Automata, (semi)groups, dualities

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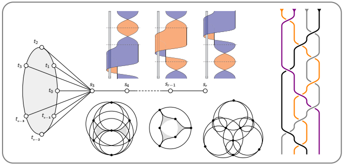
MealyM
JCJC-12-JS02-012-01

habilitation à diriger des recherches
amphithéâtre Alan Turing
10 juillet 2017

Braid (semi)groups
& Garside theory



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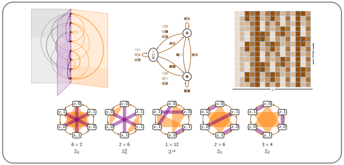

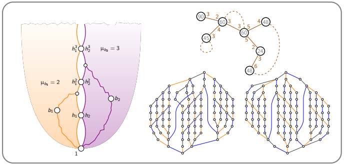

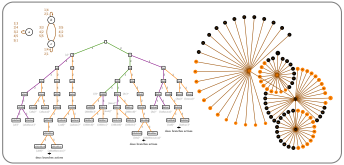

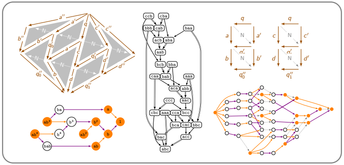

Quadratic normalisations
Thurston *vs* Mealy automata

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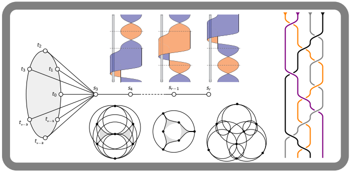

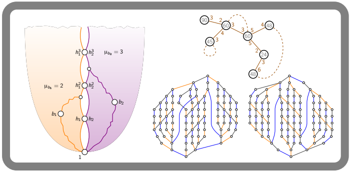

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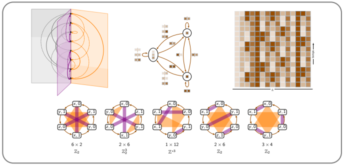

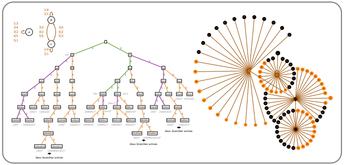

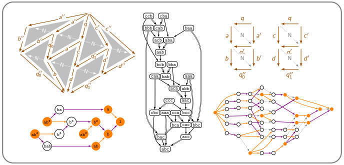

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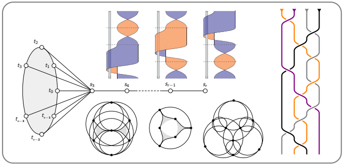

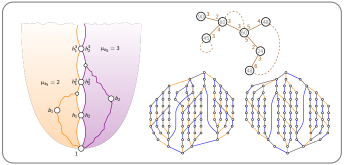

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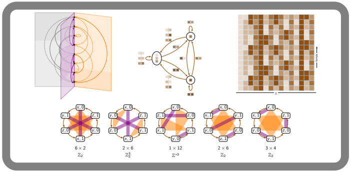

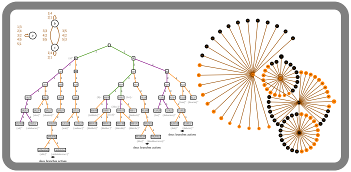

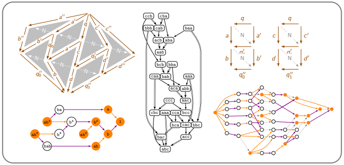

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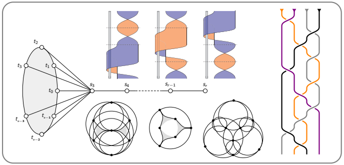

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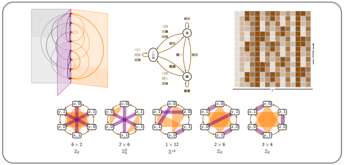

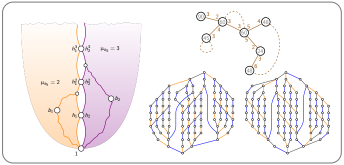

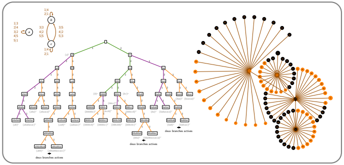

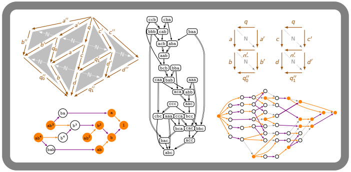

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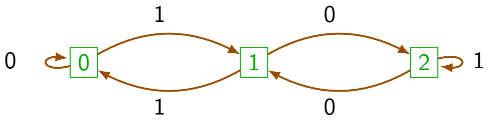
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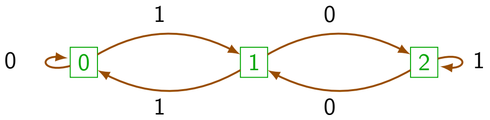

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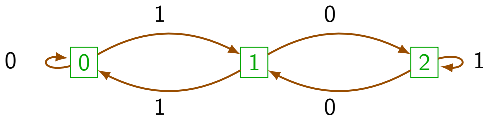

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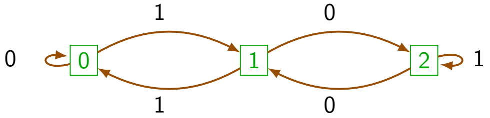
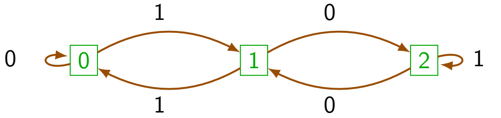
$0 \longrightarrow 0$

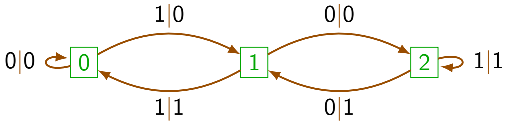


0 1 0 0 1 1

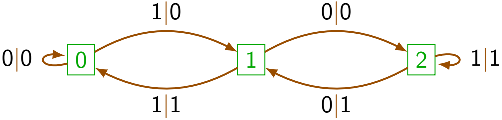
$0 \longrightarrow 0 \longrightarrow 1$



$$0 \xrightarrow{0} 0 \xrightarrow{1} 1 \xrightarrow{0} 2 \xrightarrow{0} 1 \xrightarrow{1} 0 \xrightarrow{1} 1$$



$$0 \xrightarrow{0} 0 \xrightarrow{1} 1 \xrightarrow{0} 2 \xrightarrow{0} 1 \xrightarrow{1} 0 \xrightarrow{1} 1$$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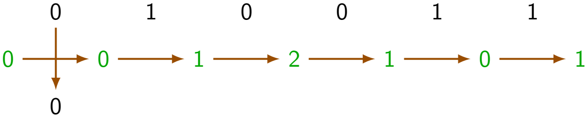
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0 0 1 2 1 0 1
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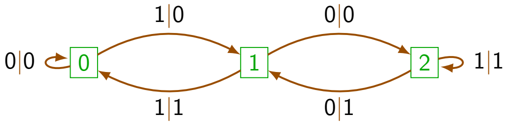
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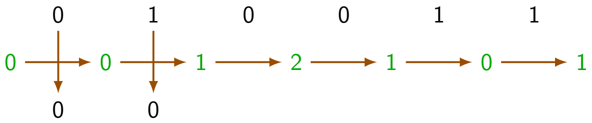
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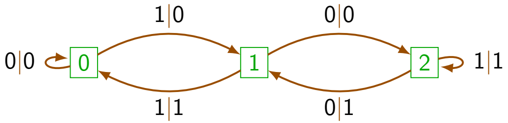
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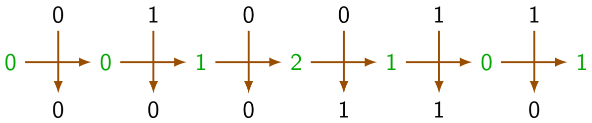
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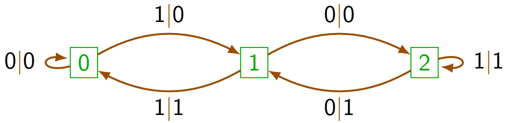

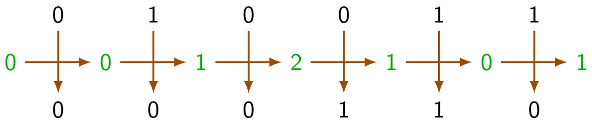

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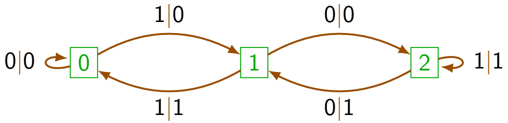
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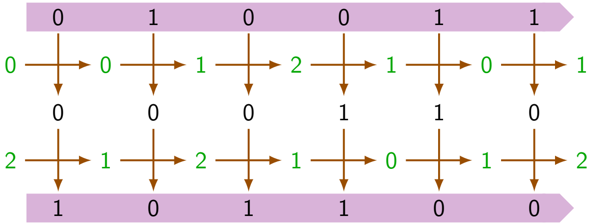
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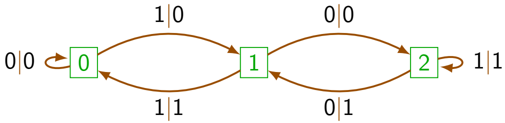

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0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 1 \\
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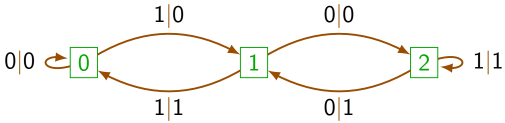
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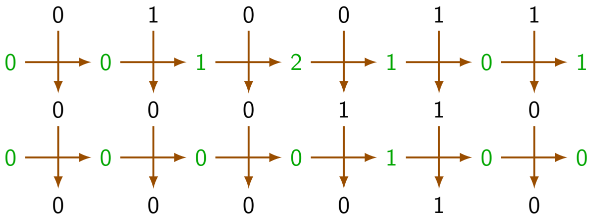
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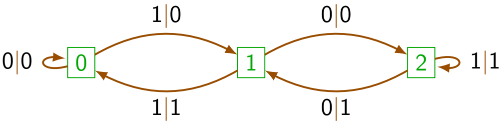
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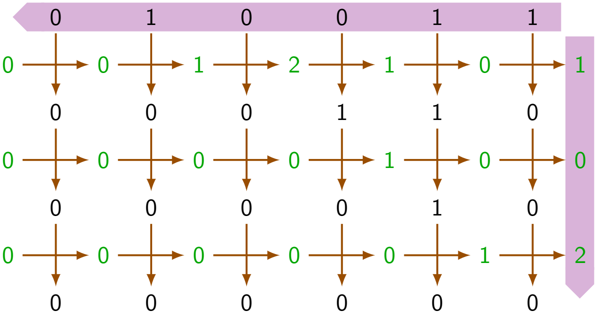
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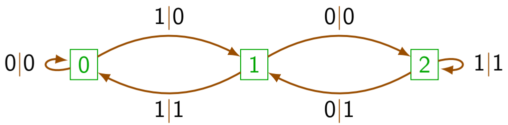

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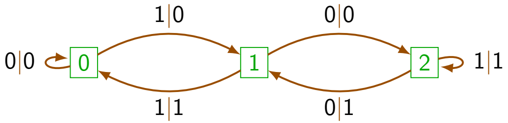
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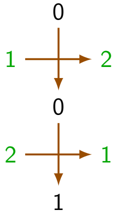
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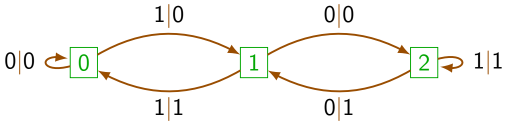
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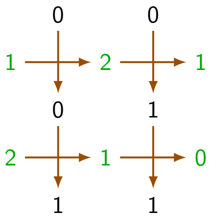
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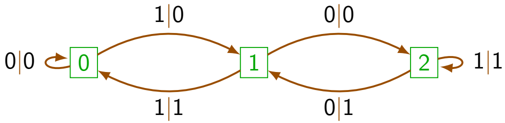
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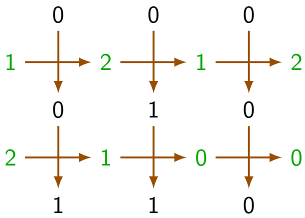
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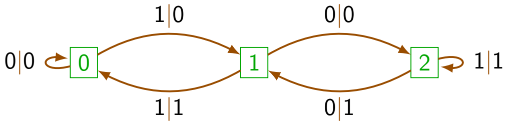
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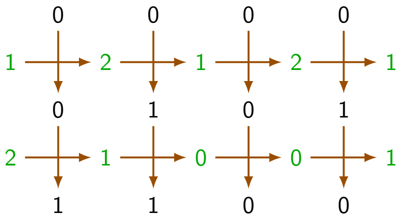
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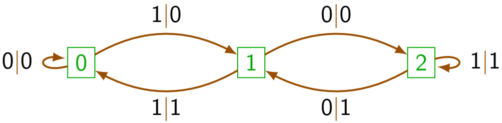
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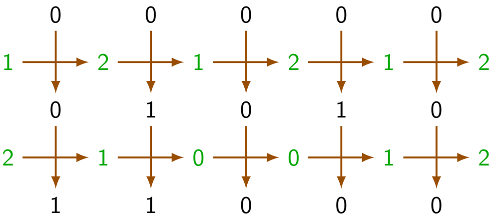
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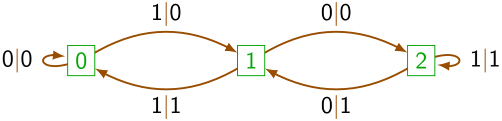

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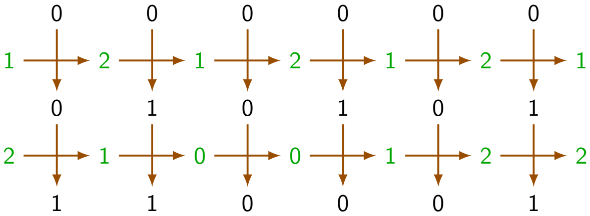

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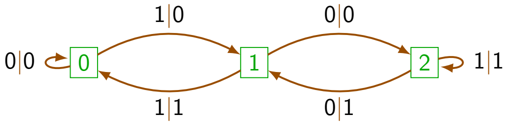
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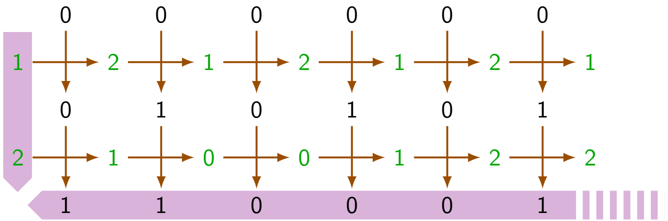
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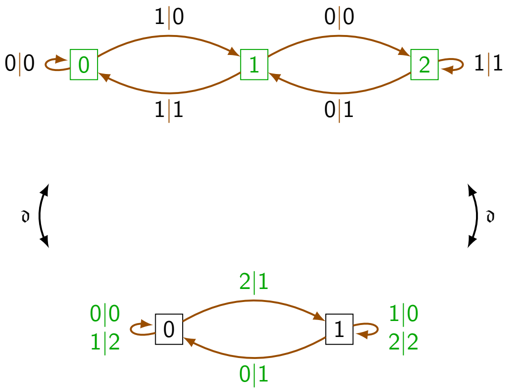
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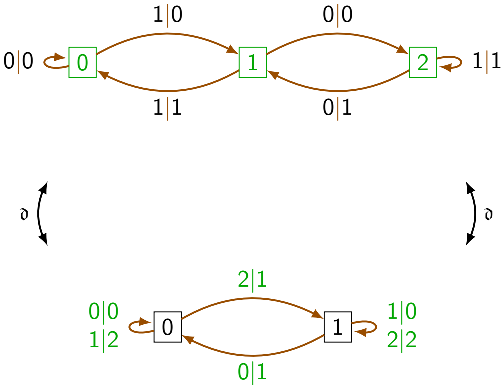
the actions induced by 0, 1, and 2
generate a rank 3 free semigroup



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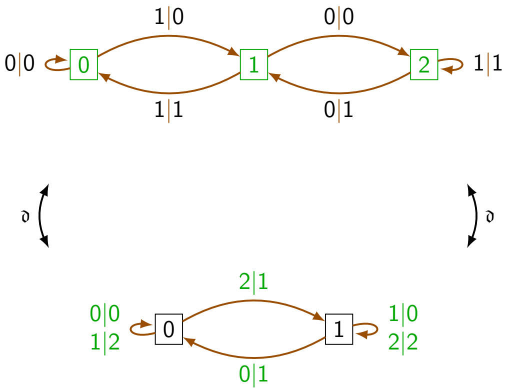
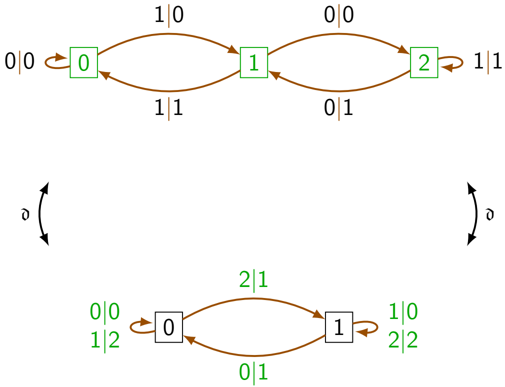


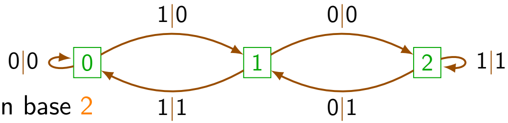
the actions induced by 0 and 1
generate a rank 2 free semigroup

the actions induced by 0, 1, and 2
generate a rank 3 free semigroup



the actions induced by 0 and 1
generate a rank 2 free semigroup
and a group isomorphic
to $\mathrm{BS}(1,2) = \langle\ \alpha, \delta : \alpha\delta = \delta\alpha^2\ \rangle$

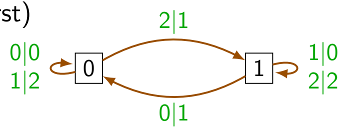
the actions induced by 0, 1, and 2
generate a rank 3 free semigroup



division by 3 in base 2
(most significant digit first)

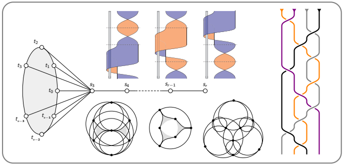
$\mathfrak{d}$

multiplication by 2 in base 3
(less significant digit first)

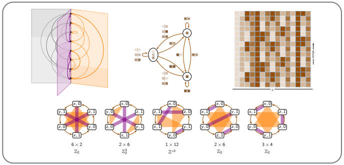


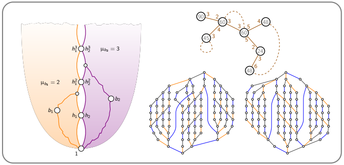
the actions induced by 0 and 1
generate a rank 2 free semigroup
and a group isomorphic
to $\mathrm{BS}(1,2) = \langle\ \alpha, \delta : \alpha\delta = \delta\alpha^2\ \rangle$

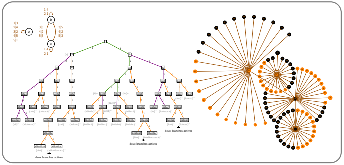
Braid (semi)groups
& Garside theory

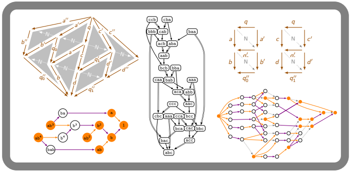


Mealy automata
& automaton (semi)groups



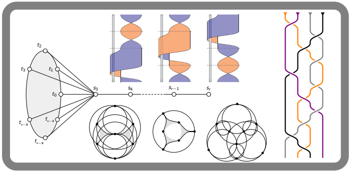






Quadratic normalisations
Thurston *vs* Mealy automata

Braid (semi)groups
& Garside theory



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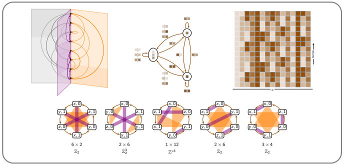

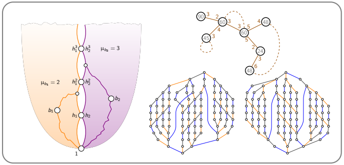

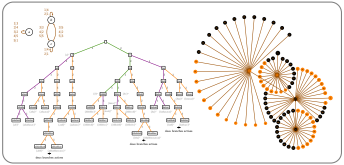

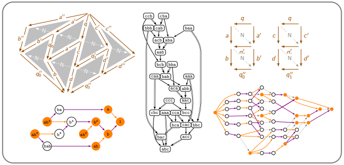

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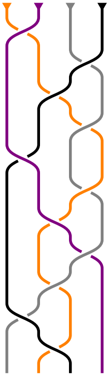
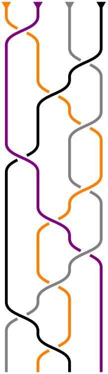
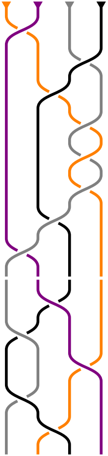


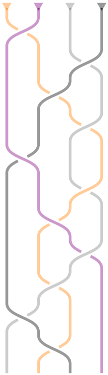


$\rightsquigarrow$



a positive braid

a negative braid



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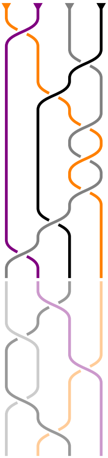

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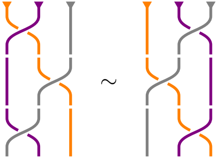
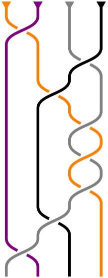
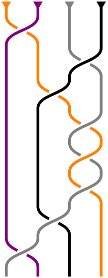




a positive braid

$\sim$



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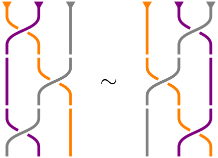
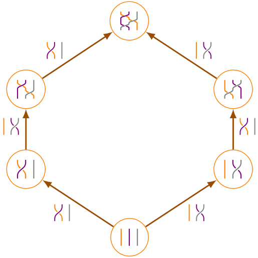
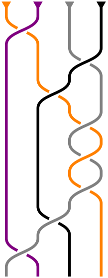
Garside 1965

The braid group $\mathbf{B}_3$ is the group of fractions of the monoid $\mathbf{B}_{3+}^{1} = \langle\, \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle_{+}^{1}$.







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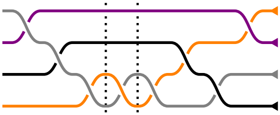

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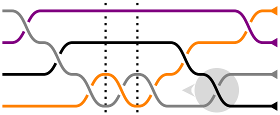

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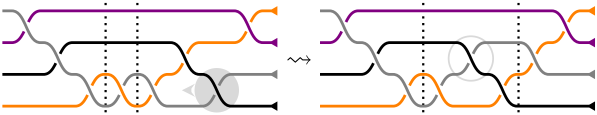

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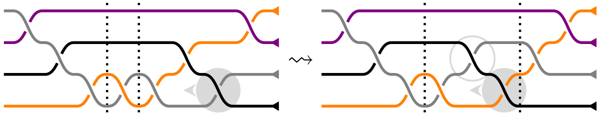

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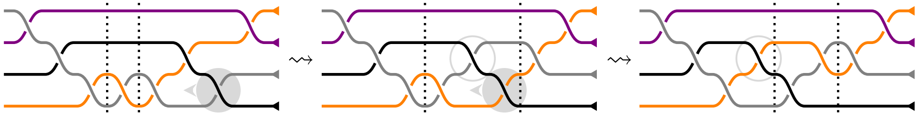

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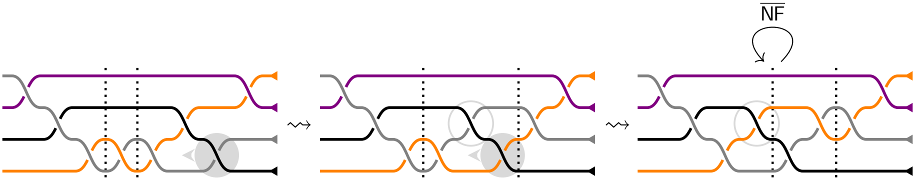

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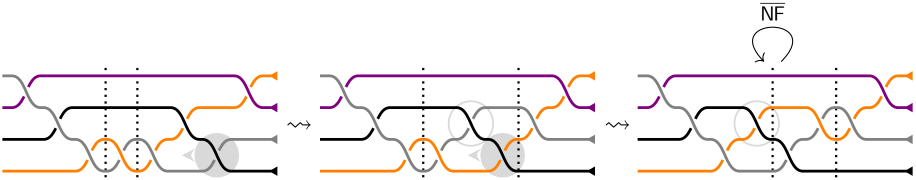

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Whenever $\mathsf{NF}(S)$ is regular, it provides a right-automatic structure for S if the language $\mathcal{L}_q = \{\ (\mathsf{NF}(a), \mathsf{NF}(aq)) : a \in S\ \}$ is regular for each $q \in \mathcal{Q}$.

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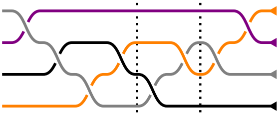

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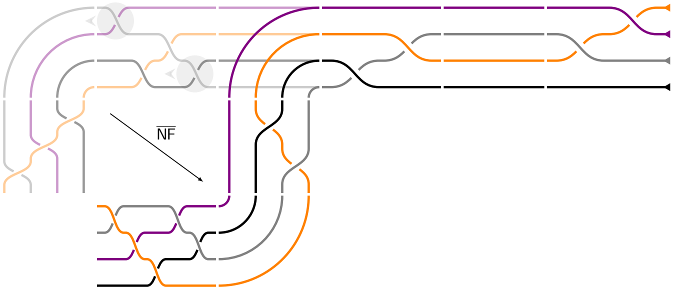

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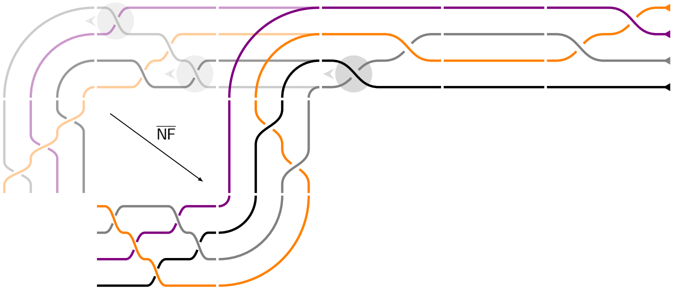

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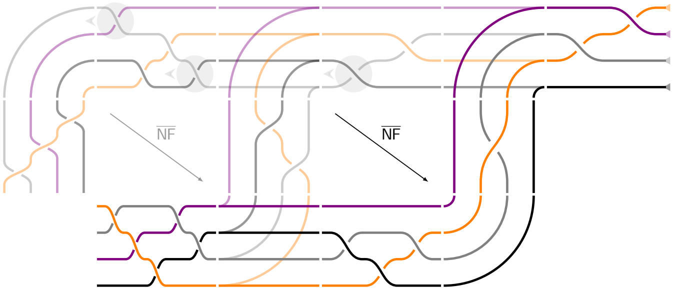

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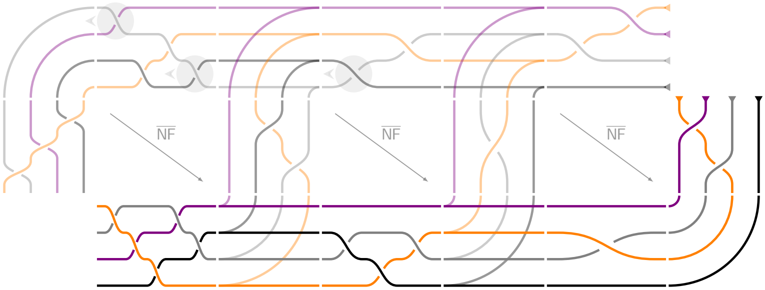

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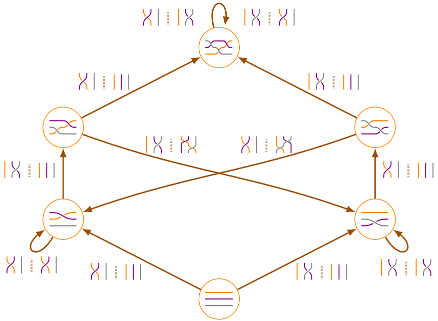

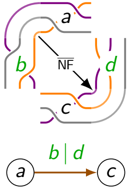

Thurston's transducer

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NF

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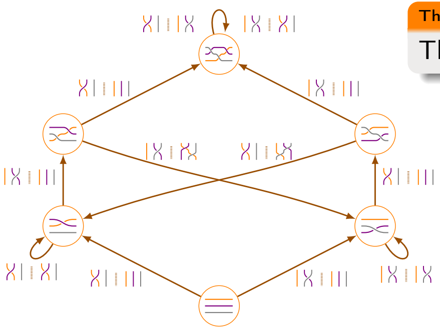
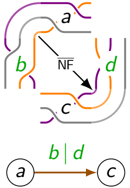


Thurston 1988

The braid groups are automatic.



Thurston's transducer

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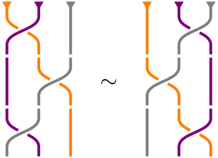
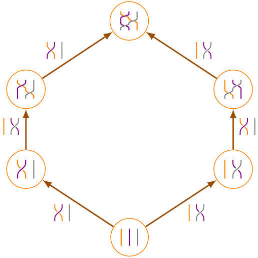
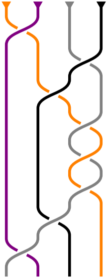


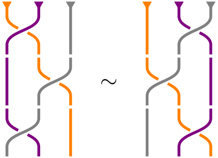


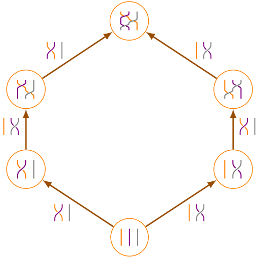


a positive braid

Garside 1965

The braid group $\mathbf{B}_3$ is the group of fractions of the monoid $\mathbf{B}_{3+}^{1} = \langle\, \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle_+^{1}$.





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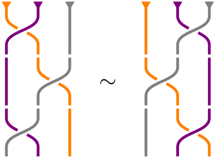

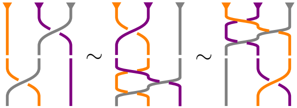

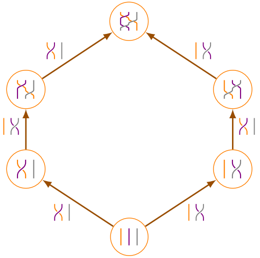

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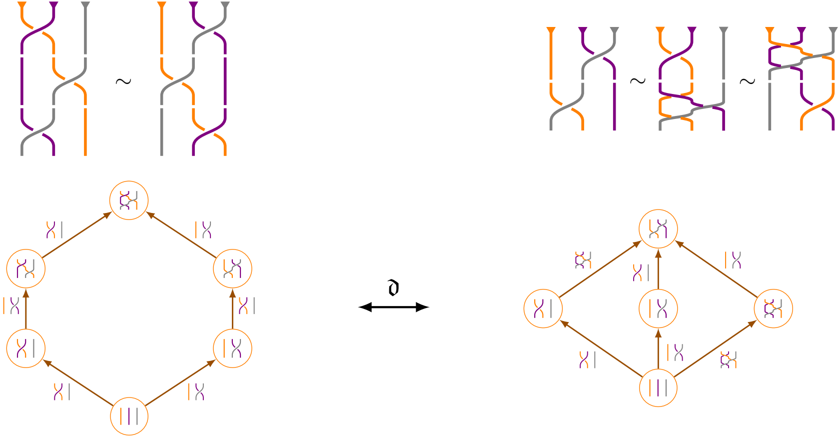

Garside 1965 **Birman Ko Lee 1998**

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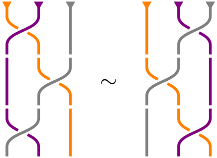

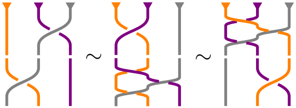

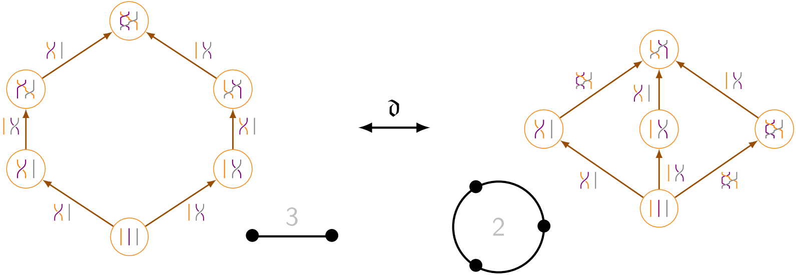

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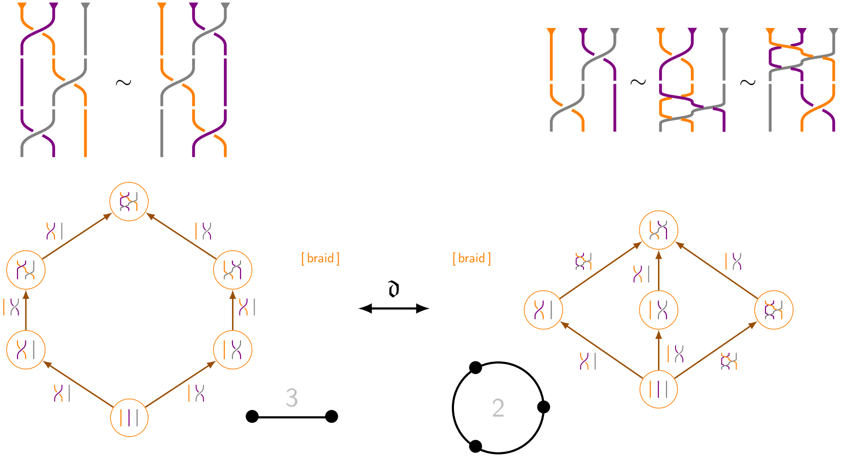

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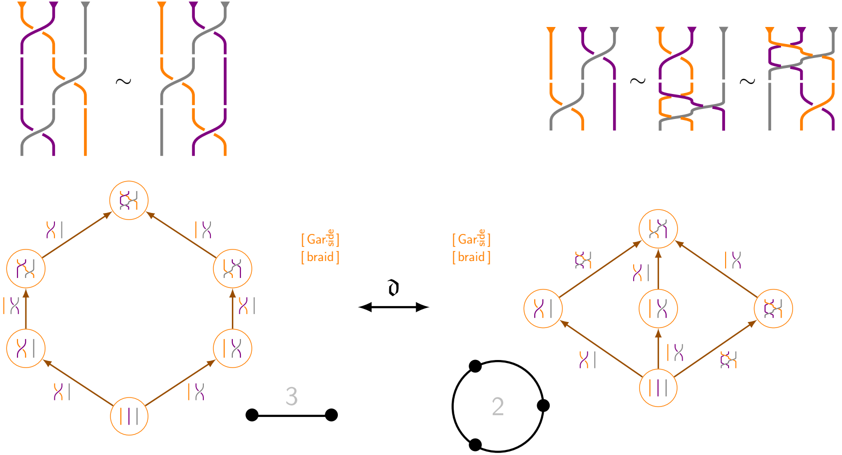

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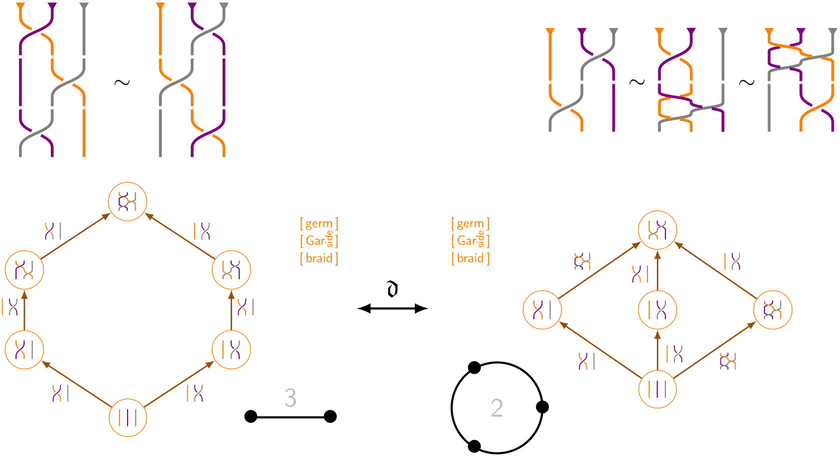

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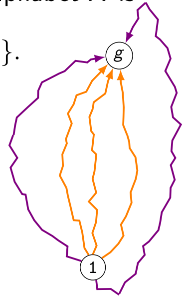

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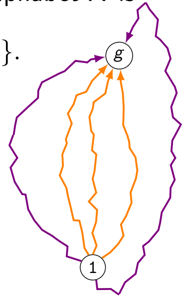

We consider the associated partial order $\preccurlyeq_X$ on G:

$$h \preccurlyeq_X g \text{ if and only if } \ell_X(h) + \ell_X(h^{-1}g) = \ell_X(g) \ .$$

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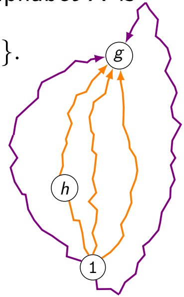

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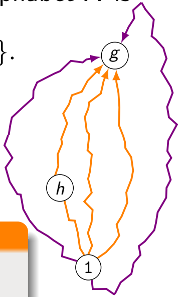

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Procedure

- ▷ Start from a finite group G.
- ▷ Choose a set X generating G as a monoid.
- ▷ Build the associated poset $(G, \preccurlyeq_X)$.
- ▷ Pick a maximal element d dominating X.
- ▷ Extract the monoid presentation $\langle\, X : R_d \,\rangle$.

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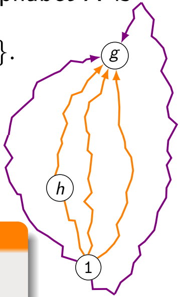

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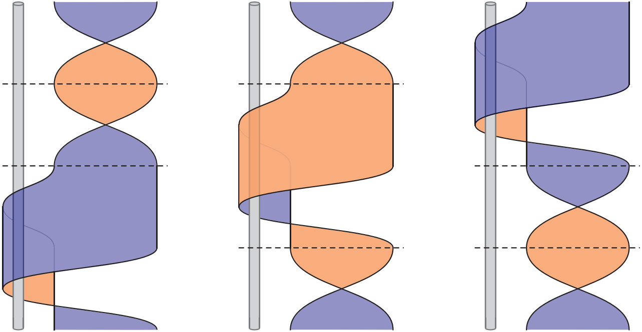
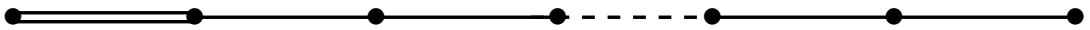
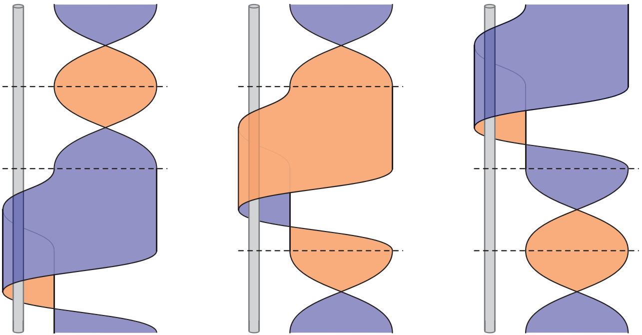
- ▷ Extract the monoid presentation $(\ X : R_d\ )$.
- ▷ Check whether or not:
  - [germ] $\langle\ X : R_d \cup x^{\mathrm{ord}(x)}\ \rangle$ is isomorphic to G;
  - [Garside] $\langle\ X : R_d\ \rangle_+^1$ is a Garside monoid;
  - [braid] $\langle\ X : R_d\ \rangle$ is isomorphic to $\mathbf{B}(G)$ (if defined).



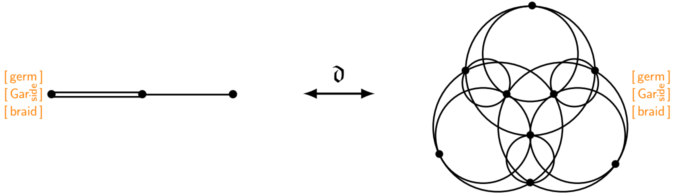




P 2002 Bessis 2003

The dual braid monoid $\mathbf{B}^{\times}(B_n)$ admits the presentation

$$\begin{aligned}\langle\ \alpha_{ts}, \beta_{ts}, \tau_t :\ & [\alpha_{ts}, \tau_s, \beta_{ts}, \tau_t] \text{ for } t > s,\\ & [\alpha_{ts}, \alpha_{sr}, \alpha_{tr}]\ ,\ [\beta_{ts}, \alpha_{sr}, \beta_{tr}]\ ,\ [\alpha_{ts}, \beta_{sr}, \beta_{tr}] \text{ for } t > s > r,\\ & [\alpha_{ts}, \tau_r]\ ,\ [\tau_t, \alpha_{sr}]\ ,\ [\beta_{tr}, \tau_s] \text{ for } t > s > r,\\ & [\alpha_{ts}, \alpha_{rq}]\ ,\ [\alpha_{ts}, \beta_{rq}]\ ,\ [\beta_{ts}, \alpha_{rq}]\ ,\\ & [\alpha_{tq}, \alpha_{sr}]\ ,\ [\beta_{tq}, \alpha_{sr}]\ ,\ [\beta_{tq}, \beta_{sr}] \text{ for } t > s > r > q\ \rangle_+^1.\end{aligned}$$



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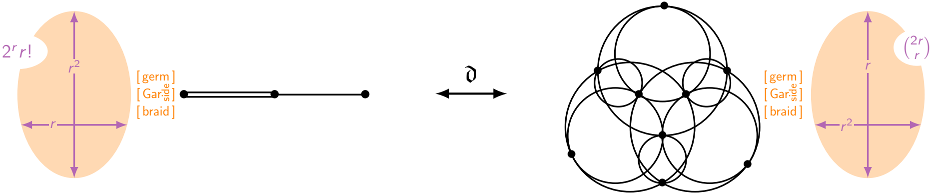

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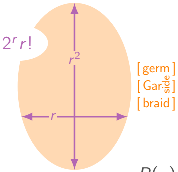
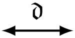
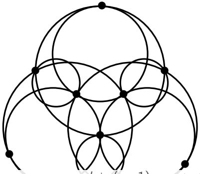
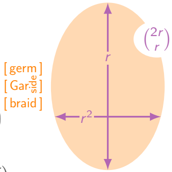






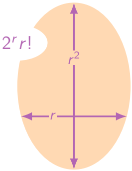


$P(q) = \prod_{k=1}^{r} (1 + q + \cdots + q^{2k-1})$

Poincaré polynomial

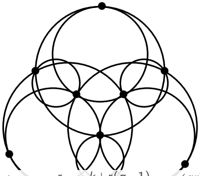
$Z(q) = \prod_{k=1}^{r} \frac{k+r(q-1)}{k} = \binom{qr}{r}$

Zeta polynomial



[germ]
[Garside]
[braid]

$\overset{\mathfrak{d}}{\longleftrightarrow}$



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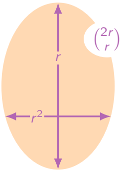

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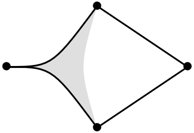

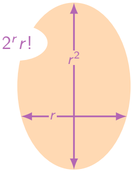

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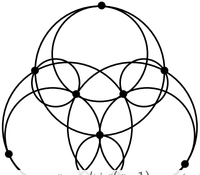

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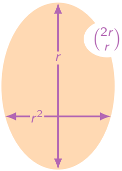

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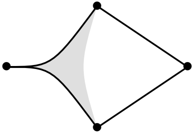

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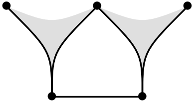

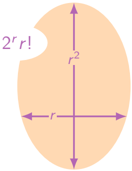

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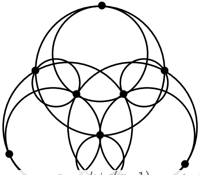

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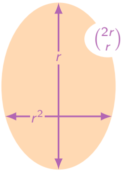

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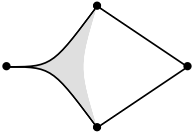

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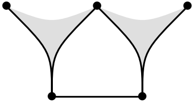

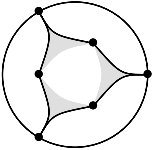

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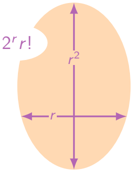

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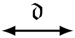

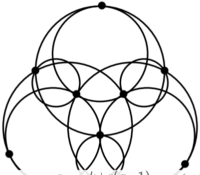

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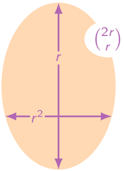

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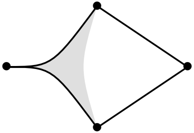

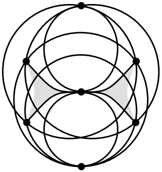

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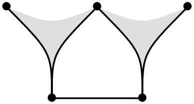

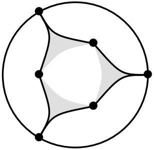

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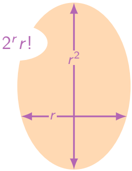

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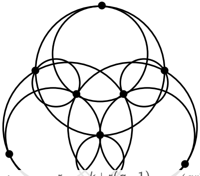
$\overset{\mathfrak{d}}{\longleftrightarrow}$



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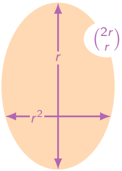

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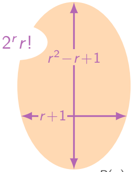

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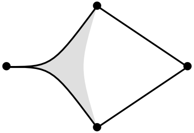

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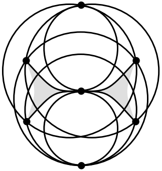

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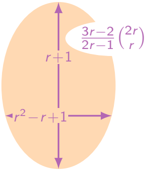

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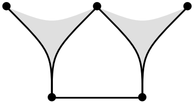

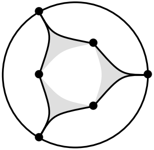

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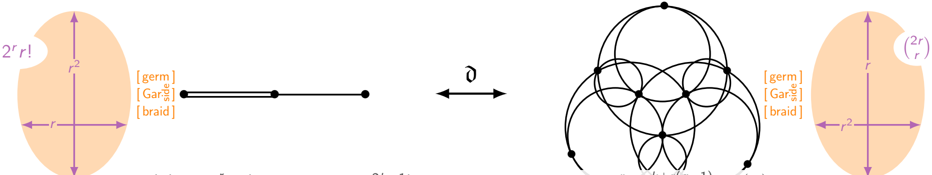

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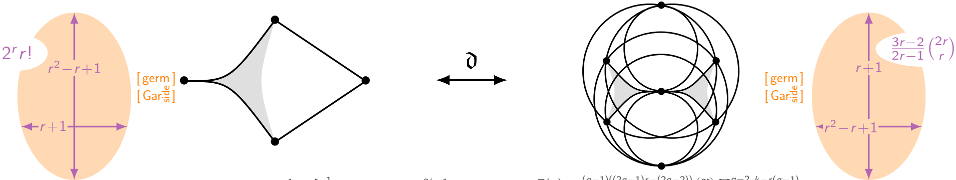

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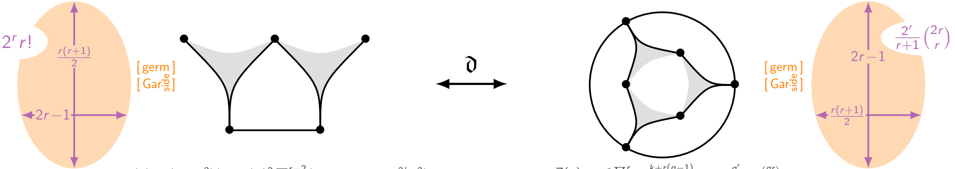
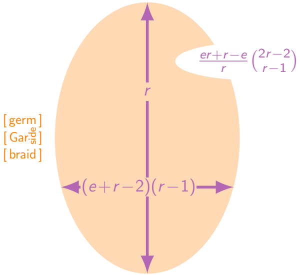


$P(q) = (1+q^2)(1+q)^{r+2}\prod_{k=2}^{r-2}(1+q+\cdots+q^{2k-2})$ $Z(q) = q^r\prod_{k=2}^{r}\frac{k+r(q-1)}{k} = \frac{q^r}{r(q-1)+1}\binom{qr}{r}$

Bessis Corran 2006

The dual braid monoid $\mathbf{B}^{\times}(e,e,r)$ satisfies

$$Z(q) = \frac{r+e(r-1)(q-1)}{r} \prod_{k=1}^{r-1} \frac{ek+e(r-1)(q-1)}{ek}.$$



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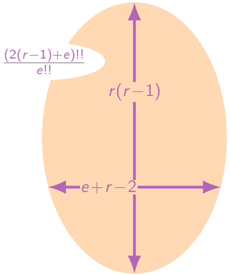

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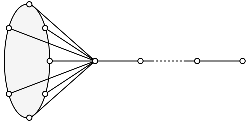

$\overset{\mathfrak{d}}{\longleftrightarrow}$

[germ]
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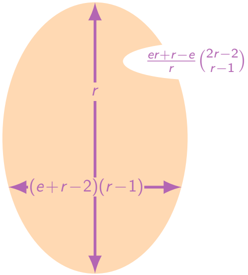

Corran P 2011

The post-classical braid monoid $\mathbf{B}^{\oplus}(e,e,r)$ satisfies

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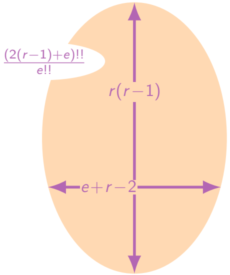

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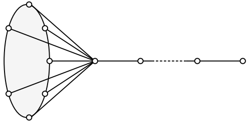

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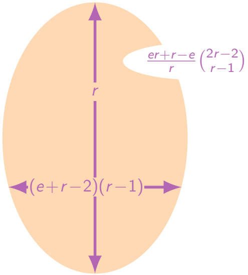

Corran P 2011 Neaime 2017

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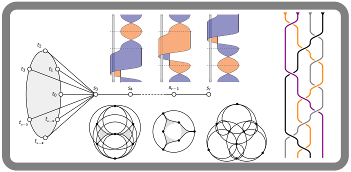
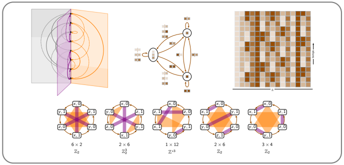
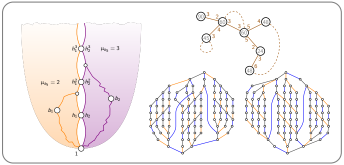
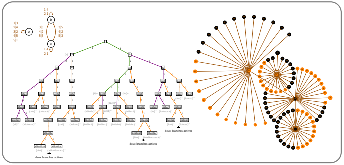
Braid (semi)groups
& Garside theory

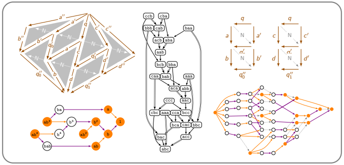


Mealy automata
& automaton (semi)groups



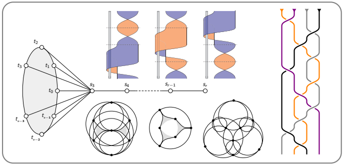






Quadratic normalisations
Thurston *vs* Mealy automata

Braid (semi)groups
& Garside theory



Mealy automata
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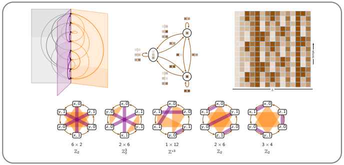
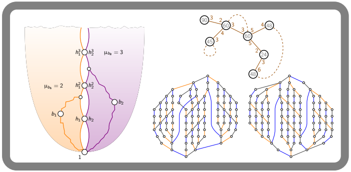
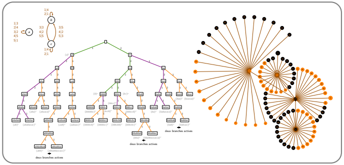
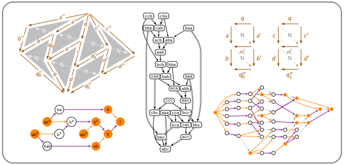






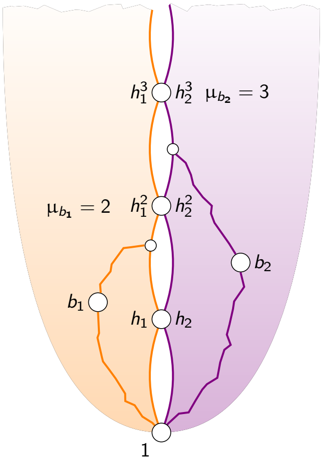


Quadratic normalisations
Thurston *vs* Mealy automata

Let M_1, M_2, H be monoids with $\begin{cases} \phi_1 : H \hookrightarrow M_1, \\ \phi_2 : H \hookrightarrow M_2. \end{cases}$

The amalgamated free product is

$$\langle\ M_1 \star M_2\ \ : \phi_1(h) = \phi_2(h), h \in H\ \rangle_+^1.$$



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P 2013

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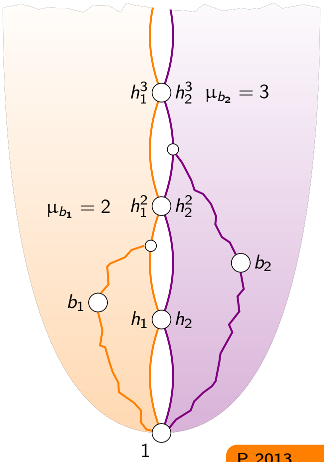

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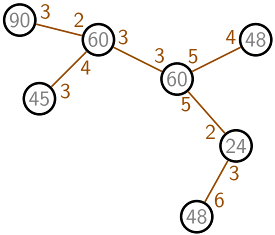
The enveloping group of $\langle\ M, t : h_1 t = t h_2\ \rangle_+^1$ is Garside iff h_1 and h_2 are n-th roots of a same Garside element.



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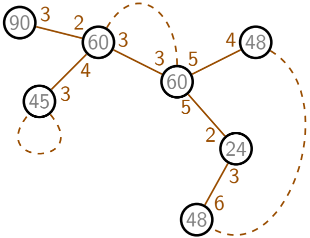

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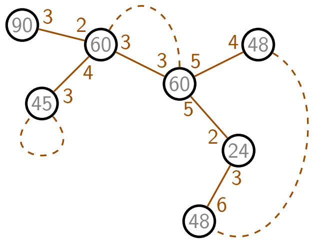
P 2013

A non-cyclic one-relator group is Garside iff its center is non-trivial.



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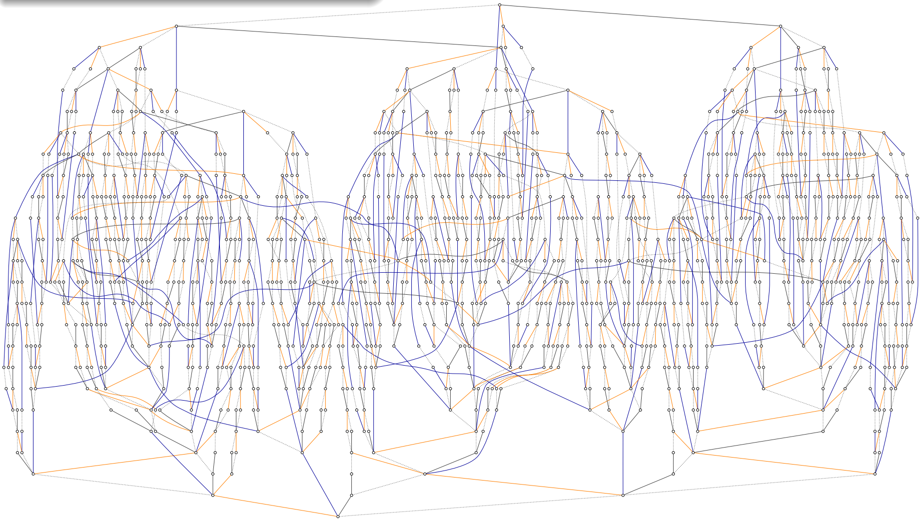
$$\langle\, a, x : x^8 a x^{-6} a^{-1} x^4 a x^{-6} a^{-1} \,\rangle$$



Braid (semi)groups
& Garside theory

Mealy automata
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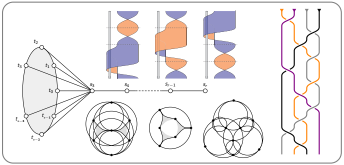

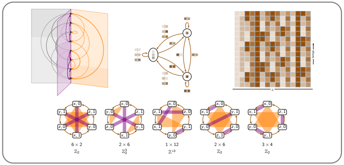

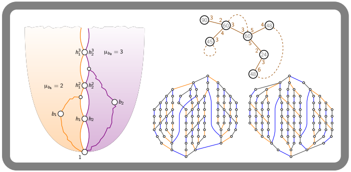

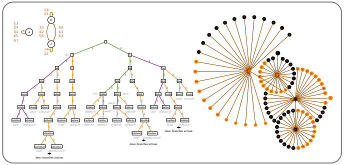

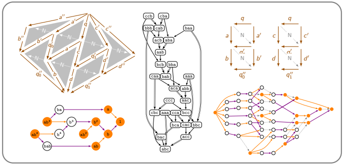

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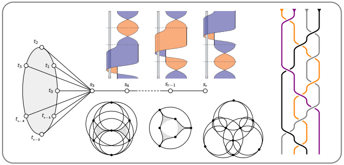
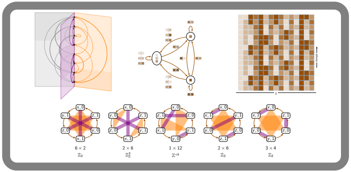
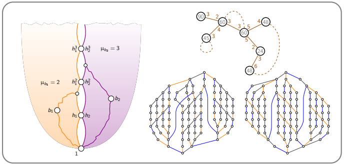
Thurston *vs* Mealy automata

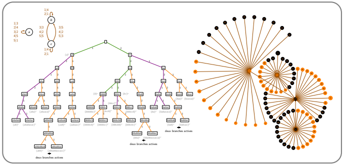
Braid (semi)groups
& Garside theory

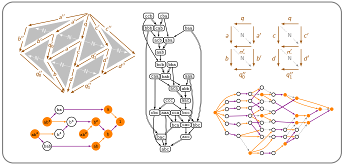
Mealy automata
& automaton (semi)groups





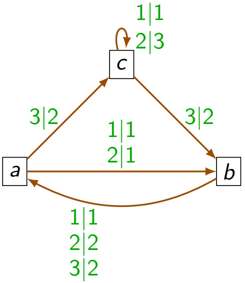






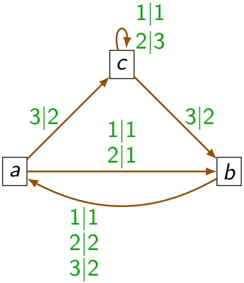
Quadratic normalisations
Thurston *vs* Mealy automata

$$\mathcal{M} = (\overset{\text{stateset}}{Q}, \overset{\text{alphabet}}{X}, \overset{\text{transition}}{\tau}, \overset{\text{output}}{\sigma})$$



$$\mathcal{M} = (\overset{\text{stateset}}{Q}, \overset{\text{alphabet}}{X}, \overset{\text{transition}}{\tau}, \overset{\text{output}}{\sigma})$$

$$\boxed{q} \xrightarrow{x \mid \sigma_q(x)} \boxed{\tau_x(q)}$$



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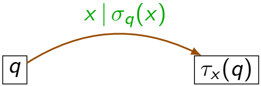

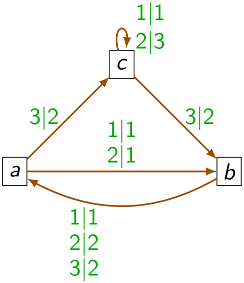

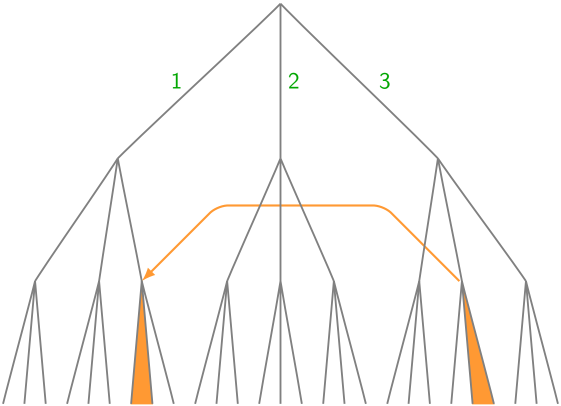

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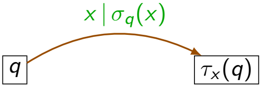
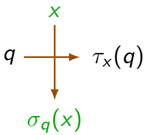
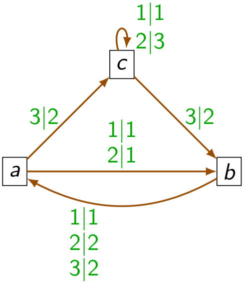
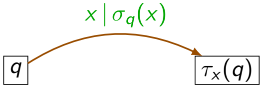


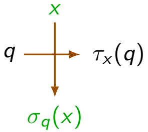


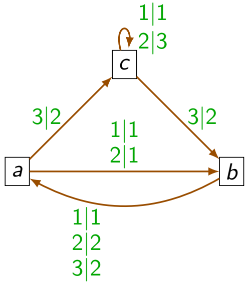


stateset alphabet transition output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$







2 2 3 1 $\cdots$

a

stateset alphabet transition output

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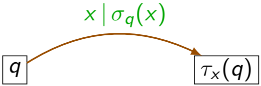

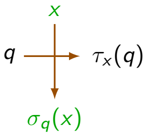

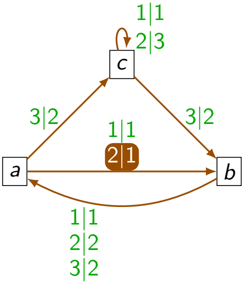

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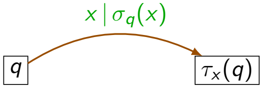

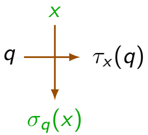

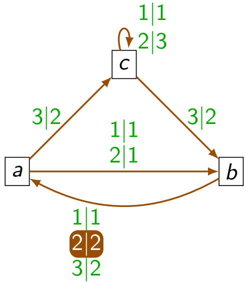

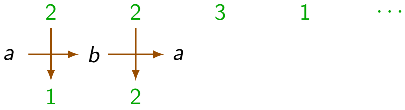

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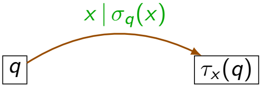
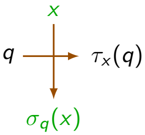
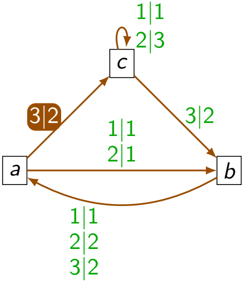
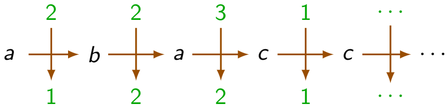
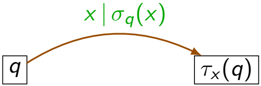


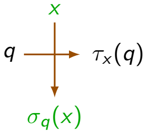


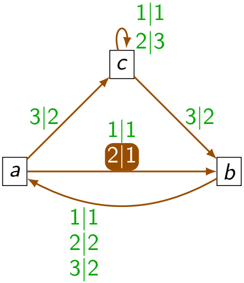


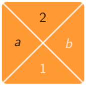


$$\mathcal{M} = (\underbrace{Q}_{\text{stateset}}, \underbrace{X}_{\text{alphabet}}, \underbrace{\tau}_{\text{transition}}, \underbrace{\sigma}_{\text{output}})$$



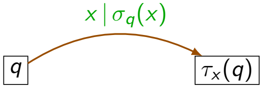


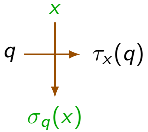


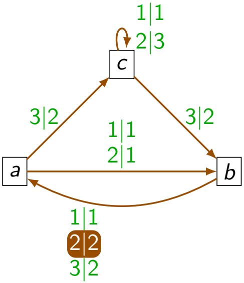


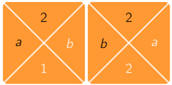
stateset alphabet transition output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$









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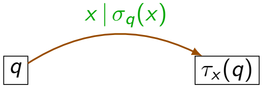

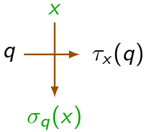

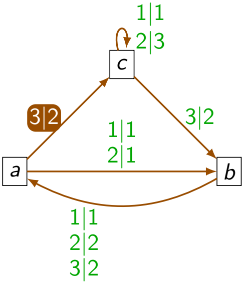

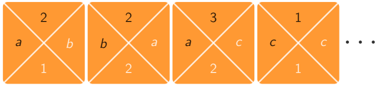

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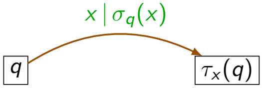
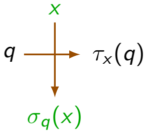
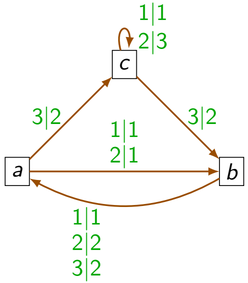


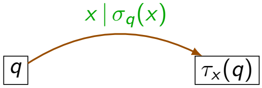


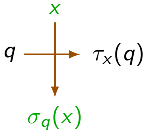


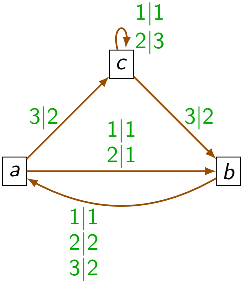
$$\sigma_a : \begin{array}{ccccc} 2 & 2 & 3 & 1 & \cdots \\ \downarrow \\ 1 & 2 & 2 & 1 & \cdots \end{array}$$

$$\mathcal{M} = (\overset{\text{stateset}}{Q}, \overset{\text{alphabet}}{X}, \overset{\text{transition}}{\tau}, \overset{\text{output}}{\sigma})$$

$$\langle\ \mathcal{M}\ \rangle_+ = \langle\ \underset{X^* \to X^*}{\sigma_q}, q \in Q\ \rangle_+$$







$$\begin{array}{lccccc} & 2 & 2 & 3 & 1 & \cdots \\ \sigma_a \downarrow & & & & & \\ & 1 & 2 & 2 & 1 & \cdots \end{array}$$

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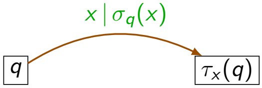

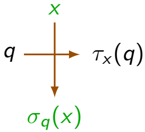

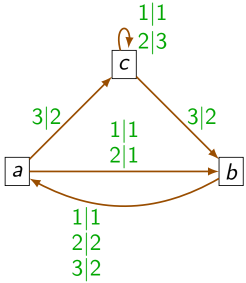

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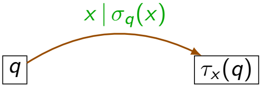

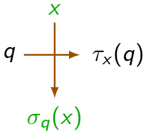

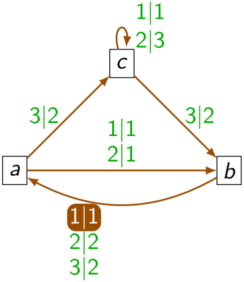

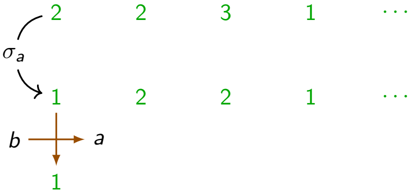

stateset alphabet transition output

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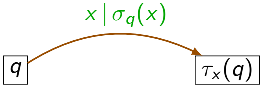

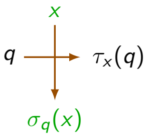

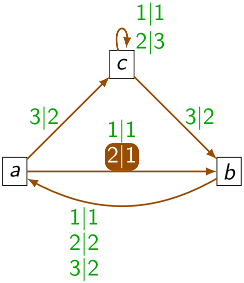

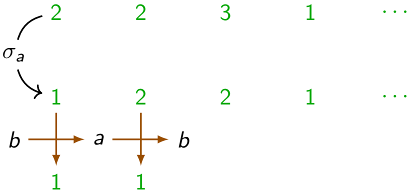

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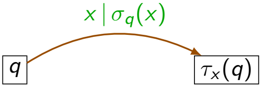

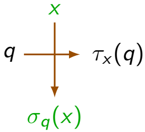

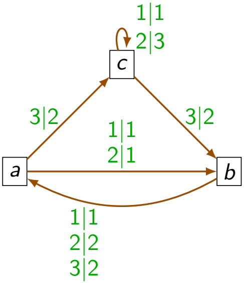

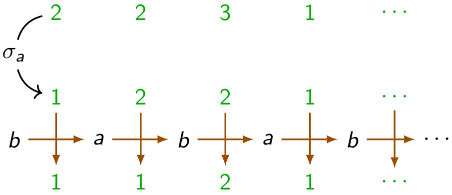

stateset alphabet transition output

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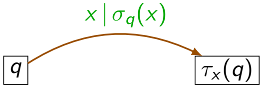

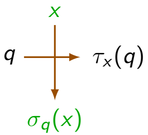

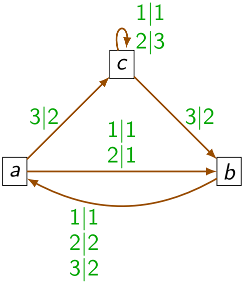

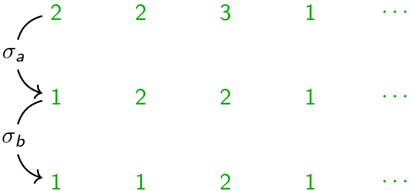

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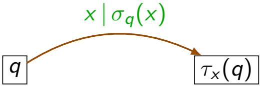
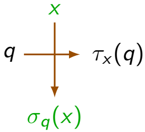
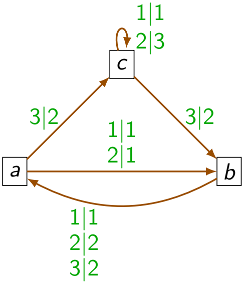
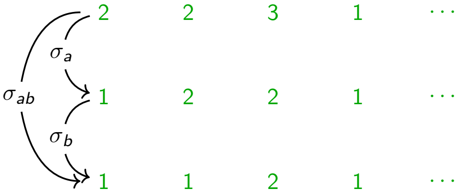
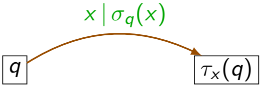


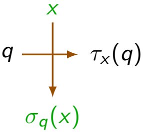


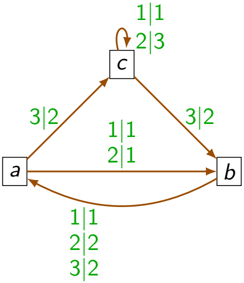


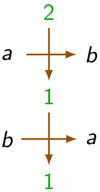


$$\mathcal{M} = (\overset{\text{stateset}}{Q}, \overset{\text{alphabet}}{X}, \overset{\text{transition}}{\tau}, \overset{\text{output}}{\sigma}) \qquad \langle\ \mathcal{M}\ \rangle_+ = \langle\ \underset{X^* \to X^*}{\sigma_q, q \in Q}\ \rangle_+$$





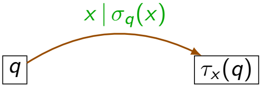


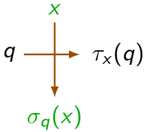


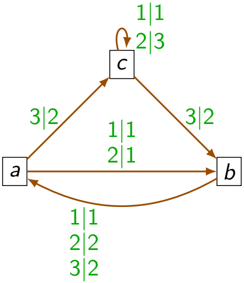
stateset alphabet transition output

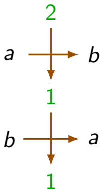
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

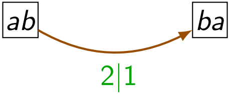
$$\langle\ \mathcal{M}\ \rangle_+ = \langle\ \underbrace{\sigma_q}_{X^* \to X^*}, q \in Q\ \rangle_+$$









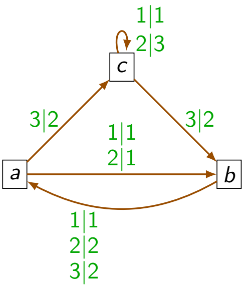


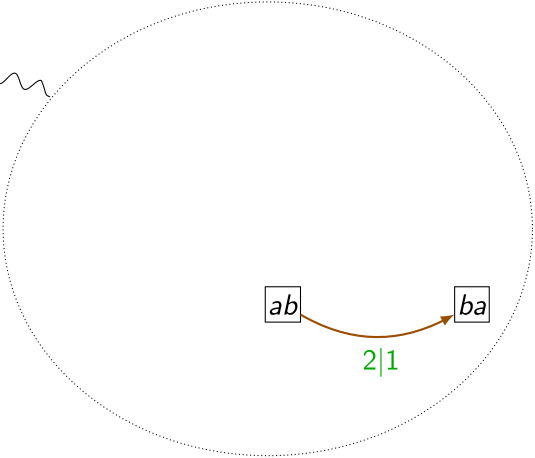
stateset alphabet transition output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$

$\mathcal{M}^2$

exponentiation





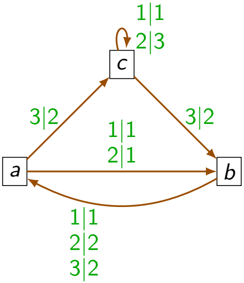
stateset alphabet transition output

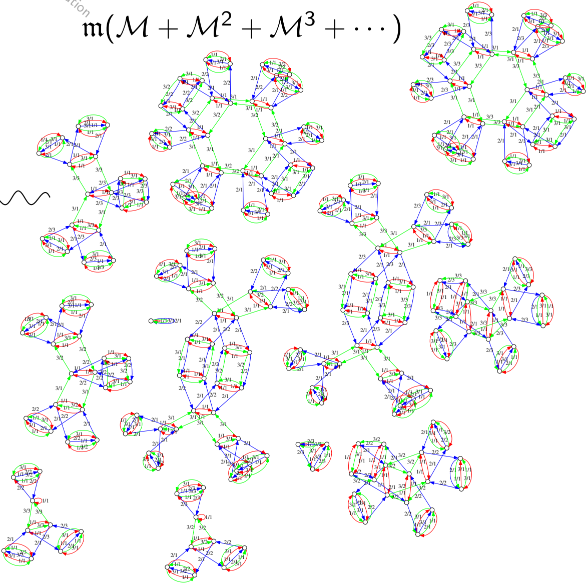
$$\mathcal{M} = (Q, X, \tau, \sigma)$$

minimisation

$$\mathfrak{m}(\mathcal{M} + \mathcal{M}^2 + \mathcal{M}^3 + \cdots)$$

completion attempt



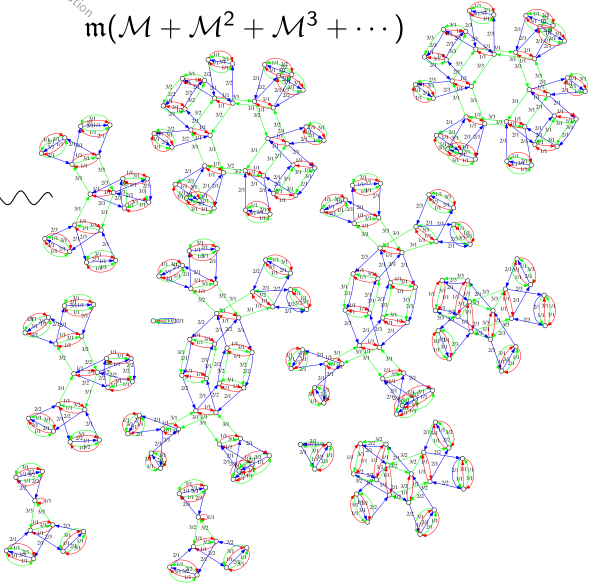


stateset alphabet transition output

$$\mathcal{M} = (Q, X, \tau, \sigma)$$

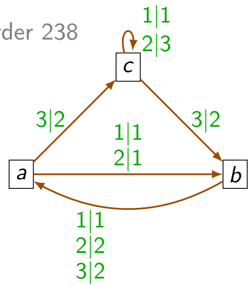
minimisation

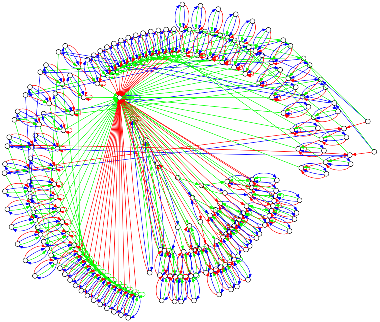
$$\mathfrak{m}(\mathcal{M} + \mathcal{M}^2 + \mathcal{M}^3 + \cdots)$$



completion attempt

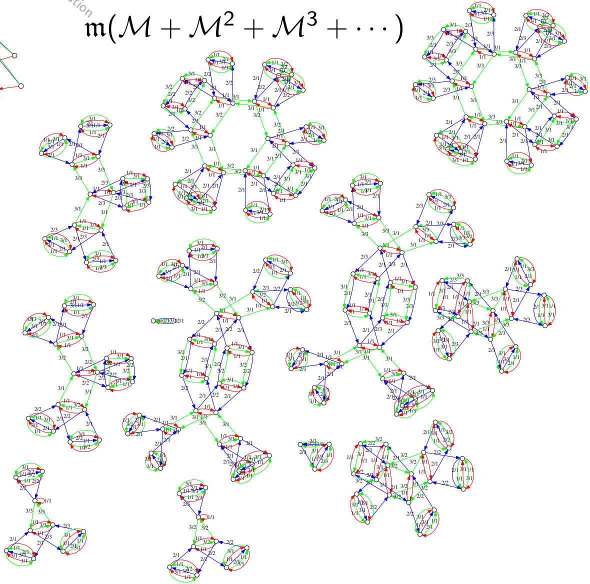
order 238



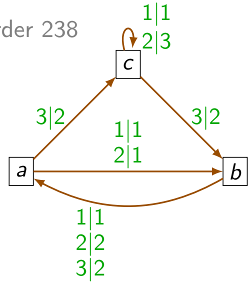


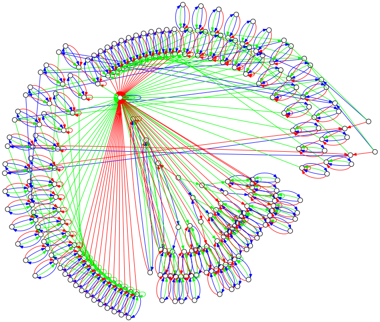
minimisation

$$\mathfrak{m}(\mathcal{M} + \mathcal{M}^2 + \mathcal{M}^3 + \cdots)$$

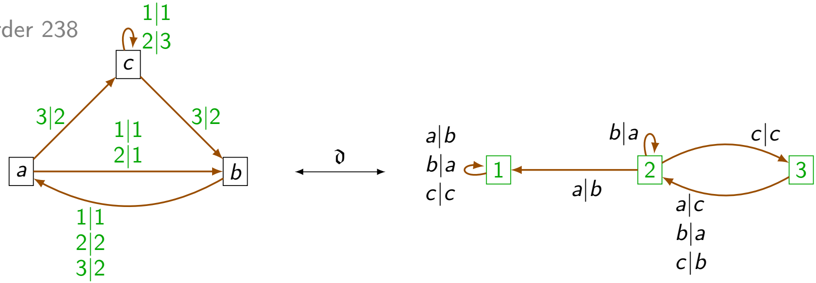


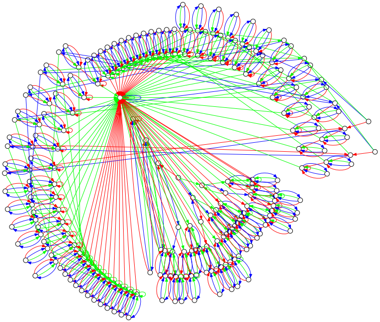
order 238





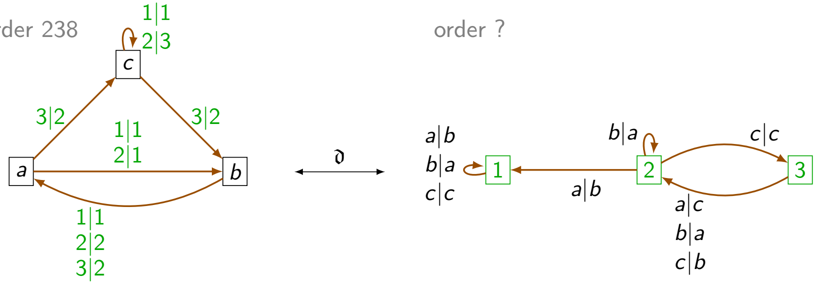
order 238

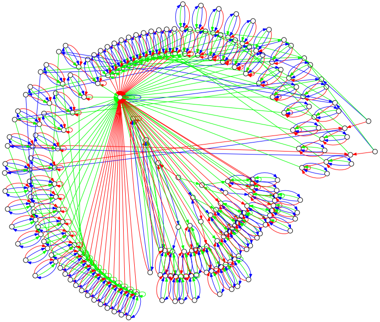




order 238

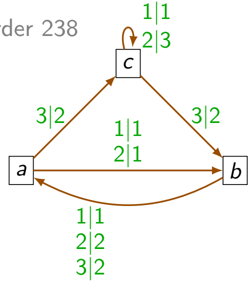
order ?



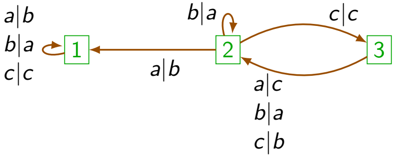


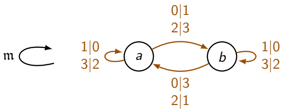
order 238

order 1 494 186 269 970 473 680 896

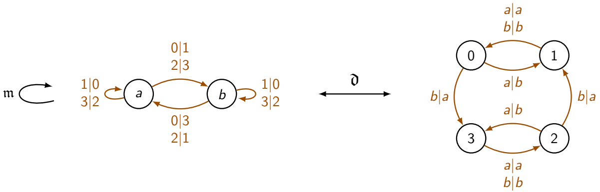


$\xleftrightarrow{\mathfrak{d}}$

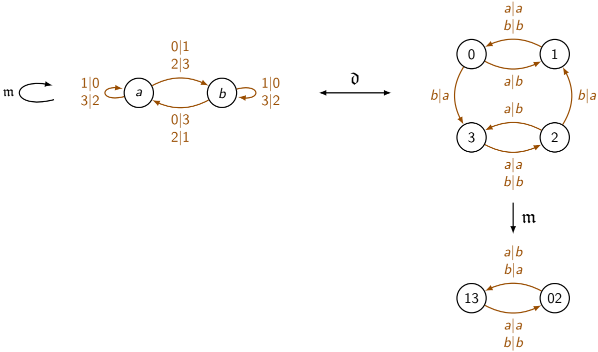




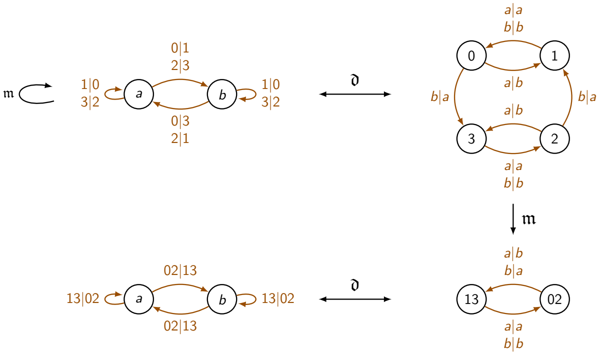
$\mathfrak{m}$
0|1
2|3
1|0
3|2
a
b
1|0
3|2
0|3
2|1

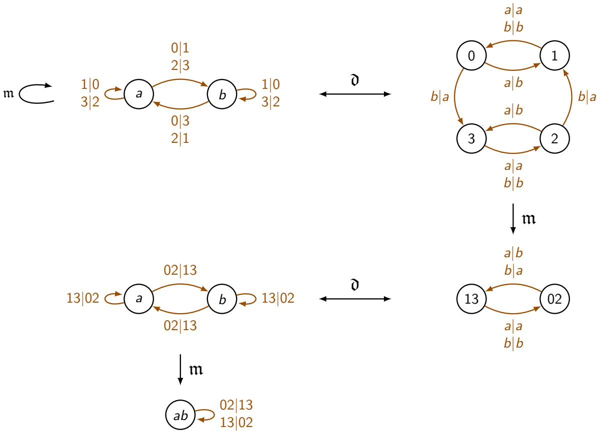


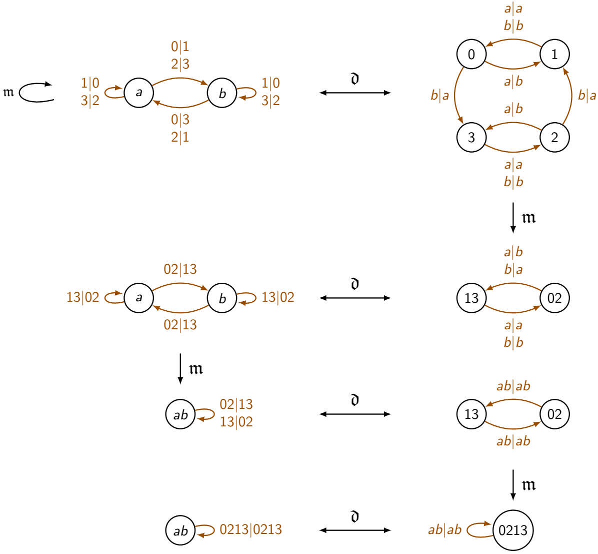
𝔪
1|0
3|2
0|1
2|3
a
b
0|3
2|1
1|0
3|2
𝔡
a|a
b|b
0
1
a|b
b|a
b|a
a|b
3
2
a|a
b|b



$\mathfrak{m}$
1|0
3|2
0|1
2|3
a
b
0|3
2|1
1|0
3|2
$\mathfrak{d}$
a|a
b|b
0
1
a|b
b|a
b|a
a|b
3
2
a|a
b|b
$\mathfrak{m}$
a|b
b|a
13
02
a|a
b|b

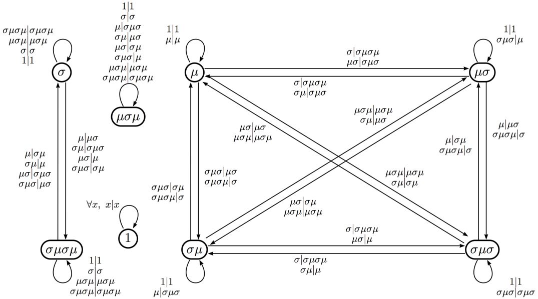

𝔪
1|0
3|2
0|1
2|3
0|3
2|1
a
b
1|0
3|2
𝔡
a|a
b|b
0
1
a|b
b|a
b|a
a|b
3
2
a|a
b|b
𝔪
13|02
02|13
02|13
a
b
13|02
𝔡
a|b
b|a
13
02
a|a
b|b


𝔪
1|0
3|2
0|1
2|3
0|3
2|1
1|0
3|2
a
b
𝔡
a|a
b|b
0
1
a|b
b|a
b|a
a|b
3
2
a|a
b|b
𝔪
a|b
b|a
13
02
a|a
b|b
02|13
13|02
02|13
13|02
𝔡
𝔪
ab
02|13
13|02



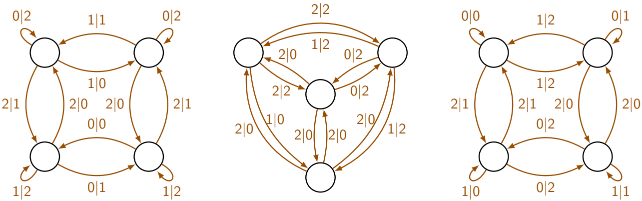
Akhavi Klimann Lombardy Mairesse P 2012

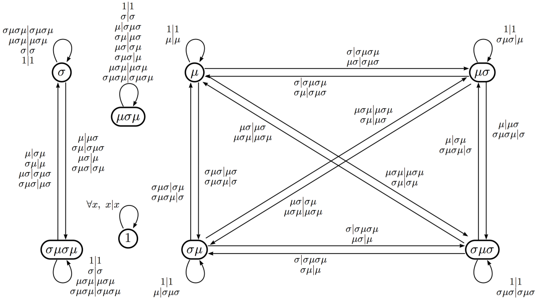
A Mealy automaton generates a finite (semi)group iff its $\mathfrak{md}$-reduced pair does.



Akhavi Klimann Lombardy Mairesse P 2012

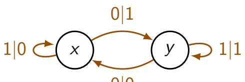
Even for bireversible automata, finiteness cannot be decide via $\mathfrak{md}$-triviality.



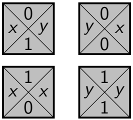


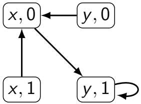
Akhavi Klimann Lombardy Mairesse P 2012

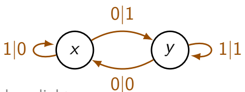
Even for bireversible automata, finiteness cannot be decide via 𝔪𝔡-triviality.



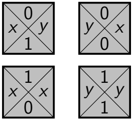
lamplighter

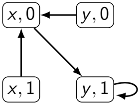


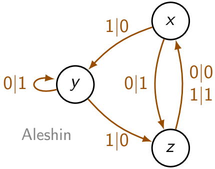




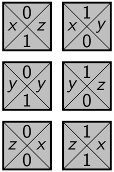
lamplighter

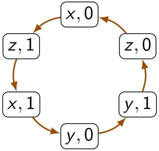


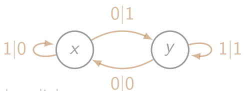




Aleshin







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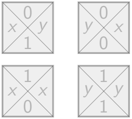
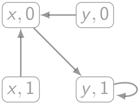
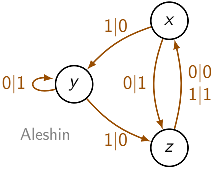
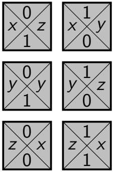
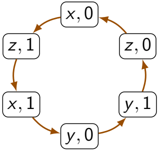
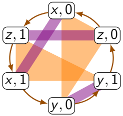






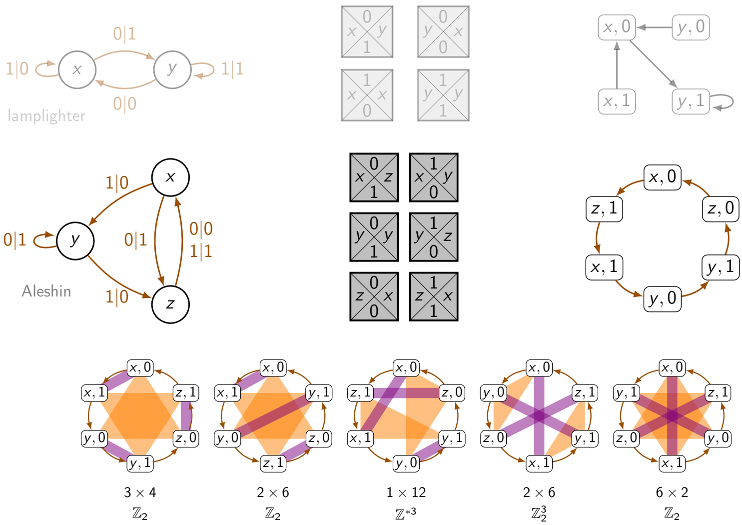


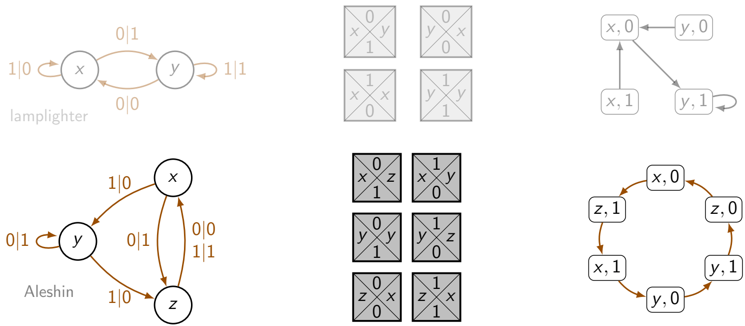


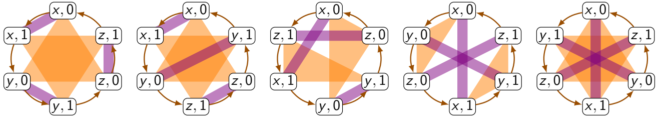


1×12

$\mathbb{Z}^{*3}$


lamplighter
Aleshin
3 × 4
$\mathbb{Z}_2$
2 × 6
$\mathbb{Z}_2$
1 × 12
$\mathbb{Z}^{*3}$
2 × 6
$\mathbb{Z}_2^3$
6 × 2
$\mathbb{Z}_2$



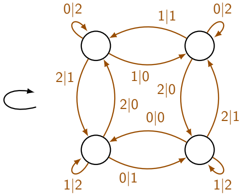


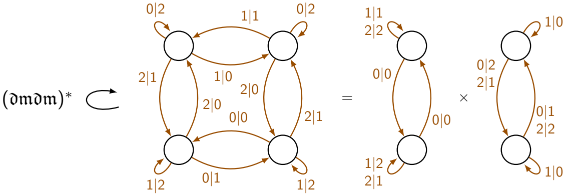
3×4 $\mathbb{Z}_2$ — 2×6 $\mathbb{Z}_2$ — 1×12 $\mathbb{Z}^{*3}$ — 2×6 $\mathbb{Z}_2^3$ — 6×2 $\mathbb{Z}_2$

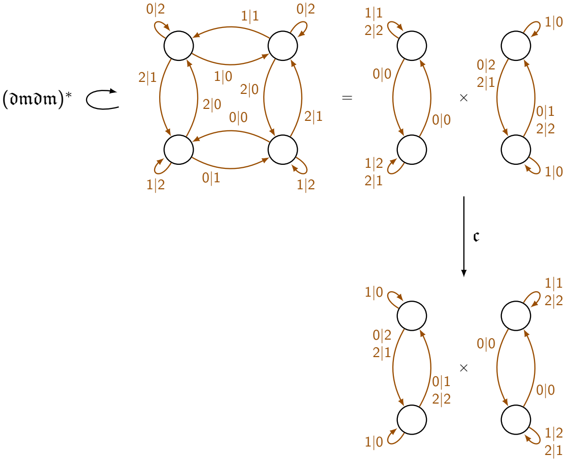
Conjecture

If a non-trivial $\mathfrak{md}$-reduced bireversible automaton is rigid, it generates an infinite group.

$(\mathfrak{dmdm})^*$




(∂𝔪∂𝔪)*
0|2
1|1
0|2
2|1
1|0
2|0
2|0
0|0
2|1
1|2
0|1
1|2
=
1|1
2|2
0|0
0|0
1|2
2|1
×
1|0
0|2
2|1
0|1
2|2
1|0


(∂m∂m)*
0|2
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1|0
2|0
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0|0
2|1
1|2
0|1
1|2
=
1|1
2|2
0|0
0|0
1|2
2|1
×
1|0
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2|1
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𝔠
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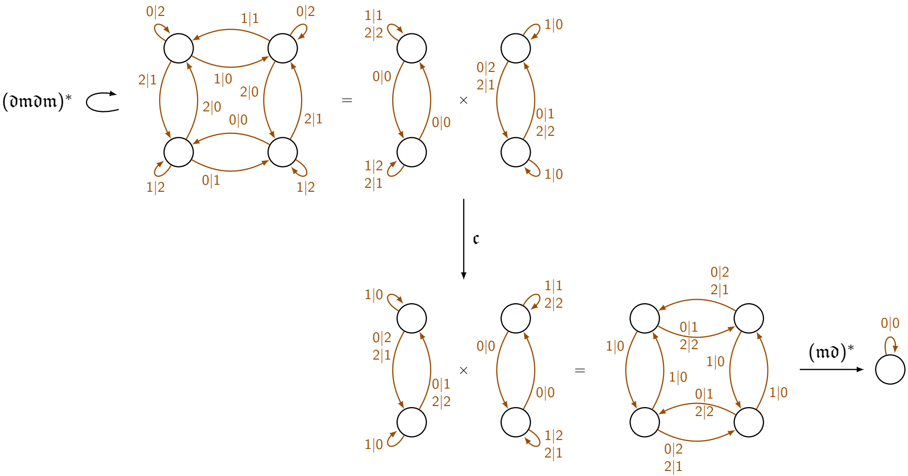
(𝔡𝔪𝔡𝔪)*
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1|1
0|2
2|1
1|0
2|0
2|0
0|0
2|1
1|2
0|1
1|2
=
1|1
2|2
0|0
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×
1|0
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×
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0|0
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=
0|2
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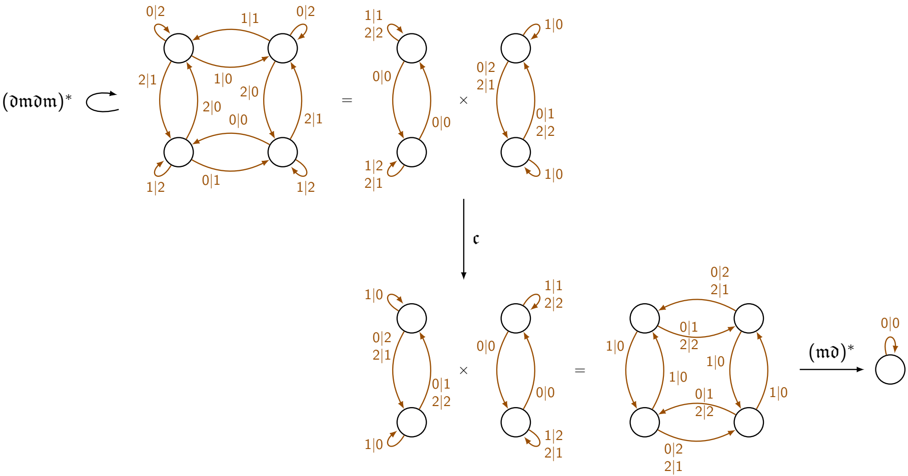
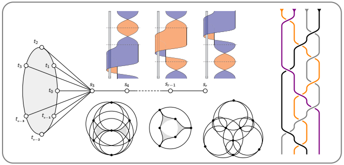
1|0
0|2
2|1
(𝔪𝔡)*
0|0



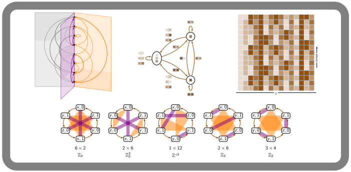
Conjecture

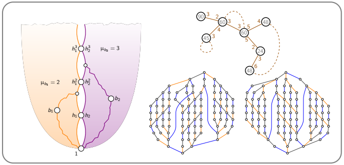
A bireversible automaton generates a finite group iff it is $\mathfrak{mdc}$-trivial.

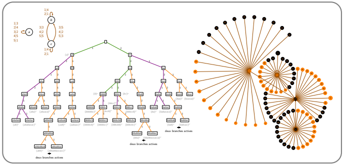
Braid (semi)groups
& Garside theory

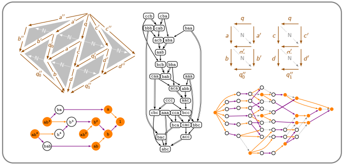


Mealy automata
& automaton (semi)groups



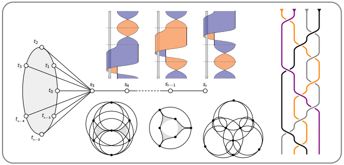




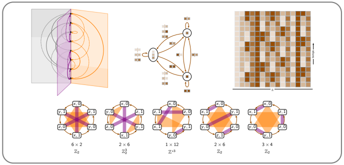


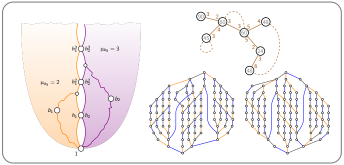
Quadratic normalisations
Thurston *vs* Mealy automata

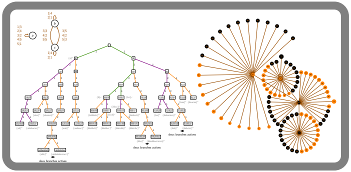
Braid (semi)groups
& Garside theory

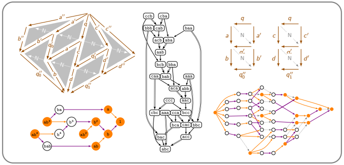


Mealy automata
& automaton (semi)groups









Quadratic normalisations
Thurston *vs* Mealy automata

Burnside 1902

Is a finitely generated torsion group necessarily finite?

Burnside 1902

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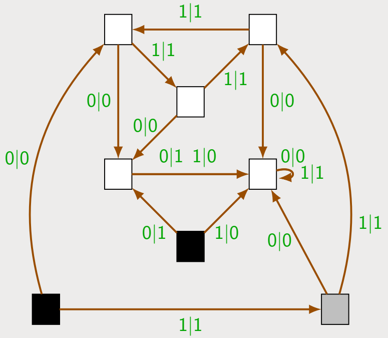
Alëšin 1972 **Grigorchuk 1980**



Burnside 1902

Is a finitely generated torsion group necessarily finite?

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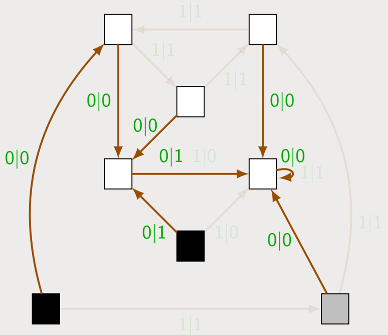

Burnside 1902

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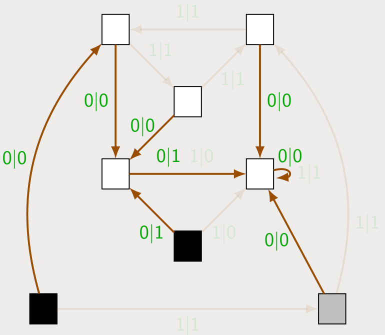
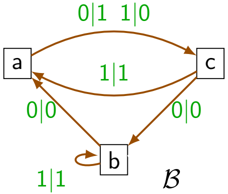


Question

And what about reversible automata?



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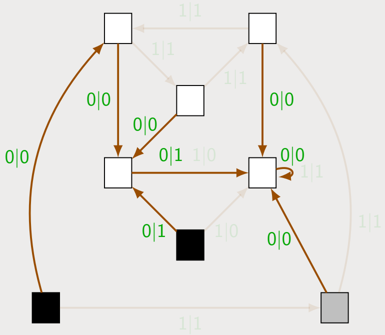

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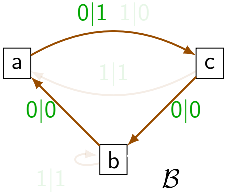

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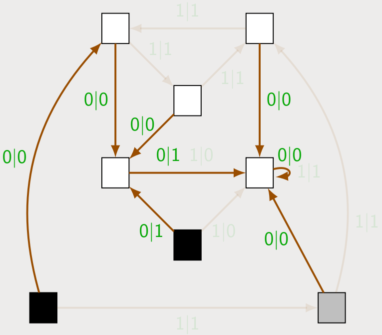

Question

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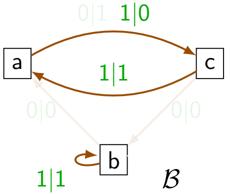

Burnside 1902

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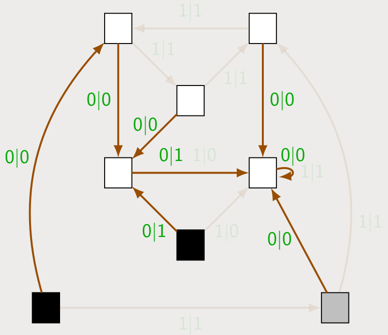

Question

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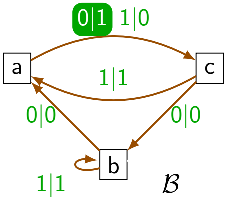
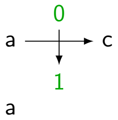
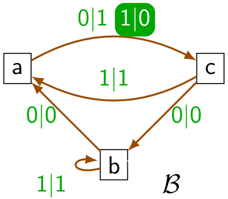
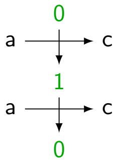
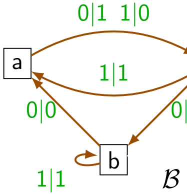
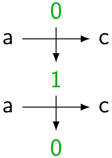

0|1
1|0
a
c
1|1
0|0
0|0
b
1|1
$\mathcal{B}$

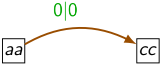

0
a
c
1
a

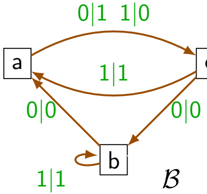

0|1 1|0
a
c
1|1
0|0
0|0
b
1|1
$\mathcal{B}$


0
a
c
1
a
c
0

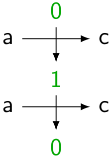

0|1 1|0
a
c
1|1
0|0
0|0
b
1|1
$\mathcal{B}$

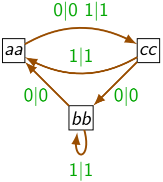

0
a
c
1
a
c
0

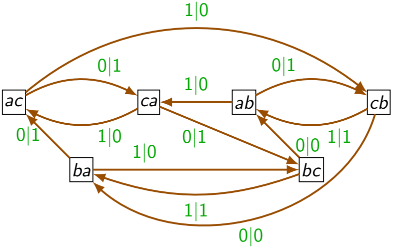

0|0
aa
cc



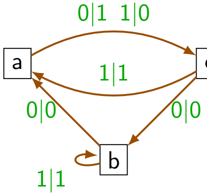
$\mathcal{B}$

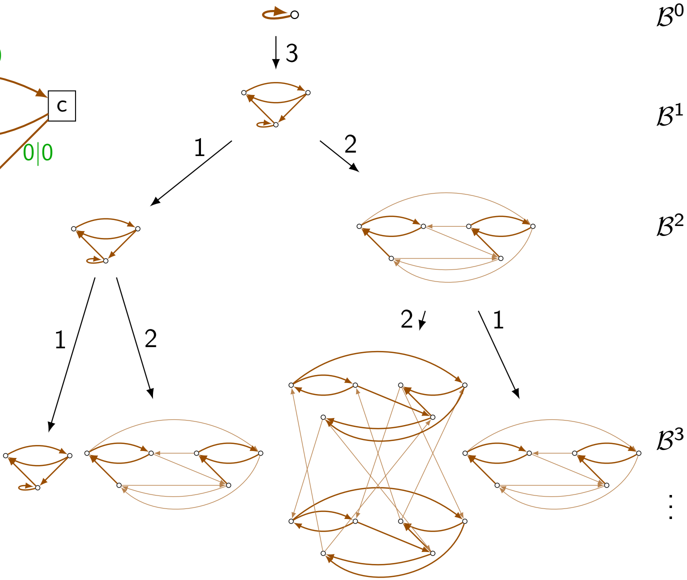


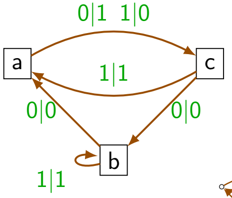


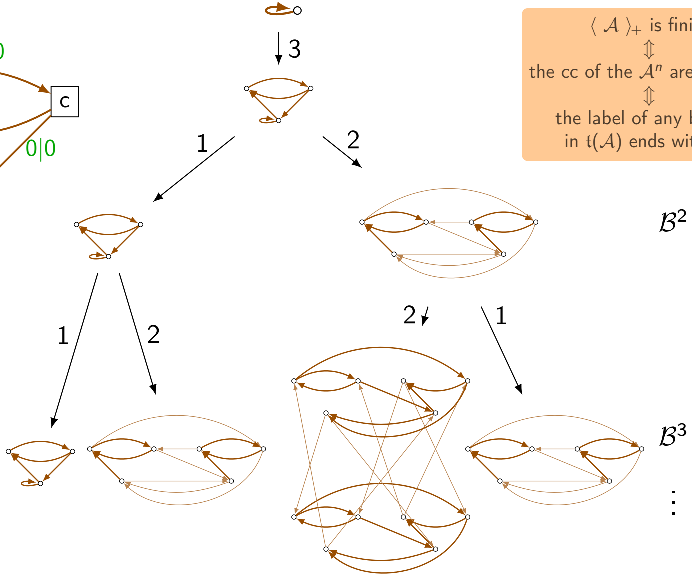


$\mathcal{B}^2$


a
c
b
0|1 1|0
1|1
0|0
0|0
1|1


3
1
2
1
2
2
1
$\mathcal{B}^0$
$\mathcal{B}^1$
$\mathcal{B}^2$
$\mathcal{B}^3$
⋮





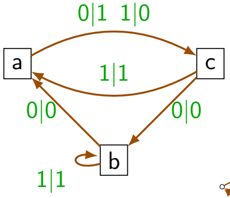
$\langle\ \mathcal{A}\ \rangle_+$ is finite

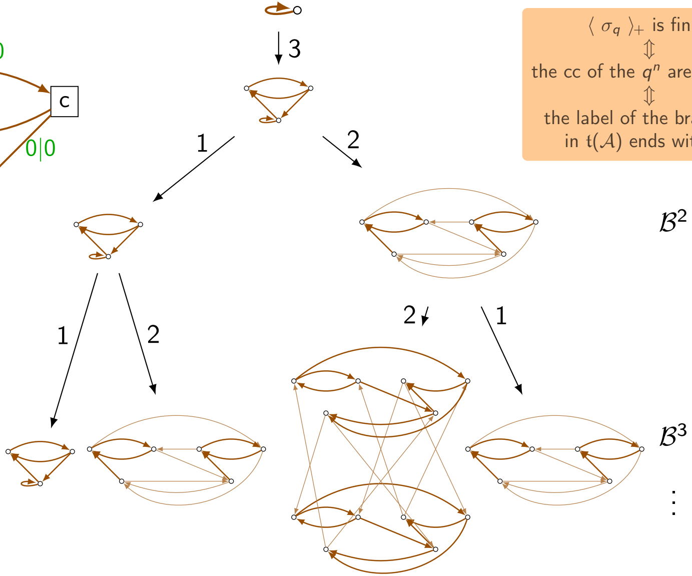
$\Updownarrow$

the cc of the $\mathcal{A}^n$ are bounded

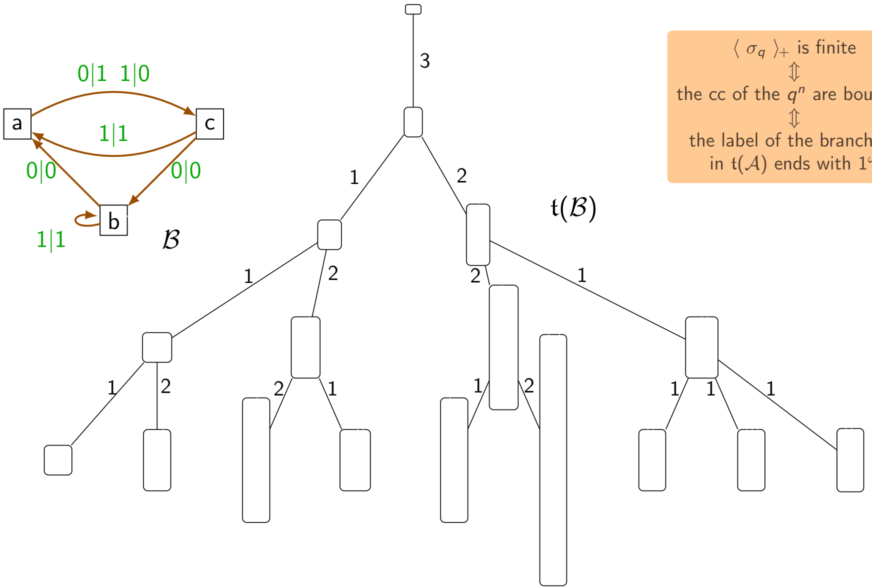
$\Updownarrow$

the label of any branch in $\mathfrak{t}(\mathcal{A})$ ends with 1^ω





$\langle\ \sigma_q\ \rangle_+$ is finite
$\Updownarrow$
the cc of the q^n are bounded
$\Updownarrow$
the label of the branch q^n in $\mathfrak{t}(\mathcal{A})$ ends with 1^ω



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$\Updownarrow$
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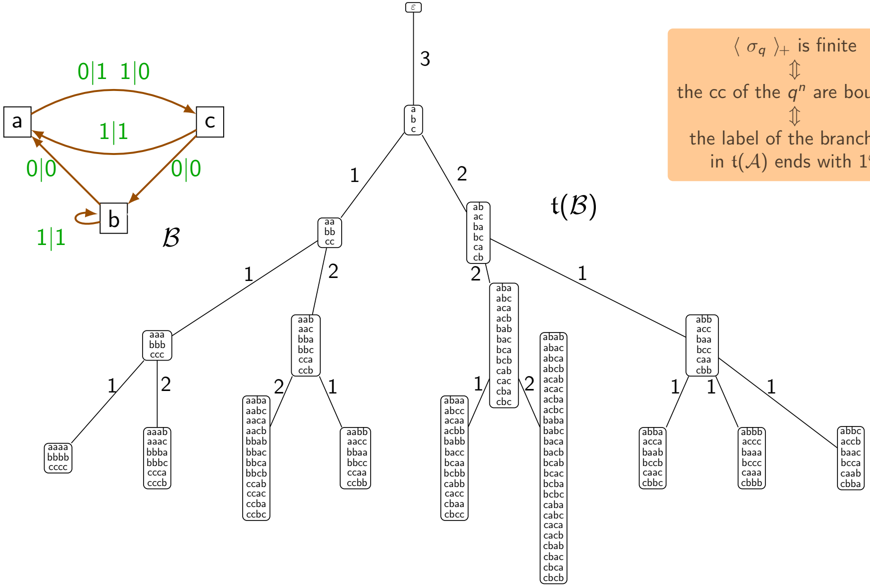

$\langle \sigma_q \rangle_+$ is finite

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the cc of the q^n are bounded

$\Updownarrow$

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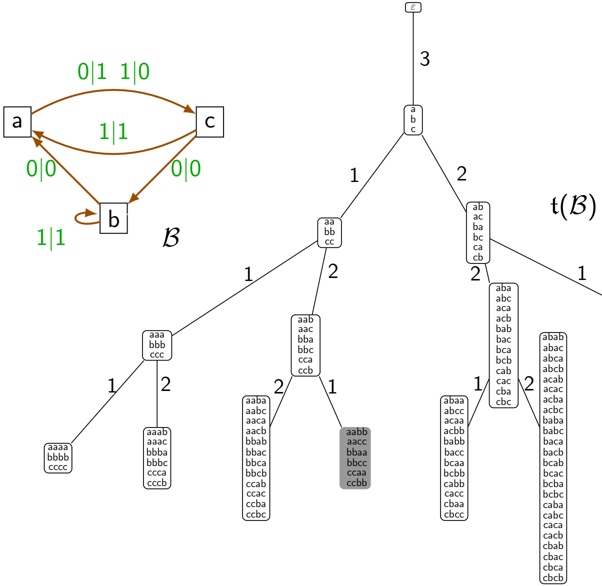

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$\Updownarrow$

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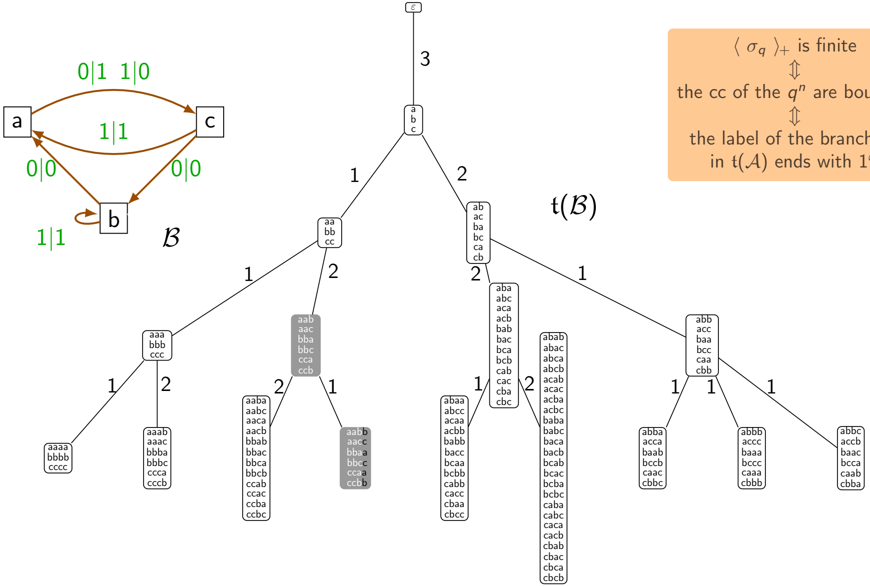

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$\Updownarrow$

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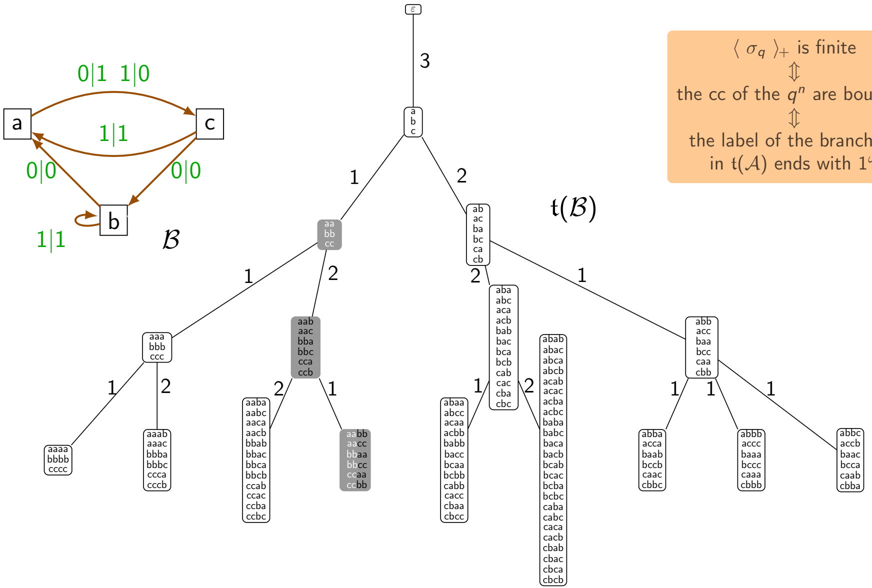

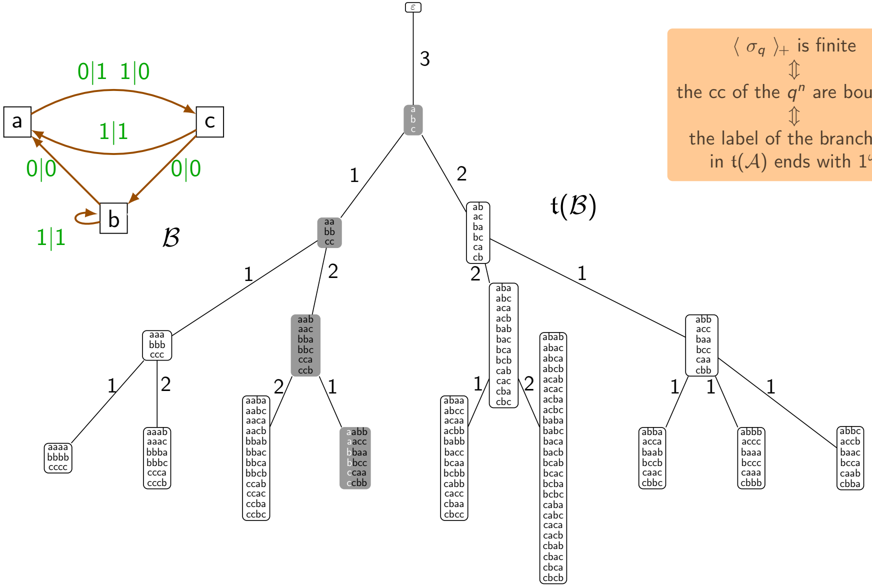

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$\Updownarrow$

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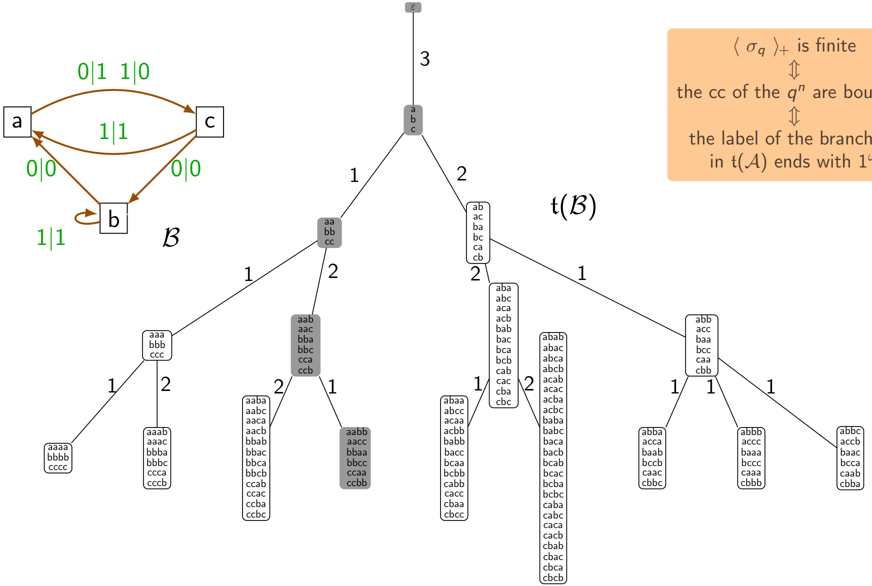


$\mathcal{B}$

$\mathfrak{t}(\mathcal{B})$

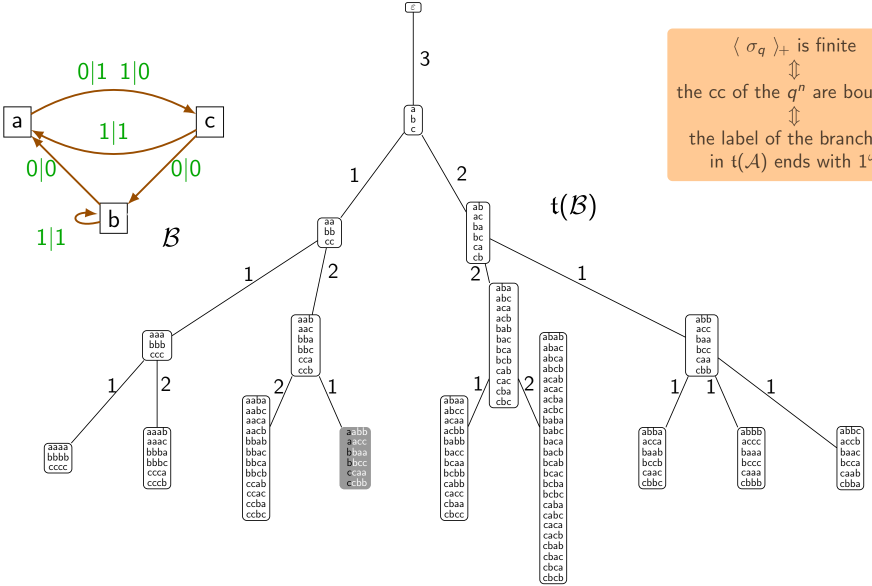
$\langle\ \sigma_q\ \rangle_+$ is finite

$\Updownarrow$

the cc of the q^n are bounded

$\Updownarrow$

the label of the branch q^n in $\mathfrak{t}(\mathcal{A})$ ends with 1^ω



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the cc of the q^n are bounded

$\Updownarrow$

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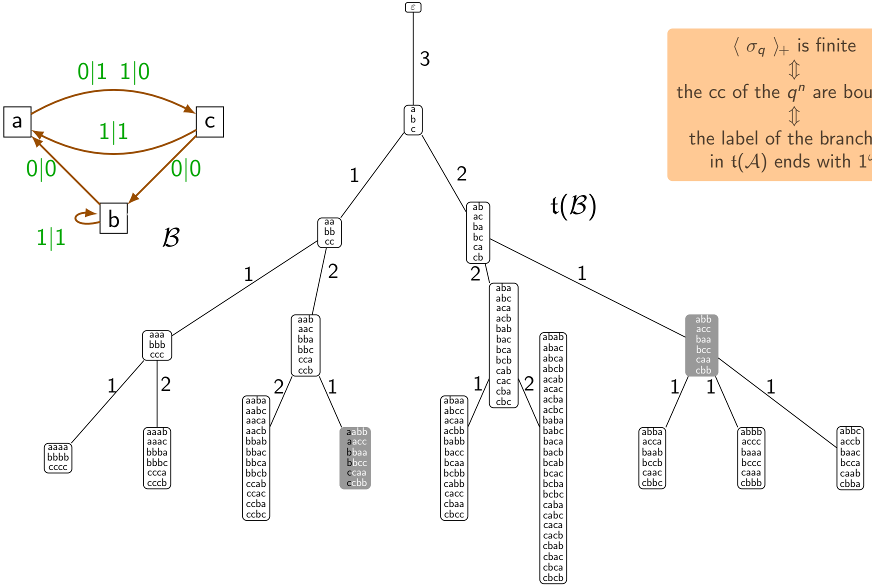

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the cc of the q^n are bounded

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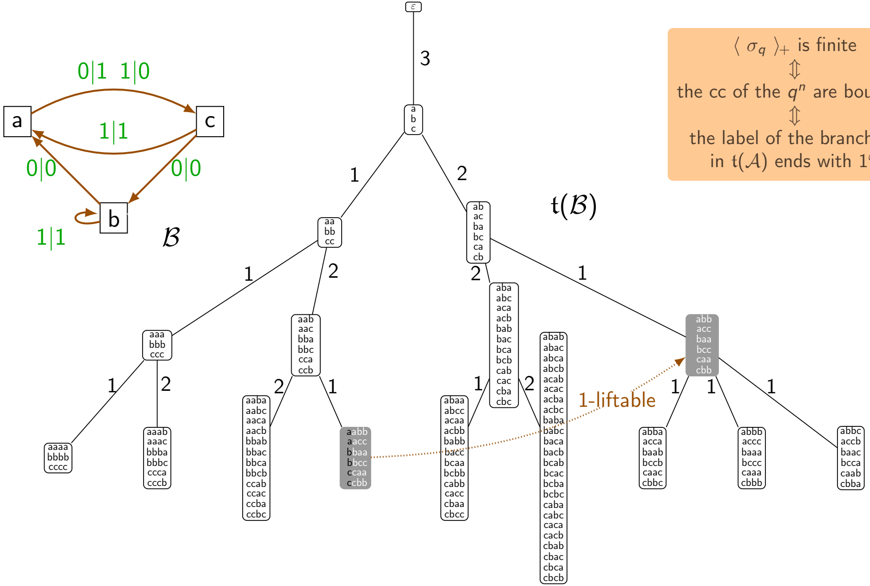

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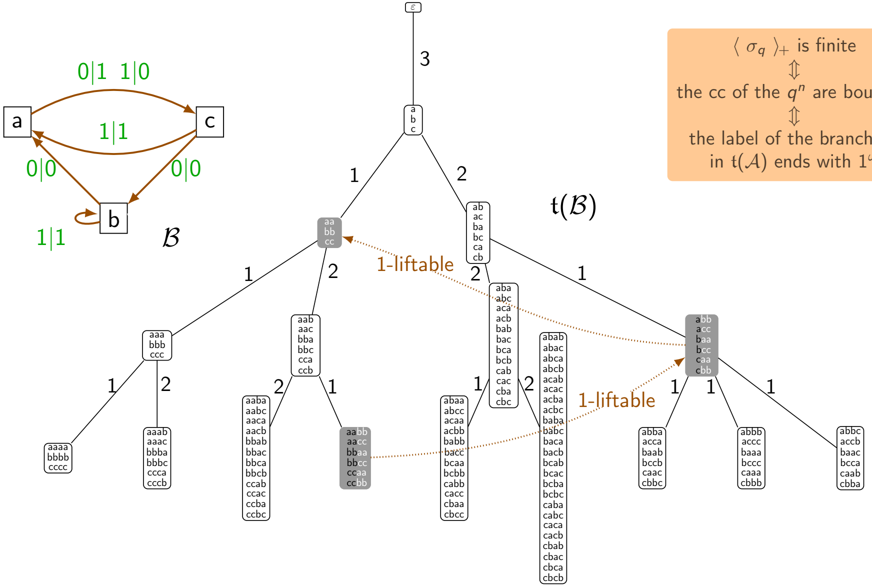

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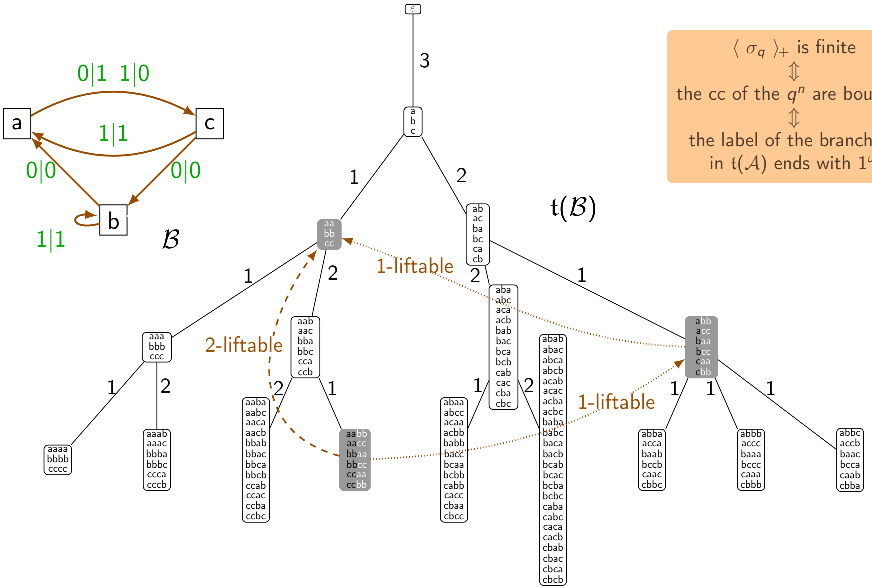

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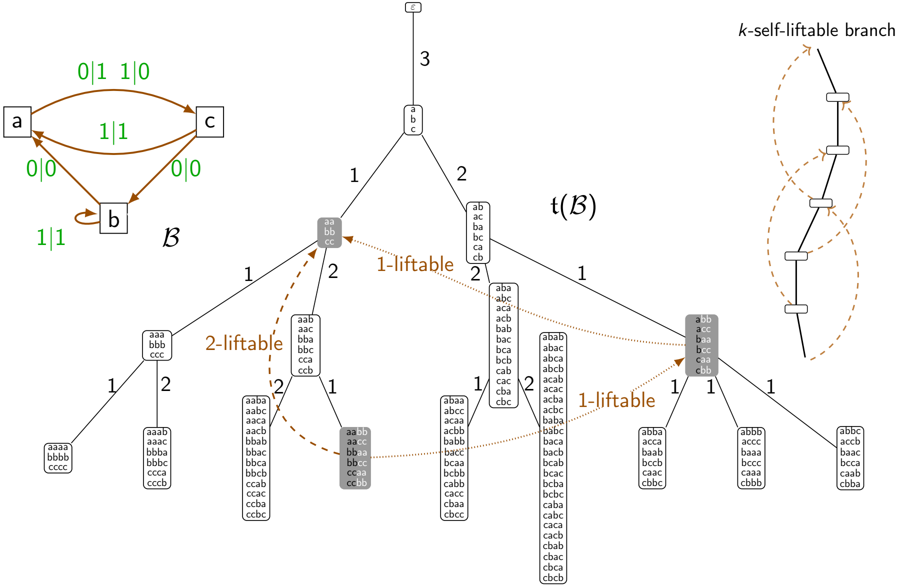
0|1 1|0
1|1
0|0
0|0
1|1
a
c
b
$\mathcal{B}$
$\mathfrak{t}(\mathcal{B})$
k-self-liftable branch
1-liftable
2-liftable
1-liftable
ε
a b c
aa bb cc
ab ac ba bc ca cb
aaa bbb ccc
aab aac bba bbc cca ccb
aba abc aca acb bab bac bca bcb cab cac cba cbc
abb acc baa bcc caa cbb
aaaa bbbb cccc
aaab aaac bbba bbbc ccca cccb
aaba aabc aaca aacb bbab bbac bbca bbcb ccab ccac ccba ccbc
aabb aacc bbaa bbcc ccaa ccbb
abaa abcc acaa acbb babb bacc bcaa bcbb cabb cacc cbaa cbcc
abab abac abca abcb acab acac acba acbc baba babc baca bacb bcab bcac bcba bcbc caba cabc caca cacb cbab cbac cbca cbcb
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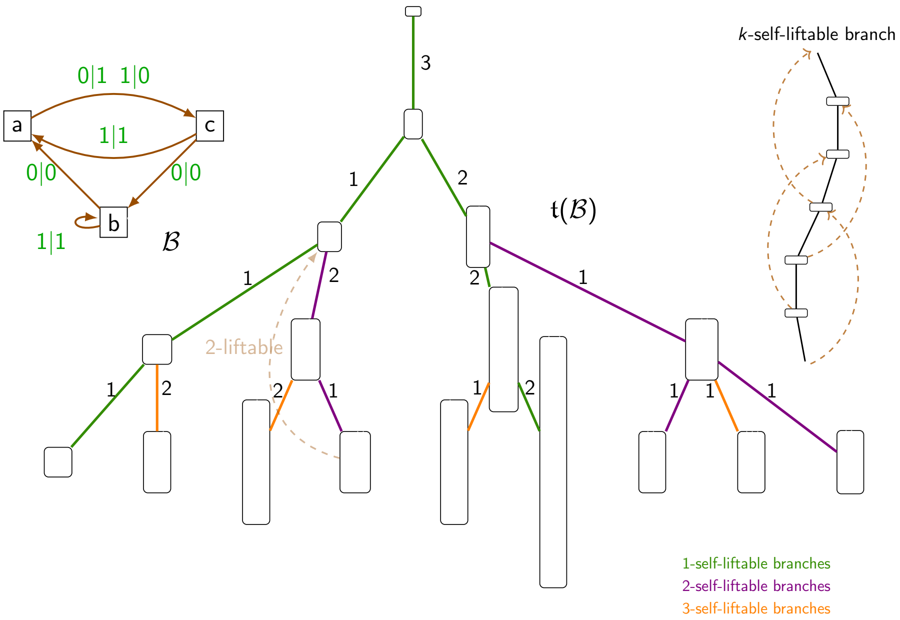
0|1 1|0
a
c
1|1
0|0
0|0
b
1|1
$\mathcal{B}$
3
1
2
$\mathfrak{t}(\mathcal{B})$
1
2
2
1
2-liftable
1
2
2
1
1
2
1
1
1
k-self-liftable branch
1-self-liftable branches
2-self-liftable branches
3-self-liftable branches

Godin P 2015

Assume that $\mathcal{A}$ is an invertible reversible q-state Mealy automaton.
Let $\lambda_{\mathcal{A},k}$ be the number of strict k-self-liftable branches in $\mathfrak{t}(\mathcal{A})$ for $k \geq 1$.
If $(\lambda_{\mathcal{A},k})_{k\geq 1} <_{\mathrm{lex}} (\pi_{q,k})_{k\geq 1}$ holds, $\langle \mathcal{A} \rangle_+$ admits elements of infinite order.

the number of q-ary words with primitive period of length k [oeis.org/A143324]

Godin P 2015

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the number of q-ary words with primitive period of length k [oeis.org/A143324]

Conjecture

Otherwise, $\langle\mathcal{A}\rangle_+$ is finite.

Godin P 2015

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the number of q-ary words with primitive period of length k [oeis.org/A143324]

Conjecture

Otherwise, $\langle\mathcal{A}\rangle_+$ is finite.

$\mathfrak{t}(\mathcal{A})$ admits a maximal number of self-liftable branches

contrapositive of the theorem

$\langle\mathcal{A}\rangle_+$ is periodic

$\langle\mathcal{A}\rangle_+$ is finite

Godin P 2015

Assume that $\mathcal{A}$ is an invertible reversible q-state Mealy automaton.
Let $\lambda_{\mathcal{A},k}$ be the number of strict k-self-liftable branches in $\mathfrak{t}(\mathcal{A})$ for $k \geq 1$.
If $(\lambda_{\mathcal{A},k})_{k\geq 1} <_{\text{lex}} (\pi_{q,k})_{k\geq 1}$ holds, $\langle\mathcal{A}\rangle_+$ admits elements of infinite order.

the number of q-ary words with primitive period of length k [oeis.org/A143324]

Conjecture

Otherwise, $\langle\mathcal{A}\rangle_+$ is finite.

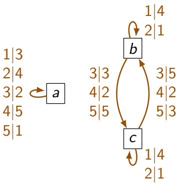
$\mathfrak{t}(\mathcal{A})$ admits a maximal number of self-liftable branches

contrapositive of the theorem

proven for special cases:
- ▷ 2-state [Klimann 2013]
- ▷ connected 3-state [Klimann P Savchuk 2015]
- ▷ no bireversible cc [Godin Klimann P 2015]
- ▷ 2-letter 4-state [Klimann P Savchuk 2015]
- ▷ connected prime-state [Godin Klimann 2016]

$\langle\mathcal{A}\rangle_+$ is periodic

$\langle\mathcal{A}\rangle_+$ is finite


1|3
2|4
3|2
4|5
5|1
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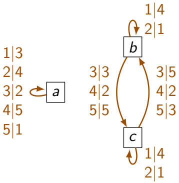
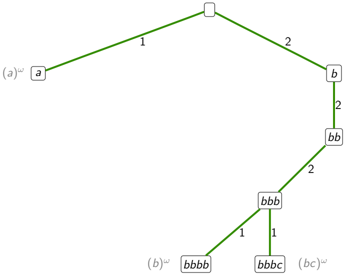
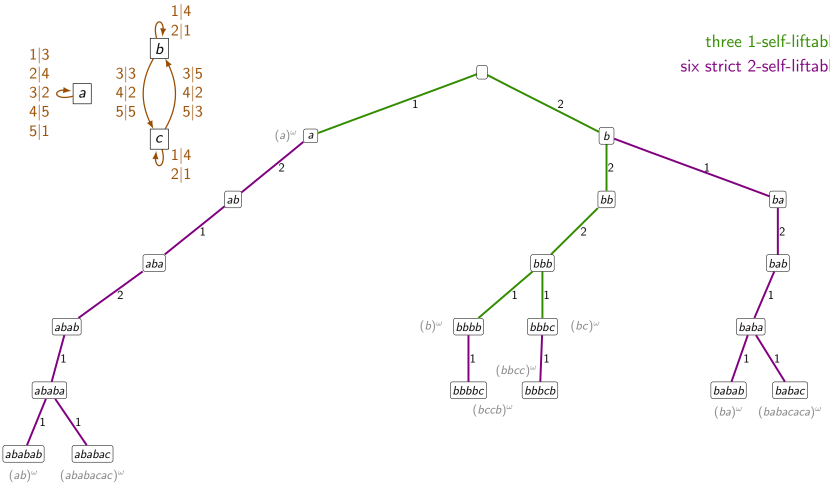
b
3|3
4|2
5|5
3|5
4|2
5|3
c
1|4
2|1



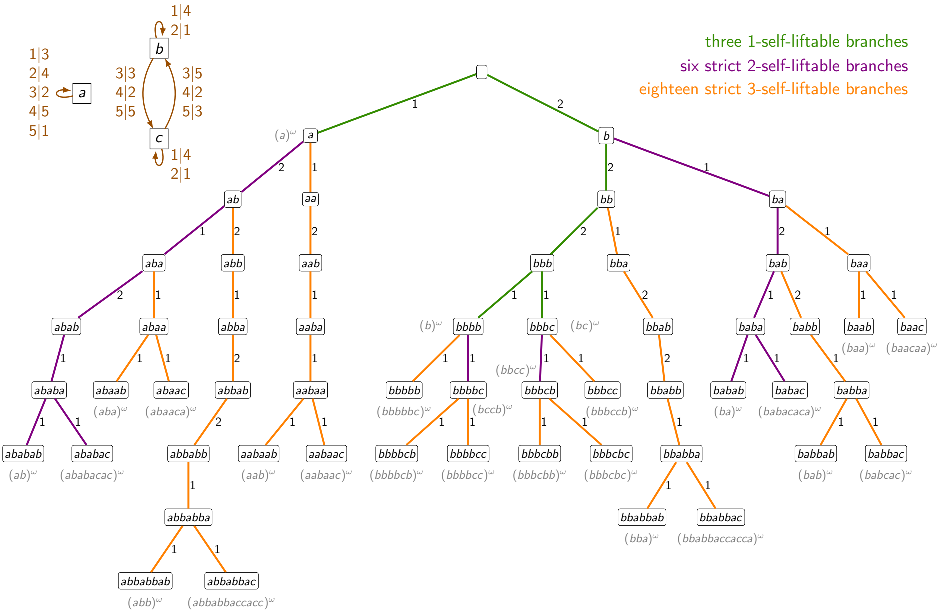


three 1-self-liftable branches



three 1-self-liftable branches

six strict 2-self-liftable branches


1|3
2|4
3|2
4|5
5|1
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b
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1|4
2|1
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4|2
5|5
3|5
4|2
5|3
1|4
2|1
three 1-self-liftable branches
six strict 2-self-liftable branches
eighteen strict 3-self-liftable branches
a
b
$(a)^\omega$
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bbbc
bbab
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babb
baab
baac
$(b)^\omega$
$(bc)^\omega$
$(baa)^\omega$
$(baacaa)^\omega$
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$(bbabbaccacca)^\omega$
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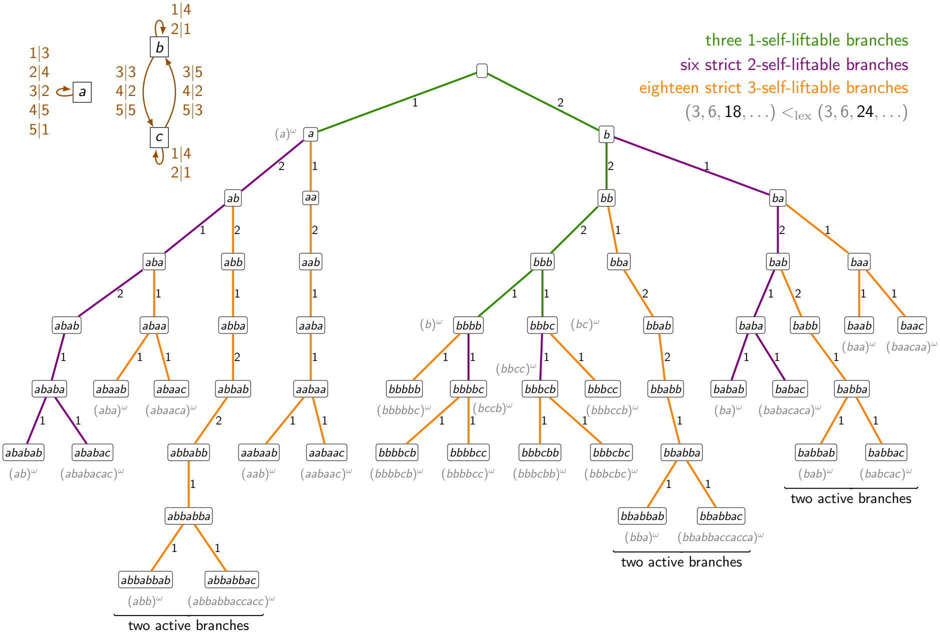
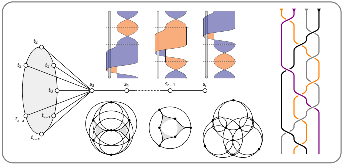
$(abb)^\omega$
$(abbabbaccacc)^\omega$

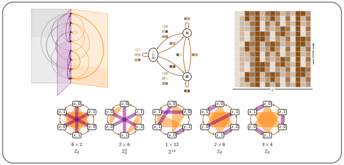
three 1-self-liftable branches
six strict 2-self-liftable branches
eighteen strict 3-self-liftable branches
$(3, 6, 18, \ldots) <_{\mathrm{lex}} (3, 6, 24, \ldots)$

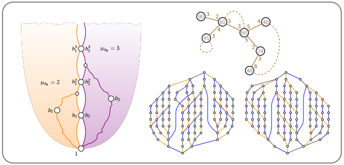


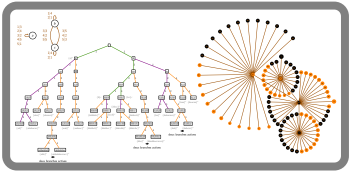
Braid (semi)groups
& Garside theory

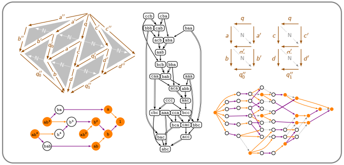


Mealy automata
& automaton (semi)groups



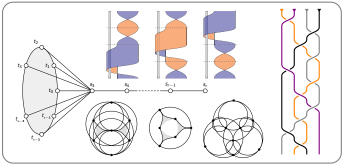


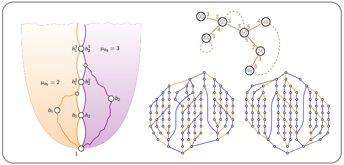




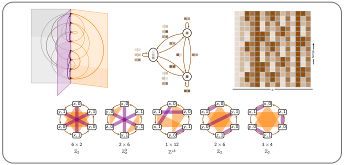
Quadratic normalisations
Thurston *vs* Mealy automata

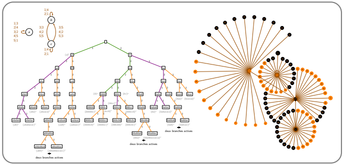
Braid (semi)groups
& Garside theory

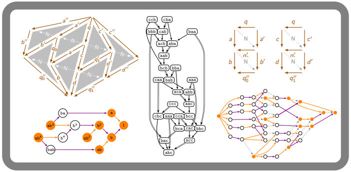




Mealy automata
& automaton (semi)groups







Quadratic normalisations
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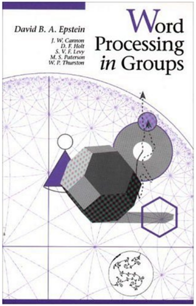
David B. A. Epstein
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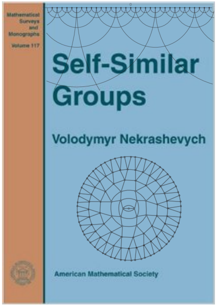
D. F. Holt
S. V. F. Levy
M. S. Paterson
W. P. Thurston
Word Processing in Groups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Volume 117
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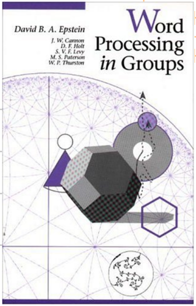
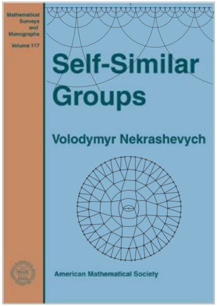
Volodymyr Nekrashevych
American Mathematical Society

geometric properties
of the Cayley graph





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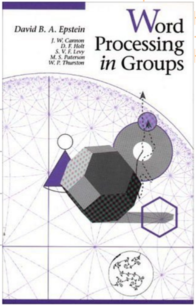
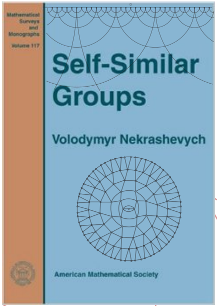
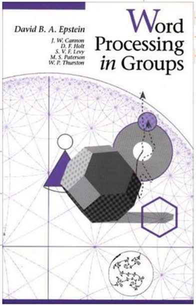


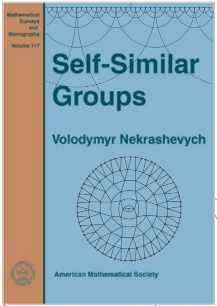


(semi)groups acting
on regular rooted trees

geometric properties of the Cayley graph

- ▷ recognize the language of normal forms
- ▷ execute the (semi)group operations





(semi)groups acting on regular rooted trees geometric properties of the Cayley graph

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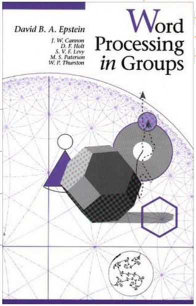
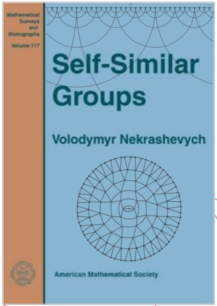




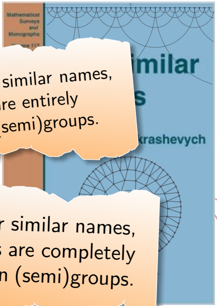
(semi)groups acting on regular rooted trees

▷ define sequential transformations

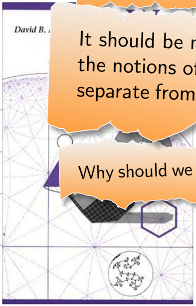
▷ represent the elements themselves

- ▷ recognize the langu
- ▷ execute the (semi)g

geometric of the Cay



We should emphasize that, despite their similar names, the notions of automaton (semi)groups are entirely separate from the notions of automatic (semi)groups.

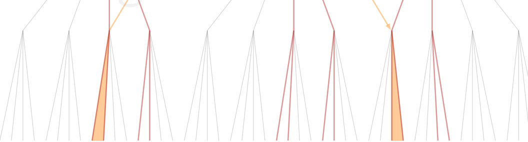


It should be noticed that, despite their similar names, the notions of automatic (semi)groups are completely separate from the notions of automaton (semi)groups.

Why should we compare Edouard Manet and Claude Monet?

- ▷ transformations
- ▷ represent the elements themselves

acting on regular rooted trees



▷ recognize the la

▷ execute the (sem

Groups defined by automata

Laurent Bartholdi *Pedro V. Silva*

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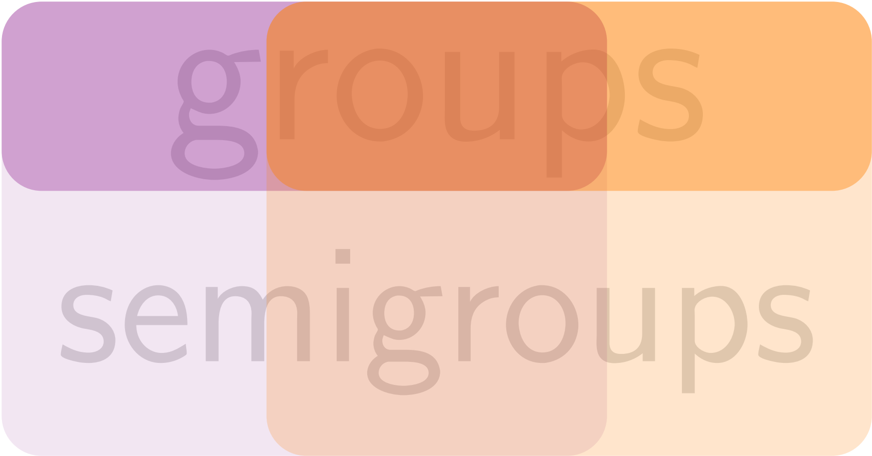
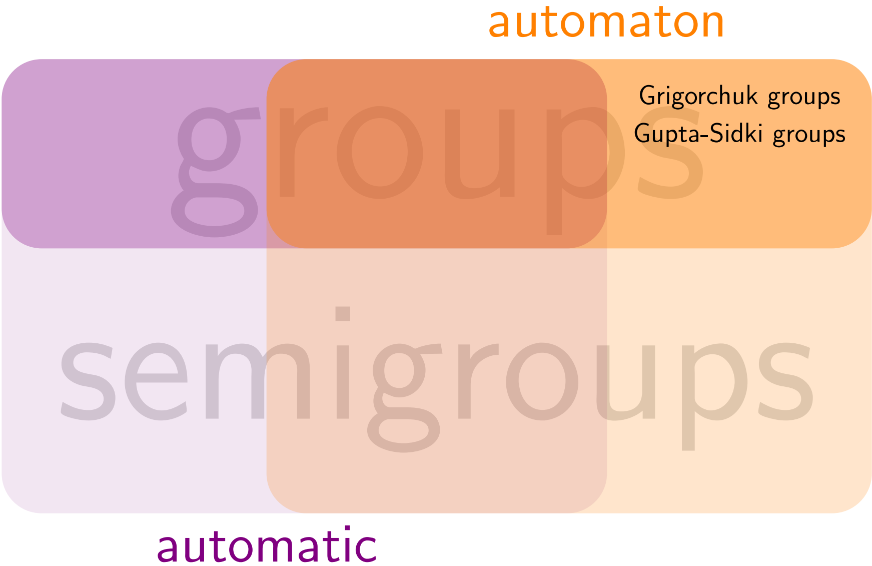


automatic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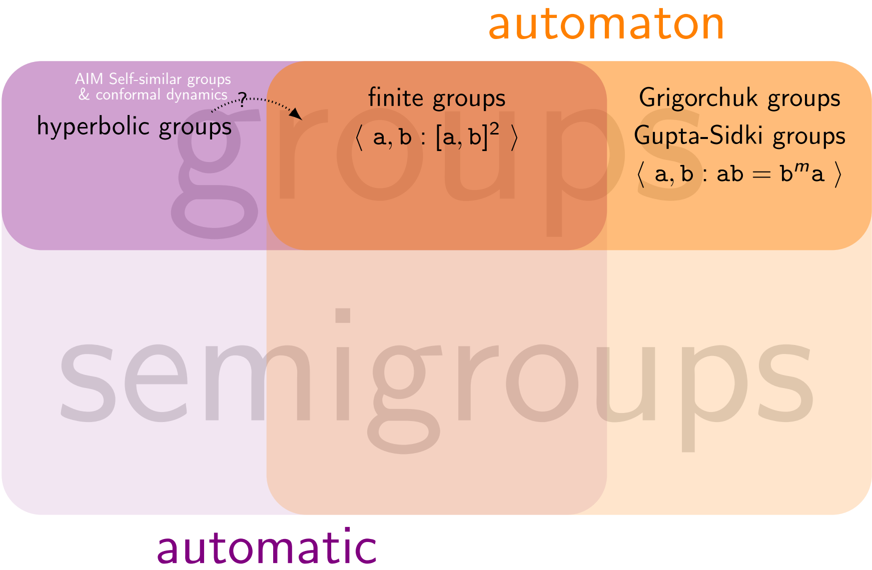
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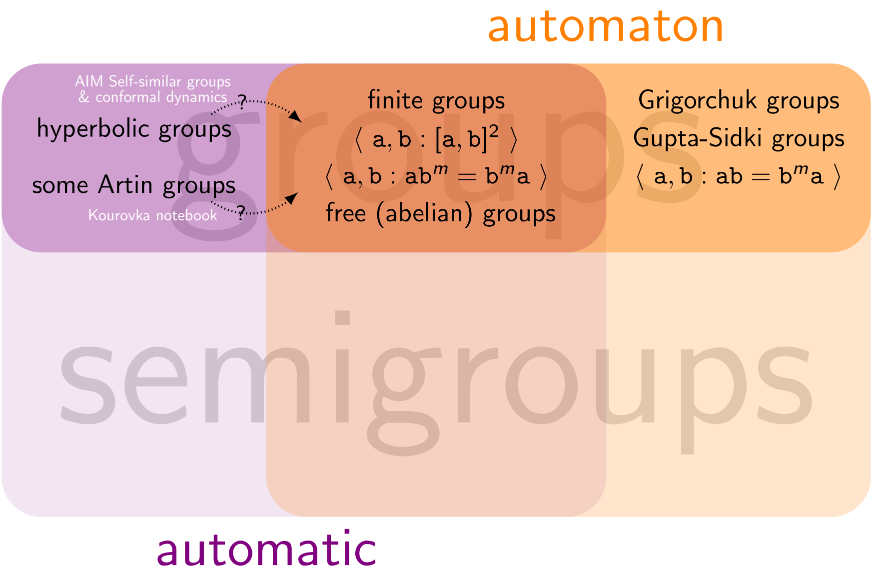
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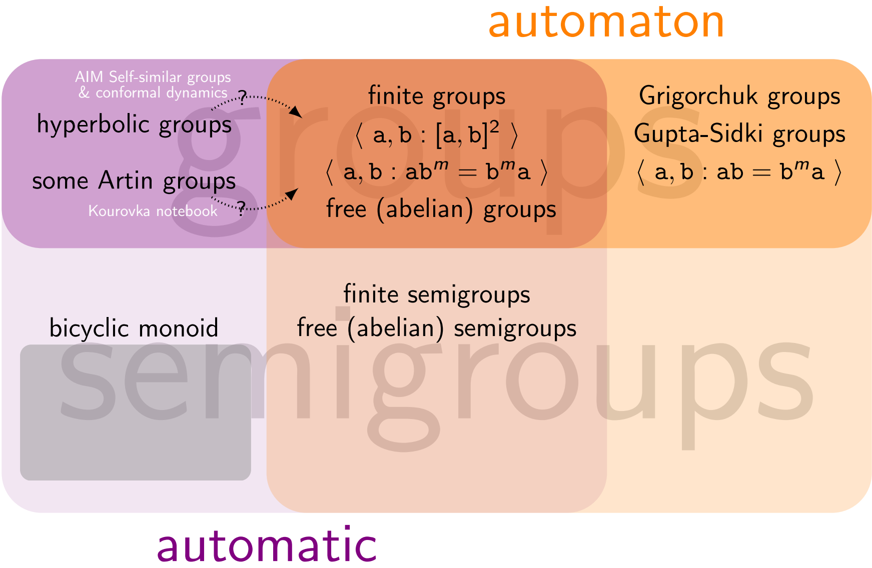
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Let $S = \langle\ \mathcal{Q} : \{w = \mathsf{NF}(w), w \in \mathcal{Q}^*\}\ \rangle_+^1$ be an automatic monoid
where NF is a normalisation associated with a Garside family $\mathcal{Q}$.

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We define the Mealy automaton $\mathcal{M}_{S,\mathcal{Q},\mathsf{NF}} = (\mathcal{Q}, \mathcal{Q}, \tau, \sigma)$ with τ and σ satisfying

$$\mathsf{NF}(ab) = \tau_a(b)\,\sigma_b(a)$$

for every $(a, b) \in \mathcal{Q}^2$, that is,

$$\begin{tikzcd} {} \arrow[d, "b"'] \arrow[dr, dotted, "\mathsf{NF}"] & {} \arrow[l, "a"'] \arrow[d, "\tau_a(b)"] \\ {} & {} \arrow[l, "\sigma_b(a)"] \end{tikzcd}$$

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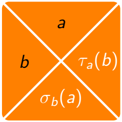

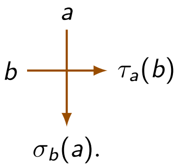

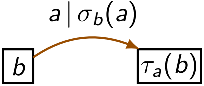

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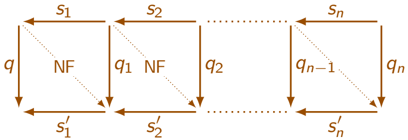
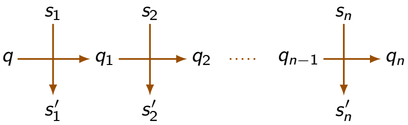
For $\mathrm{NF}(s) = s_n \cdots s_1$ and $\mathrm{NF}(sq) = q_n s'_n \cdots s'_1$, we obtain diagrammatically:





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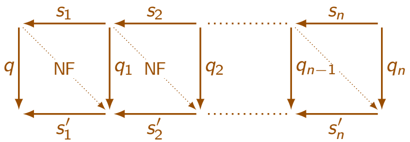

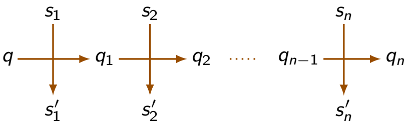

We deduce $\sigma_q(s_1 \cdots s_n) = s'_1 \cdots s'_n$ for any $q \in \mathcal{Q}$.

Theorem [P 2015]

$$S \cong \langle\ \mathcal{M}_{S,\mathcal{Q},\mathrm{NF}}\ \rangle_+^1$$

$$\langle\ a, b : ab = a\ \rangle_+^1$$

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$$\mathcal{Q} = \{a, b, 1\}$$

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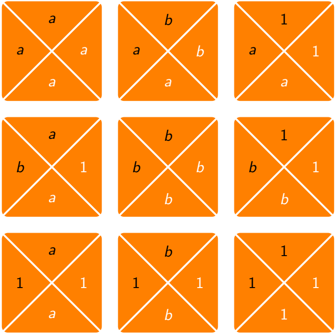

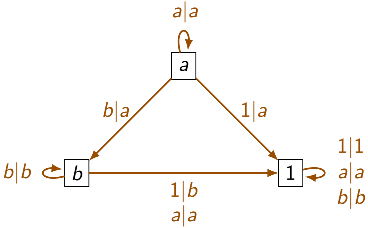

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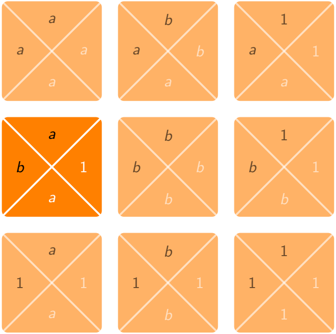

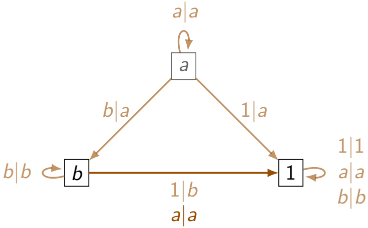

Baumslag-Solitar Artin-Krammer

$$\mathrm{BS}^1_+(1,0) = \quad \langle\, a, b : ab = a \,\rangle^1_+ \quad = \mathrm{AK}^1_+\left(\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right)$$

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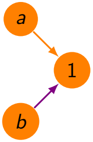

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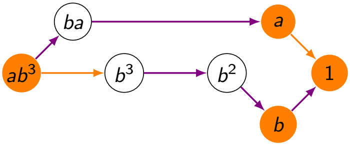

Baumslag-Solitar | Artin-Krammer

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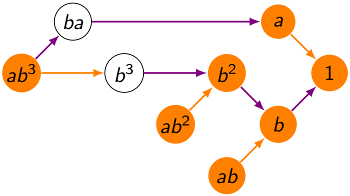

Baumslag-Solitar $\qquad\qquad\qquad\qquad$ Artin-Krammer

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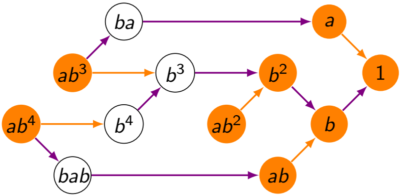

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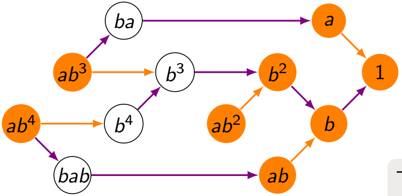

There exists a group-embeddable automaticon monoid whose enveloping group is not an automaticon group

Baumslag-Solitar Artin-Krammer

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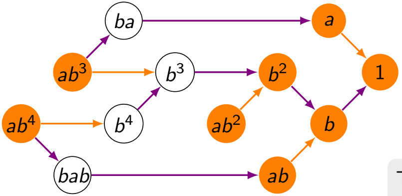

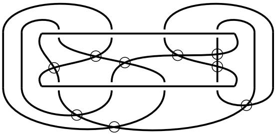

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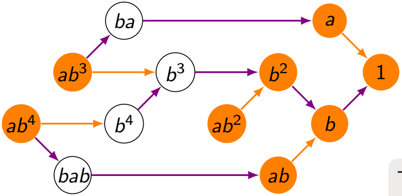

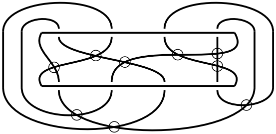

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Hoffmann 2001 P 2015

$\mathrm{BS}^1_+(m,n)$ is an automaticon monoid

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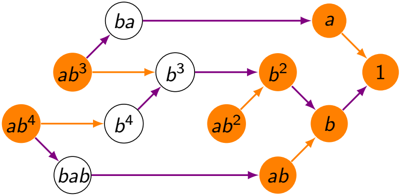

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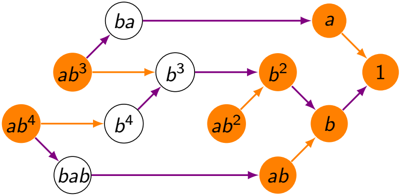

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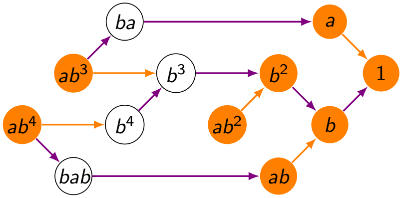

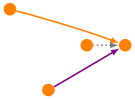

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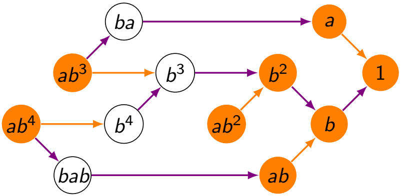

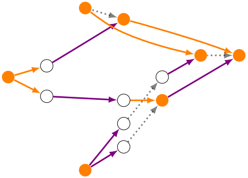

Hoffmann 2001   P 2015

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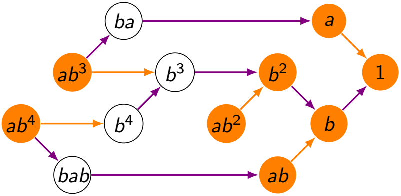

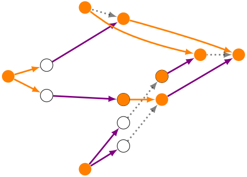

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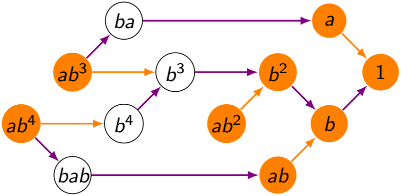

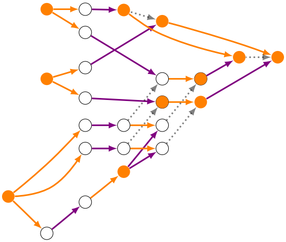

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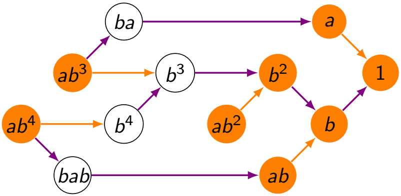

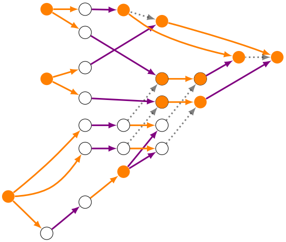

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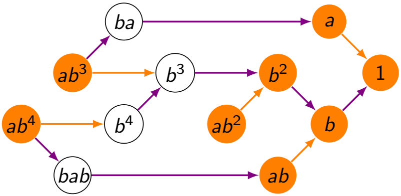

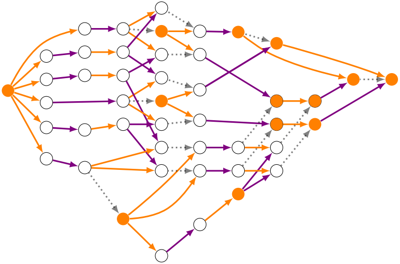

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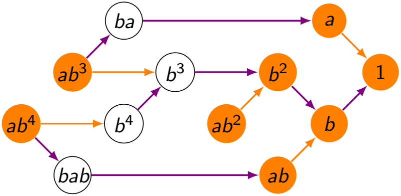

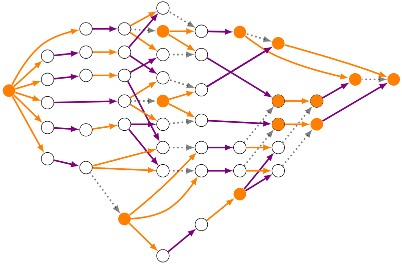

Hoffmann 2001 P 2015

$\mathrm{BS}_+^1(m,n)$ is an automaticon monoid

Dehornoy Guiraud 2016 P 2016

$\mathrm{AK}_+^1(\Gamma)$ is an automaticon monoid

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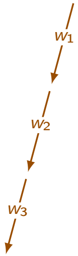

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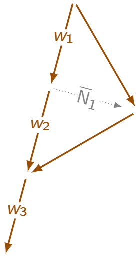

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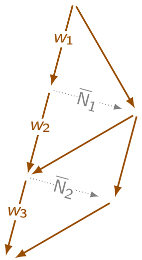

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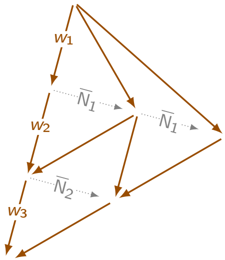

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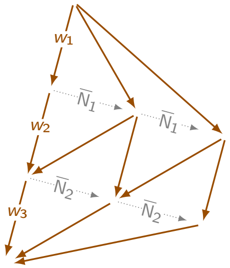

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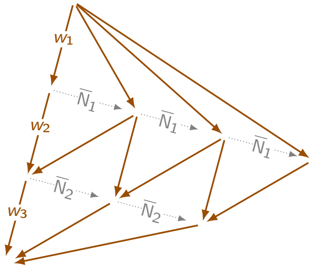

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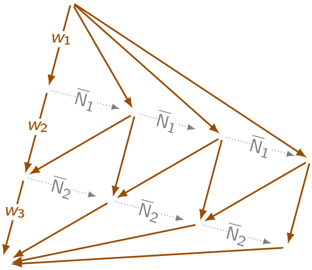

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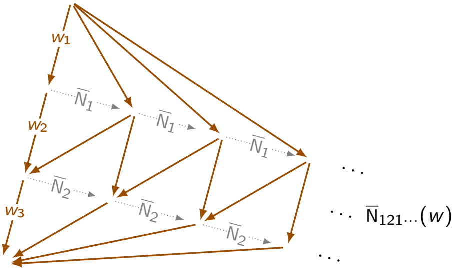

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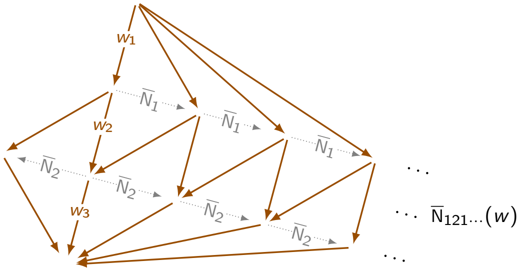

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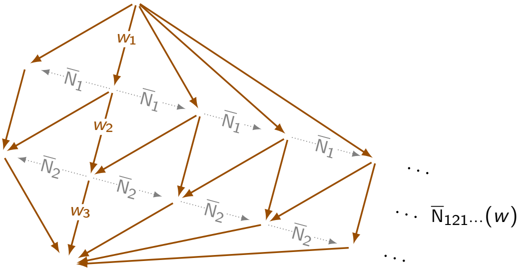

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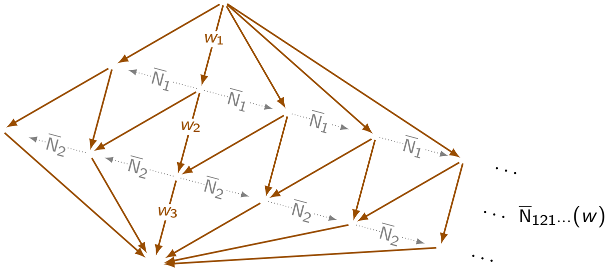

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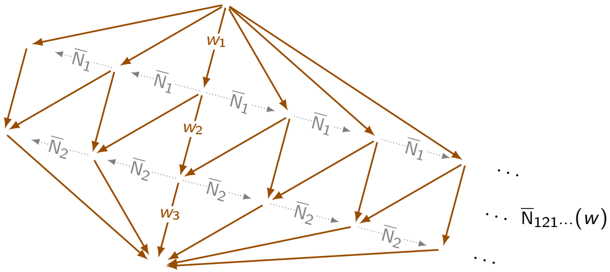

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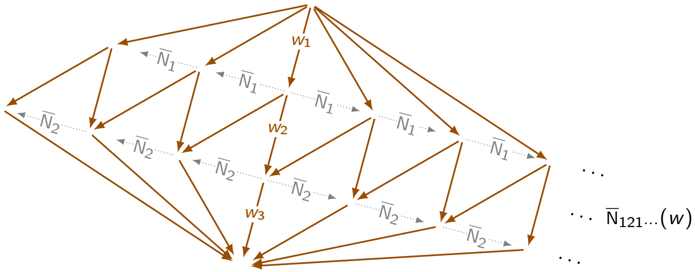

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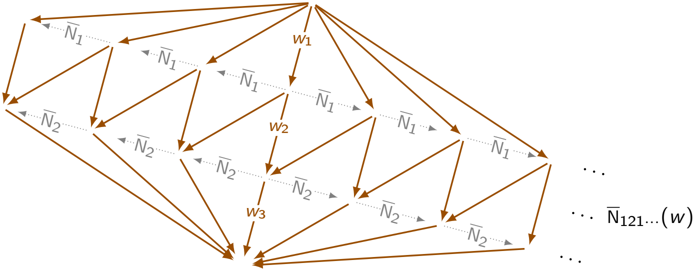

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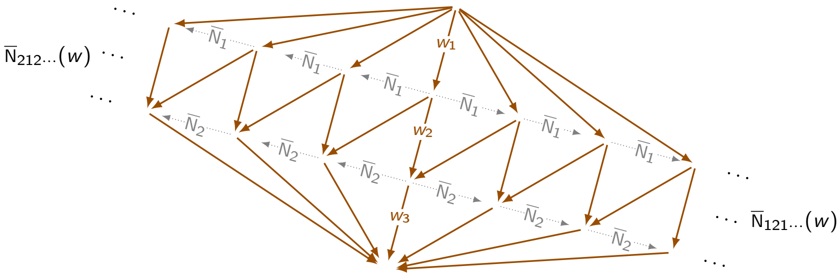

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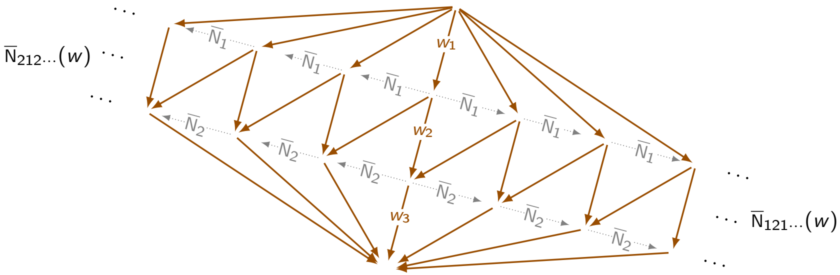

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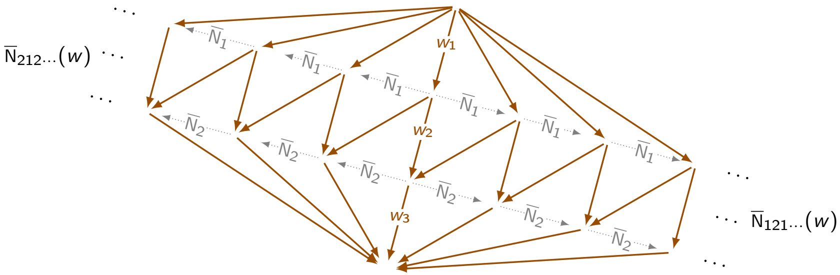

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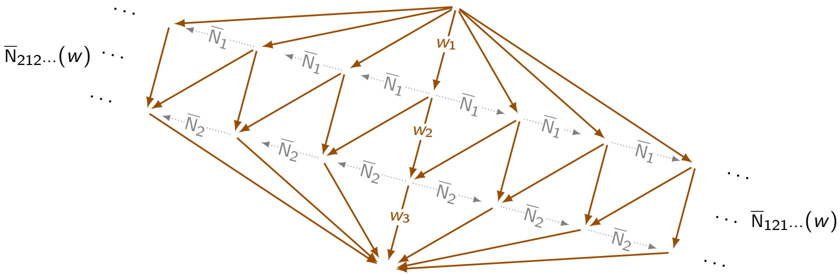

$$\left(\max_{w \in \mathcal{Q}^3} \min\{\, \ell : \mathsf{N}(w) = \overline{\mathsf{N}}_{\underbrace{212\ldots}_{\text{length } \ell}}(w)\}, \max_{w \in \mathcal{Q}^3} \min\{\, \ell : \mathsf{N}(w) = \overline{\mathsf{N}}_{\underbrace{121\ldots}_{\text{length } \ell}}(w)\} \right).$$

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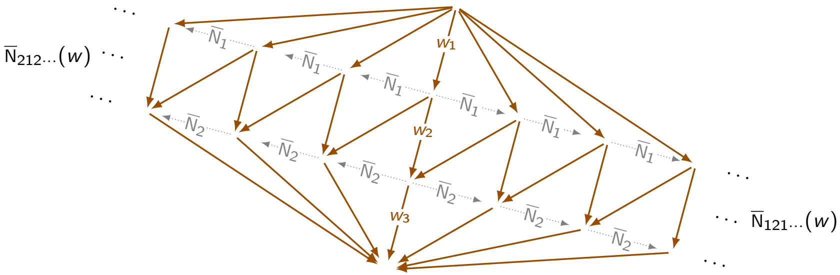

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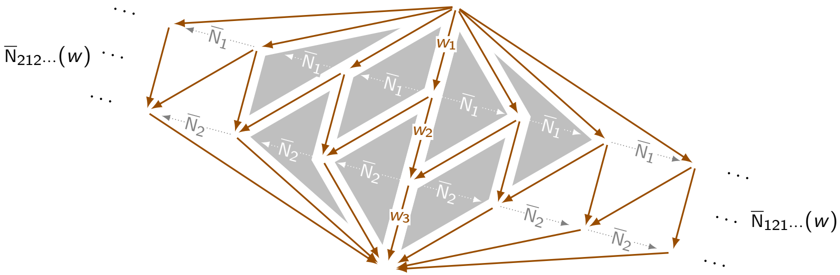

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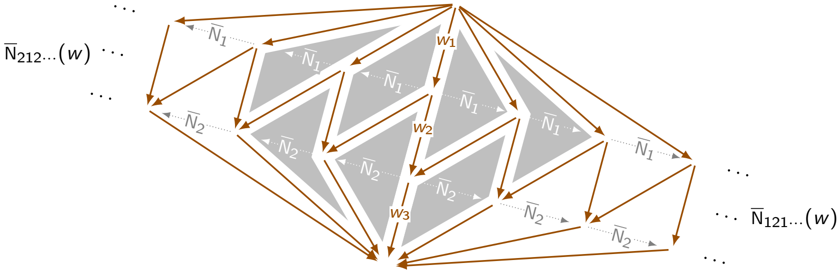

Lemma [P 2016] top-

$$S \cong \langle\, \mathcal{M}_{S,\mathcal{Q},\mathsf{N}} \,\rangle_+^1 / \,?$$

Proposition [P 2016] bottom-approximation

$$\langle\, \mathcal{M}_{S,\mathcal{Q},\mathsf{N}} \,\rangle_+^1 \cong S/\,? \iff (\mathcal{Q}, \mathsf{N}) \text{ satisfies } (\quad)$$

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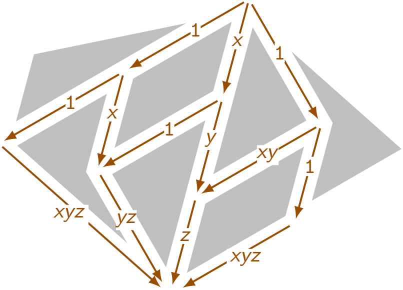

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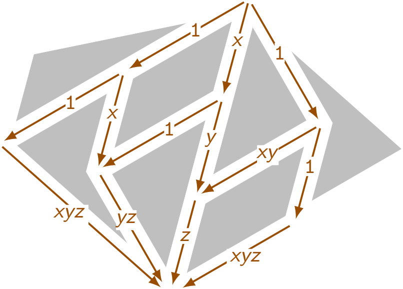

The bicyclic monoid $\mathbf{B} = \langle\ \mathtt{a}, \mathtt{b} : \mathtt{ab} = 1\ \rangle_+^1$ is not an automaton monoid:

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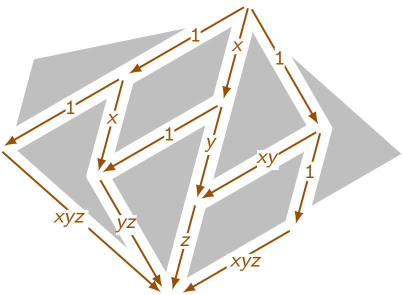

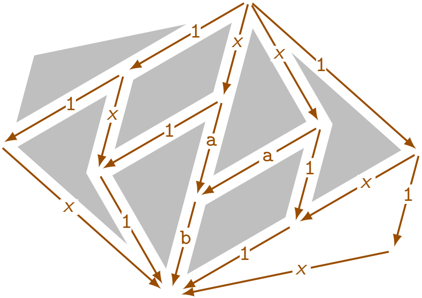

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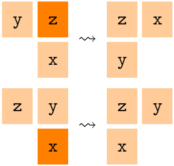

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$$\begin{array}{ccc} 5 & 6 & 6 \\ 4 & 2 & 4 \\ 3 & & 3 \\ & & 1 \end{array}$$

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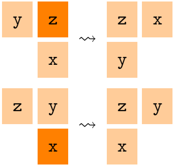

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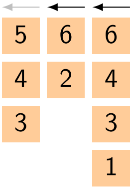

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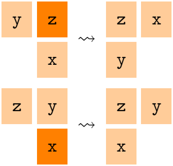

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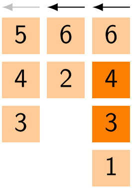

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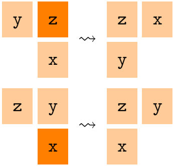

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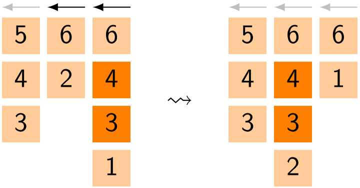

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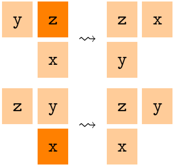

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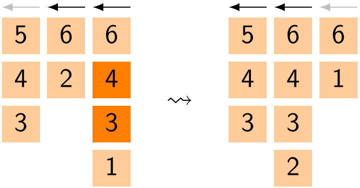

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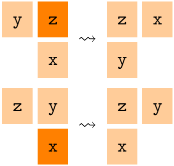

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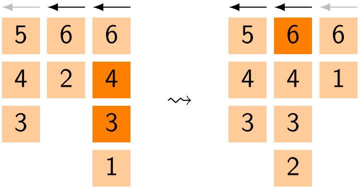

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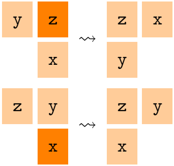

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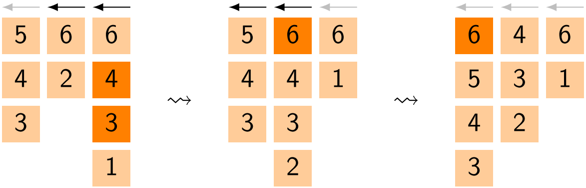

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$$\mathbf{P}_n = \left\langle 1 < \cdots < n : \begin{array}{ll} \mathtt{zxy} = \mathtt{xzy} & \text{for } \mathtt{x} \leq \mathtt{y} < \mathtt{z} \\ \mathtt{yxz} = \mathtt{yzx} & \text{for } \mathtt{x} < \mathtt{y} \leq \mathtt{z} \end{array} \right\rangle_{+}^{1}.$$

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Then $\mathbf{P}_n$ is also generated by the family $\mathcal{Q}$ of columns, defined to be strictly decreasing products of elements of $\{1, \ldots, n\}$.

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$\mathbf{P}_n$ is an automatic monoid

$$\begin{array}{ccc} 5 & 6 & 6 \\ 4 & 2 & 4 \\ 3 & & 3 \\ & & 1 \end{array} \rightsquigarrow \begin{array}{ccc} 5 & 6 & 6 \\ 4 & 4 & 1 \\ 3 & 3 & \\ & 2 & \end{array} \rightsquigarrow \begin{array}{ccc} 6 & 4 & 6 \\ 5 & 3 & 1 \\ 4 & 2 & \\ 3 & & \end{array}$$

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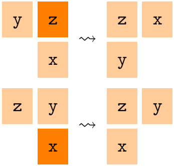

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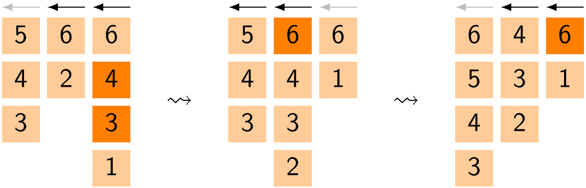

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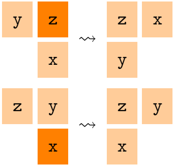

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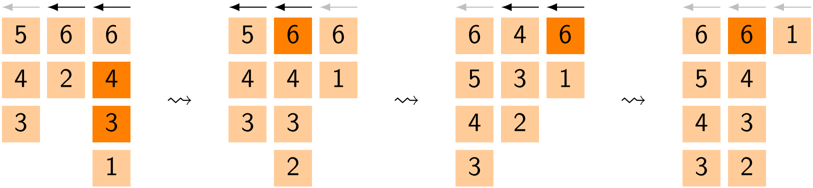

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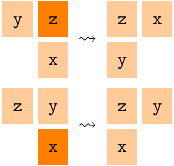

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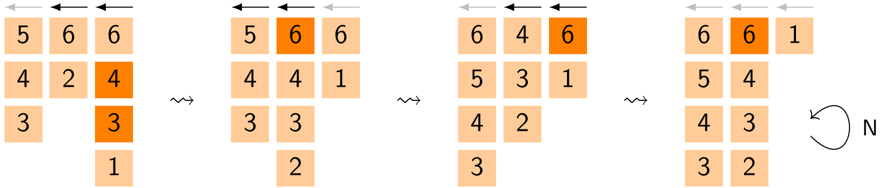

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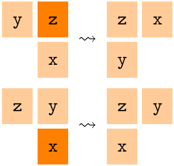

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P 2016

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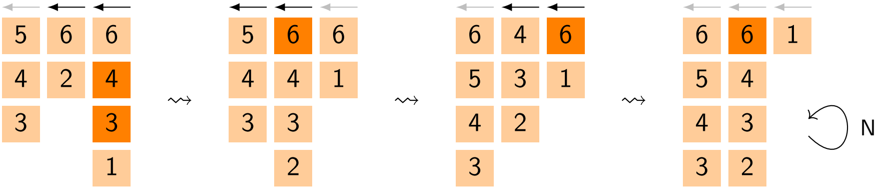

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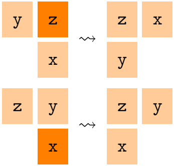

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P 2016

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The Chinese monoid of rank n is

$$\mathbf{C}_n = \langle\, 1 < \cdots < n : \mathrm{zyx} = \mathrm{zxy} \;\; = \mathrm{yzx} \quad \text{for } \mathrm{x} \leq \mathrm{y} \leq \mathrm{z} \,\rangle_+^1 .$$

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Cain Gray Malheiro 2014

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P 2016

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The Chinese monoid of rank n is

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Then $\mathbf{C}_n$ is generated by $\mathcal{Q} = \{\mathrm{x} : n \geq \mathrm{x} \geq 1\} \cup \{\mathrm{yx} : \mathrm{y} > \mathrm{x}\}$.

Cain Gray Malheiro 2016

$\mathbf{C}_n$ is an automatic monoid

The plactic monoid of rank n is

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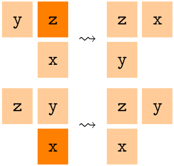

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Cain Gray Malheiro 2016

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P 2016

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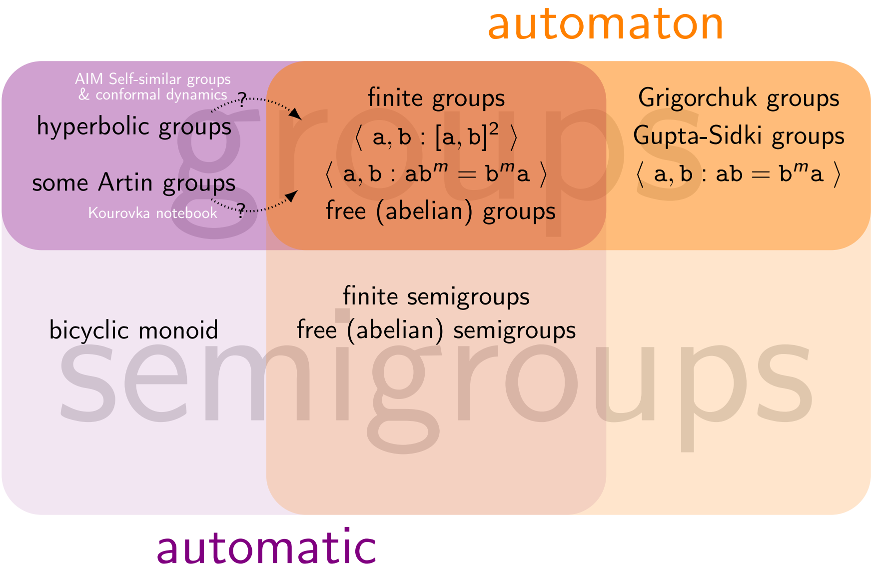
automaton
AIM Self-similar groups
& conformal dynamics
?
hyperbolic groups
some Artin groups
?
Kourovka notebook
groups
finite groups
$\langle\, a, b : [a, b]^2 \,\rangle$
$\langle\, a, b : ab^m = b^m a \,\rangle$
free (abelian) groups
Grigorchuk groups
Gupta-Sidki groups
$\langle\, a, b : ab = b^m a \,\rangle$
bicyclic monoid
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free (abelian) semigroups
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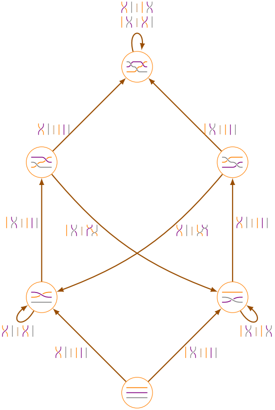
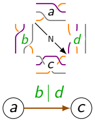
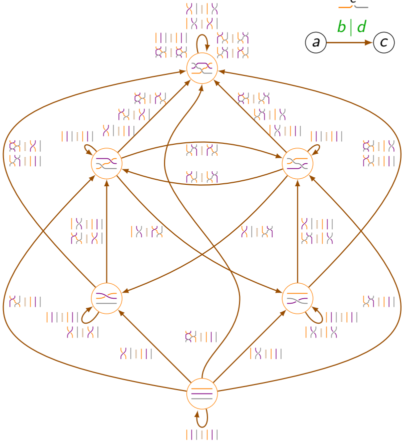
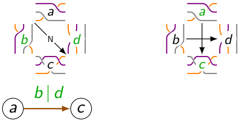


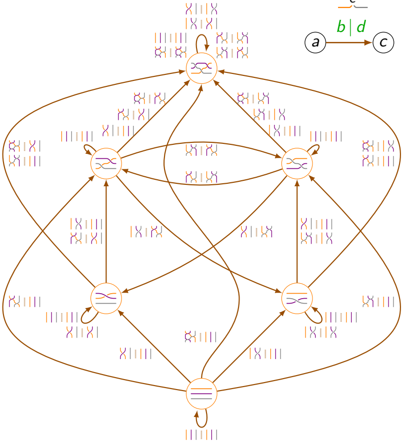
Thurston transducer





(completed)
Thurston transducer





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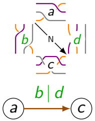

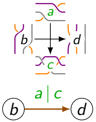

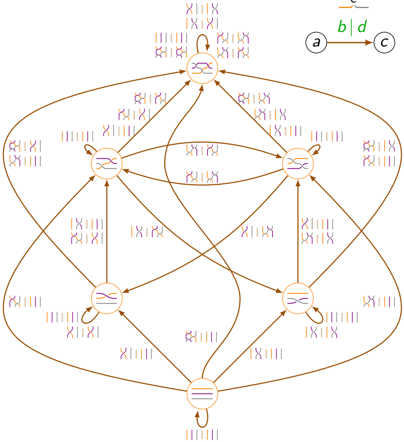

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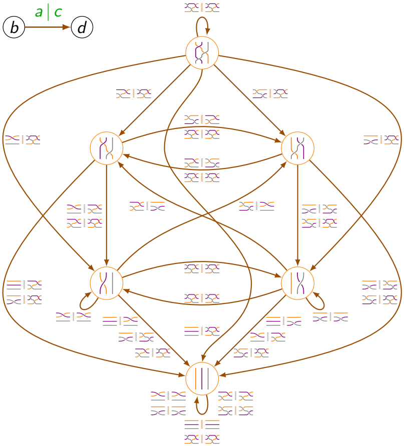
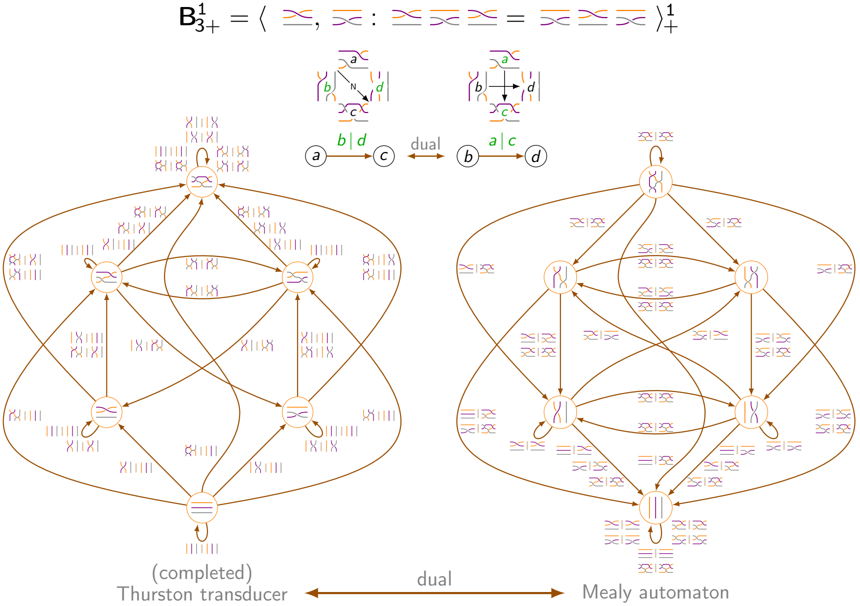
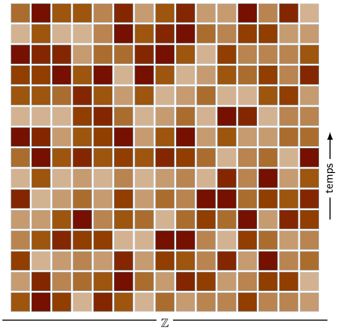
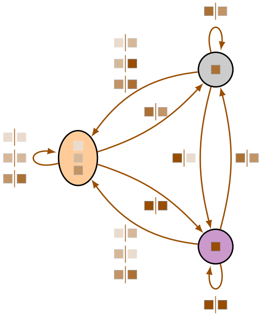


Mealy automaton


$B^1_{3+} = \langle$, : = $\rangle^1_+$
a
b
N
d
c
b | d
dual
a | c
(completed)
Thurston transducer
dual
Mealy automaton

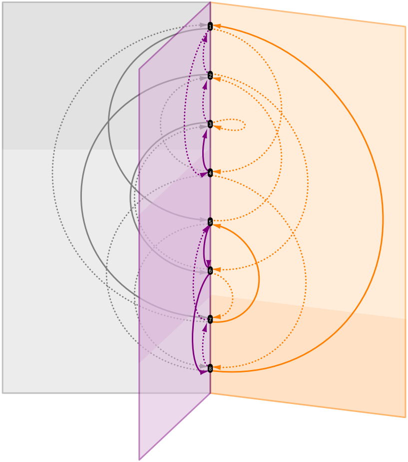


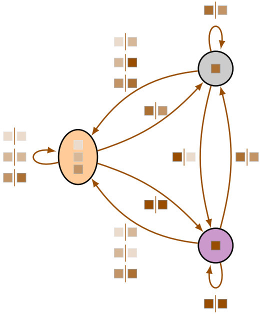


adapter au nouvel alphabet $\{1, \dots, 8\}$

Question

Is the finiteness problem for reset automaton groups decidable ?





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