of left-handed duality adapted to linearly distributive chiralities. The main technical result of the paper appears in §5.4. We establish that every linearly distributive chirality equipped with a left-handed duality defines a dialogue chirality in the sense of §3.1. Finally, we conclude in §5.5 by showing that the notion of linearly distributive chirality (and of left-handed duality) coincides with the traditional notion of linearly distributive category (and of duality) in the "depolarized" case when the two sides $\mathscr{A}$ and $\mathscr{B}$ of the chirality coincide, and the two functors L and R are equal to the identity functor. We also explain in §5.6 how this depolarization result can be expressed using a category of left-handed exact pairings.

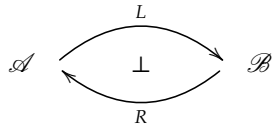
5.1 Definition

We define below the notion of *right-handed* linearly distributive chirality. Since we do not use the *left-handed* variant of the notion here, and unless necessary, we find convenient to shorten the terminology and to simply call it linearly distributive chirality.

Definition 9 (linearly distributive chirality) *A right-handed* linearly distributive chirality *is a pair of monoidal categories*

$$(\mathscr{A}, \parr, \mathit{true}) \qquad\qquad (\mathscr{B}, \otimes, \mathit{false})$$

equipped with an adjunction



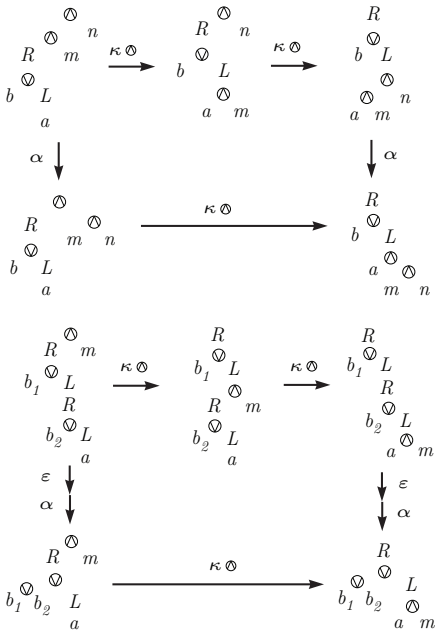
together with two bimodulations

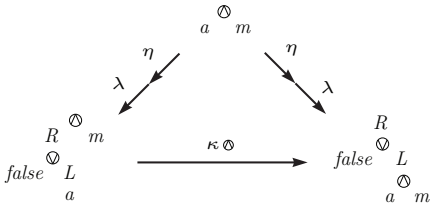
$$\begin{array}{lcccc} \kappa^{\parr} & : & R(b \otimes L(a)) \parr m & \longrightarrow & R(b \otimes L(a \parr m)) \\ \kappa^{\otimes} & : & L(a \parr R(b \otimes n)) & \longrightarrow & L(a \parr R(b)) \otimes n \end{array} \tag{48}$$

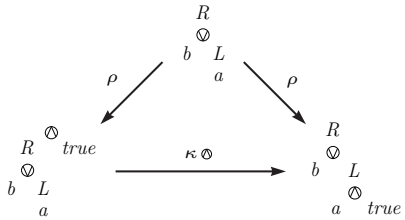
between

- *the $\mathscr{B}$-modulation $T_b : a \mapsto R(b \parr L(a))$ and the $\mathscr{A}$-modulation $S_m : a \mapsto a \parr m$ on the category $\mathscr{A}$,*
- *the $\mathscr{B}$-comodulation $K_n : b \mapsto b \otimes n$ and the $\mathscr{A}$-comodulation $L_a : b \mapsto L(a \parr R(b))$ on the category $\mathscr{B}$.*

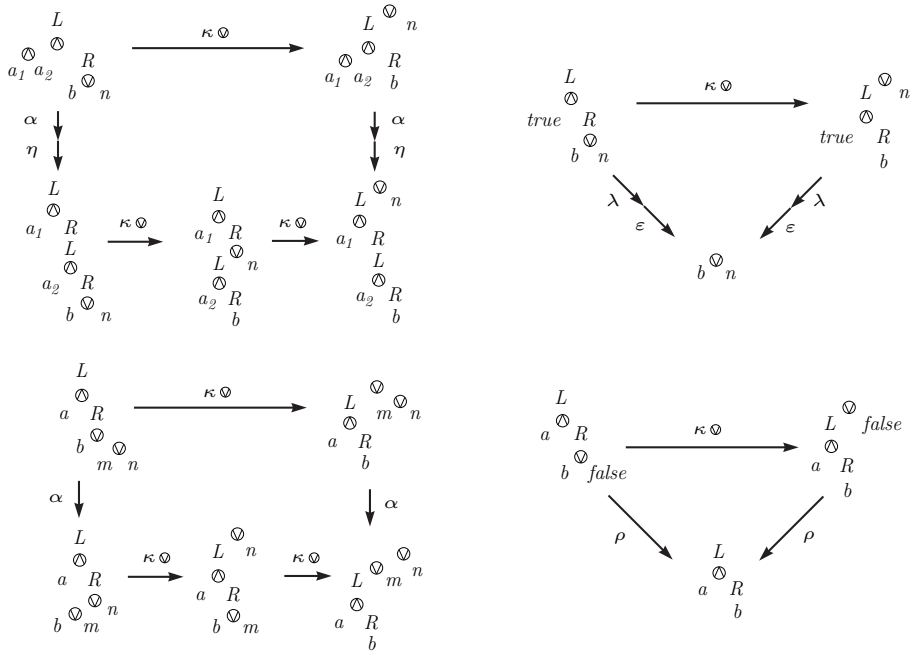
Each definition of a bimodulation requires four diagrams to commute. We find useful to review the $2 \times 4 = 8$ diagrams in turn. First of all, the fact that $\kappa^{\parr}$ defines a bimodulation means that the four diagrams



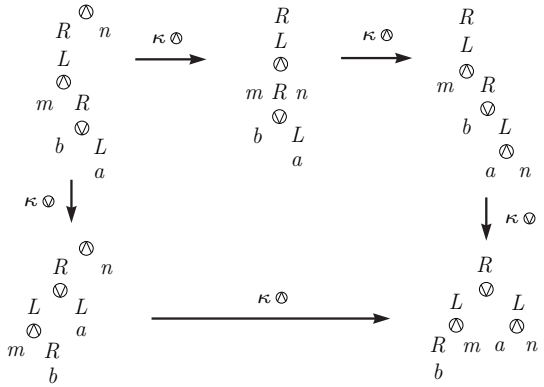




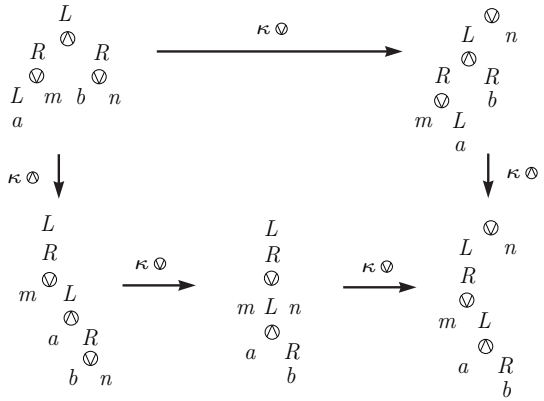
commute for all objects a, m, n in $\mathscr{A}$ and b, b_1, b_2 in $\mathscr{B}$. Symmetrically, the fact that $\kappa^{\otimes}$ defines a bimodulation means that the four diagrams



commute for all objects a, a_1, a_2 in $\mathscr{A}$ and b, m, n in $\mathscr{B}$. Besides this series of $2 \times 4 = 8$ coherence diagrams, we ask that the diagram



commutes for all objects a, m, n of the category $\mathscr{A}$ and all object b of the category $\mathscr{B}$. Symmetrically, we ask that the diagram



commutes for any object a of the category $\mathscr{A}$ and all objects b, m, n of the category $\mathscr{B}$.

In contrast to the $2 \times 4 = 8$ coherence diagrams which are prescribed by the two bimodulations $\kappa^{\parr}$ and $\kappa^{\otimes}$, the algebraic nature of these last two coherence diagrams remains somewhat mysterious at the current stage. In particular, these two diagrams are not justified by the previous discussions on bimodulations since their purpose is precisely to relate the two bimodulations $\kappa^{\parr}$ and $\kappa^{\otimes}$. We know from §4.5 that every right-handed dialogue chirality is equipped with two such bimodulations. It is not difficult to check that these last two diagrams are also satisfied by any right-handed dialogue chirality. From this follows that

Proposition 15 *Every right-handed dialogue chirality defines a right-handed linearly distributive chirality.*

This result leads us to an important question, which is to understand what additional structure on a linearly distributive chirality turns it into a dialogue chirality. The next three sections (from §5.2 to §5.4) are devoted to clarifying this specific point using the notion of left-handed duality.

Remark. Before carrying on, we would like to observe that every linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ becomes a linearly distributive chirality $(\mathscr{B}^{op(1)}, \mathscr{A}^{op(1)})$ after applying the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(1)}$. This preservation property is another instance of the invariance principle discussed in §4.4 which states that *Prover* and *Denier* are given the same argumental power in a logical dispute.

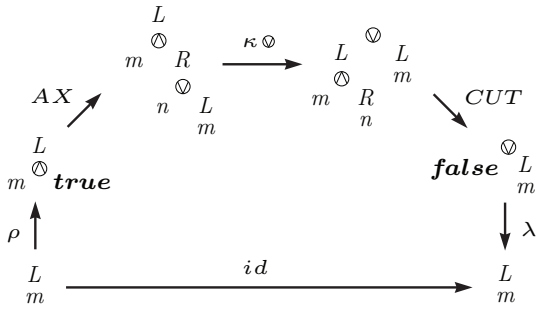
5.2 Left-handed exact pairing

We introduce a notion of left-handed exact pairing $x \dashv y$ which generalizes the notion of exact pairing $x \dashv y$ formulated in §2.2 for linearly distributive categories.

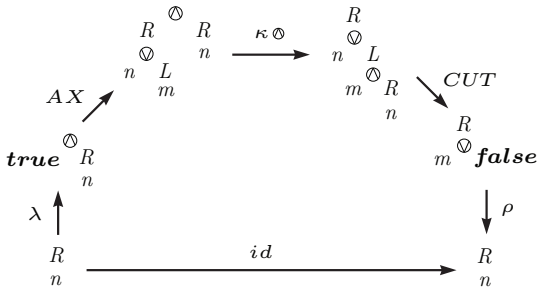
Definition 10 (exact pairing) *A left-handed exact pairing $m \dashv n$ in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ consists of:*

- *an object m of the category $\mathscr{A}$ and an object n of the category $\mathscr{B}$*
- *a pair of morphisms* $\mathbf{AX} : \mathit{true} \longrightarrow R\,(n \parr L\,m)$ *and* $\mathbf{CUT} : L\,(m \otimes R\,n) \longrightarrow \mathit{false}$

making the following diagrams commute:



(49)



(50)

One novelty compared to the original definition of exact pairing is that the two objects m and n are *not* objects of the same category anymore, since m is an object of $\mathscr{A}$ and n is an object of $\mathscr{B}$. The following statement refines Proposition 1 and adapts it to linearly distributive chiralities:

Proposition 16 *Every left-handed exact pairing $m \dashv n$ in a right-handed linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ induces an adjunction*

$$L(m \parr -) \quad \dashv \quad R(n \otimes -)$$

where the functor

$$L(m \parr -) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

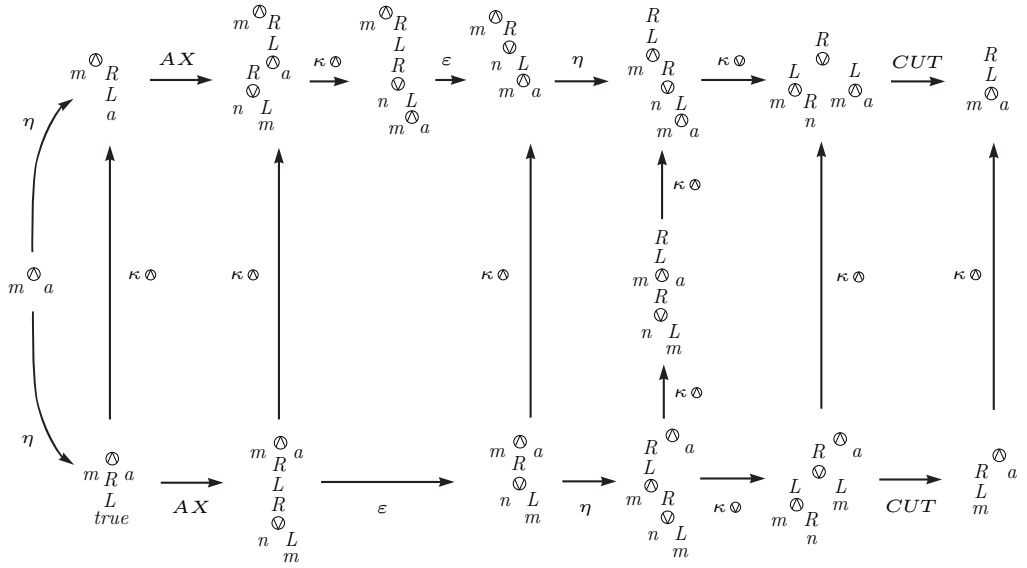
is left adjoint to the functor

$$R(n \otimes -) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}.$$

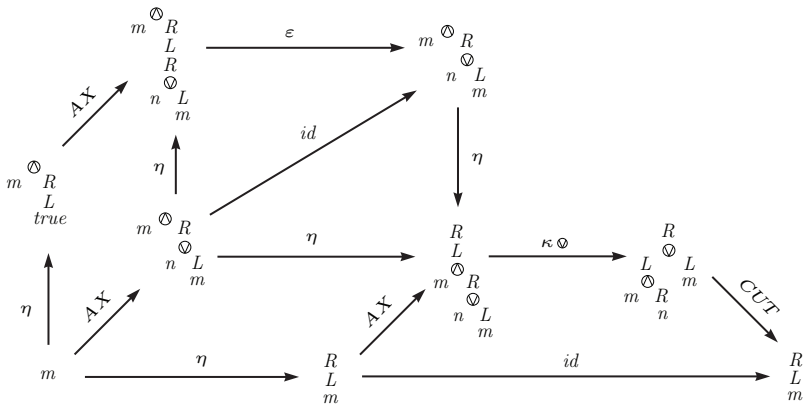
Proof. We have seen in Proposition 14 that such an adjunction may be alternatively formulated as a transjunction between the functors $m \parr -$ and $n \otimes -$ accross the adjunction $L \dashv R$. We find convenient to use this formulation here. We thus introduce the two categorical combinators **axiom** and **cut** characterized by the fact that the diagrams below commute:

$$\begin{array}{ccc} L(R(n \otimes Lm) \parr a) & \xrightarrow{\kappa^{\parr}} & LR(n \otimes L(m \parr a)) \\ {\scriptstyle \mathbf{AX}}\uparrow & & \downarrow{\scriptstyle \varepsilon} \\ L(a) & \xrightarrow{\quad \mathbf{axiom} \quad} & n \otimes L(m \parr a) \end{array} \qquad \begin{array}{ccc} RL(m \parr R(n \otimes b)) & \xrightarrow{\kappa^{\otimes}} & R(L(m \parr Rn) \otimes b) \\ {\scriptstyle \eta}\uparrow & & \downarrow{\scriptstyle \mathbf{CUT}} \\ m \parr R(n \otimes b) & \xrightarrow{\quad \mathbf{cut} \quad} & R(b) \end{array} \tag{51}$$

In order to establish that this pair of combinators defines a transjunction, there simply remains to check that the two coherence diagrams (a) and (b) formulated in (40) do indeed commute. The reader may look at a similar instantiation in (41) of these diagrams (a) and (b) in the case of left-handed dialogue chiralities. To that purpose, one constructs the diagram chase below



completed by the following one:



The reader will easily deduce from these two commutative diagrams that the coherence diagram (*a*) commutes. The fact that the other coherence diagram (*b*) commutes is proved in exactly the same way: one replaces each diagram chase by its image under the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(1)}$ described in §4.4. We take advantage here of the fact that the definition of left-handed exact pairing is itself invariant under this symmetry: indeed, a left-handed exact pairing $m \dashv n$ in the linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is the same thing as a left-handed exact pairing $n \dashv m$ in the linearly distributive chirality $(\mathscr{B}^{op(1)}, \mathscr{A}^{op(1)})$ where the morphisms **AX** and **CUT** have been interverted. This establishes that the pair of morphisms **axiom** and **cut** defines a transjunction accross the adjunction $L \dashv R$. By Proposition 14, this establishes that we have just constructed an adjunction between the functors $L(m \parr -)$ and $R(n \otimes -)$. This concludes the proof of Proposition 16.

Remark. Given a left-handed exact pairing $h \dashv \hbar$ in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$, the statement just established (Prop. 16) provides us with a simple recipe to construct a new linearly distributive chirality with same components $\mathscr{A}$ and $\mathscr{B}$ but with negation functors $\tilde{L}$ and $\tilde{R}$ defined this time as

$$\tilde{L} = a \mapsto L(h \parr a) : \mathscr{A} \to \mathscr{B} \qquad \tilde{R} = b \mapsto R(\hbar \otimes b) : \mathscr{B} \to \mathscr{A}.$$

The adjunction $\tilde{L} \dashv \tilde{R}$ is provided by the construction (51) while the new distributivity laws $\tilde{\kappa}^{\parr}$ and $\tilde{\kappa}^{\otimes}$ are easily deduced from the old distributivity laws $\kappa^{\parr}$ and $\kappa^{\otimes}$. This construction combined with the observations of §5.5 (Prop. 19 and Prop. 20) enables us to construct a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ from any exact pairing $h \dashv \hbar$ in a linearly distributive category $\mathscr{C}$. The monoidal category $\mathscr{A}$ is equal in that case to the category $\mathscr{C}$ equipped with the tensor product $(\otimes)$ while the monoidal category $\mathscr{B}$ is equal to the category $\mathscr{C}$ equipped with the cotensor product $(\parr)$.

5.3 Left-handed duality in linearly distributive chiralities

A left-handed duality in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is defined as a monoidal equivalence

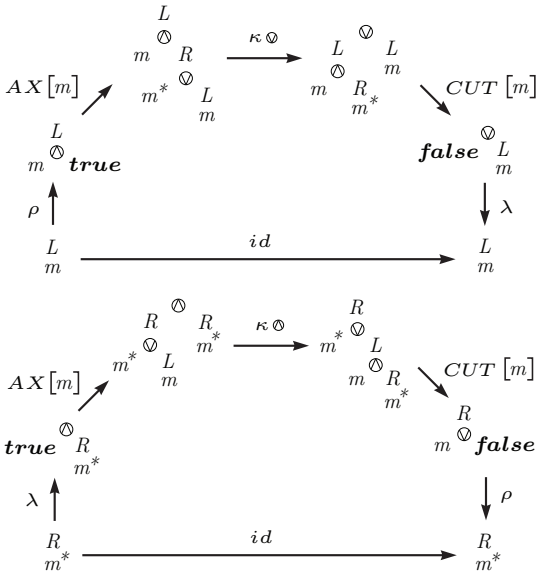
$$\mathscr{A} \quad \underset{{}^*(-)}{\overset{(-)^*}{\rightleftarrows}} \quad \mathscr{B}^{op(0,1)} \qquad \text{(monoidal equivalence)} \tag{52}$$

together with two families of morphisms

$$\mathbf{AX}[m] \quad : \quad true \quad \longrightarrow \quad R\,(m^* \otimes L\,(m))$$

$$\mathbf{CUT}[m] \quad : \quad L\,(\,m \parr R\,(\,m^*\,)\,) \quad \longrightarrow \quad \mathit{false}$$

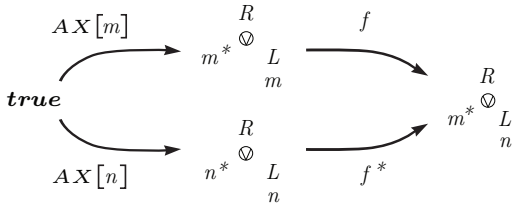
parametrized by the objects m of the category $\mathscr{A}$. These morphisms are required to make a series of 4×2 coherence diagrams commute. Each pair of coherence diagrams consists of a diagram and of its image under the symmetry $\mathscr{A} \leftrightarrow \mathscr{B}^{op(1)}$. The first pair of coherence diagrams expresses the fact that the combinators **AX**$[m]$ and **CUT**$[m]$ define a left-handed exact pairing $m \dashv m^*$ in the linearly distributive chirality $(\mathscr{A},\mathscr{B})$ in the sense elaborated in §5.2:



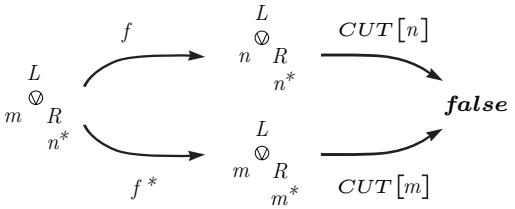
(53)

(54)

The second pair of coherence diagrams expresses the fact that the two combinators **AX**$[-]$ and **CUT**$[-]$ are natural:

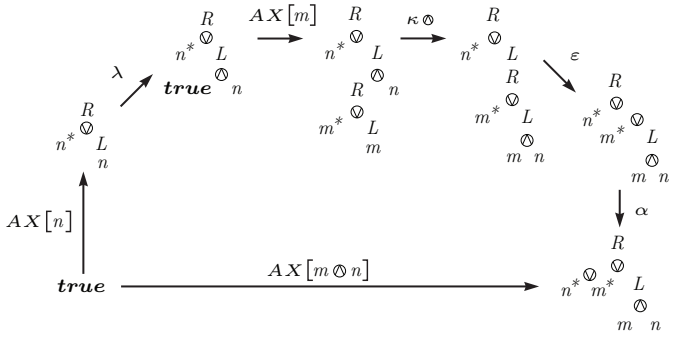


(55)

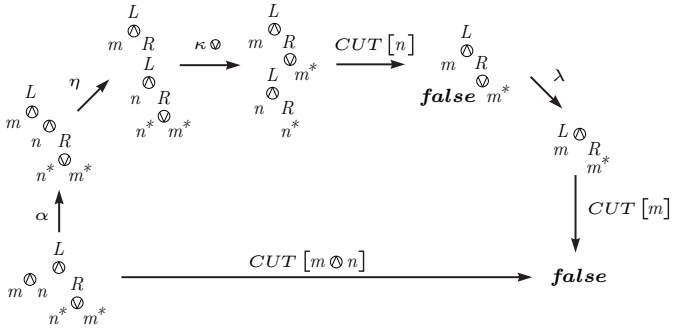


(56)

The third pair of coherence diagrams expresses the fact that the combinators **AX**[−] and **CUT**[−] are monoidal:

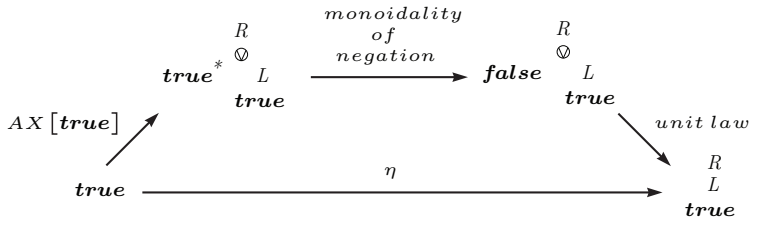


(57)

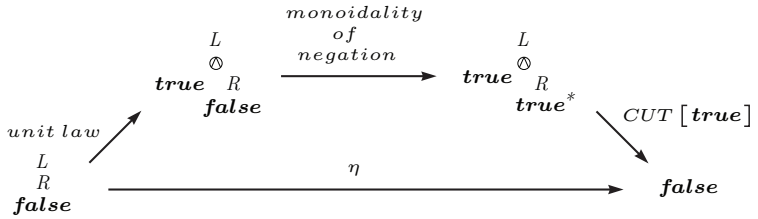


(58)

The fourth and last pair of coherence diagrams expresses the fact that the combinators **AX**[−] and **CUT**[−] are monoidal, this time with respect to the units:



(59)



(60)

As expected, these diagrams are required to commute for all objects m, n and all morphisms $f : m \to n$ of the category $\mathscr{A}$.

5.4 Main statement

At this stage, we are ready to establish the main result of the paper for the notion of *right-handed* linearly distributive chirality introduced in Def. 9.

Proposition 17 *Every linearly distributive chirality* $(\mathscr{A}, \mathscr{B})$ *equipped with a left-handed duality defines a left-handed dialogue chirality.*

This statement provides a natural justification for the two notions of linearly distributive chirality and of left-handed duality, which have been carefully carved in order to establish the result. We start by establishing an important preliminary property:

Proposition 18 *In a right-handed linearly distributive chirality $(\mathscr{A},\mathscr{B})$ equipped with a left-handed duality, the functor*

$$L(m \parr -) \quad : \quad \mathscr{A} \quad \longrightarrow \quad \mathscr{B}$$

is left adjoint to the functor

$$R(m^* \otimes -) \quad : \quad \mathscr{B} \quad \longrightarrow \quad \mathscr{A}$$

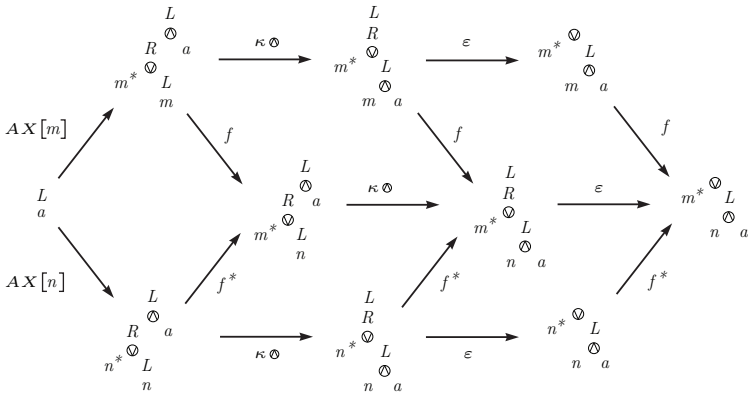
for every object m of the category $\mathscr{A}$. The family of adjunctions is moreover natural in the parameter m.

Proof. First of all, we define the two categorical combinators **axiom**$[-]$ and **cut**$[-]$ in such a way as to make the diagrams below commute:

$$\begin{array}{ccc} L(R(m^* \otimes Lm) \parr a) & \xrightarrow{\kappa^{\parr}} & LR(m^* \otimes L(m \parr a)) \\ {\scriptstyle \mathbf{AX}[m]}\uparrow & & \downarrow{\scriptstyle \varepsilon} \\ L(a) & \xrightarrow{\mathbf{axiom}[m]} & m^* \otimes L(m \parr a) \end{array}$$

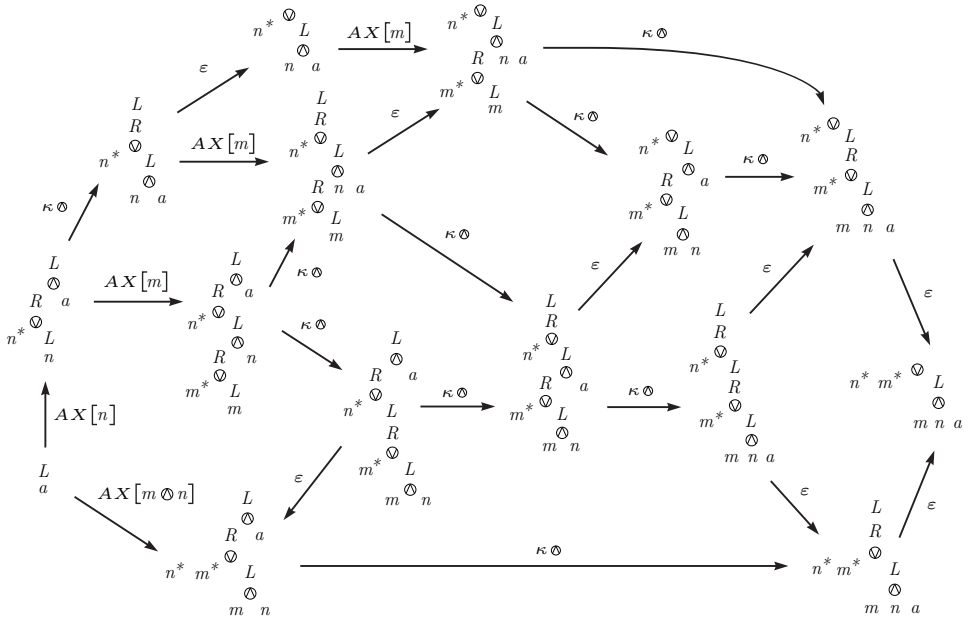
$$\begin{array}{ccc} RL(m \parr R(m^* \otimes b)) & \xrightarrow{\kappa^{\otimes}} & R(L(m \parr R(m^*)) \otimes b) \\ {\scriptstyle \eta}\uparrow & & \downarrow{\scriptstyle \mathbf{CUT}[m]} \\ m \parr R(m^* \otimes b) & \xrightarrow{\mathbf{cut}[m]} & R(b) \end{array}$$

The two first coherence diagrams mean that the morphisms **AX**$[m]$ and **CUT**$[m]$ define a left-handed exact pairing $m \dashv m^*$ in the linearly distributive chirality $(\mathscr{A},\mathscr{B})$. We deduce from Proposition 1 that the pair of morphisms **axiom**$[m]$ and **cut**$[m]$ defines a transjunction $(m \parr -) \multimap (m^* \otimes -)$ accross the adjunction $L \dashv R$ for every object m of the category $\mathscr{A}$. This establishes the first part of the statement. The fact that the resulting family of adjunctions $L(m \parr -) \dashv R(m^* \otimes -)$ is natural in m is easily deduced from the fact that the families of combinators **axiom**$[-]$ and **cut**$[-]$ are natural in m. Typically, the naturality of the family **axiom**$[-]$ in m is established by the diagram chase below:



This concludes the proof of Proposition 18.

Proof of Proposition 17. At this stage, we have proved that the functor $L(m \parr -)$ is left adjoint to the functor $R(m^* \otimes -)$ for every object m of the category $\mathscr{A}$, and that the family of adjunctions is natural. So, in order to establish the property, there only remains to show the additional fact that the family **axiom**$[-]$ is monoidal in the sense of Proposition 13. This is proved by the diagram chase below:



This concludes the proof that the family of transjunctions **axiom**[−] and **cut**[−] provides a left-handed dialogue chirality structure to the linearly distributive chirality $(\mathscr{A}, \mathscr{B})$.

Remark. The reader will observe that only four of the 4×2 coherence diagrams required of a left-handed duality are used in the proof of Proposition 17: namely equations (53), (54), (55) and (57). We keep the four other coherence diagrams in the definition of a left-handed duality because each of them makes perfect sense, either as the mirror image of another coherence diagram in the case of (56) and (58) ; or as the degenerate case for units of another coherence diagram in the case of (59) and (60). On the other hand, it follows from Proposition 17 that the four coherence diagrams (56), (58), (59) and (60) are automatically valid when the coherence diagrams (53), (54), (55) and (57) hold in a linearly distributive chirality.

5.5 A comparison with linearly distributive categories

Following the general philosophy of depolarization, we would like to recover the two notions of "linearly distributive category" and of "duality in a linearly distributive category" by depolarizing the corresponding notions in tensorial logic. In order to proceed with the comparison, we formally introduce the notion of *depolarized* linearly distributive chirality.

Definition 11 *A linearly distributive chirality is called* self-dual *when its two sides $\mathscr{A}$ and $\mathscr{B}$ are equal as categories (but not necessarily as monoidal categories). A linearly distributive chirality is called* depolarized *when it is self-dual and when the two functors L and R are the identity functors.*

A particular case of such depolarization occurs when the double negation monad $A \mapsto \neg\neg A$ of a given symmetric dialogue category $\mathscr{C}$ is idempotent (or equivalently commutative). In that case, the category $\mathscr{A}$ of algebras of the double negation monad coincides with the Kleisli category of the monad, and is $*$-autonomous. The category $\mathscr{B}$ may be thus taken as equal to $\mathscr{A}$ with the functor L and R defined as the identity. As a matter of fact, it is essentially immediate to check that

Proposition 19 *A linearly distributive category is the same thing as a depolarized linearly distributive chirality.*

The proof is based on a direct comparison of the coherence diagrams for linearly distributive category in §2.1 and for linearly distributive chirality in §5.1. It is easy to show that there is a one-to-one relationship between them, where each diagram §2.1 is obtained by depolarizing the corresponding diagram in §5.1. Similarly,

Proposition 20 *The notion of duality in a linearly distributive category coincides with the notion of left-handed duality in a depolarized linearly distributive chirality.*

Proof. Suppose given a linear distributive category $\mathscr{C}$ and the corresponding depolarized linearly distributive chirality $(\mathscr{A},\mathscr{B})$ with $\mathscr{A} = (\mathscr{C},\otimes,\mathbf{1})$ and $\mathscr{B} = (\mathscr{C},\parr,\bot)$. The correspondence between the notion of duality in $\mathscr{C}$ and the notion of left-handed duality in $(\mathscr{C},\mathscr{C})$ is immediate when one takes the definition of duality in a linearly distributive category as reformulated in §2.6, Proposition 11. Under that correspondence, the monoidal equivalence (52) between $\mathscr{A}$ and $\mathscr{B}^{op(0,1)}$ coincides with the monoidal equivalence (26) between $(\mathscr{C},\otimes,\mathbf{1})$ and $(\mathscr{C},\parr,\bot)^{op(0,1)}$, while the coherence diagrams are associated in the following way:

(13) ↔ (53) (54) (19) ↔ (55) (56) (23) ↔ (57) (24) ↔ (58) (25) ↔ (59) (60)

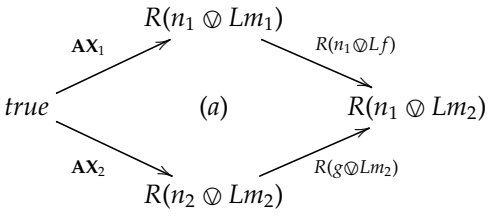
This dictionary clarifies in what sense the notion of "left-handed duality" in a linearly distributive chirality refines the traditional notion of "duality" in a linearly distributive category.

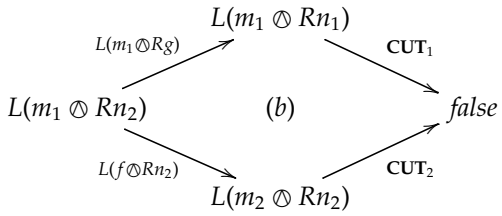
5.6 The category of left-handed exact pairings

To conclude the section, we construct the category $(\mathscr{A},\mathscr{B})^{\mathbf{ep}}$ of left-handed exact pairings in a right-handed linearly distributive chirality $(\mathscr{A},\mathscr{B})$. The objects of the category $(\mathscr{A},\mathscr{B})^{\mathbf{ep}}$ are the left-exact exact pairings $m \dashv n$ of the chirality $(\mathscr{A},\mathscr{B})$, and its morphisms

$$(f,g) \quad : \quad m_1 \dashv n_1 \quad \longrightarrow \quad m_2 \dashv n_2$$

are the pairs (f,g) of morphisms $f : m_1 \to m_2$ in the category $\mathscr{A}$ and $g : n_2 \to n_1$ in the category $\mathscr{B}$ such that the two diagrams below commute:





One also retrieves the two projection functors (17) encountered in §2.5 and adapted to the 2-sided framework of linearly distributive chiralities:

$$\begin{array}{rccc} \pi_{\text{left}} & : & (\mathscr{A},\mathscr{B})^{\mathbf{ep}} & \longrightarrow & \mathscr{A} \\ & & m \dashv n & \mapsto & m \end{array} \qquad \begin{array}{rccc} \pi_{\text{right}} & : & (\mathscr{A},\mathscr{B})^{\mathbf{ep}} & \longrightarrow & \mathscr{B}^{op} \\ & & m \dashv n & \mapsto & n \end{array} \tag{61}$$

One main difference with the construction of the category $\mathscr{C}^{\mathbf{ep}}$ associated to a linearly distributive category $\mathscr{C}$ is that the morphism $f : m_1 \to m_2$ does not determine the morphism $g : n_2 \to n_1$, and conversely, in the case of a linearly distributive chirality. Compare this with Proposition 4 in §2.5. Except for that, the situation is very similar. In particular, we establish a variant of Proposition 5:

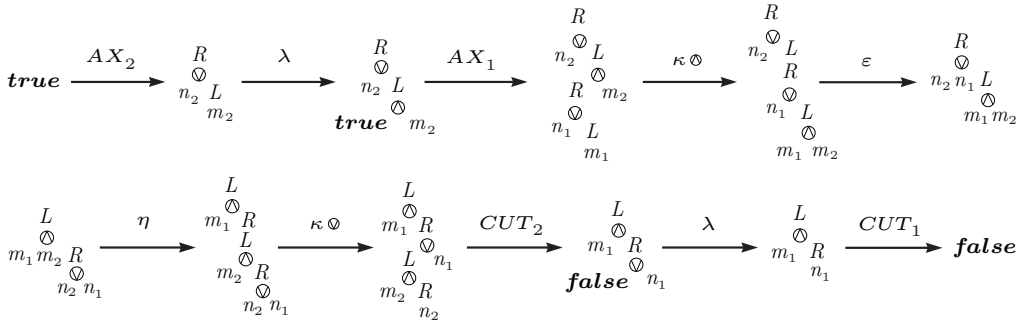
Proposition 21 *Every pair of left-handed exact pairings $m_1 \dashv n_1$ and $m_2 \dashv n_2$ induces a left-handed exact pairing*

$$m_1 \with m_2 \dashv n_2 \parr n_1$$

with structure morphisms

$$\mathbf{AX} : \mathit{true} \longrightarrow R((n_2 \parr n_1) \parr L(m_1 \with m_2)) \qquad \mathbf{CUT} : L((m_1 \with m_2) \with R(n_2 \parr n_1)) \longrightarrow \mathit{false}$$

defined as the composite:



From this follows that for every right-handed linearly distributive chirality $(\mathscr{A}, \mathscr{B})$,

Proposition 22 *The category $(\mathscr{A}, \mathscr{B})^{\mathbf{ep}}$ is monoidal, with tensor product $\otimes$ defined in Prop. 21.*

With the tensor product of the category $(\mathscr{A}, \mathscr{B})^{\mathbf{ep}}$ defined in this way, the two projection functors defined in (61) become monoidal functors:

$$\pi_{\text{left}} : (\mathscr{A}, \mathscr{B})^{\mathbf{ep}} \longrightarrow (\mathscr{A}, \with, \mathit{true}) \qquad \pi_{\text{right}} : (\mathscr{A}, \mathscr{B})^{\mathbf{ep}} \longrightarrow (\mathscr{B}, \parr, \mathit{false})^{op(0,1)}$$

The reader will observe that

Proposition 23 *The monoidal category $(\mathscr{A}, \mathscr{B})^{\mathbf{ep}}$ coincides with the monoidal category $\mathscr{C}^{\mathbf{ep}}$ equipped with its tensor product $\otimes$ in a depolarized linearly distributive chirality $(\mathscr{A}, \mathscr{B}) = (\mathscr{C}, \mathscr{C})$.*

This leads us to another formulation of left-handed dualities in a right-handed linearly distributive chirality $(\mathscr{A}, \mathscr{B})$, following the conceptual track of Proposition 10:

Proposition 24 *A left-handed duality in a linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ is alternatively defined as a pair of monoidal functors*

$$\mathit{rightdual} : (\mathscr{A}, \with, \mathit{true}) \longrightarrow (\mathscr{A}, \mathscr{B})^{\mathbf{ep}} \qquad \mathit{leftdual} : (\mathscr{B}, \parr, \mathit{false})^{op(0,1)} \longrightarrow (\mathscr{A}, \mathscr{B})^{\mathbf{ep}}$$

defining sections (in the monoidal sense) of the monoidal projection functors (17).

In that formulation, the two monoidal functors $(-)^*$ and ${}^*(-)$ are recovered by applying the same recipe (18) and (20) as in §2.6 and by defining them as:

$$\begin{array}{rcll} (-)^* & = & \pi_{\text{right}} \circ \mathit{rightdual} & : (\mathscr{A}, \with, \mathit{true}) \longrightarrow \mathscr{C}^{\mathbf{ep}} \longrightarrow (\mathscr{B}, \parr, \mathit{false})^{op(0,1)} \\ {}^*(-) & = & \pi_{\text{left}} \circ \mathit{leftdual} & : (\mathscr{B}, \parr, \mathit{false})^{op(0,1)} \longrightarrow \mathscr{C}^{\mathbf{ep}} \longrightarrow (\mathscr{A}, \with, \mathit{true}) \end{array}$$

This alternative formulation of left-handed dualities in linearly distributive chiralities provides us with another means to establish Proposition 20. Indeed, it appears that one recovers the definition of "duality" in Proposition 10 from the definition of "left-handed duality" formulated in Proposition 24 in the specific (and degenerate) case of a depolarized linearly distributive chirality $(\mathscr{A}, \mathscr{B}) = (\mathscr{C}, \mathscr{C})$.

6 Concluding remarks

Despite the work done in this paper, we are still halfway of the complete story. There is indeed a serious technical obstruction to a full resolution of the original problem. The reason is that we investigate in this paper the most primitive notion of dialogue category where the tensor product $\otimes$ is not equipped with any pivotal or braiding structure. This lack of structure implies that there are two *intrinsically different* choices of adjunctions $L \dashv R$. The first choice $L(x) = \bot \multimap x$ and $R(x) = x \multimap \bot$ induces a left-handed dialogue chirality (see §3.1) whereas the second choice $L(x) = x \multimap \bot$ and $R(x) = \bot \multimap x$ induces a right-handed dialogue chirality (see §4.3). Although this distinction between left-handed and right-handed dialogue chiralities is legitimate and even fundamental, it induces a subtle mismatch in the present paper:

- on the one hand, we observe (see Prop. 15 in §5.1) that, as should be expected, every *right-handed* dialogue chirality defines a *right-handed* linearly distributive chirality,
- then, we establish in Proposition 17 (see §5.4) that in the presence of a left-handed duality, every right-handed linearly distributive chirality defines a *left-handed* dialogue chirality, rather than a *right-handed* dialogue chirality as one would have naively thought...

This mismatch explains why we keep Proposition 17 (in §5.4) as unsatisfactory as it stands, and why we do not even attempt to establish that a dialogue chirality is *the same thing* as a linearly distributive chirality equipped with an appropriate notion of left-handed and/or right-handed duality. A tentative solution would be to shift to an extended notion of dialogue category $\mathscr{C}$ with a *binary* rather than a *unary* negation, equipped with a two-sided version of currification

$$\varphi_{m,x,n} \quad : \quad \mathscr{C}(m \otimes x \otimes n, \bot) \quad \cong \quad \mathscr{C}(x, m \multimap \bot \circ\!\!- n)$$

The solution is appealing but probably not sufficient. We thus prefer to explore a more sophisticated direction, based on the idea that the left and the right negations $x \multimap \bot$ and $\bot \circ\!\!- x$ should coincide up to isomorphism. This track leads us to develop the notion of *pivotal dialogue category* where the two negations are related by a natural isomorphism

$$\mathbf{turn}_A \quad : \quad A \multimap \bot \quad \longrightarrow \quad \bot \circ\!\!- A$$

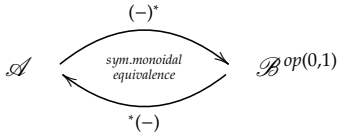
satisfying a carefully chosen coherence diagram. This pivotal structure leads us to a topological refinement of traditional game semantics, where the control flow of the tensorial net represented by the functors L and R may be "twisted" in the same way as a ribbon. Following the principle of depolarization, one recovers the familiar notion of cyclic proof-net of non-commutative linear logic [16] when one specializes to the situation of a self-dual dialogue chirality where the two sides $\mathscr{A}$ and $\mathscr{B}$ coincide, and where the two functors L and R are the identity. These issues are treated in the companion paper [14].

However, we understand that the reader may find frustrating to have done such a long journey without getting in the end a perfectly clear resolution of the questions discussed in the introduction. We thus mention that a satisfactory answer exists in the specific case of a *symmetric* dialogue category. Let us start by the case of $*$-autonomous categories. Recall that a linearly distributive category $\mathscr{C}$ is called symmetric when its tensor product is equipped with a symmetry.

Proposition 25 (Cockett-Seely) *A symmetric $*$-autonomous category $\mathscr{C}$ is the same thing as a symmetric linearly distributive category $\mathscr{C}$ equipped with a duality.*

This statement is a symmetric variant of Prop. 3 in §2.4. It may be adapted to the case of dialogue chiralities in the following way. We declare that a dialogue chirality $(\mathscr{A}, \mathscr{B})$ is *symmetric* when the monoidal categories $(\mathscr{A}, \otimes, \mathit{true})$ and $(\mathscr{B}, \otimes, \mathit{false})$ are both equipped with

a symmetry, and when the original equivalence of monoidal categories (11) is an equivalence of *symmetric* monoidal categories:



$$\mathscr{A} \qquad \mathscr{B}^{op(0,1)} \tag{62}$$

Note that the definition of symmetric dialogue chirality $(\mathscr{A}, \mathscr{B})$ does not depend on the orientation of the original dialogue chirality, which may be equivalently taken left-handed or right-handed. Similarly, we declare that a linearly distributive chirality $\mathscr{C}$ is *symmetric* when the monoidal categories $(\mathscr{A}, \parr, \mathit{true})$ and $(\mathscr{B}, \otimes, \mathit{false})$ are both equipped with a symmetry. Again, the linearly distributive chirality $\mathscr{C}$ may be equivalently taken left-handed or right-handed in the definition. The key theorem in the symmetric case is that:

Proposition 26 *A symmetric dialogue chirality $(\mathscr{A}, \mathscr{B})$ is the same thing as a symmetric linearly distributive chirality $(\mathscr{A}, \mathscr{B})$ equipped with a duality.*

Here, by duality in a symmetric linearly distributive chirality, we mean a left-handed duality in the sense of §5.3. Of course, because the underlying dialogue chirality is symmetric, a duality may be equivalently defined as a "right-handed duality", given by a pair of combinators

$$\overline{\mathbf{AX}}[m] \quad : \quad \mathit{true} \quad \longrightarrow \quad R\,(\,L\,(\,m\,) \parr m^*\,)$$

$$\overline{\mathbf{CUT}}[m] \quad : \quad L\,(\,R\,(\,m^*\,) \otimes m\,) \quad \longrightarrow \quad \mathit{false}$$

satisfying the expected coherence properties. This statement closes the circle in the case of symmetric dialogue categories. Indeed, a straightforward adaptation of [12] establishes that the notion of symmetric dialogue chirality discussed here is equivalent to the original notion of symmetric dialogue category, in an appropriate 2-categorical sense.

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