S-adic expansions related to continued fractions

By

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Abstract

We consider S-adic expansions associated with continued fraction algorithms, where an S-adic expansion corresponds to an infinite composition of substitutions. Recall that a substitution is a morphism of the free monoid. We focus in particular on the substitutions associated with regular continued fractions (Sturmian substitutions), and with Arnoux–Rauzy, Brun, and Jacobi–Perron (multidimensional) continued fraction algorithms. We also discuss the spectral properties of the associated symbolic dynamical systems under a Pisot type assumption.

§ 1. Introduction

We consider in this paper S-adic expansions associated with substitutions provided by (multidimensional) continued fraction algorithms, in the continuation of [23]. We focus in particular on the substitutions associated with regular continued fractions (Sturmian substitutions), and with Arnoux–Rauzy, Brun, and Jacobi–Perron (multidimensional) continued fraction algorithms. As new contributions with respect to [23], the present paper presents spectral results related to the existence of rotation factors for Pisot type S-adic systems.

We recall that an S-adic expansion corresponds to an infinite composition of substitutions. More precisely, an infinite word u is said to admit an $\mathcal{S}$-adic expansion if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_{n-1}(a_n),$$

where $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ is a sequence of substitutions that belong to the set $\mathcal{S}$, and $(a_n)_{n\in\mathbb{N}}$ a sequence of letters with $a_n \in \mathcal{A}_n$ for all n. Without reference to the set of

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substitutions, we use the generic term S-adic expansion. If the substitutions are known to belong to some set $\mathcal{S}$, we use the term $\mathcal{S}$-adic expansion. There is a deep parallelism between subshifts associated with such expansions (under natural assumptions like primitivity, see Section 2.2) and Bratteli–Vershik systems endowed with adic transformations, hence the terminology 'adic', with the letter S referring to 'substitution'. Indeed, substitutive symbolic dynamical systems correspond to stationary Bratteli diagrams, S-adic symbolic dynamical systems to non-stationary ones, and S-adic dynamical systems having a finite set of substitutions $\{\sigma_n\}$, assumed furthermore to be positive, correspond to finite (topological) rank systems. This connection between adic models and substitutions has been widely investigated; see e.g. [49], or [26] and the references therein. As an example, explicit constructions of adding machines associated with substitutions for Denjoy systems in the framework of continued fractions are given in [76]. The main difference with the usual Bratteli–Vershik viewpoint is that we work here with measure-preserving dynamical systems and not in a topological dynamics framework. Recall that it was shown in [70] that the Vershik adic construction provides a one-to-one correspondence between minimal Cantor systems and properly ordered Bratteli diagrams: any Cantor minimal system admits a Bratteli–Vershik representation (via topological conjugacy).

The S-adic systems considered here are associated with continued fraction algorithms that produce matrices with nonnegative entries: we consider these matrices as incidence matrices of substitutions. We consider mostly algorithms under an additive form: it makes it easier to associate with them substitutions. Indeed the produced matrices have entries that belong to $\{0, 1\}$. For more details on this approach, see e.g. [22]. We focus here on algorithms such as considered in [74] having exponential convergence (their second Lyapunov exponent is negative). We are in a so-called S-adic Pisot framework. We thus consider spectral properties of the associated symbolic dynamical systems.

Let us briefly sketch the contents of this paper. Section 2 is devoted to the basic notions on substitutions and S-adic systems that will be needed here. Section 3 introduces the substitutions and S-adic systems associated with continued fractions. We focus on Pisot S-adic systems in Section 4, and on their spectral properties.

§ 2. First definitions

§ 2.1. Words and substitutions

Let $\mathcal{A}$ be finite set of *letters*, called *alphabet*. A finite *word* is an element of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$. We will work here both with one-sided words and two-sided words (it is easier from a combinatorial viewpoint to generate one-sided words in an

S-adic way but working with two-sided words makes the shift invertible).

A *substitution* σ over the alphabet $\mathcal{A}$ is a non-erasing endomorphism of the free monoid $\mathcal{A}^*$ (the image of a letter is never equal to the empty word, it contains at least one letter).

For $i \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, $|w|_i$ stands for the number of occurrences of the letter i in the word w. Let us denote by d the cardinality of $\mathcal{A}$. Let σ be a substitution. Its *incidence matrix* $M_\sigma = (m_{i,j})_{1 \leq i,j \leq d}$ is defined as the square matrix with entries $m_{i,j} = |\sigma(j)|_i$ for all i, j. A substitution is said *primitive* if there exists a power of its incidence matrix whose entries are all positive. We say that σ is *unimodular* if $\det(M_\sigma) = \pm 1$. The following notion is natural in the framework of Bratteli diagrams: a substitution over $\mathcal{A}$ is said *proper* if there exist two letters $b, e \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $\sigma(a)$ begins with b and ends with e.

The set $\mathcal{A}^{\mathbb{Z}}$ is equipped with the product topology of the discrete topology on each copy of $\mathcal{A}$, it is a compact space. This topology is defined by the usual distance: for $u \neq v \in \mathcal{A}^{\mathbb{Z}}$, $d(u, v) = 2^{-\min\{n \in \mathbb{N};\ u_{|n|} \neq v_{|n|}\}}$. The same holds analogously for $\mathcal{A}^{\mathbb{N}}$.

A word $w_1 \cdots w_\ell$ is a *factor* of the word u if there exists k such that $u_k \cdots u_{k+\ell-1} = w_1 \cdots w_\ell$. A word $u = (u_n)_n \in \mathcal{A}^{\mathbb{N}}$ (or in $\mathcal{A}^{\mathbb{Z}}$) is *uniformly recurrent* if every word occurring in u occurs in an infinite number of positions with bounded gaps, that is, if for every factor w, there exists s such that for every n, w is a factor of $u_n \ldots u_{n+s-1}$. The set of factors $\mathcal{L}_u$ of an infinite word u is called its *language*. A word u is said to be *linearly recurrent* if there exists a constant C such that every factor of length Cn contains every factor of length n. The (*factor*) *complexity function* of an infinite word u counts the number of distinct factors of a given length. We recall that linearly recurrent words have at most linear factor complexity [49].

Let Σ stand for the (left) *shift* acting on $\mathcal{A}^{\mathbb{Z}}$ (or on $\mathcal{A}^{\mathbb{N}}$), that is, $\Sigma((u_n)_n) = (u_{n+1})$. One associates with any infinite word in $\mathcal{A}^{\mathbb{Z}}$ (or in $\mathcal{A}^{\mathbb{N}}$) the symbolic dynamical system (X_u, Σ), where X_u is the closure of the orbit of u under the shift. We also associate such a symbolic system (X_σ, Σ) with a primitive substitution σ by considering the symbolic system X_u associated with any periodic word u (that is, a word u fixed by some power of σ): $X_\sigma := X_u$. By primitivity, X_σ does not depend on the choice of u. For more details, see e.g. [86]. More generally a *subshift* (also called *shift*) (X, Σ) of $\mathcal{A}^{\mathbb{N}}$ (respectively $\mathcal{A}^{\mathbb{Z}}$) is a closed shift invariant subset of $\mathcal{A}^{\mathbb{N}}$ (respectively $\mathcal{A}^{\mathbb{Z}}$). Its language $\mathcal{L}_X$ is the set of its factors, that is, the set of factors of words in X.

Let u be a word in $\mathcal{A}^{\mathbb{N}}$. The *frequency* of a letter i in u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0 u_1 \cdots u_{n-1}$ divided by n. The vector f whose components are given by the frequencies of the letters (if they exist) is called the *letter frequency vector* of u. The word u has *uniform letter frequencies* if, for every letter i of u, the number of occurrences of i in $u_k \cdots u_{k+n-1}$

divided by n has a limit when n tends to infinity, uniformly in k. Similarly, we can define the frequency and the uniform frequency of a factor, and we say that u has *uniform frequencies* if all its factors have uniform frequency. This extends in a natural way to two-sided words and shifts.

A probability measure μ on X_u is said invariant if $\mu(\Sigma^{-1}A) = \mu(A)$ for every measurable set $A \subset X_u$. An invariant probability measure on X_u is *ergodic* if any shift-invariant measurable set has either measure 0 or 1. The property of having uniform factor frequencies for a shift is equivalent to unique ergodicity. For more details on invariant measures and ergodicity, we refer to [86] and [26, Chap. 7].

Recall that if σ is a primitive substitution, then (X_σ, Σ) is minimal, linearly recurrent, uniquely ergodic. Hence, any of its elements has at most linear factor complexity. For more details, see [86, 53].

Let $u \in \mathcal{A}^{\mathbb{N}}$ and assume that each letter i has frequency f_i in u. The *discrepancy* of u is $\Delta(u) = \limsup_{i\in\mathcal{A},\ n\in\mathbb{N}} ||u_0 u_1 \dots u_{n-1}|_i - nf_i|$. The quantity $\Delta(u)$ is considered e.g. in [1, 2]. A word $u \in \mathcal{A}^{\mathbb{N}}$ is said to be *C-balanced* if for any pair v, w of factors of the same length of u, and for any letter $i \in \mathcal{A}$, one has $||v|_i - |w|_i| \le C$. It is said *balanced* if there exists $C > 0$ such that it is C-balanced. If u has letter frequencies, then u is balanced if and only if its discrepancy $\Delta(u)$ is finite. It is also said to have *bounded deviation* (the term ‘deviation’ refers here to the ergodic averages, that is, the Birkhoff sums associated with the indicator function of the cylinders associated with letters).

Let us recall that an algebraic integer $\alpha > 1$ is a *Pisot–Vijayaraghavan number* or a *Pisot number* if all its algebraic conjugates λ other than α itself satisfy $|\lambda| < 1$. According to Perron–Frobenius’ theorem (see e.g. [92]), if a substitution is primitive, then its incidence matrix admits a dominant eigenvalue (it dominates strictly in modulus the other eigenvalues) that is (strictly) positive. It is called its *Perron–Frobenius eigenvalue*, or else its *expansion* factor. A primitive substitution is said to be *Pisot* if its expansion number is a Pisot number. A primitive substitution is said *Pisot irreducible* if the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number. Recall that primitive Pisot substitutions are balanced, and have finite discrepancy (see e.g. [1, 2]). See also [41, 89] for similar results for primitive tiling spaces.

§ 2.2. S-adic shifts

Let $\mathcal{S}$ be a set of substitutions. Let $s = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, with $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, be a sequence of substitutions, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters with $a_n \in \mathcal{A}_n$ for all n. The infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is said to admit $(\sigma_n, a_n)_n$ as an *$\mathcal{S}$-adic expansion*[1] if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n).$$

The sequence s is called the *directive sequence. We work here under the assumption that all substitutions in $\mathcal{S}$ are defined on the same alphabet $\mathcal{A}$.*

Let us stress the fact that any word admits many possible S-adic expansions, such as illustrated by the now classical example by J. Cassaigne (see [23, Remark 3] and Section 3.6). We now introduce several properties of S-adic expansions that induce relevant properties for the generated words.

An S-adic expansion with directive sequence $(\sigma_n)_n$ is said *weakly primitive* if, for each n, there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive. It is said *strongly primitive* if the set of substitutions $\{\sigma_n\}$ is finite, and if there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive, for each n.

Assume we are given a (weakly) primitive directive sequence $s = (\sigma_n)_{n\in\mathbb{N}}$. Let u be an infinite word of the form $u = \bigcap_n \sigma_0 \cdots \sigma_n(\mathcal{A}^{\mathbb{N}})$, where the substitutions σ_n of the directive s are defined on the alphabet $\mathcal{A}$. According to [11], such a word is called a *limit word* of the directive sequence s (the intersection which defines u is reduced to a unique infinite word by primitivity of s). We define the shift $(X(s), \Sigma)$ generated by s as $X(s) := X_u$, for u limit word of s, and its language $\mathcal{L}(s)$ as $\mathcal{L}(s) := \mathcal{L}_u$. One checks that these definitions do not depend on u by Theorem 2.1 below. For a discussion on the way (two-sided) subshifts can be associated with directive sequences of substitutions (without any primitivity assumption), see [13] where the notions of global (one can desubstitute infinitely often) and local S-adic subshifts (defined in terms of language) are developed.

Theorem 2.1 ([53]). *If an infinite word u admits a weakly primitive S-adic expansion, then u is uniformly recurrent and the shift (X_u, Σ) is minimal. If moreover u admits a strongly primitive S-adic expansion, then (X_u, T) is also uniquely ergodic and it has at most linear factor complexity.*

Furthermore, an infinite word is linearly recurrent if and only if it admits a strongly primitive and proper $\mathcal{S}$-adic expansion, where an $\mathcal{S}$-adic expansion is said to be proper *if the substitutions in $\mathcal{S}$ are proper.*

Analogous results exists in the framework of Vershik adic maps; see e.g. [48] for the case of strongly primitive systems: Cantor minimal systems with topological finite rank are either expansive or topologically conjugate to an odometer.

Recall that for a primitive matrix M (with non-negative entries), the cones $M^n\mathbb{R}^d_+$ nest down to a single line directed by the Perron–Frobenius eigendirection at an expo-

[1] We recall that, without reference to the set of substitutions, we use the generic term S-adic expansion. If the substitutions are known to belong to some set $\mathcal{S}$, we use the term $\mathcal{S}$-adic expansion.

nential convergence speed (see e.g. [92]). Recall also that the situation for invariant measures for S-adic systems is not as simple as it can be for substitutive dynamical systems, for which primitivity implies unique ergodicity. This is well understood since Keane's counterexample for unique ergodicity for 4-interval exchanges [71]: weak primitivity does not imply unique ergodicity.

We recall here a handy characterization of unique ergodicity (see [23, Theorem 5.7]). Let $X(s)$ be an S-adic shift with directive sequence $s = (\sigma_n)_n$. Denote by $(M_n)_n$ the associated sequence of incidence matrices. We assume that the directive sequence s is *everywhere growing*, that is, for any sequence of letters $(a_n)_n$, one has $\lim_{n\to+\infty} |\sigma_0 \cdots \sigma_{n-1}|(a_n) = +\infty$. The S-adic dynamical system $(X(s), \Sigma)$ is uniquely ergodic if and only if, for each k, the limit cone

$$C^{(k)} = \bigcap_{n\to\infty} M_k \cdots M_n \mathbb{R}_+^d$$

is one-dimensional. Here d stands for the cardinality of the alphabet $\mathcal{A}$ on which the substitutions are defined. See also [34, 56, 101] for analogous rests for Bratteli–Vershik adic maps.

The following condition is then a sufficient condition for the sequence of cones $M_0 \cdots M_n \mathbb{R}_+^d$ to nest down to a single line as n tends to infinity (for square matrices with non-negative entries).

Theorem 2.2 ([59, pp. 91–95]). *Let $(M_n)_n$ be a sequence of non-negative integer matrices of size d. Assume that there exist a strictly positive matrix B and indices $j_1 < k_1 \leq j_2 < k_2 \leq \cdots$ such that $B = M_{j_1} \cdots M_{k_1-1} = M_{j_2} \cdots M_{k_2-1} = \cdots$. Then,*

$$\bigcap_{n\in\mathbb{N}} M_0 \cdots M_{n-1} \mathbb{R}_+^d = \mathbb{R}_+ f \quad \textit{for some positive vector } f \in \mathbb{R}_+^d.$$

This vector f, when normalized so that the sum of its coordinates equals 1, is called the *generalized right eigenvector* associated with the S-adic expansion of $u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n)$. In fact, this vector f is the *letter frequency vector* of u. The above condition of Theorem 2.2 implies that u admits uniform letter frequencies (by [23, Theorem 5.7]), and even unique ergodicity (in other words, uniform factor frequencies).

We now introduce further dynamics acting on the set of directive sequences. Let $\mathcal{S}$ be a finite set of substitutions, and let (D, Σ, μ) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic shift equipped with a probability measure μ. Here Σ stands for the shift acting on D. We will assume D to have positive entropy, and it will be here a sofic shift or a shift of finite type. We call such a shift an *$\mathcal{S}$-adic system.* Since we will work with continued fraction algorithms, it might prove useful also to work with countable sets of substitutions $\mathcal{S}$, and thus with shifts (D, Σ, μ) defined on a countable alphabet $\mathcal{S}$ (we then loose compactness for $\mathcal{S}^{\mathbb{N}}$). This formalism is inspired by the study of interval exchanges in connection with the Teichmüller flow, see e.g. [98, 102]. The main difference here is that the second Lyapunov exponent is negative [103].

§ 3. Continued fractions

We introduce in this section the continued fraction algorithms we will work with and associated sets of substitutions.

§ 3.1. General definitions

We consider here unimodular continued fraction algorithms by following the formalism introduced in [74] which covers most classical unimodular types of algorithms, such as discussed in [95, 36, 91].

Let $d \geq 1$. A d-dimensional unimodular continued fraction algorithm associates with $\alpha = (\alpha_1, \cdots, \alpha_{d-1}) \in [0,1]^{d-1}$ a sequence of matrices $(A^{(n)})_{n\in\mathbb{N}}$ with values in $GL(d,\mathbb{Z})$. Matrices $A^{(n)}$ play the role of partial quotients and the matrices $A^{(1)} \cdots A^{(n)}$ produce convergents. These latter products provide Diophantine approximations (via their column vectors) of the direction $(\alpha, 1)$ by points of the lattice $\mathbb{Z}^d$. The rational approximations are obtained by using the following projection[2]

$$\pi\colon \mathbb{R}^d \setminus \{(x_1, \cdots, x_d) \mid x_d = 0\} \to \mathbb{R}^{d-1},\ (x_1, \cdots, x_d) \mapsto (x_1/x_d, \cdots, x_{d-1}/x_d).$$

The last element of each column of $A^{(1)} \cdots A^{(n)}$ is a denominator for the associated simultaneous rational approximations.

Such an algorithm producing a sequence of matrices $(A^{(n)})_{n\in\mathbb{N}}$ is usually defined in dynamical terms. We will have here mostly a measure-theoretical viewpoint: the algorithms will be defined a.e. with respect to the Lebesgue measure on $[0,1]^{d-1}$.

Let $X \subset [0,1]^{d-1}$. (Usually X is $[0,1]^{d-1}$ but some algorithms can also be defined on sets of the form $\{x = (x_1, \cdots, x_{d-1}) \in [0,1]^{d-1} \mid 0 \leq x_1 \leq \cdots \leq x_{d-1} \leq 1\}$.) A d-dimensional *continued fraction map* over X is given by measurable maps

$$T\colon X \to X,\ A\colon X \to GL(d,\mathbb{Z}),\ \theta\colon X \to \mathbb{R}_+$$

that satisfy the following: for a.e. $x \in X$, one has

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \theta(x)A(x)\begin{bmatrix} T(x) \\ 1 \end{bmatrix}.$$

The associated *continued fraction algorithm* consists in iteratively applying the map T on a vector $x \in X$ which yields the matrices $A^{(n)} := A(T^n(x))$ for $n \geq 1$.

[2]Note one can also choose to work directly on the projective space $\mathbb{P}(\mathbb{R}^d)$ by associating with each element $[y_1 : y_2 : \cdots : y_{d-1} : y_d]$ the representative defined by $\max y_i = 1$ and by working with projectivizations of matrices in $GL(d,\mathbb{Z})$.

Let

$$A_n(x) = A(x)A(T(x))\dots A(T^{n-1}(x)), \quad \theta_n(x) = \theta(x)\theta(T(x))\dots\theta(T^{n-1}(x)).$$

One has

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \theta_n(x)A_n(x)\begin{bmatrix} T^n(x) \\ 1 \end{bmatrix}.$$

The map $A\colon X \to GL(d,\mathbb{Z})$ is a matrix cocycle. Indeed one has the following cocycle type relation

$$A_{m+n}(x) = A_m(x)A_n(T^m x).$$

The map θ can be considered as an arithmetic uniformization map. In the regular continued fraction case (see Section 3.2), one has $\theta_n(x) = x\,T_G(x)\cdots T_G^{n-1}(x) = |q_{n-1}x - p_{n-1}|$, where p_n/q_n stands for the n-th convergent of x, and $T_G\colon [0,1] \to [0,1]$, $x \mapsto \{1/x\}$, if $x \neq 0$, with $T_G(0) = 0$. More generally, if $l_n(x)$ stands for the vector provided by the last line of $A_n(x)^{-1}$, then $\theta_n(x) = \langle l_n(x), (x,1)\rangle$.

Such an algorithm is said Markovian or 'without memory'. Indeed, the $(n+1)$th step of the algorithm only depends on the map T and on the value $T^n(\alpha)$, contrary for example to lattice reduction or LLL algorithms, such as developed e.g. in [55, 54, 73, 68, 35].

An algorithm is said to be *positive* if the linear cocyle A takes its values in the set of matrices with nonnegative entries. The algorithm thus produces a sequence of basis of $\mathbb{R}^d$ that all determine a homogeneous cone in $\mathbb{R}^d$ that contains the ray $\{\lambda(\alpha,1) \mid \lambda \geq 0\}$. The convergence (weak or strong) has to do with the fact that the vectors of the basis have to converge (in angle or in distance) to the ray $(\alpha, 1)$.

An algorithm is said to be *additive* in [74] if all the matrices belong to a finite set. We even consider here positive algorithms whose matrices have coefficients that belong to $\{0,1\}$: the linear cocyle A takes its values in the set matrices with entries in $\{0,1\}$.

The associated maps are assumed to be measurable piecewise continuous maps. Usually T is piecewisely an homography, and is called *linear simplex-splitting algorithm* in [74]: if the algorithm is assumed to be positive, the homogeneous cone $\{(y_1,\cdots,y_d) \in \mathbb{R}_+^d \mid y_d = \max y_i\}$ is partitioned into a countable number of homogeneous integral subcones (via the matrices produced by A) and the corresponding maps are fractional linear transformations provided (via a 'projectivization' step) by the inverses of the matrices produced by A. For the existence of an absolutely continuous invariant measure, see e.g. [15, Lemma 2.1].

We then can associate with the matrices produced by a positive algorithm substitutions whose incidence matrices coincide with them. In other words, the continued fraction algorithm produces directive sequences, and thus S-adic words and shifts. The connection between continued fraction algorithms and S-adic shifts then goes through frequencies: an expansion of the continued fraction algorithm produces an S-adic shift which is such that its letter frequency vector (under suitable assumptions that provide its existence) admits this particular expansion. The continued fraction algorithm can be seen as a renormalization process that acts on the frequencies of words.

Observe also that the interest of this measure-theroretical dynamical approach is that under rather mild assumption, Kingman's subadditive ergodic theorem [72] or Oseledet's multiplicative ergodic theorem [83] can be applied. Ergodic invariant measures of the continued fraction algorithm can then be transported to the set of directive sequences.

§ 3.2. Sturmian words

The first example of continued fraction substitutions, namely Sturmian substitutions, is provided by the regular continued fraction algorithm. Recall that the Gauss map

$$T_G\colon [0,1] \to [0,1],\ x \mapsto \{1/x\},\ \text{if } x \neq 0,\ \text{and } T_G(0) = 0$$

produces the digits in the continued fraction algorithm. Consider the continued fraction expansion of $x \in (0,1)$, i.e., $x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$; one has $a_n = \left[\frac{1}{T^{n-1}(x)}\right]$ for $n \geq 1$. Matricially, this can be written as

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = x \begin{bmatrix} 0 & 1 \\ 1 & \left[\frac{1}{x}\right] \end{bmatrix} \begin{bmatrix} T(x) \\ 1 \end{bmatrix}.$$

The Gauss map produces infinitely many matrices ($[1/x]$ takes generically infinitely many values), this will thus prove to be more convenient to work with its additive version, the Farey map T_F, is defined on $[0,1]$ as

$$T_F(x) = \frac{x}{1-x} \ \text{ if } x \leq 1/2, \quad T_F(x) = \frac{1-x}{x} \ \text{ if } x \geq 1/2.$$

Its linear form is defined on the cone $\mathbb{R}^2_+ \setminus \{0\}$ as follows

$$T_F^{(L)} : \begin{cases} (A,B) \mapsto (A-B,B), \ \text{if } A \geq B \\ (A,B) \mapsto (A,B-A), \ \text{if } A \leq B. \end{cases}$$

Let

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Let $(A_1, B_1) = T_F^{(L)}(A, B)$. One has

$$\begin{bmatrix} A \\ B \end{bmatrix} = M_1 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \text{ if } A \geq B, \qquad \begin{bmatrix} A \\ B \end{bmatrix} = M_2 \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} \text{ if } A \leq B.$$

If one works directly with the Farey map T_F, things are more complicated to describe matricially than when one works with the linear form: this is why we favor linear descriptions of algorithms in the following. Indeed, let $x = \inf(A, B)/\max(A, B)$. If $A \geq B$, one has $x = B/A$, and

$$\begin{bmatrix} 1 \\ x \end{bmatrix} = \frac{1}{1 + T_F(x)} M_1 \begin{bmatrix} 1 \\ T_F(x) \end{bmatrix} \text{ if } x \leq 1/2, \quad \begin{bmatrix} 1 \\ x \end{bmatrix} = x M_1 \begin{bmatrix} T_F(x) \\ 1 \end{bmatrix} \text{ if } x \geq 1/2,$$

and if $A \leq B$, then $x = A/B$, and

$$\begin{bmatrix} x \\ 1 \end{bmatrix} = \frac{1}{1 + T_F(x)} M_2 \begin{bmatrix} T_F(x) \\ 1 \end{bmatrix} \text{ if } x \leq 1/2, \quad \begin{bmatrix} x \\ 1 \end{bmatrix} = x M_2 \begin{bmatrix} 1 \\ T_F(x) \end{bmatrix} \text{ if } x \geq 1/2.$$

Let us thus come back to the linear form. Consider the substitutions μ_1 and μ_2 defined over the alphabet $\mathcal{A}_2 = \{1, 2\}$ as

$$\mu_1 \colon 1 \mapsto 1, 2 \mapsto 21 \quad \mu_2 \colon 1 \mapsto 12, 2 \mapsto 2.$$

They have respectively as incidence matrices M_1 and M_2. Let (i_n) be a sequence in $\{1, 2\}^{\mathbb{N}}$ (it will provide the directive sequence $(\mu_{i_n})_n$).One checks that the following limit

$$u = \lim_{n \to \infty} \mu_{i_0} \mu_{i_1} \cdots \mu_{i_{n-1}}(1)$$

exists. Furthermore, if the sequence $(i_n)_{n \geq 0}$ is not ultimately constant, then the directive sequence $(\mu_{i_n})_n$ is weakly primitive. A *Sturmian word* is an infinite word whose set of factors coincides with the set of factors of an infinite word of the previous form, with the sequence $(i_n)_{n \geq 0}$ being not ultimately constant, that is, an element of a minimal symbolic dynamical system (X_u, Σ) generated by a word u of the previous form, with $(i_n)_n$ not constant. For more on Sturmian words, see e.g. [75, 79, 85].

A *Sturmian substitution* is a substitution such that the image of any Sturmian word is a Sturmian word. Sturmian substitutions are known to be exactly the substitutions that belong to the monoid generated by μ_1 and $\tilde{\mu}_1 \colon 1 \mapsto 1, 2 \mapsto 12$, together with the permutation that exchanges the letters (one checks that $\tilde{\mu}_2 \colon 1 \mapsto 21, 2 \mapsto 2$ also belongs to this monoid). Moreover, any fixed point of a Sturmian substitution is a Sturmian word. For more details, see for instance [75, Section 2.3].

As an example the fixed point $u = \lim_{n \to \infty} \sigma^n(1)$ of the Fibonacci substitution σ defined by $\sigma(1) = 21$ and $\sigma(2) = 1$ is a Sturmian word. Consider indeed the square of σ. One has $\sigma^2 = \tilde{\mu}_1 \tilde{\mu}_2$. Hence $u = \lim_{n \to \infty} (\tilde{\mu}_1 \tilde{\mu}_2)^n(1)$.

Consider a Sturmian word of the form $u = \lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_{n-1}}(1)$ with directive sequence $s = (\mu_{i_n})_n$. If the sequence $(i_n)_n$ is not ultimately constant, then one checks that the conditions of Theorem 2.2 apply, and that the frequencies of letters exist for u and for the (uniquely ergodic) shift symbolic system (X_u, Σ) it generates. Let $f = (f_1, f_2)$ be the letter frequency vector of u (that is, the generalized right eigenvector provided by $\bigcap_{n\in\mathbb{N}} M_{i_0}\cdots M_{i_{n-1}}\mathbb{R}^d_+$.) One recovers the sequence $(i_n)_n$ from the continued fraction expansion of f_1/f_2. This allows one to deduce numerous properties of Sturmian words from the continued fraction expansion of f_1/f_2. For instance, a Sturmian word is linearly recurrent if and only if its partial quotients are bounded [79, 53].

§ 3.3. Arnoux–Rauzy words

Arnoux and Rauzy introduced in [9] a generalization of Sturmian words to higher size alphabets: these words are now called Arnoux–Rauzy words, or else, strict episturmian words (see e.g. the survey [61]). Arnoux–Rauzy words are particular codings of interval exchanges. In particular, they have factor complexity $(d-1)n+1$ when defined on an alphabet of cardinality d. Nevertheless, they do not behave like generic interval exchanges. Some Arnoux–Rauzy words might have bounded deviation, other not. However, when $d = 2$, Sturmian words are known to be 1-balanced [75] (they have bounded deviation); they even are exactly the 1-balanced infinite words that are not eventually periodic. The combinatorial and spectral behaviors of Arnoux–Rauzy words have been described very accurately in [43, 42]. In particular, there exist Arnoux–Rauzy words that are (measure-theoretically) weak mixing. Note that Arnoux–Rauzy words are widely studied, in the word combinatorics community, but also in the interval exchange community, for their connections with systems of isometries of thin type such as introduced by Dynnikov in [45]. Systems of isometries [60] are a natural generalization of interval exchange transformations and interval translation mappings. The Arnoux–Rauzy continued fraction algorithm that will be described below has moreover the particularity to be defined on a set of zero mesure for $d = 3$, called Rauzy gasket; see [10, 45, 44] and the references therein; see also [17] which proves that the Hausdorff dimension of the Rauzy gasket is less than 2, and [18] which constructs a natural invariant measure for the Rauzy gasket (an invariant measure of maximal entropy) using thermodynamical formalism.

Let $\mathcal{A} = \{1, 2, \ldots, d\}$. The set of *elementary Arnoux–Rauzy substitutions* is defined as $\mathcal{S}_{AR} = \{\mu_i \mid i \in \mathcal{A}\}$ where

$$\mu_i : \; i \mapsto i, \; j \mapsto ji \text{ for } j \in \mathcal{A} \setminus \{i\}\,.$$

One recovers Sturmian words in the case $d = 2$. An *Arnoux–Rauzy word* [9] is an infinite word in $\mathcal{A}^{\mathbb{N}}$ whose set of factors coincides with the set of factors of a sequence of the

form
$$\lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_n}(1),$$
where the sequence $(i_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in $\mathcal{A}$ occurs infinitely often in $(i_n)_{n\geq 0}$. Under this latter assumption, the directive sequence $(\mu_{i_n})_n$ is weakly primitive. Note that the sequence (i_n) associated with an Arnoux-Rauzy word u is uniquely determined. The sequence $(\mu_{i_n})_n$ is called the directive sequence of u.

The symbolic dynamical systems generated by Arnoux–Rauzy words for which the directive sequence $(\mu_{i_n})_{n\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in $\mathcal{A}$ occurs infinitely often in $(i_n)_{n\geq 0}$ are minimal and uniquely ergodic. Minimality comes from Theorem 2.1, and unique ergodicity comes from Theorem 2.2. We can thus consider the letter frequency vector f associated with the shift generated by the directive sequence $(\mu_{i_n})_n$.

Let $f^{(n)}$ stand for the letter frequency vector f associated with the shift generated by the directive sequence $\Sigma^n((i_k)_k) = (i_{n+k})_{k\geq 0}$. For all n, there exist $\lambda_n \in \mathbb{R}_+$ such that
$$\lambda_n f = M_{i_1}\cdots M_{i_n} f^{(n)}.$$
The coefficient λ_n can be expressed in terms of the coefficients of the matrix $(M_{i_1}\cdots M_{i_n})^{-1}$; indeed one has $1/\lambda_n = \langle (M_{i_1}\cdots M_{i_n})^{-1} f, (1,\cdots,1)\rangle$.

The Rauzy gasket discussed above is defined as the set of frequencies of Arnoux–Rauzy words. We recall that this set has zero measure for $d = 3$.

The continued fraction algorithm which acts as a renormalisation map is thus defined in its linear version on the cone $\mathbb{R}^3_+ \setminus \{0\}$ as follows for the $d = 3$ case (see e.g. [9]):
$$\begin{cases} (A,B,C) \mapsto (A-B-C, B, C) \text{ if } A > B+C \\ (A,B,C) \mapsto (A, B-A-C, C) \text{ if } B > A+C \\ (A,B,C) \mapsto (A, B, C-A-B) \text{ if } C > A+B. \end{cases}$$
The algorithm is not defined if one has equality between a coordinate and the sum of two other coordinates, but we only consider these algorithms in a measure-theoretical sense.

Note that if one considers transposes of the substitutions, then one gets the so-called fully subtractive algorithm (see e.g. [91]), which can be described as follows: one subtracts the smallest entry to the other ones. For more on this algorithm and on variations, see [57].

There is a simple characterization of primitivity for finite products of Arnoux–Rauzy substitutions.

Theorem 3.1. *[14] Let $A = A_{i_1}\cdots A_{i_n}$ be a product of incidence matrices of Arnoux–Rauzy substitutions in dimension d. The matrix A is primitive if and only if* *all letters in* $\{1, \cdots, d\}$ *occur in* $(i_1, \cdots, i_n)$*. Moreover, if the matrix is primitive, then it is Pisot irreducible.*

Here again there is a simple characterization of linearly recurrent Arnoux–Rauzy words. A directive sequence $(\mu_{i_n})_n \in \{1, 2, \cdots, n\}^{\mathbb{N}}$ that contains each μ_i infinitely often is said to have *bounded strong partial quotients* if every substitution in $(\mu_{i_n})_n$ occurs with bounded gaps.

Proposition 3.2. *An Arnoux–Rauzy word is linearly recurrent if and only if it has bounded strong partial quotients.*

Proof. This proof comes from [29]. Let u be an Arnoux–Rauzy word with directive sequence $(\mu_{i_n})_n$. It is easy to check that strong partial quotients have to be bounded for an Arnoux–Rauzy word u to be linearly recurrent. Conversely, we cannot apply directly Theorem 2.1 since the substitutions are not proper. Nevertheless, one can deduce linear recurrence from [53] by noticing that the largest difference between two consecutive occurrences of a word of length 2 in $u^{(k)}$ is uniformly bounded (with respect to k), where $u^{(k)}$ is associated with $(\mu_{i_n})_{n \geq k}$. □

§ 3.4. Brun substitutions

We now consider an algorithm, namely Brun algorithm, that is defined for every vector of frequencies, contrarily to the Arnoux–Rauzy algorithm. Together with Jacobi–Perron algorithm, it is one of the most classical algorithms studied in this framework. Brun algorithm is closely related to the modified Jacobi–Perron algorithm, introduced in [84]: this latter algorithm is a two-point extension of Brun algorithm.

One efficient way to describe Brun algorithm is to consider its linear version (see e.g. [40]): it consists in subtracting the second largest entry to the largest. We focus here on the two-dimensional case for the sake of simplicity. Its 3-dimensional linear and additive form is thus given as follows on the cone $\mathbb{R}^3_+ \setminus \{0\}$:

$$\begin{cases} (A, B, C) \mapsto (A, B, C - B), \text{ if } B \leq C - B, \\ (A, B, C) \mapsto (A, C - B, B), \text{ if } A \leq C - B \leq C, \\ (A, B, C) \mapsto (C - B, A, B), \text{ if } C - B \leq A. \end{cases}$$

Its projectivized additive form T_B is defined on the set $\Delta_2 := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq x_2 \leq 1\}$ as follows

$$T_B\colon \begin{cases} (x_1, x_2) \mapsto \left(\frac{x_1}{1-x_2}, \frac{x_2}{1-x_2}\right), \text{ for } x_2 \leq \frac{1}{2}, \\ (x_1, x_2) \mapsto \left(\frac{x_1}{x_2}, \frac{1-x_2}{x_2}\right), \text{ for } \frac{1}{2} \leq x_2 \leq 1 - x_1, \\ (x_1, x_2) \mapsto \left(\frac{1-x_2}{x_2}, \frac{x_1}{x_2}\right), \text{ for } 1 - x_1 \leq x_2. \end{cases}$$

The Brun matrices are

$$\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&1&1 \end{pmatrix}, \quad \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&1 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0\\ 0&0&1\\ 1&0&1 \end{pmatrix}.$$

The set of *Brun substitutions* $\mathcal{S}_{BR}$ is defined on the alphabet $\mathcal{A}_3 = \{1, 2, 3\}$ as $\mathcal{S}_{BR} = \{\beta_1, \beta_2, \beta_3\}$ with

$$\beta_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \qquad \beta_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \beta_3 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases}$$

Their incidence matrices coincide with the three Brun matrices above.

There is again a simple characterization of primitivity for finite products of Brun matrices (see [14]). Let $B = B_{i_1} \cdots B_{i_n}$ be a product of Brun matrices of size 3. Then, B is primitive if and only if the matrix B_3 occurs in the product. If B is primitive, then it is Pisot irreducible.

There is no canonical choice for Brun substitutions. In particular one could choose to flip the letters in the definition of the β_i's by mapping letters on 32 instead of mapping them on 23. Experimental studies have confirmed that consistency in the ordering of letters yields better approximation results (see e.g. [25] for a discussion on experimental studies that have been conducted). There exist also in the literature various sets of Brun substitutions according to the fact that one wants to use substitutions or so-called dual substitutions: the differences one can encounter among these substitutions comes from the fact that the substitutions might be defined with respect to the Brun matrices or with respect to their transposes. Some choices are given e.g. in [14, 24, 30, 46].

§ 3.5. Jacobi–Perron substitutions

The Jacobi–Perron algorithm is not defined in terms of ordered positions. We focus here also on the two-dimensional case for the sake of simplicity. It is defined on the cone $\{(A, B, C) \in \mathbb{R}^3_+ \setminus \{0\} \mid 0 \leq A, B \leq C\}$ as follows:

$$(A, B, C) \mapsto (B - [B/A]\, A, C - [C/A]\, A, A) \ \text{ if } A \neq 0.$$

Its projectivized form is defined on $[0, 1]^2$ as

$$T_{JP} \colon (x_1, x_2) \mapsto (\{x_2/x_1\}, \{1/x_1\}).$$

The Jacobi–Perron matrices are of the form

$$M_{a,b} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & a \\ 0 & 1 & b \end{pmatrix}$$

for non-negative integers a and b, and $a \leq b$, $b \geq 1$.

The set of *Jacobi–Perron substitutions* $\mathcal{S}_{JP}$ is defined on the alphabet $\mathcal{A}_3 = \{1, 2, 3\}$ as $\mathcal{S}_{JP} = \{\sigma_{a,b}\}$ with

$$\sigma_{a,b} \colon 1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 12^a3^b.$$

This algorithm is not complete but nevertheless Markovian. Let $(x_1^{(n)}, x_2^{(n)}) = T_{JP}^n(x_1, x_2)$, $a_n = \left[\frac{x_2^{(n-1)}}{x_1^{(n-1)}}\right]$, $b_n = \left[\frac{1}{x_1^{(n-1)}}\right]$, for all n. The sequence $(a_n, b_n)_{n \geq 1}$ is the Jacobi–Perron expansion of some vector (x_1, x_2) if only if for every $n \geq 1$, we have $0 \leq a_n \leq b_n$, $b_n \neq 0$, and $a_n = b_n$ implies $a_{n+1} \neq 0$ (see e.g. [82, 91]).

By [52], every finite product of Jacobi–Perron substitutions is primitive and Pisot irreducible if $0 \leq a_n \leq b_n$ and $b_n \neq 0$ for all $n \geq 1$.

Here again there exist also Jacobi–Perron substitutions associated with the transpose matrices $M_{a,b}$ such as considered in [31, 30]. This depends on the chosen application. Note also that we can decompose Jacobi–Perron algorithm into additive steps.

§ 3.6. Elementary substitutions

We work here with continued fractions algorithms which can be decomposed, under their linear form, as successions of steps that consist in subtracting an element to another one, together with permutations of entries. Their associated matrices (more precisely the non-negative ones produced by the linear cocyle A) can thus be decomposed as products of elementary matrices together with permutation matrices.

We consider elementary matrices M_{ij} defined as follows for $i \neq j$ ($i, j \in \{1, 2, \cdots, d\}$): the entries of index (i, j) or (k, k) ($k \in \{1, 2, \cdots, d\}$) are equal to 1, and M_{ij} has zero entries elsewhere. The image by M_{ij} of the column vector $(a_1, \cdots, a_d)$ is the vector whose ith entry is equal to $a_i + a_k$, and whose other coordinates are unchanged. The inverse matrix M_{ij}^{-1} thus performs the following: the ith entry is replaced by $a_i - a_j$ and the other ones are unchanged. We now introduce the substitutions that have as incidence matrices the elementary matrices M_{ij}, for $i, j \in \{1, 2, \cdots, d\}$, with $i \neq j$:

$$\sigma_{ij} \colon j \mapsto ij,\ k \mapsto k \text{ for } k \neq j \qquad \widetilde{\sigma}_{ij} \colon j \mapsto ji,\ k \mapsto k \text{ for } k \neq j.$$

We consider the set $\mathcal{S}_e = \{\sigma_{ij} \mid i \neq j,\ 1 \leq i, j \leq d\} \cup \{\widetilde{\sigma}_{ij} \mid i \neq j,\ 1 \leq i, j \leq d\}$ of elementary substitutions. We also consider the set $\mathcal{P}$ of substitutions whose incidence matrices are permutations. For instance, for $d = 2$, the substitutions in $\mathcal{P}$ are

$$1 \mapsto 1, 2 \mapsto 2 \quad \text{and} \quad 1 \mapsto 2,\ 2 \mapsto 1.$$

These substitutions are free group automorphisms: when extended to morphisms of the free group generated by $\mathcal{A}$, they are invertible. They are furthermore positive free group automorphisms since they map letters in $\mathcal{A}$ on words that contain only occurrences of positive letters.

We say that a positive automorphism of the free group on $\mathcal{A}$ is *tame* if it belongs to the submonoid generated by the set of permutations $\mathcal{P}$ of $\mathcal{A}$ together with the set of elementary substitutions $\mathcal{S}_e$ of $\mathcal{A}$. Recall that the group of all automorphisms (positive or not) is generated by the elementary Nielsen automorphisms [81].

Sturmian substitutions are known to be tame: they are generated by elementary substitutions together with permutations. Every two-letter positive automorphism is also a product of permutations and elementary substitutions [100]. In other words, a 2-letter substitution generates a Sturmian word if and only if it is a free group automorphism (for more detail, see e.g. [75] and the references therein). The connections between invertible substitutions, products of elementary substitutions (and permutations) and generalized Sturmian words does not hold anymore on a larger size alphabet. The monoid of positive automorphisms is not finitely generated as soon as the alphabet has at least three generators. Recall indeed that there exist three-letter invertible substitutions that cannot be decomposed as products of invertible substitutions, according to [96].

More generally, S-adic expansions that are produced by elementary substitutions on an alphabet of size at least 3 do not behave like in the Sturmian case, in particular with respect to factor complexity. Recall that an S-adic expansion defined by the directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ is said to be everywhere growing if for any sequence of letters $(a_n)_n$, one has $\lim_{n\to+\infty} |\sigma_0\cdots\sigma_{n-1}|(a_n) = +\infty$. According to [23, Theorem 4.3], any everywhere growing $\mathcal{S}$-adic word u whose directive sequence takes its values in a finite set $\mathcal{S}$ of substitutions has zero entropy, that is, $\lim_{n\to\infty} \frac{\log p_u(n)}{n} = 0$, where p_u stands for the factor complexity of u. In particular, any everywhere growing $\mathcal{S}'_e$-adic word u (e.g., any weakly primitive $\mathcal{S}'_e$-adic word) whose directive sequence takes its values in the set of elementary substitutions and permutations $\mathcal{S}'_e = \mathcal{S}_e \cup \mathcal{P}$ has zero entropy. Nevertheless, we will see below how to get high factor complexity (among zero entropy words and shifts).

Recall that for any word $u \in \{1,2\}^{\mathbb{N}}$, the word $3u$ admits an $\mathcal{S}'_e$-adic representation whose directive sequence takes its values in the set $\mathcal{S}'_e = \mathcal{S}_e \cup \mathcal{P}$ on a three-letter alphabet, according to the following construction due to J. Cassaigne. Take the two tame substitutions

$$\widetilde{\sigma}_{13}\colon 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 31$$

$$\widetilde{\sigma}_{23}\colon 1 \mapsto 1,\ 2 \mapsto 2,\ 3 \mapsto 32.$$

One has $3u = \lim_{n\to\infty} \widetilde{\sigma}_{u_0}\cdots\widetilde{\sigma}_{u_n}(3)$. Note however that the word $3u$ is not recurrent.

The following construction is due to J. Cassaigne and J. Leroy. This construction is inspired by similar constructions in [47, 50]. For any word v in $\{1,2\}^*$, let

$$\alpha_v \colon 1 \mapsto 3v1, 2 \mapsto 3v2, 3 \mapsto 3v.$$

One checks that α_v is tame. Its incidence matrix is primitive as soon as v contains the letters 1 and 2. We fix an increasing sequence of positive integers $(k_n)_n$. Let $(v_n)_n$ be a sequence of words such that, for all n, v_n contains all the words of length k_n over the alphabet $\{1,2\}$. We consider the uniformly recurrent $\mathcal{S}'_e$-adic word $u = \lim_{n\to\infty} \alpha_{v_0} \cdots \alpha_{v_n}(3)$. Note that $|\alpha_{v_0} \cdots \alpha_{v_n}(1)| = |\alpha_{v_0} \cdots \alpha_{v_n}(2)|$, for all n. Set $\ell_n := |\alpha_{v_0} \cdots \alpha_{v_n}(1)| = |\alpha_{v_0} \cdots \alpha_{v_n}(2)|$. Take j with $1 \leq j \leq k_{n+1}$. The number of factors of u of length $j\ell_n$ is at least 2^j. Indeed, all the words of length j occur in v_{n+1} and their respective images by $\alpha_{v_1} \cdots \alpha_{v_n}$ all have the same length and are distinct.

For any d, it remains to choose the sequence $(k_n)_n$ with a sufficiently large growth satisfying $2^{k_{n+1}} > (k_{n+1}\,\ell_n)^d$, for n large enough, to get a uniformly recurrent $\mathcal{S}'_e$-adic word $\lim_{n\to\infty} \alpha_{v_0} \cdots \alpha_{v_n}(3)$ which admits an everywhere growing $\mathcal{S}'_e$-adic representation and which has a factor complexity that is not of polynomial order d. These examples illustrate the fact that $\mathcal{S}'_e$-adic words, with substitutions in the directive sequence being tame, can have various combinatorial behaviors. As examples of such $\mathcal{S}'_e$-adic words, see the family of tree words such as developed in [33]. This family includes Arnoux-Rauzy words and codings of interval exchanges.

Note that the S-adic expansions produced by continued fraction algorithms beneficiate from their ergodic properties. Both algorithms (Brun and Jacobi–Perron) are known to have an invariant ergodic probability measure equivalent to the Lebesgue measure (see for instance [91]). However, this measure is not known explicitly for Jacobi–Perron (the density of the measure is shown to be a piecewise analytical function in [38]), whereas it is known explicitly for Brun [8, 58, 63]. Brun algorithm is a space-ordering algorithm according to the terminology introduced in [64]. (Note that it is called ordered Jacobi–Perron in [63].) Furthermore, each step of Brun algorithm produces only one partial quotient. This helps in computing the natural extension and the invariant measure of Brun algorithm (see e.g. [8] which shows in a very efficient way how to determine the invariant measure of Brun algorithm thanks to the natural extension). Contrary to Brun algorithm, the role played by the first two entries is not determined by a comparison between both parameters in Jacobi–Perron case; this might explain the fact that an explicit realization of the natural extension of this algorithm is still not known.

§ 4. Pisot case

We introduce in this section an S-adic counterpart to Pisot substitutions via the notion of Pisot S-adic systems and we discuss their spectral properties.

§ 4.1. S-adic systems and Lyapunov exponents

Consider a positive continued fraction algorithm (T, A, θ). We associate with it an $\mathcal{S}$-adic system as follows. Let $\mathcal{S}$ be the set of substitutions produced by the algorithm. These substitutions have as incidence matrices the matrices that lie in the image of the matrix cocycle A. We define D as the closure in $\mathcal{S}^{\mathbb{N}}$ of the set of directive sequences associated with orbits under the map A, that is, D is the closure of the set of sequences $(\gamma_n)_n$ for which there exists x such that γ_n has for incidence matrix $A(T^n x)$, for all n. We also consider an ergodic shift invariant measure μ on D. We get an ergodic shift (D, Σ, μ), with $D \subset \mathcal{S}^{\mathbb{N}}$. Recall that Σ stands for the shift.

We now introduce Lyapunov exponents associated with the linear cocyle map A. For simplicity we also use the notation A for the following map

$$A : D \to \mathrm{GL}(d, \mathbb{Z}),\ \gamma = (\gamma_n)_{n\in\mathbb{N}} \mapsto M_{\gamma_0}$$

where M_{γ_0} is the incidence matrix of γ_0. We assume that the map A is log-*integrable*[3], that is,

$$\int_X \log\max(\|A(\gamma)\|, \|A(\gamma)^{-1}\|) d\mu(\gamma) < \infty.$$

If the matrices $A(\gamma)$ are bounded (e.g. if the set $\mathcal{S}$ is finite) and unimodular, then this condition is automatically satisfied. We then define the Lyapunov exponents $\theta_1^\mu, \theta_2^\mu, \dots, \theta_d^\mu$ (d stands for the cardinality of the alphabet on which the substitutions in $\mathcal{S}$ are defined) as the μ-a.e. limit of

$$\theta_1^\mu + \theta_2^\mu + \cdots + \theta_k^\mu = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^k \left(A(\gamma)A(\Sigma\gamma)\cdots A(\Sigma^{n-1}\gamma)\right)\|$$

for $1 \le k \le d$, where $\wedge^k$ denotes the k-fold wedge product. In particular, the first Lyapunov exponent θ_1^μ is the μ-a.e. limit

$$\theta_1^\mu = \lim_{n\to\infty} \frac{\log \|A_n(\gamma)\|}{n},$$

with $A_n(\gamma) := M_{\gamma_0} \cdots M_{\gamma_{n-1}}$.

Remark that we get a (forward) random dynamical system according to the terminology of [6]: T (the continued fraction map) is the base transformation (it is not

[3]We also say that (D, μ) is log-integrable.

invertible here) and A is the generator; we consider $\mathcal{S}$-adic systems for which directive sequences are one-sided (D is one-sided, Σ is not invertible). Oseledet's theorem [83] yields in this one-sided framework a filtration, and not a splitting. If we need a splitting of the space, we need to consider an invertible shift, hence a natural extension for T. This is the approach developed in [12] where Markov partitions are associated with 'Pisot' non-stationary biinfinite sequences of toral automorphisms (these are the linear Anosov families of [7]) associated with multidimensional continued fraction algorithms, such as Brun. The S-adic Pisot assumption (such as defined below) yields non-stationary hyperbolic dynamics.

§ 4.2. S-adic Pisot systems and shifts

We introduce in this section the notion of S-adic Pisot systems. We first recall the following result from [23]. A finite product of substitutions $\gamma_0 \dots \gamma_{k-1}$ is said *positive* if the associated matrix $M_{\gamma_0} \dots M_{\gamma_{k-1}}$ is positive. Recall that if $\gamma = (\gamma_n)$, then $A_n(\gamma) := M_{\gamma_0} \cdots M_{\gamma_{n-1}}$ for positive integer n. Recall also that if the finite word w is a factor of the subshift D, then the cylinder $[w]$ is the set of infinite sequences in D having w as a prefix.

Theorem 4.1. *Let $\mathcal{S}$ be a set of substitutions with invertible incidence matrices, and let (D, Σ, μ), with $D \subset \mathcal{S}^{\mathbb{N}}$, be an ergodic shift with respect to the shift invariant probability measure μ. Let $\mathcal{A}$ stand for the alphabet of the substitutions in $\mathcal{S}$, and d stand for its cardinality. Assume that there exists a finite word in $\mathcal{S}^*$ such that the matrix of the associated product of substitutions is positive and whose associated cylinder has positive measure for μ. Then, for μ-almost every sequence $\gamma \in D$, the corresponding $\mathcal{S}$-adic subshift $X_D(\gamma)$ is uniquely ergodic. Furthermore, one has $\theta_1^\mu > 0$ and $\theta_1^\mu > \theta_2^\mu$.*

Let us denote by $f(\gamma) = (f_i(\gamma))_{i \in \mathcal{A}}$ the generalized right eigenvector of a μ-generic sequence γ. For μ-almost every $\mathcal{S}$-adic sequence in D, $X_D(\gamma)$ is weakly convergent:

$$\lim_{n\to\infty} \max_{i\in\mathcal{A}} \frac{1}{n} \log \mathrm{d}\left(\frac{A_n(\gamma)e_i}{\|A_n(\gamma)e_i\|_1}, f(\gamma)\right) = \theta_2^\mu - \theta_1^\mu,$$

where $(e_1, \cdots, e_d)$ stands for the canonical basis of $\mathbb{R}^d$.

Moreover, if $\theta_2^\mu < 0$, then, for μ-almost every $\mathcal{S}$-adic sequence in D, $X_D(\gamma)$ is strongly convergent:

$$\lim_{n\to\infty} \max_{i\in\mathcal{A}} \frac{1}{n} \log \mathrm{d}(A_n(\gamma)e_i, f(\gamma)) = \theta_2^\mu,$$

and for μ-almost all γ in D, $X_D(\gamma)$ has bounded deviation, that is, there exists a constant $C = C(\gamma)$ such that for every letter $i \in \mathcal{A}$, every word u in $X_D(\gamma)$ and every n, we have

$$||u_0 \dots u_{n-1}|_i - n f_i(\gamma)| \leq C.$$

In particular, each word in $X_D(\gamma)$ is C-balanced.

The positivity condition of Theorem 4.1 is in the flavour of condition H_5 in [74]; see also [14]. The quantity $1 - \frac{\theta_2^\mu}{\theta_1^\mu} = \frac{1}{\theta_1^\mu}(\theta_1^\mu - \theta_2^\mu)$ is expressed in [74] as the uniform approximation exponent for unimodular continued fractions algorithms; see also [19, 20]. For a thorough study of the Lyapounov exponents of the Jacobi–Perron algorithm (which also applies to Brun algorithm), see [38, 39]. Note that the log-integrability for the accelerated (i.e., multiplicative) version comes from the the comparability of the invariant measure with respect to Lebesgue measure (see e.g. [74, Theorem 1.1]). This yields in particular the a.e. exponential (strong) convergence of Brun [58, 78, 90] and of Jacobi–Perron algorithm [39] (see also [74]): there exists $\delta > 0$ s.t. for a.e. (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}, \ |\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by the algorithm. For a criterium for the simplicity of the Lyapunov spectrum, see [16, 77].

We now can introduce the S-adic counterpart of the notion of Pisot irreducible substitution following [14, 23]. Recall that a substitution is said *Pisot irreducible* if the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number.

Definition 4.2 (S-adic Pisot system)**.** Let $\mathcal{S}$ be a set of substitutions with invertible incidence matrices over the alphabet $\mathcal{A}$, and let (D, Σ, μ) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic, shift equipped with the probability measure μ. We also assume log-integrability for (D, μ). Let $(\theta_i^\mu)_{i \in \mathcal{A}}$ stand for its Lyapunov exponents. We say that (D, Σ, μ) satisfies the *Pisot condition* if

$$\theta_1^\mu > 0 > \theta_2^\mu.$$

According to Theorem 4.1, for a.e. γ, $(X_D(\gamma))$ is uniquely ergodic, minimal, and C-balanced.

The analog of algebraic irreducibility is then the following. According to [29], the directive sequence γ is said to be *algebraically irreducible* if, for each $k \in \mathbb{N}$, the characteristic polynomial of $M_{\gamma_k} \cdots M_{\gamma_\ell}$ is irreducible for all sufficiently large ℓ. Recall that the the sequence γ is said to be (weakly) *primitive* if, for each $k \in \mathbb{N}$, $M_{\gamma_k} \cdots M_{\gamma_\ell}$ is a positive matrix for *some* $\ell > k$.

As examples of Pisot irreducible systems, one has Arnoux–Rauzy (in any dimension), and Brun S-adic systems, when $d = 3$, by [14], for a large choice of measures μ.

§ 4.3. Rotation factors and the Pisot condition

We now consider spectral properties of S-adic Pisot shifts. Recall that Pisot irreducible substitutions are assumed to have pure discrete spectrum: this is the so-called Pisot substitution conjecture. For more details, see e.g. [3].

We first recall a classical statement in topological dynamics relating bounded deviation for ergodic averages associated with a continuous function f, that is, bounded sums $\sum_{n=0}^{N} f(T^n x)$, and the fact that f is a coboundary. This will allow us to exhibit (topological) eigenfunctions. For more on the connections between this statement and bounded remainder sets, see the survey [69].

Theorem 4.3 (Gottschalk–Hedlund [62]). *Let X be a compact metric space and $T\colon X \to X$ be a minimal homeomorphism. Let $f\colon X \to \mathbb{R}$ be a continuous function. Then f is a coboundary, that is,*

$$f = g - g \circ T$$

for a continuous function g if and only if there exists $C > 0$ such that

$$|\sum_{n=0}^{N} f(T^n(x))| < C$$

for all N and all x.

We will apply it here to the shift Σ acting on the two-sided shift spaces $X_D(\gamma)$. Recall that the directive sequence of substitutions γ is said to be algebraically irreducible if, for each $k \in \mathbb{N}$, the characteristic polynomial of $M_{\gamma_k \cdots \gamma_\ell}$ is irreducible for all sufficiently large ℓ.

Theorem 4.4. *Let $\mathcal{S}$ be a set of substitutions with invertible incidence matrices on an alphabet $\mathcal{A}$ of cardinality d, and let (D, Σ, μ), with $D \subset \mathcal{S}^{\mathbb{N}}$, be an ergodic shift with respect to the shift invariant probability measure μ. We assume that (D, Σ, μ) satisfies the Pisot condition. Let $\theta_1^\mu > 0$ stand for its first Lyapunov exponent. Then, for μ almost every γ, if γ is moreover algebraically irreducible, then $(X_D(\gamma), \Sigma)$ admits as a topological factor a minimal translation on the torus $\mathbb{T}^{d-1}$. In particular, it is not weakly mixing.*

Proof. Let (X, Σ) be a topological dynamical system. Recall that a non-zero complex-valued continuous function in $\mathcal{C}(X)$ is an *eigenfunction* for S if there exists $\lambda \in \mathbb{C}$ such that $\forall x \in X,\ f(Sx) = \lambda f(x)$. The system (X, Σ) is said to have *topological discrete spectrum* if the eigenfunctions span $\mathcal{C}(X)$.

According to Theorem 4.1, for μ a.e. $\gamma \in D$, the shift space $X_D(\gamma)$ is minimal. We consider the two-sided version of $X_D(\gamma)$, that is, the set of two-sided words in $\mathcal{A}^{\mathbb{Z}}$ whose language is equal to the language of $X_D(\gamma)$. We keep the same notation for the two-sided version of $X_D(\gamma)$. The shift Σ acting on the two-sided shift $(X_D(\gamma), \Sigma)$ is thus a minimal homeomorphism acting on it.

Let γ be a generic point such that two-sided space $X_D(\gamma)$ is minimal and C-balanced, for some C (that depends on γ). Let a be a letter of the alphabet $\mathcal{A}$ on which the substitutions of $\mathcal{S}$ are defined. Consider the continuous map $f = \mathbf{1}_{[a]}(x) - f_a \mathbf{1}$ defined on $X_D(\gamma)$, where $\mathbf{1}_{[a]}$ is the characteristic function of the cylinder set of sequences $(u_n)_{n\in\mathbb{Z}} \in X_D(\gamma)$ such that $u_0 = a$, and where $\mathbf{1}$ stands for the constant function taking value 1. Since f has bounded deviation by C-balancedness, we can apply Theorem 4.3: there exists a continuous function g such that $f = g - g \circ \Sigma$. Note that $e^{2i\pi \mathbf{1}_{[a]}(v)} = 1$ for any $v \in X_D(\gamma)$. This yields

$$\exp^{2i\pi g\circ\Sigma} = \exp^{2i\pi f_a} \exp^{2i\pi g},$$

where f_a stands for the frequency of the letter a. Hence $\exp^{2i\pi g}$ is a continuous eigenfunction associated with the eigenvalue $\exp^{2i\pi f_a}$.

We assume γ algebraically irreducible. The proof follows [29]. Let us prove that the coordinates of the letter frequency vector $f = (f_a)_{a\in\mathcal{A}}$ are rationally independent. Let $z \in \mathbb{Z}^d \setminus \{0\}$ such that $\langle f, z\rangle = 0$. Recall that $A_n(\gamma) := M_{\gamma_0} \cdots M_{\gamma_{n-1}}$. One has $\langle {}^t A_n(\gamma)\, z, e_i\rangle = \langle z, A_n(\gamma)\, e_i\rangle$ is bounded (uniformly in n) for each $i \in \mathcal{A}$. Indeed, by strong convergence, one has $\lim_{n\to\infty} \frac{\log \mathrm{d}(A_n(\gamma)e_i, \mathbb{R}f(\gamma))}{n} = \theta_2^{\mu} < 0$ and $\langle f, z\rangle = 0$. The vectors ${}^t A_n(\gamma)\, f$ take furthermore integer values. The values taken by $||{}^t A_n(\gamma)\, f||$ are thus bounded. Hence, there exist k and infinitely many $\ell > k$ such that ${}^t A_k(\gamma)\, f = {}^t A_\ell(\gamma)\, f$. One has ${}^t A_k(\gamma)\, f \neq 0$ by algebraic irreducibility. Hence, it is an eigenvector of ${}^t(M_k \cdots M_{\ell-1})$ associated with the eigenvalue 1. This contradicts the fact that $M_k \cdots M_{\ell-1}$ has irreducible characteristic polynomial for large ℓ.

Let $\mathcal{A} = \{1, 2, \cdots, d\}$. We consider the toral translation R_f by the vector $= (f_1, \ldots, f_{d-1})$ on the d-dimensional torus $\mathbb{T}^{d-1} = \mathbb{R}^{d-1}/\mathbb{Z}^{d-1}$:

$$R_f(x_1, \cdots, x_{d-1}) = (x_1 + f_1, \cdots, x_{d-1} + f_{d-1}) \text{ modulo } 1.$$

This translation is minimal since the coordinates of the letter frequency vector $f = (f_a)_{a\in\mathcal{A}}$ are rationally independent. Since its spectrum, i.e., $\exp(2i\pi \sum_{a\in\mathcal{A}} \mathbb{Z} f_a)$, is included in the continuous spectrum of $X_D(\gamma)$, this translation is a topological factor of $X_D(\gamma)$. □

If we drop the algebraic irreducibility condition in Theorem 4.3, there still exists a non-trivial rotation factor but it is not necessarily defined on $\mathbb{T}^{d-1}$.

Note that more can be said on the two-letter case: the Pisot conjecture in this two-letter S-adic setting has been proved in [32]. Recall that the two-letter substitutive case has been solved in [65] together with [21], and in [66].

§ 4.4. Arnoux–Rauzy and Brun words

We now end the present paper by recalling results from [29]: in dimension $d = 3$, we even have (measure-theoretical) pure discrete spectrum for a.e. Arnoux–Rauzy and Brun S-adic shifts.

Theorem 4.5 ([29]). *Let $\mathcal{S}_{AR}$ be the set of Arnoux–Rauzy substitutions over three letters and consider the shift $(\mathcal{S}_{AR}^{\mathbb{N}}, \Sigma, \mu)$ for some shift invariant ergodic probability measure μ that assigns positive measure to each cylinder. Then $(\mathcal{S}_{AR}^{\mathbb{N}}, \Sigma, \mu)$ satisfies the Pisot condition. Moreover, for μ-almost all sequences $\gamma \in \mathcal{S}_{AR}^{\mathbb{N}}$, the $\mathcal{S}_{AR}$-adic shift (X_γ, Σ) is (measure-theoretically) isomorphic to a translation on the two-dimensional torus $\mathbb{T}^2$, that is, (X_γ, Σ) has (measure-theoretically) pure discrete spectrum.*

Let $\mathcal{S}_{BR}$ be the set of Brun substitutions over three letters, and consider the shift $(\mathcal{S}_{BR}^{\mathbb{N}}, \Sigma, \mu)$ for some shift invariant ergodic probability measure μ that assigns positive measure to each cylinder. Then $(\mathcal{S}_{BR}^{\mathbb{N}}, \Sigma, \mu)$ satisfies the Pisot condition. Moreover, for μ-almost all sequences $\gamma \in \mathcal{S}_{BR}^{\mathbb{N}}$, the $\mathcal{S}_{BR}$-adic shift (X_γ, Σ) is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^2$.

As an example of a measure satisfying the assumptions of Theorem 4.5 for Arnoux-Rauzy shifts, consider the measure of maximal entropy for the suspension flow of the Rauzy gasket constructed in [18]. The suspension flow is a renormalization flow obtained analogously as for the Teichmüller flow in the classical case of interval exchanges [97]: it is based on a roof function associated with the cocyle which is the first return time to some subsimplex of the parameter space and on the accelerated version of the Arnoux-rauzy algorithm (which is proved to be log-integrable in [18]). Hence, with respect to this Gibbs measure, a.e. Arnoux-Rauzy word has pure discrete spectrum.

Let us conclude with the case of Brun shifts. Consider the measure $\mu := m \circ \phi^{-1}$ on $\mathcal{S}_{BR}^{\mathbb{N}}$: here m is the (ergodic) invariant measure absolutely continuous with respect to Lebesgue measure of the additive version of Brun algorithm (it is a finite measure), ϕ is the natural measure-theoretic isomorphism between (Δ_2, T_B, m) and $(\mathcal{S}_{BR}^{\mathbb{N}}, \Sigma, \nu)$, where $\Delta_2 = \{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 \leq x_1 \leq x_2 \leq 1\}$, and T_B is the projectivized additive form of Brun algorithm introduced in Section 3.4. The measure μ is shift-invariant, ergodic and gives positive measure to each cylinder. For more details, see [29, Proof of Theorem 8]. In other words, by Theorem 4.5, the $\mathcal{S}_{BR}$-adic shifts associated with Brun algorithm provide natural symbolic codings of almost all rotations on the torus $\mathbb{T}^2$.

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