HW5 Randomized algorithms & random structures

MPRI 1.24 Tue. Dec. 6, 2016 - **Due on Tue. Dec. 13**

You are asked to complete the exercise(s) marked with a [★] and to send me your solutions at:
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using subject "1.24-HW5" for your email and naming your file "1-24-HW5-your_name.pdf".
(or drop it in my mail box at the 4th floor of Sophie Germain) on **Tue. Dec. 13**.

■ **Exercise 1 (FPT algorithm for spotting k disjoint triangles).** [★]
Given $G = (V, E)$ an undirected graph ($n = |V|$ and $m = |E|$) and k an integer, we are looking for k vertex-disjoint triangles in G. Note that this problem is NP-complete when k is part of the input. We are looking for an algorithm of time complexity $O(f(k)n^a m^b)$ where the exponents a and b are constant, independent of k. Such an algorithm is called FPT for Fixed Parameter Tractable, which means that the complexity is a fixed-degree polynomial in the size of the input for any fixed value of the parameter k. Consider the following randomized algorithm:

Algorithm 1 FPT randomized algorithm for k disjoint triangles

- Choose independently for each vertex u a color $c_u \in \{1, \dots, 3k\}$ uniformly at random.
- **return** "Yes" if there is a *colorful* solution, i.e. a set of k triangles whose $3k$ vertices use exactly once each color; **return** "I don't know" otherwise.

► ***Question* 1.1)** *Show that if G contains k disjoint triangles, then the probability that the algorithm answer "Yes" is at least e^{-3k}.*
▷ Hint. *use that : $k! \geqslant (k/e)^k$ for all k.*

► ***Question* 1.2)** *How many times should you run this algorithm to improve success probability to $1/2$?*

In order to check whether a colorful solution exists, we propose to try all permutations π on $\{1, \dots, 3k\}$ and check if there is are triangles of colors $(\pi_1, \pi_2, \pi_3), \dots, (\pi_{3k-2}, \pi_{3k-1}, \pi_{3k})$.

► ***Question* 1.3)** *Describe an algorithm that decides if such a collection of triangles exists. (Explicit the exact data structure you are using). What is the overall expected time complexity in k, n and m, of the algorithm that uses this method to return k disjoint triangles with probability at least $1/2$ if they exists in G? What is the time complexity if k is fixed?*