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A New Hierarchy for Automaton Semigroups

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We define a new strict and computable hierarchy for the family of automaton semigroups, which reflects the various asymptotic behaviors of the state-activity growth. This hierarchy extends that given by Sidki for automaton groups, and also gives new insights into the latter. Its exponential part coincides with a notion of entropy for some associated automata.

We prove that the Order Problem is decidable whenever the state-activity is bounded. The Order Problem remains open for the next level of this hierarchy, that is, when the state-activity is linear. Gillibert showed that it is undecidable in the whole family.

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We extend the aforementioned hierarchy via a semi-norm making it more coarse but somehow more robust and we prove that the Order Problem is still decidable for the first two levels of this alternative hierarchy.

Keywords: Automaton; semigroup; entropy; hierarchy; decision problem.

1. Introduction

The family of automaton groups and semigroups has provided a wide playground to various algorithmic problems in computational (semi)group theory [1, 3–5, 7, 8, 12–15]. While many undecidable questions in the world of (semi)groups remain undecidable for this family, the underlying Mealy automata provide a combinatorial leverage to solve the Word Problem for this family, and various other problems in some important subfamilies. Recall that a Mealy automaton is a letter-to-letter, complete, deterministic transducer with same input and output alphabet, so each of its states induces a transformation from the set of words over its alphabet into itself. Composing these Mealy transformations leads to so-called automaton (semi)groups, and the Word Problem can be solved using a classical technique of minimization.

The Order Problem is one of the current challenging problems in computational (semi)group theory. On the one hand, it was proven to be undecidable for automaton semigroups by Gillibert [12]. On the other hand, Sidki introduced a polynomial hierarchy for invertible Mealy transformations in [27] and, with Bondarenko, Bondarenko, and Zapata in [8], solved the Order Problem for its lowest two levels (bounded invertible automata).

Our main contributions in this paper are the following: an activity-based hierarchy for possibly non-invertible Mealy transformations (Sec. 3), extending Sidki construction [27] to non-necessarily-invertible transformations; and a study of the algorithmic properties in the lowest two levels of the hierarchy, namely Mealy automata with bounded activity. We prove:

Theorem (see Sec. 5). *The* Order Problem *is decidable for bounded Mealy transformations; namely, there is an algorithm that, given a bounded Mealy automaton with a distinguished initial state, decides whether the transformation τ that it defines has infinite order, and if not finds the minimal $r > s$ satisfying $\tau^r = \tau^s$.*

Our strategy of proof follows closely that of Sidki [27] and Bondarenko, Bondarenko, Sidki and Zapata [8], with some crucial differences. On the one hand, a naive count of the number of nontrivial states of a transformation yields neither a useful invariant, nor a hierarchy stable under multiplication; on the other hand, the structure of cyclic semigroups (e.g. $\langle\, a \mid a^m = a^{m+n} \,\rangle_+$ has index m and period n) is more complex than that of cyclic groups (e.g. $\langle\, a \mid a^m \,\rangle$ has order m).

Moreover, in Sec. 6, we propose an extended version of the activity such that the class of Mealy automata without cycles with exit — studied by Antonenko and Russyev [2, 23], who proved that they only generate finite (semi)groups — form the very first level (i.e. transformations with finitary extended activity) of this alternative hierarchy and for which the ORDER PROBLEM remains decidable for the second level (i.e. transformations with bounded extended activity). It is worthwhile noting that this also gives a nontrivial extension for the group case.

2. Notions from Automata and Graph Theory

This section gathers some basics about automata, especially some links between automata, Mealy automata, automaton semigroups, and finite-state transformations. We refer the reader to handbooks for graph theory [22], automata theory [24], and automaton (semi)groups [7].

A *non-deterministic finite-state automaton* (**NFA** for short) is given by a directed graph with finite vertex set Q, a set of edges Δ labeled by an alphabet Σ, and two distinguished subsets of vertices $I \subseteq Q$ and $F \subseteq Q$. The vertices of the graph are called *states* of the automaton and its edges are called *transitions*. The elements of I and F are called respectively *initial* and *final* states. A transition from the state p to the state q with label x is denoted by $p \xrightarrow{x} q$.

A **NFA** is *deterministic* — **DFA** for short—(resp. *complete*) if for each state q and each letter x, there exists at most (resp. at least) one transition from q with label x. Given a word $\mathbf{w} = w_1 w_2 \cdots w_n \in \Sigma^*$ (where the w_i are letters), a *run with label* $\mathbf{w}$ in an automaton (**NFA** or **DFA**) is a sequence of consecutive transitions

$$q_1 \xrightarrow{w_1} q_2 \xrightarrow{w_2} q_3 \to \cdots \to q_n \xrightarrow{w_n} q_{n+1}.$$

Such a run is said *successful* whenever q_1 is an initial state and q_{n+1} a final state. A word in Σ^* is *recognized* by an automaton if it is the label of at least one successful run. The *language* recognized by an automaton is the set of words it recognizes. A **DFA** is *coaccessible* if each state belongs to some run ending at a final state.

Let $\mathcal{A}$ be a **NFA** with stateset Q. The Rabin–Scott powerset construction [20] returns, in a nutshell, the (co)accessible **DFA** — denoted by $\det(\mathcal{A})$—with states corresponding to subsets of Q, whose initial state is the subset of all initial states of $\mathcal{A}$ and whose final states are the subsets containing at least one final state of $\mathcal{A}$; its transition labeled by x from a state $S \subseteq 2^Q$ leads to the state $\{q \mid \exists p \in S, p \xrightarrow{x} q \text{ in } \mathcal{A}\}$. Notice that the size of the resulting **DFA** might therefore be exponential in the size of the original **NFA**.

Given a language $L \subseteq \Sigma^*$, its *entropy* is

$$h(L) = \limsup_{\ell \to \infty} \frac{1}{\ell} \log \#(L \cap \Sigma^\ell).$$

This quantity appears in various situations, in particular for subshifts [18] and for finite-state automata [10]. We shall recall how to compute it with matrices. Notice that, in all the cases studied here, the languages will be prefix-closed (actually all paths in a graph) so we can replace lim sup by lim, which will always exist [18, 21].

To any **NFA** $\mathcal{A}$ with n states, associate its *transition matrix* $A = \{A_{i,j}\}_{i,j} \in \mathbb{N}^{n\times n}$ where $A_{i,j}$ is the number of transitions from i to j. Let furthermore $v \in \mathbb{N}^n$ be the row vector with '1' at all positions in I and $w \in \mathbb{N}^n$ be the column vector with '1' at all positions in F. Then $vA^\ell w$ is the number of successful runs in $\mathcal{A}$ with length ℓ. Assuming furthermore that $\mathcal{A}$ is deterministic, $vA^\ell w$ is the cardinality of $L \cap \Sigma^\ell$, where L is the language recognized by $\mathcal{A}$. Moreover, we have a good understanding of the possible behaviour of this set. We say that a function f grows exponentially if $\exp(C_1 k) \leq f(k) \leq \exp(C_2 k)$ for some $C_1, C_2 \in \mathbb{R}_*^+$; in particular, the functions $k \mapsto \exp(Ck)k^d$ grow exponentially.

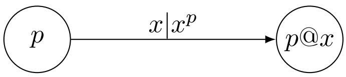
Proposition 1 ([27, Theorem 6]). *Let A be an $m \times m$ matrix with non-negative integer entries, and define the functions $f_{i,j}(k) = A^k_{i,j}$. Then for each pair (i,j), the function $f_{i,j}$ either grows exponentially or is a polynomial function of degree at most $m-1$.*

Since the transition matrix of an automaton $\mathcal{A}$ is non-negative, it admits a positive real eigenvalue of maximal absolute value, which is called its Perron eigenvalue and is written $\lambda(\mathcal{A})$. Therefore, assuming that $\mathcal{A}$ is coaccessible, we get the following.

Proposition 2 ([26, Theorem 1.2]). *Let $\mathcal{A}$ be a coaccessible **DFA** recognizing the language L. We have $h(L) = \log \lambda(\mathcal{A})$.*

2.1. *Mealy automata*

A *Mealy automaton* is a **DFA** over an alphabet of the form $\Sigma \times \Sigma$. If an edge's label is (x, y), one calls x the *input* and y the *output*, and denotes the transition by $p \xrightarrow{x|y} q$. Such a Mealy automaton is assumed to be complete and deterministic in its inputs: for every state p and every letter x, there exists exactly one transition from p with input letter x. We denote by x^p its corresponding output letter and by $p@x$ its target state:



A given Mealy automaton with stateset Q and alphabet Σ admits a Mealy *subautomaton* with stateset $Q' \subseteq Q$ and alphabet $\Sigma' \subseteq \Sigma$ provided that it satisfies

$$\forall\, q \in Q', \forall\, x \in \Sigma', q@x \in Q' \quad \text{and} \quad x^q \in \Sigma'.$$

In particular, a subautomaton with the same alphabet as the original automaton and whose stateset consists in a single state is called a *sink*.

A crucial point with Mealy automata is that states act on letters and letters on states. Such actions can be composed in the following way: for all $p \in Q$, $\mathbf{q} \in Q^*$, $x \in \Sigma$, $\mathbf{u} \in \Sigma^*$, we have

$$x^{\mathbf{q}p} = (x^{\mathbf{q}})^p \quad \text{and} \quad p@(\mathbf{u}x) = (p@\mathbf{u})@x.$$

We extend recursively the actions of states on letters and of letters on states (see just below left). Compositions can be more easily understood via an alternative representation by a *cross-diagram* [1] (below right).

For all $p \in Q$, $\mathbf{q} \in Q^*$, $x \in \Sigma$, $\mathbf{u} \in \Sigma^*$, we have:

$$(\mathbf{u}x)^{\mathbf{q}} = \mathbf{u}^{\mathbf{q}} x^{\mathbf{q}@\mathbf{u}}$$

and

$$(\mathbf{q}p)@\mathbf{u} = \mathbf{q}@\mathbf{u} \cdot p@\mathbf{u}^{\mathbf{q}}.$$

$$\begin{array}{ccccc} & \mathbf{q} & & p & \\ \mathbf{u} & \longrightarrow & \mathbf{u}^{\mathbf{q}} & \longrightarrow & \mathbf{u}^{\mathbf{q}p} \\ & \mathbf{q}@\mathbf{u} & & p@\mathbf{u}^{\mathbf{q}} & \\ x & \longrightarrow & x^{\mathbf{q}@\mathbf{u}} & & \\ & \mathbf{q}@\mathbf{u}x & & & \end{array}$$

The mappings defined above are length-preserving and prefix-preserving. Note that in particular the image of the empty word is itself.

From an algebraic point of view, the composition gives a semigroup structure to the set of transformations $\mathbf{u} \mapsto \mathbf{u}^{\mathbf{q}}$ for $\mathbf{q} \in Q^*$. This semigroup is called *the semigroup generated* by the Mealy automaton $\mathcal{M}$ and denoted by $\langle \mathcal{M} \rangle_+$. An *automaton semigroup* is a semigroup which can be generated by a Mealy automaton. Any element of such an automaton semigroup induces a so-called *finite-state* transformation.

Conversely, for any length- and prefix-preserving transformation t of Σ^* and any word $\mathbf{u} \in \Sigma^*$, we denote by $\mathbf{u}^t$ the image of $\mathbf{u}$ by t, and by $t@\mathbf{u}$ the unique transformation s of Σ^* satisfying $(\mathbf{uv})^t = \mathbf{u}^t\mathbf{v}^s$ for any $\mathbf{v} \in \Sigma^*$. Whenever $Q(t) = \{t@\mathbf{u} : \mathbf{u} \in \Sigma^*\}$ is finite, the transformation t is said to be *finite-state* and admits a unique (minimal) *associated Mealy automaton* $\mathcal{M}_t$ with stateset $Q(t)$.

We also use the following convenient notation to define a finite-state transformation t: for each state $s \in Q(t)$, we write an equation (traditionally called *wreath recursion* in the algebraic theory of automata) of the following form

$$s = (s@x_1, \ldots, s@x_{|\Sigma|})\sigma_s,$$

where $\sigma_s = [{x_1}^s, \ldots, {x_{|\Sigma|}}^s]$ denotes the transformation on Σ induced by s.

We consider the semigroup $\mathrm{FEnd}(\Sigma^*)$ of those finite-state transformations of Σ^*.

Example 3. The transformation $t_0 = (\mathbb{1}, t_0)[2,2]$ belongs to $\mathrm{FEnd}(\{1,2\}^*)$ with $Q(t_0) = \{\mathbb{1}, t_0\}$. See Examples 11 and 16 for further details about t_0.

Example 4. The transformation $p = (q,r)[1,1]$ with $q = (r, \mathbb{1})$ and $r = (r,r)[2,2]$ also belongs to $\mathrm{FEnd}(\{1,2\}^*)$ with $Q(p) = \{\mathbb{1}, p, q, r\}$. See Fig. 1 for $\mathcal{M}_p$.

3. An Activity-Based Hierarchy for FEnd(Σ^*)

In this section, we define a suitable notion of activity for finite-state endomorphisms, together with two norms, from which we build a new hierarchy. We will prove its strictness and its computability in Sec. 4.

Sidki defined in [27] the activity of a finite-state automorphism $t \in \mathrm{FAut}(\Sigma^*)$ as

$$\theta_t : n \mapsto \#\{\mathbf{u} \in \Sigma^n : \ t@\mathbf{u} \neq \mathbb{1}\}.$$

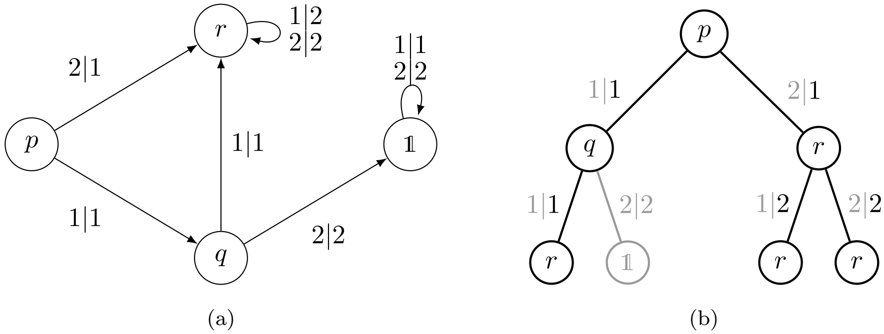


Fig. 1. (a) The Mealy automaton $\mathcal{M}_p$ for the transformation p from Example 4 satisfies $\alpha_p(0) = \alpha_p(1) = 1$ and $\alpha_p(2) = 2$. (b) The transformation p induces 3 nontrivial transformations on level 2: the leftmost one is associated with the output 11, the middle right one with 12 and the rightmost one with 12, hence nontrivial transformations can be reached by runs with only two different output words.

Using this notion of activity θ for transformations from $\mathrm{FEnd}(\Sigma^*) \smallsetminus \mathrm{FAut}(\Sigma^*)$ happens to inevitably lead to a stalemate: the associated classes of those transformations with fixed degree polynomial activity θ would not be closed under composition (see for instance Example 11 below).

For any element $t \in \mathrm{FEnd}(\Sigma^*)$, we define its *activity* (see Fig. 1) as

$$\alpha_t : n \mapsto \#\{\mathbf{v} \in \Sigma^n : \exists \mathbf{u} \in \Sigma^n, t@\mathbf{u} \neq \mathbb{1} \text{ and } \mathbf{u}^t = \mathbf{v}\}.$$

It is straightforward that our new notion of activity α coincides with Sidki's activity θ in the case of automorphisms.

This definition (still) requires an identity element, and would become trivial for any endomorphism t with $\mathbb{1} \notin Q(t)$. This possible issue will be addressed in Sec. 6, but we first describe this simpler case for legibility.

We also define two norms on $\mathrm{FEnd}(\Sigma^*)$. When α_t has polynomial growth, namely when the set $D = \{d : \lim_{n\to\infty} \frac{\alpha_t(n)}{n^d} = 0\}$ is nonempty, then we define $\|t\|_{\mathrm{p}} = \min D - 1$ (from Proposition 1, $\|t\|_{\mathrm{p}}$ is an integer greater than or equal to -1). Otherwise, the value of $\lim_{n\to\infty} \frac{\log \alpha_t(n)}{n}$ is denoted by $\|t\|_{\mathrm{e}}$.

We then define the following classes of finite-state transformations:

$$\mathrm{SPol}(d) = \{t \in \mathrm{FEnd}(\Sigma^*) : \|t\|_{\mathrm{p}} \leq d\} \quad \text{and}$$
$$\mathrm{SExp}(\lambda) = \{t \in \mathrm{FEnd}(\Sigma^*) : \|t\|_{\mathrm{e}} \leq \lambda\}.$$

Notice that, as a corollary of Proposition 1, the activity α either is ultimately equivalent to a polynomial function or grows exponentially, whence the classification proposed.

In addition, we call those elements belonging to $\mathrm{SPol}(0)$ *bounded* transformations, and those elements of $\mathrm{SPol}(-1)$ *finitary* transformations. Theses sets are respectively the ones where activity is bounded by a constant and ultimately 0.

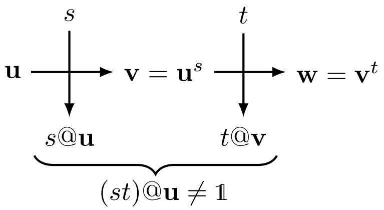
We shall see in Theorem 7 that they yield a strict and computable hierarchy for $\mathrm{FEnd}(\Sigma^*)$. The following basic lemma is crucial:

Lemma 5. *For each $n \geq 0$, the map $t \mapsto \alpha_t(n)$ is subadditive.*

Proof. Assume $s, t \in \mathrm{FEnd}(\Sigma^*)$.

For any $\mathbf{u} \in \Sigma^n$ with $(st)@\mathbf{u} \neq \mathbb{1}$, we have either $s@\mathbf{u} \neq \mathbb{1}$ or $t@\mathbf{u}^s \neq \mathbb{1}$ (or both).

We deduce $\alpha_{st}(n) \leq \alpha_s(n) + \alpha_t(n)$ for every $n \geq 0$.



□

We deduce that $\|.\|_{\mathrm{p}}$ and $\|.\|_{\mathrm{e}}$ are max-subadditive: $\|st\|_{\mathrm{p}} \leq \max\{\|s\|_{\mathrm{p}}, \|t\|_{\mathrm{p}}\}$ and $\|st\|_{\mathrm{e}} \leq \max\{\|s\|_{\mathrm{e}}, \|t\|_{\mathrm{e}}\}$ hold for any $s, t \in \mathrm{FEnd}(\Sigma^*)$. This proves the following.

Proposition 6. *Let Σ be an alphabet. For every integer $d \geq -1$,* $\mathrm{SPol}(d)$ *is a subsemigroup of* $\mathrm{FEnd}(\Sigma^*)$. *So is* $\mathrm{SExp}(\lambda)$ *for every $0 \leq \lambda \leq \log \#\Sigma$.*

As an easy corollary of Proposition 6, the subadditivity property allows us to compute the hierarchy class of the semigroup generated by any given Mealy automaton by considering only its generators.

Theorem 7. *Let Σ be an alphabet of size at least 2. The elements of the semigroup* $\mathrm{FEnd}(\Sigma^*)$ *can be graded according to the following hierarchy: for any integers d_1, d_2 with $-1 < d_1 < d_2$ and any reals λ_1, λ_2 with $0 < \lambda_1 < \lambda_2 < \log \#\Sigma$, we have:*

$$\begin{aligned} &\mathrm{SPol}(-1) \subsetneq \cdots \subsetneq \mathrm{SPol}(d_1) \subsetneq \cdots \subsetneq \mathrm{SPol}(d_2) \subsetneq \cdots \subsetneq \mathrm{SExp}(0) \\ &\subseteq \mathrm{SExp}(\lambda_1) \subseteq \cdots \subseteq \mathrm{SExp}(\lambda_2) \subseteq \cdots \subseteq \mathrm{SExp}(\log \#\Sigma). \end{aligned}$$

Moreover, if $\lambda_1 < \lambda_2$ are Perron eigenvalues of some non-negative integral matrices with 1-norm at most $\#\Sigma$, then we have $\varnothing \neq \mathrm{SExp}(\log \lambda_1) \subsetneq \mathrm{SExp}(\log \lambda_2)$.

The proof of the previous result is postponed to the end of Sec. 4 on page 1078. The class $\mathrm{SExp}(0)$ coincides with the infinite union $\bigcup_{d \geq -1} \mathrm{SPol}(d)$, whose corresponding automorphisms subclass is denoted by $\mathrm{Pol}(\infty)$ in [27].

4. Structural Characterization of the Activity Norm

From [27, Proposition 10], we know that the finite-state automorphisms which have polynomial activity are exactly those whose underlying automaton does not contain entangled — or mutually reachable — cycles (except on the trivial state). Moreover, the degree of the polynomial is given by the longest chain of cycles in the automaton. The first claim remains true for finite-state endomorphisms, but things are a bit more involved for the second one (see Example 11).

To any minimal Mealy automaton $\mathcal{M}$ with stateset Q and alphabet Σ, we associate its *output-pruned* automaton $\mathcal{M}^{\text{out}}$ defined as the **NFA** with stateset $Q\smallsetminus\{\mathbb{1}\}$ (all states being final) and alphabet Σ, and whose transitions are simply given, for $p, q \in Q\smallsetminus\{\mathbb{1}\}$, by

$$p \xrightarrow{y} q \in \mathcal{M}^{\text{out}} \quad \Leftrightarrow \quad p \xrightarrow{x|y} q \in \mathcal{M}.$$

According to the context, we shall identify a transformation $t \in \mathrm{FEnd}(\Sigma^*)$ with the state of $\mathcal{M}_t$, and with the corresponding state of $\mathcal{M}_t^{\text{out}}$.

Lemma 8. *The activity of a transformation $t \in \mathrm{FEnd}(\Sigma^*)$ is the number of paths starting from t in the (non-complete) deterministic automaton $\det(\mathcal{M}_t^{\text{out}})$ constructed via the Rabin–Scott construction.*

Proof. Let $t \in \mathrm{FEnd}(\Sigma^*)$ with $\mathcal{M}_t$ its associated Mealy automaton. Let us count the words $\mathbf{v} \in \Sigma^n$ for which there is a word $\mathbf{u} \in \Sigma^n$ with $t@\mathbf{u} \neq \mathbb{1}$ and $\mathbf{u}^t = \mathbf{v}$. For $n = 1$, $\alpha_t(1)$ is exactly the number of different outputs from the state t that do not lead to a trivial state of $\mathcal{M}_t$. Now for $\mathbf{v} \in \Sigma^n$, if $\mathcal{E}$ denotes the set of those states accessible from t by reading $\mathbf{v}$ (this corresponds to the Rabin–Scott powerset construction) in $\mathcal{M}_t^{\text{out}}$, the number of ways to extend $\mathbf{v}$ without getting into a trivial state in $\mathcal{M}_t$ corresponds to the number of outputs of the state $\mathcal{E}$ in $\det(\mathcal{M}_t^{\text{out}})$, whence the result. $\square$

Whether the activity of a given $t \in \mathrm{FEnd}(\Sigma^*)$ is polynomial or exponential can be decided by only looking at the cycle structure of $\mathcal{M}_t^{\text{out}}$. Any cycle considered throughout this paper is simple: no repetitions of vertices or edges are allowed. A *chain of cycles* is a sequence of cycles such that each cycle is reachable from its predecessor. Two cycles are *mutually reachable* if both are reachable from the other.

Proposition 9. *A transformation $t \in \mathrm{FEnd}(\Sigma^*)$ has exponential activity if and only if it can reach two mutually reachable cycles with distinct labels in $\mathcal{M}_t^{\text{out}}$.*

Proof. From [11] or from [27, Proposition 10], the statement holds for $\det(\mathcal{M}_t^{\text{out}})$. We shall prove that it still holds for $\mathcal{M}_t^{\text{out}}$, by showing that $\det(\mathcal{M}_t^{\text{out}})$ admits two mutually reachable cycles if and only if so does $\mathcal{M}_t^{\text{out}}$.

Assume $\det(\mathcal{M}_t^{\text{out}})$ has two mutually reachable cycles. Then there exist two loops $\mathbf{v} \neq \mathbf{w}$ (with $\mathbf{v}^k = \mathbf{w}^\ell \Rightarrow k = \ell = 0$) on some state $\mathcal{E}$ of $\det(\mathcal{M}_t^{\text{out}})$. Hence for some $p \in \mathcal{E}$, there exist k, ℓ such that, in $\mathcal{M}^{\text{out}}$, $\mathbf{v}^k$ and $\mathbf{w}^\ell$ are loops on p.

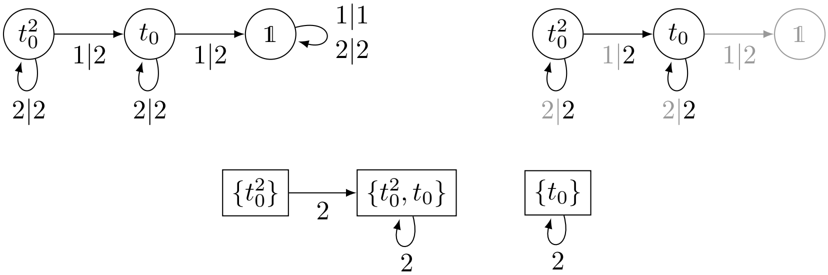
Let now $\mathbf{v} \neq \mathbf{w}$ be two loops on some state p of $\mathcal{M}^{\text{out}}$. Then, for any word $\mathbf{m} \in \{\mathbf{v}, \mathbf{w}\}^*$, there is a path starting from $\{p\}$ to some state $\mathcal{E}_{\mathbf{m}} \ni p$ of $\det(\mathcal{M}_t^{\text{out}})$. Since the stateset of $\det(\mathcal{M}_t^{\text{out}})$ is finite, each path labeled by $\mathbf{m}^*$ loops in $\det(\mathcal{M}_t^{\text{out}})$, and since there is an infinity of such words $\mathbf{m} \in \{\mathbf{v}, \mathbf{w}\}^*$ (even choosing primitive words), theses loops cannot be all pairwise disjoints. $\square$

Since a state which can reach d cycles in the determinized output-pruned automaton has polynomial activity of degree $d-1$, the subadditivity of the activity — see Lemma 5 — together with Proposition 9 give the following.

Corollary 10. *Let $\mathcal{M}$ be a Mealy automaton. The transformations of $\langle \mathcal{M} \rangle_+$ are all of polynomial activity if and only if there are no mutually reachable cycles in the automaton* $\det(\mathcal{M}^{\text{out}})$*. Moreover the degree of the* (*polynomial*) *activity corresponds to the longest chain of cycles in* $\det(\mathcal{M}^{\text{out}})$ *minus 1.*

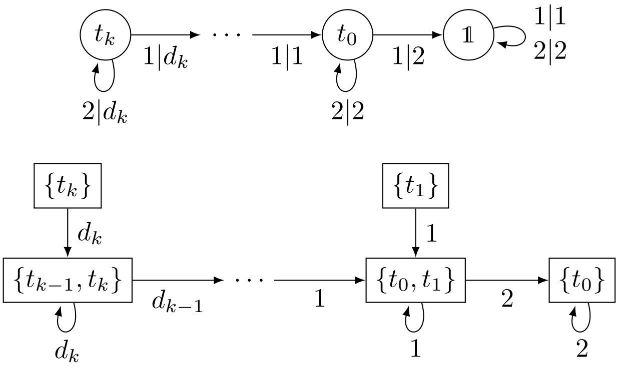
We deduce the decidability of the membership problem for $\mathrm{SPol}(d)$.

Example 11. Consider the transformation $t_0 = (\mathbb{1}, t_0)[2,2]$ from Example *3,* its square t_0^2, and the associated automata $\mathcal{M}_{t_0^2}$ (left), the output-pruned automaton $\mathcal{M}_{t_0^2}^{\text{out}}$ (right, in black), and the determinized output-pruned automaton $\det(\mathcal{M}_{t_0^2}^{\text{out}})$ (below):



Note that, before determinization, two disjoint cycles are accessible from the state t_0^2. In the determinized version, $\{t_0\}$ and $\{t_0^2\}$ both access to only one cycle, and we conclude $\{t_0, t_0^2\} \subset \mathrm{SPol}(0)$. By Proposition 6, we actually knew the full inclusion $\langle t_0 \rangle_+ \subset \mathrm{SPol}(0)$.

By defining further $t_k = (t_{k-1}, t_k)[d_k, d_k] \in \mathrm{FEnd}(\{1,2\}^*)$ with $d_k = \frac{3+(-1)^k}{2}$ for $k > 0$, we obtain a family satisfying $t_k \in \mathrm{SPol}(k) \setminus \mathrm{SPol}(k-1)$ for $k > 0$, that witnesses the strictness of the polynomial part of the hierarchy from Theorem 7.



Using Proposition 2, we obtain the following explicit formula for the norm $\|\cdot\|_{\mathrm{e}}$.

Proposition 12. *Let t be a transformation of* $\mathrm{FEnd}(\Sigma^*)$*. The norm $\|t\|_{\mathrm{e}}$ is the logarithm of the Perron eigenvalue of the transition matrix of* $\det(\mathcal{M}_t^{\text{out}})$:

$$\|t\|_{\mathrm{e}} = \log \lambda(\det(\mathcal{M}_t^{\text{out}})).$$

Proof. By Lemma 8, the activity of t counts the number of paths in $\det(\mathcal{M}_t^{\text{out}})$. Since all its states are final by definition, this automaton is coaccessible and the cardinality of the recognized language when putting $\{t\}$ as the initial state is exactly the activity of t. Therefore by Proposition 2, we have

$$\|t\|_{\mathrm{e}} = \lim_{\ell\to\infty} \frac{\log \alpha_t(\ell)}{\ell} = \lim_{\ell\to\infty} \frac{1}{\ell} \log \sum_{t'=1}^{n} (A^{\ell})_{t,t'} = h(L) = \log \lambda(\det(\mathcal{M}_t^{\text{out}})),$$

where $A = (A_{i,j})_{i,j}$ is the adjacency matrix of $\det(\mathcal{M}_t^{\text{out}})$. □

We are now ready to give the proof for the main result from Sec. 3.

Proof of Theorem 7. The strictness for the polynomial part is obtained from Example 11. Now, as the norm $\|.\|_{\mathrm{e}}$ is the logarithm of the maximal eigenvalue of a matrix with integer coefficients, the classes $\mathrm{SExp}(\lambda)$ increase only when e^{λ} is an algebraic integer that is the largest zero of its minimal polynomial, i.e. a root of a Perron number. Now consider an algebraic number e^{λ}. Using [17, Theorem 1], there exists a matrix of $\mathbb{N}^{n\times n}$ which admits e^{λ} as Perron eigenvalue. Let d be its 1-norm (i.e. its maximum column-sum), hence there exists a classical n-state automaton $\mathcal{A}_{\lambda}$ over some alphabet Σ of size d whose adjacency matrix has Perron eigenvalue e^{λ}. We use this automaton as $\det(\mathcal{M}_{\lambda}^{\text{out}})$, and then build $\mathcal{M}_{\lambda}$ from it in the following way:

- if the automaton $\mathcal{A}_{\lambda}$ is not complete, then we add a sink state and direct all missing arrows to it (this does not change the alphabet);
- for every state p but the sink, we choose a nontrivial permutation σ_p and put $\sigma_p(i)$ as input for the arrow starting from p whose output is i (i.e. $p \xrightarrow{\sigma_p(i)|i} p'$), hence p induces a nontrivial transformation;
- if we added a sink e, then we label its transitions so that it induces the identity (i.e. $e \xrightarrow{i|i} e$).

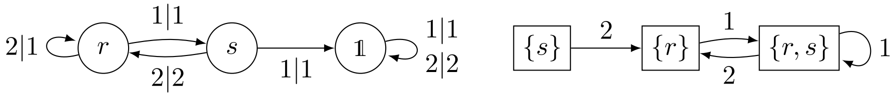
From this construction, we get an invertible Mealy automata $\mathcal{M}_{\lambda}$ satisfying $\det(\mathcal{M}_{\lambda}^{\text{out}}) = \mathcal{M}_{\lambda}^{\text{out}}$, hence the announced result.

Furthermore, each of these numbers is the norm of some finite-state endomorphism (actually, automorphism), by using [17, Theorem 3] and the fact that, for any matrix, we can associate a complete (invertible) automaton whose activity is given by this matrix by adding a sink state corresponding to the identity and completing the automaton with edges to this sink. It is also known that Perron numbers are dense in $[1,\infty)$, which gives us the strictness for the exponential part: $\lambda_1 < \lambda_2$ implies $\mathrm{SExp}(\lambda_1) \subsetneq \mathrm{SExp}(\lambda_2)$.

Finally, the growth rate can be computed with any precision $0 < \delta < 1$ in time $\Theta(-\log(\delta)\cdot n)$, where n is the number of states of the determinized automaton [25]. □

Notice moreover that, using [17, Theorem 3], the actual exponential activity of an endomorphism has to be the logarithm of some Perron number.

Example 13. Consider the transformations $r = (s, r)[1, 1]$ and $s = (\mathbb{1}, r)$ with common associated automata $\mathcal{M}$ (on the left) and $\det(\mathcal{M}^{\text{out}})$ (on the right):



The adjacency matrix of $\det(\mathcal{M}^{\text{out}})$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. We find that $\alpha_r(n) = \alpha_s(n+1)$ corresponds to the n-th Fibonacci number. We deduce $\|r\|_{\text{e}} = \|s\|_{\text{e}} = \log\varphi$ where φ is the golden ratio, hence $r, s \in \text{SExp}(\log\varphi)$.

5. The Orbit Signalizer Graph and the Order Problem

This section is devoted to the Order Problem: can one decide whether a given element generates a finite semigroup? The latter is known to be undecidable for general automaton semigroups [12] (and groups [5, 13]) and decidable for Pol(0) [8]. We give a general construction that associates a graph to any transformation in $\text{FEnd}(\Sigma^*)$, and show that, if finite, this graph allows us to compute the index and period of the transformation. We show that this graph is finite for elements from SPol(0), and solve the Order Problem in this manner.

Let Σ be an alphabet. We define the *orbit signalizer graph* Φ for $\text{FEnd}(\Sigma^*)$ as the following (infinite) graph. The vertices are the pairs of elements in $\text{FEnd}(\Sigma^*)$. For each letter $x \in \Sigma$, there is an arrow from the source (s, t) with label $(x : m, \ell)$ where m and ℓ are the minimal integers (with $\ell > 0$) satisfying

$$x^{st^{m+\ell}} = x^{st^m},$$

and with target $(r@x, t^\ell@x^r)$ for $r = st^m$. The parameters m and ℓ correspond respectively to the *index* and to the *period* of the orbit of x under the action of st^ω, see Fig. 2.

In what follows, the intuition is roughly to generalize Fig. 2, by considering a path π instead of the letter x: such a construction leads also to a so-called tadpole

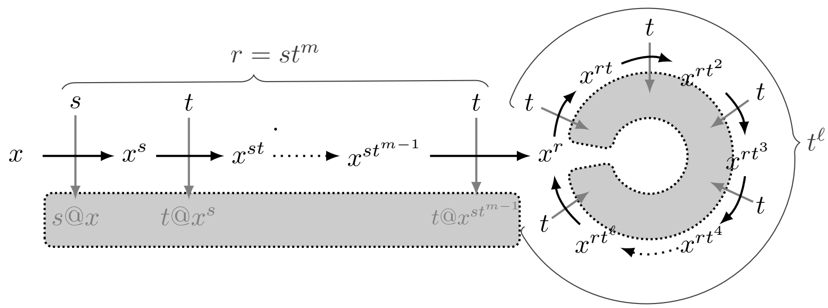


Fig. 2. The cross-diagram associated with the orbit of some letter $x \in \Sigma$ under the action of st^ω. The index m and period ℓ will complete the label of the x-arrow away from the vertex (s, t) in the graph Φ. Each of the two gray zones indicates an entry of the corresponding target vertex $(r@x, t^\ell@x^r)$ with $r = st^m$.

graph, whose path-part has length i (actually, we will need two approximations of this length, namely i^- and i^+), and whose cycle-part has length p. The main challenge here is to be able to keep the construction finite, when possible.

The *inf-index-cost*, *sup-index-cost*, and the *period-cost* of a given walk π on Φ

$$\pi : (s,t) \xrightarrow{x_1 : m_1, \ell_1} \cdots \xrightarrow{x_{|\pi|} : m_{|\pi|}, \ell_{|\pi|}} (s', t')$$

are respectively defined ($\delta_{m_k,0}$ being the Kronecker symbol) by:

$$i^-(\pi) = \sum_{1 \le k \le |\pi|} \left((1 - \delta_{m_k,0}) \left((m_k - 1) \left(\prod_{1 \le j < k} \ell_j \right) + 1 \right) \right),$$

$$i^+(\pi) = \sum_{1 \le k \le |\pi|} \left(\prod_{1 \le j < k} \ell_j \right) m_k, \quad \text{and} \quad p(\pi) = \prod_{1 \le i \le |\pi|} \ell_i.$$

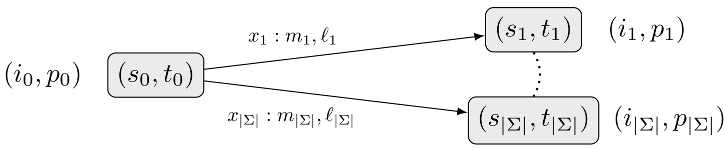
The basic intuition for these costs is that, according to Fig. 2 to words (see Fig. 4), the period after reading a new letter is the former period multiplied by the size of the cycle; and the index is roughly the previous index plus the previous period times the size of the handle, but this approximation is too coarse since the actual looping might happen within a period, so we have to introduce i^-.

For any $t \in \mathrm{FEnd}(\Sigma^*)$, we define the *orbit signalizer graph* $\Phi(t)$ as the subgraph of Φ accessible from the source vertex $(\mathbb{1}, t)$. The *inf-index-cost*, *sup-index-cost*, and the *period-cost* of $t \in \mathrm{FEnd}(\Sigma^*)$ are then respectively defined by

$$i_t^- = \sup_{\pi \text{ on } \Phi(t)} i^-(\pi), \quad i_t^+ = \sup_{\pi \text{ on } \Phi(t)} i^+(\pi), \quad \text{and} \quad p_t = \operatorname*{lcm}_{\pi \text{ on } \Phi(t)} p(\pi).$$

Proposition 14. *The semigroup generated by an element* $t \in \mathrm{FEnd}(\Sigma^*)$ *is finite if and only if its index-costs* $i_t^\pm$ *and its period-cost* p_t *are finite. In that case, we have* $\langle\, t \,\rangle_+ = \langle\, t : t^{i_t} = t^{i_t + p_t} \,\rangle_+$ *for some index* i_t *with* $i_t^- \le i_t \le i_t^+$.

Proof. Let $\Sigma = \{x_1, \ldots, x_{|\Sigma|}\}$. Let (s_0, t_0) be a vertex in Φ and (s_k, t_k) its successor vertex with arrow $x_k : m_k, \ell_k$ for $1 \le k \le |\Sigma|$.



For $0 \le k \le |\Sigma|$, let $(i_k, p_k) \in \{\omega, 0, 1, 2, \ldots\} \times \{\omega, 1, 2, 3 \ldots\}$ denote the possible minimal pair of ordinals (with $p_k > 0$) satisfying

$$s_k t_k^{i_k} = s_k t_k^{i_k + p_k}.$$

Whenever there is at least one successor with $(i_k, p_k) = (\omega, \omega)$, (s_0, t_0) satisfies also $(i_0, p_0) = (\omega, \omega)$, and so does any of its predecessors. Otherwise, we claim

$$\max_{1 \le k \le |\Sigma|} \big(m_k + \max(0, \ell_k (i_k - 1) + 1) \big) \le i_0 \le \max_{1 \le k \le |\Sigma|} (m_k + \ell_k i_k)$$

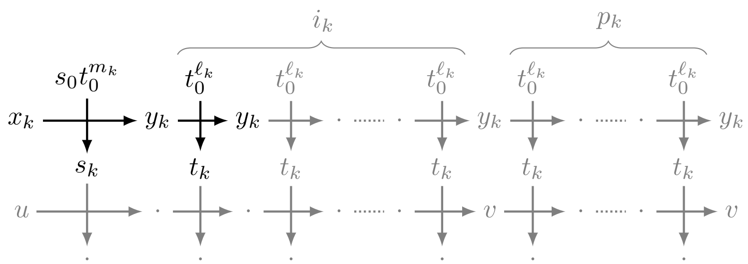
and

$$p_0 = \operatorname*{lcm}_{1 \le k \le |\Sigma|} \ell_k p_k.$$

Indeed, for $1 \le k \le |\Sigma|$ and for any $u \in \Sigma^*$, we have

$$y_k v = (x_k u)^{s_0 t_0^{m_k + \ell_k i_k}} = (y_k v)^{t_0^{\ell_k p_k}}$$

with $y_k = {x_k}^{s_0 t_0^{m_k}}$ and $v = u^{s_k t_k^{i_k}}$, as illustrated by the cross-diagram:



We conclude using an induction on the length of the paths. ☐

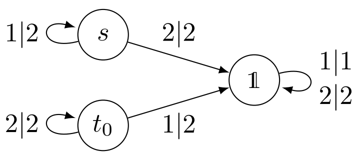
In the previous proposition, the quantities i^- and i^+ are distinguished in order to provide explicit bounds on the size of the index. However both are simultaneously either finite or infinite. Moreover we can easily have a dichotomy:

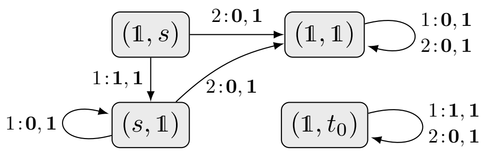
Theorem 15. *The* ORDER PROBLEM *is decidable for any element* $t \in \mathrm{FEnd}(\Sigma^*)$ *with a finite orbit signalizer graph* $\Phi(t)$*.*

Proof. Since $\Phi(t)$ is a graph with outdegree $\#\Sigma > 0$ by construction, its finiteness implies the existence of cycles. Consider the simple cycles (there is only a finite number of these). One can compute the index-costs $i^-(\kappa)$ and $i^+(\kappa)$ and the period-cost $p(\kappa)$ of each such cycle κ. Whenever $i^-(\kappa) > 0$ or $p(\kappa) > 1$ for some cycle κ, then t has infinite order, and finite order otherwise. ☐

In fact, we stress that this decision process can be easily performed on the orbit signalizer graph directly: if $m > 0$ or $\ell > 1$ appears on some cycle then the order is infinite.

Example 16. The transformations $s = (s, \mathbb{1})[2,2]$ and $t_0 = (\mathbb{1}, t_0)[2,2]$ (on the left) admit respective graphs $\Phi(s)$ and $\Phi(t_0)$ (on the right):





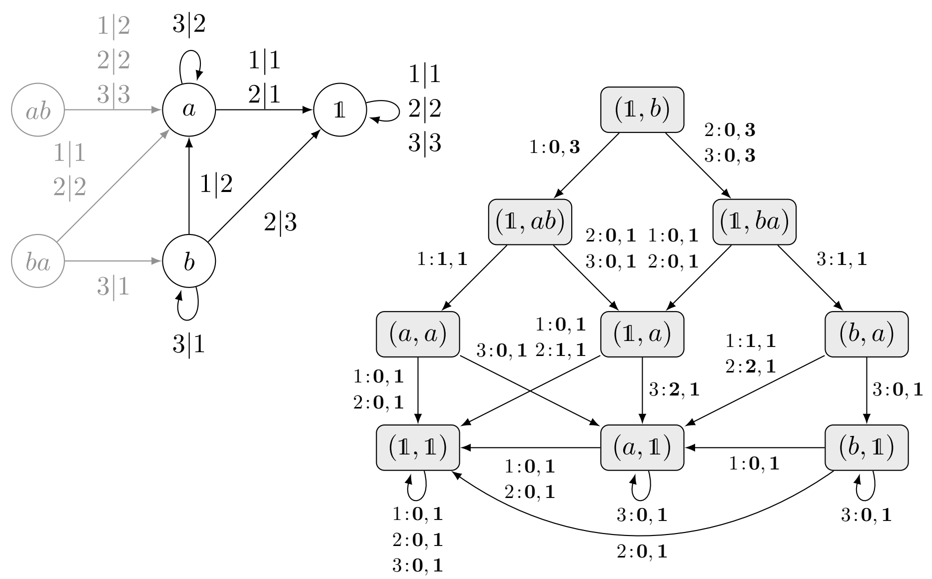


Fig. 3. The Mealy automaton $\mathcal{M}_b$ and the graph $\Phi(b)$ from Example 17.

According to Proposition 14, they generate the finite monoid $\langle s : s^2 = s\rangle_+$ and the free monoid $\langle t_0 : \ \rangle_+$.

Example 17. The transformation $b = (a, \mathbb{1}, b)[2, 3, 1]$ from $\mathrm{SPol}(1) \smallsetminus \mathrm{SPol}(0)$ with $a = (\mathbb{1}, \mathbb{1}, a)[1, 1, 2]$ admits the finite graph $\Phi(b)$ displayed in Figure 3, in which we can read that both ab and ba have period 1, and that b has thus period $p_b = 3$. According to Proposition 14 again, the index of b satisfies $7 \le i_b \le 9$, and can be explicitly computed as $i_b = 8$.

Example 18. To compute the size of the orbit of 232 under the action of b, we first make b act on 2. This leads us to the state $(\mathbb{1}, ba)$ with coefficients $m_1 = 0$ and $\ell_1 = 3$. Then applying ba to 3 leads to (b, a) with $m_2 = 1$ and $\ell_2 = 1$. Finally, the orbit of a on 2^b gives us $m_3 = 2$ and $\ell_3 = 1$. We find $p = 3 \times 1 \times 1 = 3$,

$$
\begin{aligned}
i_3^- &= 0, & i_3^+ &= 0,\\
i_2^- &= m_3 + \max(\ell_3(i_3^- - 1) + 1, 0) & i_2^+ &= m_3 + \ell_3 i_3^+\\
&= 2 + \max(1(0-1) + 1, 0) = 2, & &= 2 + 1 \times 0 = 2,\\
i_1^- &= m_2 + \max(\ell_2(i_2^- - 1) + 1, 0) & i_1^+ &= m_2 + \ell_2 i_2^+\\
&= 1 + \max(1(2-1) + 1, 0) = 3, & &= 1 + 1 \times 2 = 3,\\
i_0^- &= m_1 + \max(\ell_1(i_1^- - 1) + 1, 0) & i_0^+ &= m_1 + \ell_1 i_1^+\\
&= 0 + \max(3(3-1) + 1, 0) = 7, & &= 0 + 3 \times 3 = 9.
\end{aligned}
$$

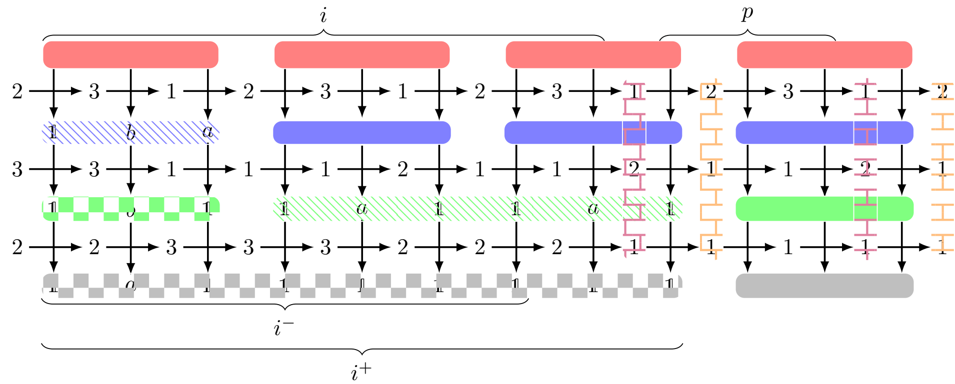


Fig. 4. Detail of a computation of the orbit of the word 232 under the action of b from Examples 17 and 18. General coefficients are found in Figure 3. This highlights the importance of i^-.

Hence we get the bounding $7 \le i \le 9$. In fact, $i = 8$ as shown in Fig. 4: considering blocks leads to approximation of the actual value.

Proposition 19. *Every bounded finite-state transformation* $t \in \mathrm{SPol}(0)$ *admits a finite orbit signalizer graph* $\Phi(t)$.

Proof. The activity α_t of $t \in \mathrm{SPol}(0)$ is uniformly bounded by some constant C:

$$\#\{\mathbf{v} \in \Sigma^n : \exists \mathbf{u} \in \Sigma^n,\ t@\mathbf{u} \neq \mathbb{1} \text{ and } \mathbf{u}^t = \mathbf{v}\} \le C \quad \text{for } n \ge 0.$$

Now the vertices of the graph $\Phi(t)$ are built as follows: with each word $\mathbf{u} \in \Sigma^*$ we associate a pair $(r(\mathbf{u}), s(\mathbf{u}))$ of words and four integers $m(\mathbf{u}), \ell(\mathbf{u}), m^*(\mathbf{u}), \ell^*(\mathbf{u})$ with $m^*(\varepsilon) = 0, \ell^*(\varepsilon) = 1$ that simultaneously and inductively defined as follows. For $\mathbf{u} = \mathbf{v}x \in \Sigma^+$ with $x \in \Sigma$, let:

$m(\mathbf{u}) + \ell(\mathbf{u})$ be the largest integer such that the images $\mathbf{u}^{t^{m^*(\mathbf{v})+\ell^*(\mathbf{v})i}}$ for $i \in \{0, \dots, m(\mathbf{u})+\ell(\mathbf{u})-1\}$ are pairwise distinct,

$m(\mathbf{u})$ minimal such that $\mathbf{u}^{t^{m^*(\mathbf{u})}} = \mathbf{u}^{t^{m^*(\mathbf{u})+\ell^*(\mathbf{u})}}$ holds.

$$\begin{aligned} m^*(\mathbf{u}) &= m^*(\mathbf{v}) + \ell^*(\mathbf{v})m(\mathbf{u}),\\ \ell^*(\mathbf{u}) &= \ell^*(\mathbf{v})\ell(\mathbf{u}),\\ r(\mathbf{u}) &= t^{m^*(\mathbf{u})}@\mathbf{u},\\ s(\mathbf{u}) &= t^{\ell^*(\mathbf{u})}@\mathbf{u}^{t^{m^*(\mathbf{u})}}. \end{aligned}$$

Note in particular $r(\mathbf{u})s(\mathbf{u}) = t^{m^*(\mathbf{u})+\ell^*(\mathbf{u})}@\mathbf{u}$. We will prove that the graph $\Phi(t)$ is finite by showing that r and s may take only finitely many values, namely are some products of at most C elements of $\mathcal{M}_t$. We shall prove, by induction on the length of $\mathbf{u}$, that

(1) the $\mathbf{u}^{t^i}$ for $i \in \{0, \dots, m^*(\mathbf{u})\}$ are pairwise distinct;

(2) the $\left(\mathbf{u}^{t^{m^*(\mathbf{u})}}\right)^{t^i} = \mathbf{u}^{t^{m^*(\mathbf{u})+i}}$ for $i \in \{0, \dots, \ell^*(\mathbf{u}) - 1\}$ are pairwise distinct;
(3) the distance $|i-j|$ between indices giving the same image $\mathbf{u}^{t^i} = \mathbf{u}^{t^j}$ is a multiple of $\ell^*(\mathbf{u})$.

Note however that the $\mathbf{u}^{t^i}$ for $i \in \{0, \dots, m^*(\mathbf{u}) + \ell^*(\mathbf{u}) - 1\}$ might not be pairwise distinct (see for instance Fig. 4).

For the first claim, assume $\mathbf{u}^{t^i} = \mathbf{u}^{t^j}$ for some $i \neq j$ both belonging to $\{0, \dots, m^*(\mathbf{u})\}$. By the induction hypothesis we cannot have i and j both in $\{0, \dots, m^*(\mathbf{v})\}$, and the distance $|i - j|$ has to be a multiple of $\ell^*(\mathbf{v})$. For $i < m^*(\mathbf{v}) \leq j$, we have $i = m^*(\mathbf{v}) - \delta$ and $j = m^*(\mathbf{v}) + \alpha\ell^*(\mathbf{v}) - \delta$, so $\mathbf{u}^{t^{m^*(\mathbf{v})}} = \mathbf{u}^{t^{m^*(\mathbf{v})+\alpha\ell^*(\mathbf{v})}}$ holds, which contradicts the definition of $m(\mathbf{u})$. For $m^*(\mathbf{v}) < i, j \leq m^*(\mathbf{u}) = m^*(\mathbf{v}) + m(\mathbf{u})\ell^*(\mathbf{v})$, we have $i = m^*(\mathbf{v}) + \alpha\ell^*(\mathbf{v}) - \delta$ and $j = m^*(\mathbf{v}) + \beta\ell^*(\mathbf{v}) - \delta$, so $\mathbf{u}^{t^{m^*(\mathbf{v})+\alpha\ell^*(\mathbf{v})}} = \mathbf{u}^{t^{m^*(\mathbf{v})+\beta\ell^*(\mathbf{v})}}$ holds, which contradicts the definition of $m(\mathbf{u})$.

For the second claim, assume $\mathbf{u}^{t^{m^*(\mathbf{u})+i}} = \mathbf{u}^{t^{m^*(\mathbf{u})+j}}$ for some $i \neq j$ both belonging to $\{0, \dots, \ell^*(\mathbf{u}) - 1\}$. By the induction hypothesis, the distance $|i - j|$ is a multiple of $\ell^*(\mathbf{v})$, so we have $i = \alpha\ell^*(\mathbf{v}) - \delta$ and $j = \beta\ell^*(\mathbf{v}) - \delta$. Therefore, $\mathbf{u}^{t^{m^*(\mathbf{v})+\alpha\ell^*(\mathbf{v})}} = \mathbf{u}^{t^{m^*(\mathbf{v})+\beta\ell^*(\mathbf{v})}}$ holds, which contradicts the definition of $\ell(\mathbf{u})$.

For the last claim, if $\mathbf{u}^{t^i} = \mathbf{u}^{t^j}$ holds, then $|i - j|$ is a multiple of $\ell^*(\mathbf{u})$ by the definition of $m(\mathbf{u})$ and $\ell(\mathbf{u})$.

Now since $r(\mathbf{u}) = t^{m^*(\mathbf{u})}@\mathbf{u} = (t@\mathbf{u})(t@\mathbf{u}^t)\cdots$ is a product of sections of t at distinct words, it is a product of at most C nontrivial states of $\mathcal{M}_t$ (which has finitely many states), and thus belongs to a finite set $\{\mathbf{q} \in Q^{\leq C}\}$. As $s(\mathbf{u})$ is also a product of sections of t at distinct words, it also belongs to a finite set, hence, the vertex set of the orbit signalizer graph $\Phi(t)$ of a bounded finite-state transformation t is finite. Its edge set is also finite since the labels belong to $\{0, \dots, \#\Sigma - 1\} \times \{1, \dots, \#\Sigma\}$. $\square$

Corollary 20. *The* Order Problem *is decidable for* SPol(0).

6. Beyond Monoids

In this section, we define an extension of the notion of activity which allows to prove that several new (semi)groups have a decidable Order Problem. The previous definition we gave for activity required an identity element which might not exist in a general semigroup. To address this problem, we increase the possible set of arrival states in the definition. In the former definition, the requirement that the sink has to be the identity element is very restrictive, and it can be relaxed to going into a subautomaton that generates a finite (semi)group. Besides particular examples, there are two distinguished classes of Mealy automata that are known to generate finite (semi)groups: $\mathfrak{md}$-trivial Mealy automata [1, 19] and Mealy automata without cycles with exit [2, 23]. We choose to focus on the latter. Let us mention that this structural class has the special feature that the finiteness is independent from the

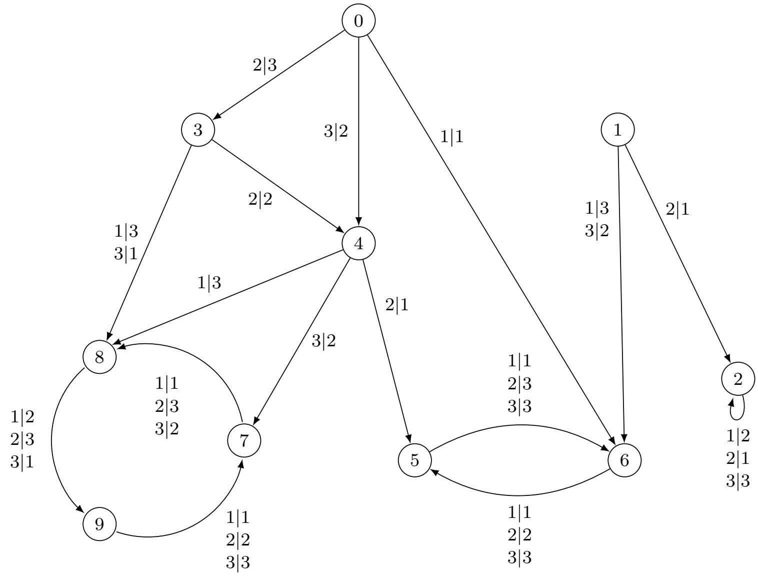


Fig. 5. A *nocywex* Mealy automaton, that is, without cycles with exit: it typically belongs to EPol(-1) by definition and generates a finite semigroup by Theorem 21.

choice of output, and that, for every automaton not in this class, one can modify outputs such that the generated semigroup becomes infinite, see [16]. Throughout the section, we shall briefly discuss how natural or reliable this choice seems.

A Mealy automaton with stateset Q and alphabet Σ is said to be *without cycles with exit—nocywex*for short—if, whenever $q \in Q$ belongs to a cycle, all transitions from q lead to the same state q', that is,

$$\mathcal{M} \text{ is } \textit{nocywex} \;\Leftrightarrow\; \forall\, q \in Q, \exists \mathbf{u} \in \Sigma^+, q@\mathbf{u} = q \Rightarrow \#\{q@x, x \in \Sigma\} = 1.$$

Informally, a *nocywex* automaton consists in a directed acyclic graph whose leaves are attached to cycles; an example is depicted in Fig. 5.

Theorem 21 ([2, 23]). *The semigroup generated by a nocywex Mealy automaton is finite.*

We define the *extended activity* of a transformation $t \in \mathrm{FEnd}(\Sigma^*)$ as

$$\varepsilon_t : n \longmapsto \#\{\mathbf{v} \in \Sigma^n : \exists \mathbf{u} \in \Sigma^n,\; t@\mathbf{u} \notin \langle \mathcal{N} \rangle_+ \text{ and } \mathbf{u}^t = \mathbf{v}\},$$

where $\mathcal{N}$ is the maximal *nocywex*subautomaton of $\mathcal{M}_t$ with stateset $N \subseteq Q(t)$ and same alphabet Σ. We can algorithmically recognize the maximal *nocywex*part, and compute the extended activity ε by pruning the whole *nocywex*part instead of pruning the single trivial sinks as in the computation of the activity α. Hence, we obtain the *output $\mathcal{N}$-pruned automaton* ${\mathcal{M}_t}^{\mathrm{out}_{\mathcal{N}}}$.

The extended activity function ε shares several properties with the original activity α. For instance, the proof of Lemma 5 readily adapts to the extended case to show that activity function ε is subadditive. This makes extended activity a *seminorm* on $\mathrm{FEnd}(\Sigma^*)$.

We define the classes EPol with the exact same definition as SPol by replacing the activity α with the extended activity ε. Notice that the class of extended finitary automata coincides with the class *nocywex*, so all elements from $\mathrm{EPol}(-1)$ generate finite semigroups. Recall that in contrast, the Mealy automaton of any element from $\mathrm{SPol}(-1) \subsetneq \mathrm{EPol}(-1)$ consists in a directed acyclic graph whose leaves are trivial sinks. We stress that the choice of *nocywex* as $\mathrm{EPol}(-1)$ is nonetheless arbitrary and that any other class which generates only finite semigroups would do.

Our main result about the class of bounded Mealy automata remains true under this extension.

Proposition 22. *The* Order Problem *is decidable for* $\mathrm{EPol}(0)$.

Proof. We are going to prove that the orbit signalizer graph $\Phi(t)$ of any transformation $t \in \mathrm{EPol}(0)$ is finite, and the result will follow by Theorem 15. The vertices of the graph $\Phi(t)$ are those pairs $(r(\mathbf{u}), s(\mathbf{u}))$ with $r(\mathbf{u}) = t^{m^*(\mathbf{u})}@\mathbf{u}$ and $s(\mathbf{u}) = t^{\ell^*(\mathbf{u})}@\mathbf{u}^{t^{m^*(\mathbf{u})}}$ where $\ell^*(\mathbf{u})$ and $m^*(\mathbf{u})$ are defined as in Proposition 19. In particular the images $\mathbf{u}^{t^i}$ for $i \in \{0, \ldots, m^*(\mathbf{u})\}$ are pairwise distinct, and so are the $\left(\mathbf{u}^{t^{m^*(\mathbf{u})}}\right)^{t^i} = \mathbf{u}^{t^{m^*(\mathbf{u})+i}}$ for $i \in \{0, \ldots, \ell^*(\mathbf{u}) - 1\}$.

We can write

$$t^{m^*(\mathbf{u})}@\mathbf{u} = t@\mathbf{u} \cdot t@\mathbf{u}^t \cdots t@\mathbf{u}^{t^{m^*(\mathbf{u})-1}}.$$

By very definition of ε_t, the number C_ℓ of factors not in $\langle \mathcal{N} \rangle_+$ is bounded by a constant C. By denoting g_i with $1 \le i \le C_\ell \le C$ the factors not in $\langle \mathcal{N} \rangle_+$ and $f_{i,j}$ the others, we obtain:

$$t^{m^*(\mathbf{u})}@\mathbf{u} = \prod_{j=0}^{j_0} f_{0,j} \cdot g_1 \cdot \prod_{j=0}^{j_1} f_{1,j} \cdot g_2 \cdot \ \cdots \ \cdot g_{C_\ell - 1} \cdot \prod_{j=0}^{j_{C_\ell - 1}} f_{C_\ell - 1, j} \cdot g_{C_\ell} \cdot \prod_{j=0}^{j_{C_\ell}} f_{C_\ell, j}.$$

Each product $f_{i,0} \ldots f_{i,j_i}$ can be expressed as an element f'_i of $\langle \mathcal{N} \rangle_+$, and we find:

$$t^{m^*(\mathbf{u})}@\mathbf{u} = f'_0 \cdot g_1 \cdot f'_1 \cdot g_2 \cdot \ \cdots \ \cdot g_{C_\ell - 1} \cdot f'_{C_\ell - 1} \cdot g_{C_\ell} \cdot f'_{C_\ell}.$$

Now as $\langle \mathcal{N} \rangle_+$ and $\{g : \exists \mathbf{u} \in \Sigma^*, \ g = t@\mathbf{u}\}$ are finite, so is the set

$$\{f'_0 g_1 f'_1 g_2 \cdots f'_{C_\ell - 1} g_{C_\ell} f'_{C_\ell} \mid C_\ell \le C, \ f'_i \in \langle \mathcal{N} \rangle_+, \ g_i \in \{g \mid \ \exists \mathbf{u} \in \Sigma^*, \ g = t@\mathbf{u}\}\},$$

so is the set of possible first coordinate of the orbit signalizer graph $\Phi(t)$. We apply the same reasoning for the second coordinate, considering $t^{\ell^*(\mathbf{u})}@\mathbf{u}^{t^{m^*(\mathbf{u})}}$ and we finally obtain the finiteness of the orbit signalizer graph $\Phi(t)$. □

Example 23. The family from Example 11 still satisfies $t_k \in \mathrm{EPol}(k) \smallsetminus \mathrm{EPol}(k-1)$ for $k > 0$, that witnesses the strictness of the polynomial part of this alternative hierarchy (see Theorem 7):

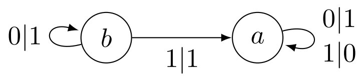
$$\mathrm{EPol}(-1) \subsetneq \cdots \subsetneq \mathrm{EPol}(d_1) \subsetneq \cdots \subsetneq \mathrm{EPol}(d_2) \subsetneq \cdots \subsetneq \mathrm{EExp}(0)$$

holds for any integers d_1, d_2 with $-1 < d_1 < d_2$.

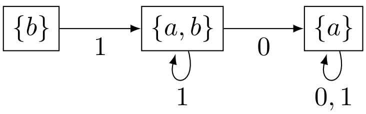
Recall that the original purpose was to handle the cases without an identity element, that is, these Mealy automata without any trivial sink. The interest of this extension turns out to be double: first it allows indeed to show that some more examples have decidable Order Problem, see Example 24, but it also makes this alternative hierarchy somehow more robust, see Example 25.

Example 24. The Mealy automaton $\mathcal{I}_2$ is known from [6] as the very smallest Mealy automaton of intermediate growth.

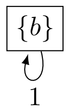
$\mathcal{I}_2$:



$\det(\mathcal{I}_2{}^{\mathrm{out}})$:

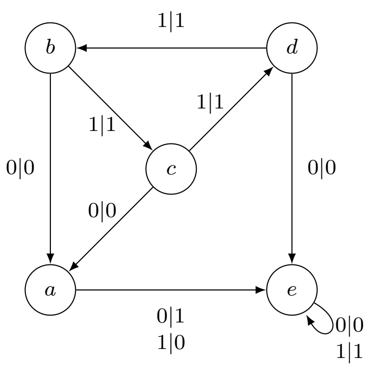


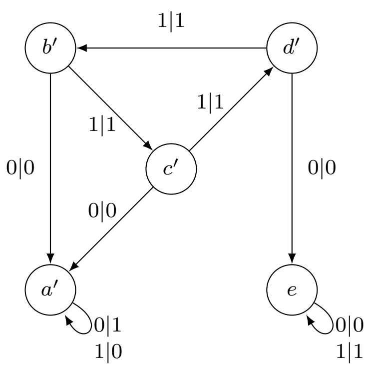
$\det(\mathcal{I}_2{}^{\mathrm{out}_{\{a\}}})$:



From the cycle structure of $\det(\mathcal{I}_2{}^{\mathrm{out}})$, we immediately deduce that $\mathcal{I}_2$ admits an exponential activity, but, since $\{a\}$ is a *nocywex*, we can construct $\det(\mathcal{I}_2{}^{\mathrm{out}_{\{a\}}})$ and deduce that $\mathcal{I}_2$ has bounded extended activity. We obtain that $\mathcal{I}_2$ belongs to EPol(0) (and incidentally to SExp(log 2)). In particular, by Proposition 22, the semigroup $\langle \mathcal{I}_2 \rangle_+$ has decidable Order Problem.

Example 25. These two Mealy automata both generate the Grigorchuk 2-group.





The leftmost Mealy automaton (the well-known one, see [27]) has a bounded activity. The rightmost one is a modified version which has an exponential activity but a bounded extended activity: it belongs to $\mathrm{EPol}(0) \setminus \bigcup_{d \geq -1} \mathrm{SPol}(d)$.

Concluding remarks. Gillibert has shown that the Order Problem is undecidable for (reset) automaton semigroups [12]: could we find/approximate the lowest class of the hierarchy in which lie automaton semigroups with undecidable Order Problem? In addition, Bondarenko and Wächter proved that the Finiteness Problem is decidable for *groups* generated by bounded automata [9], and it would be challenging to know whether their methods extend to the semigroup case.

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