Bijective counting of one-face maps on surfaces.

Guillaume Chapuy*, SFU

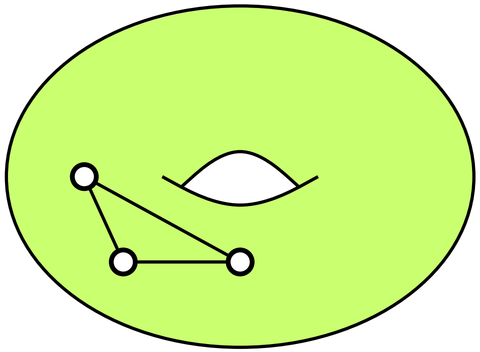
* PIMS-CNRS postdoc

Discrete Math seminar, UBC, 2009.

Orientable surfaces

Map of genus g

= graph drawn (without edge-crossings) on a surface of genus g, such that each face is homeomorphic to a disk.



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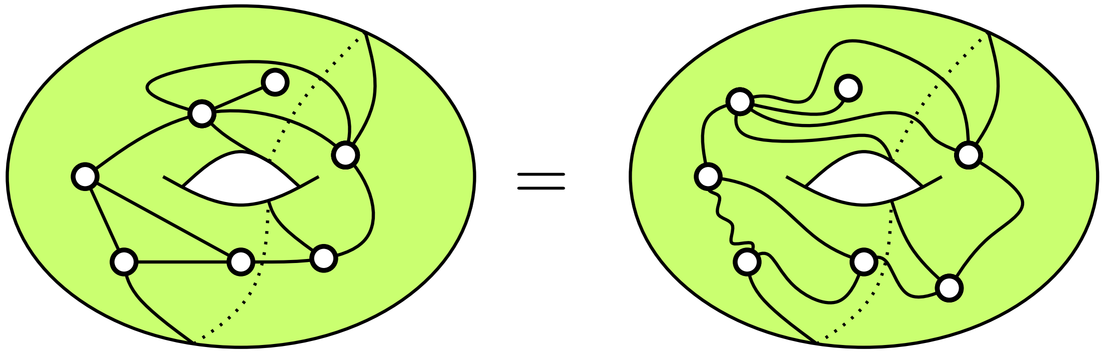

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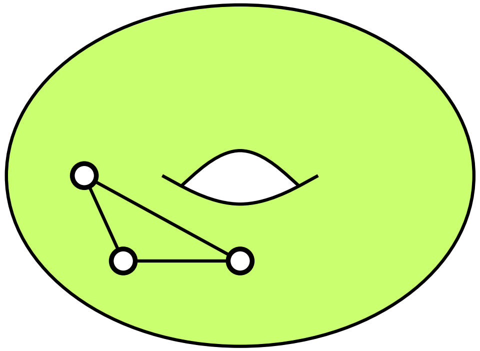

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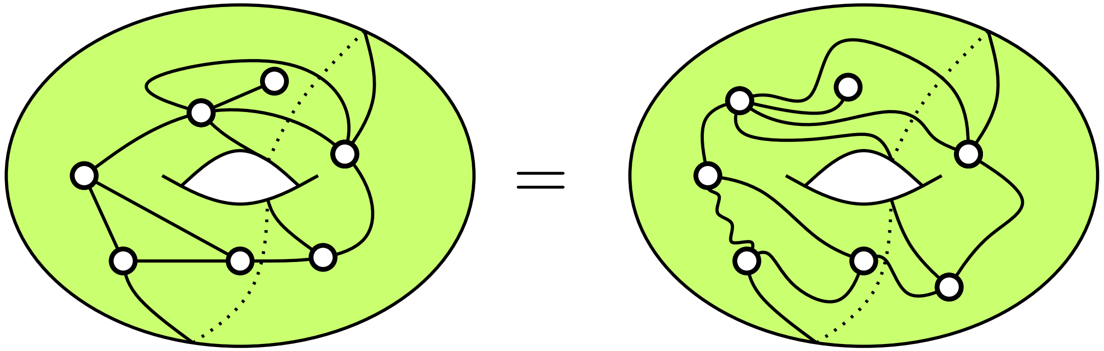

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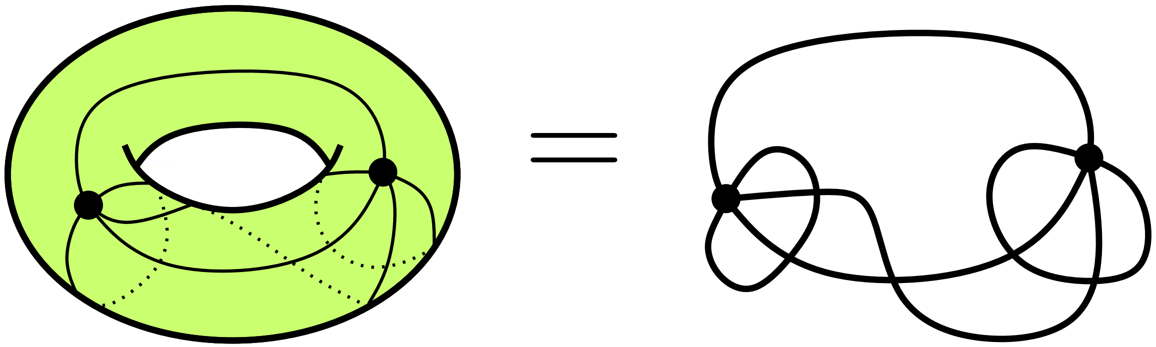
Maps are combinatorial objects:

Map = graph + rotation system around each vertex.



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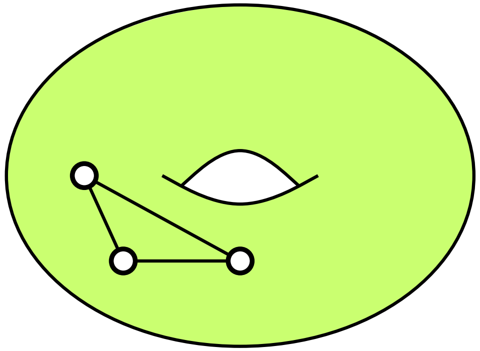

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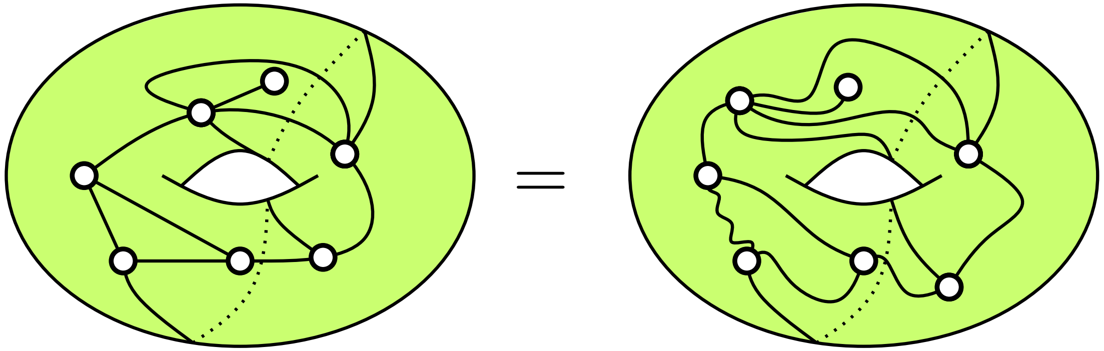

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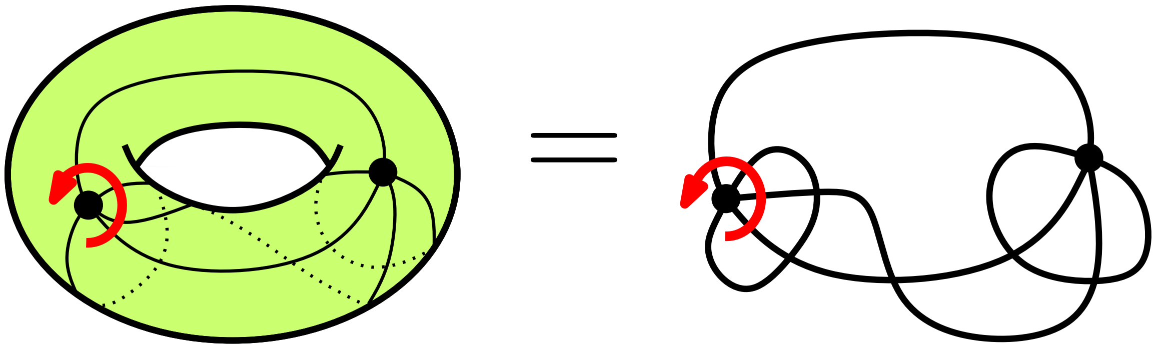

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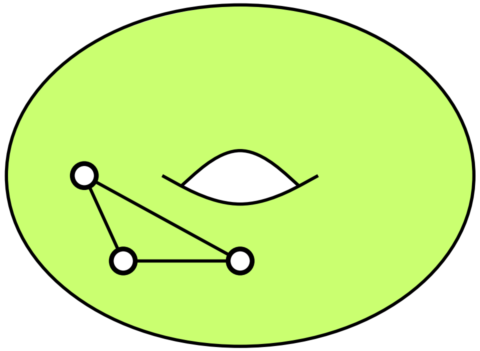

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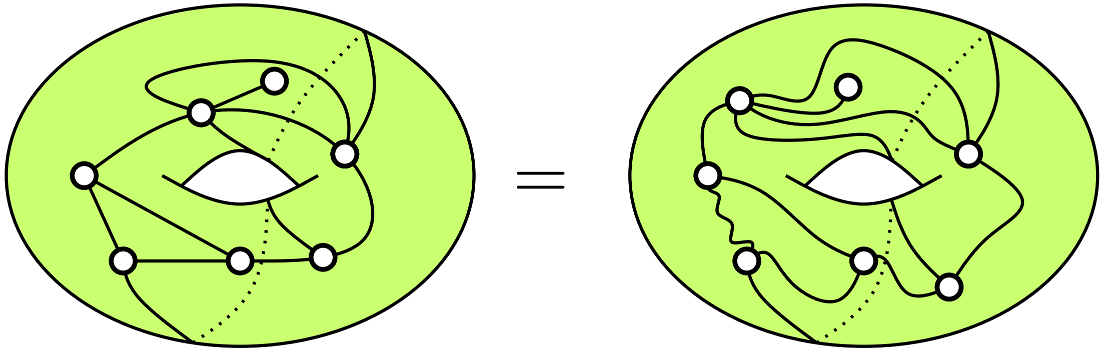

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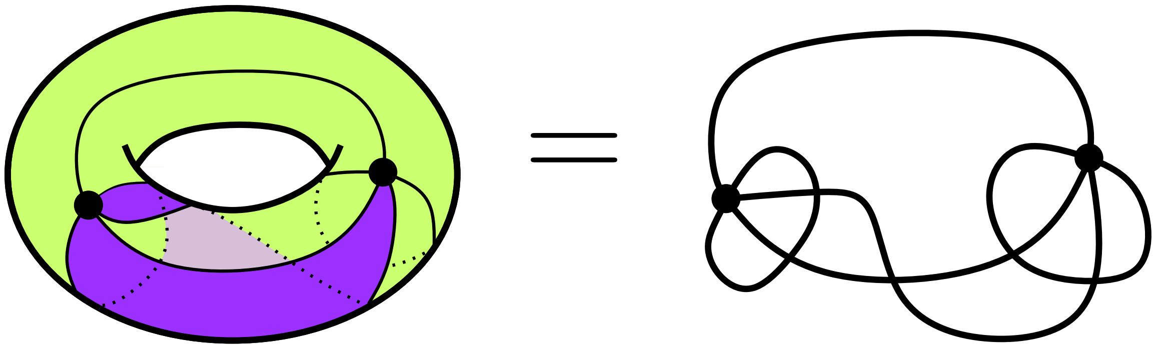

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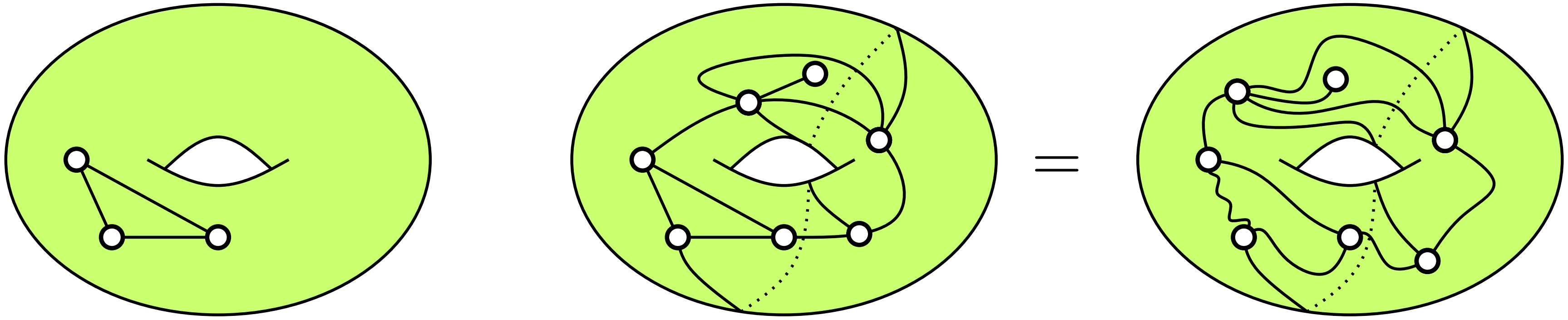

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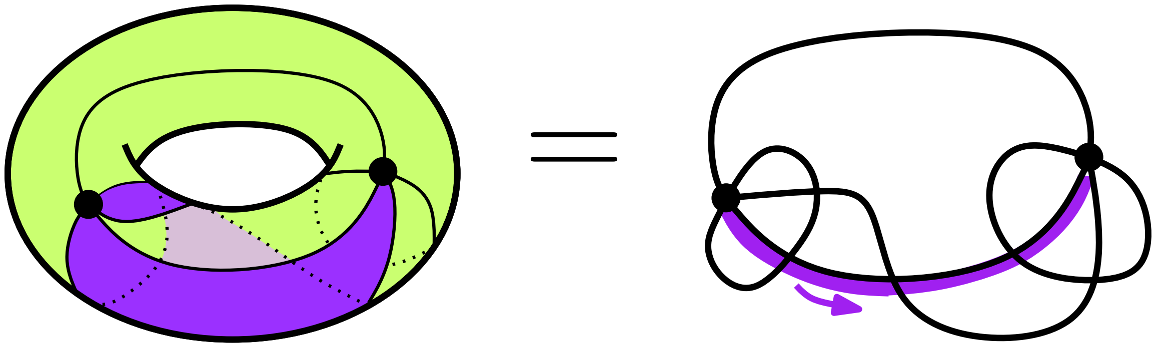

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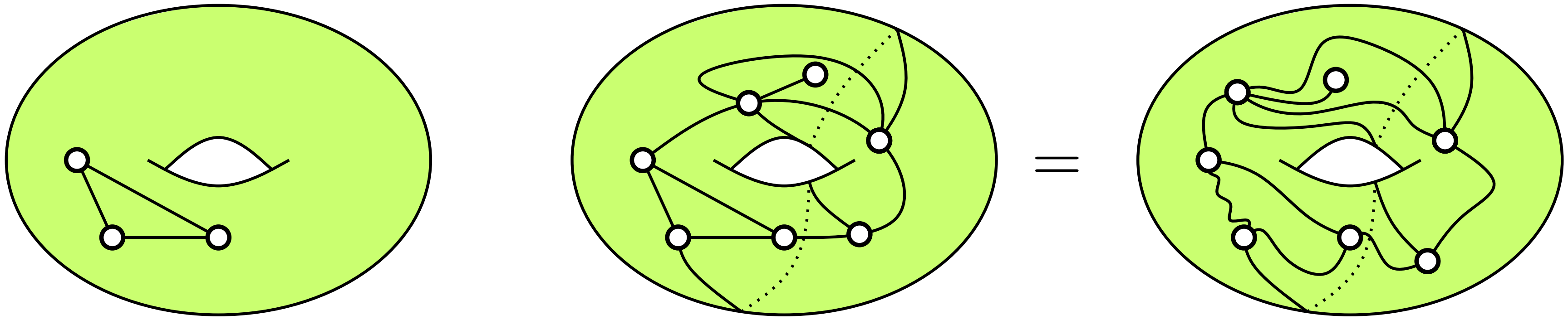

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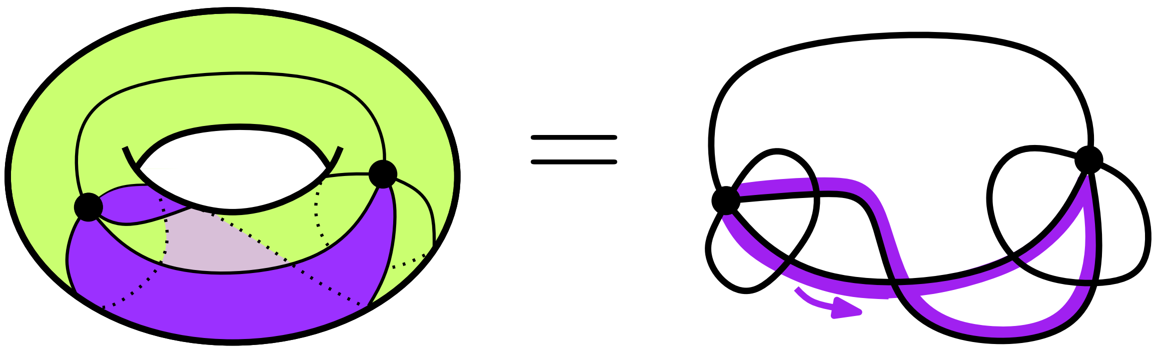

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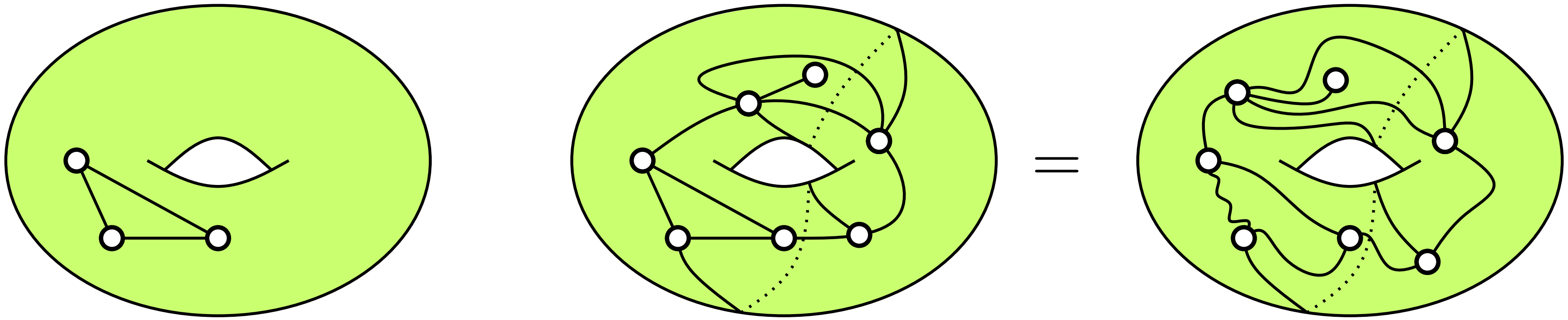

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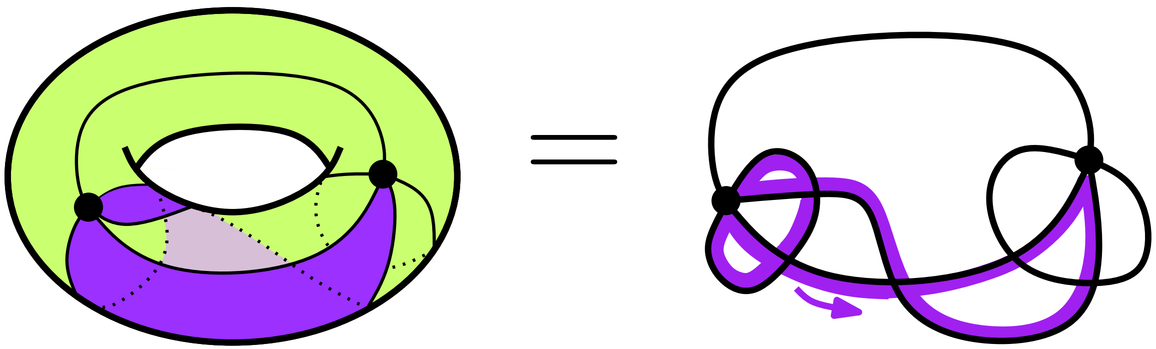

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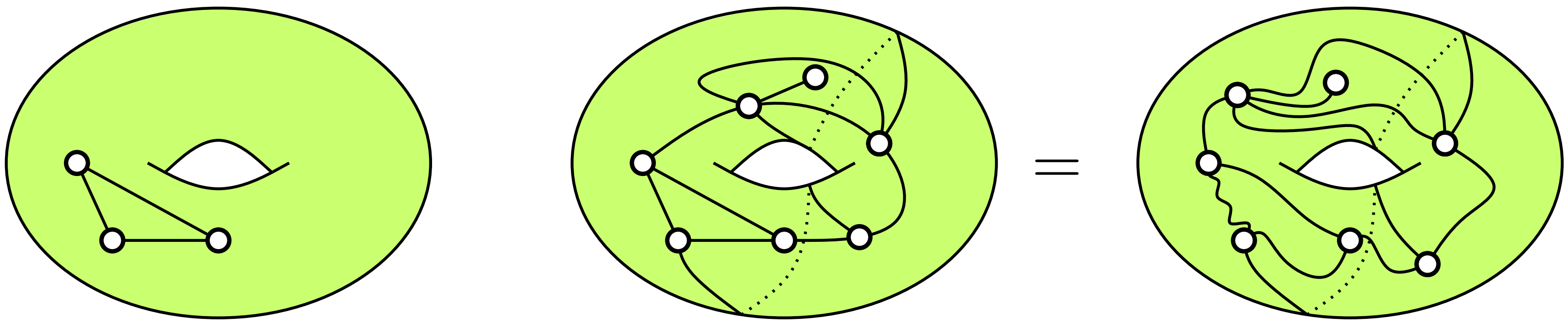

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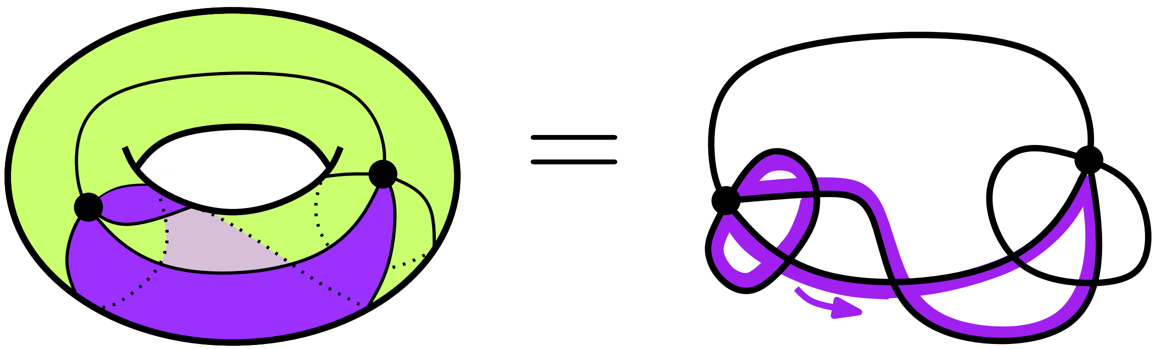

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Euler's formula gives the genus combinatorially:

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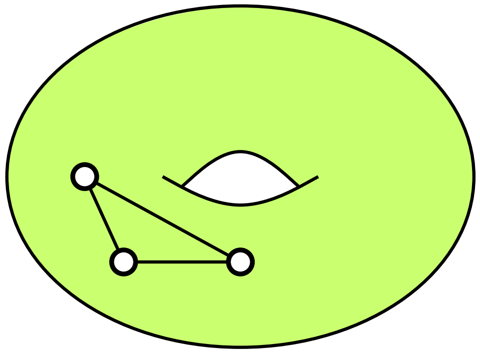

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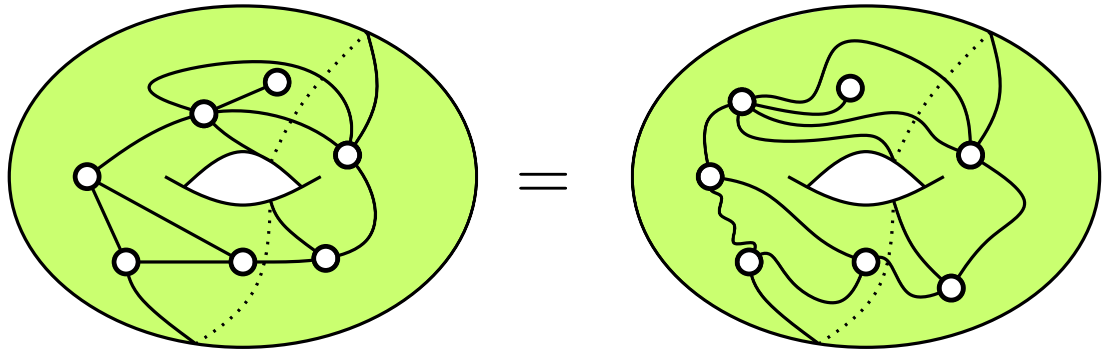

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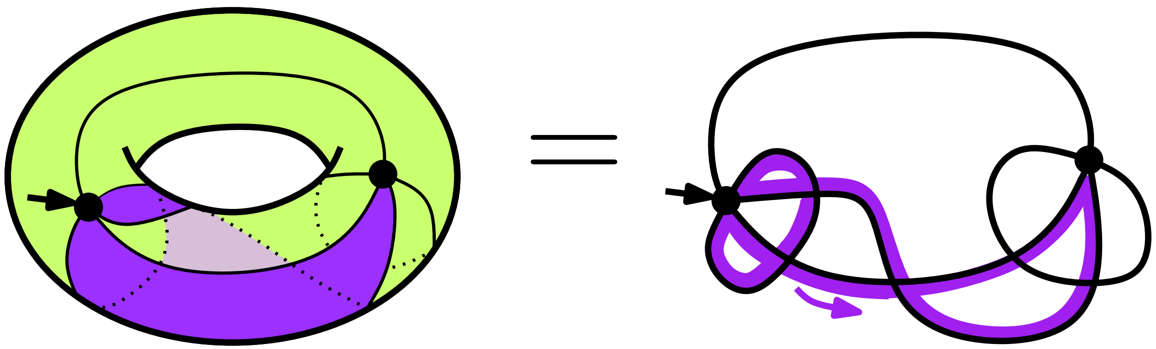

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Rooted map = a corner is distinguished

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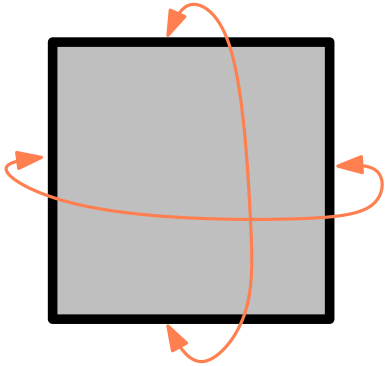

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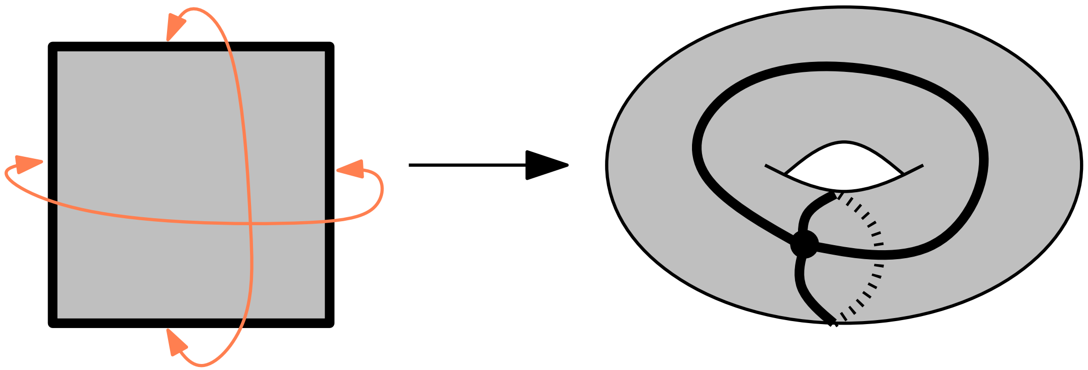

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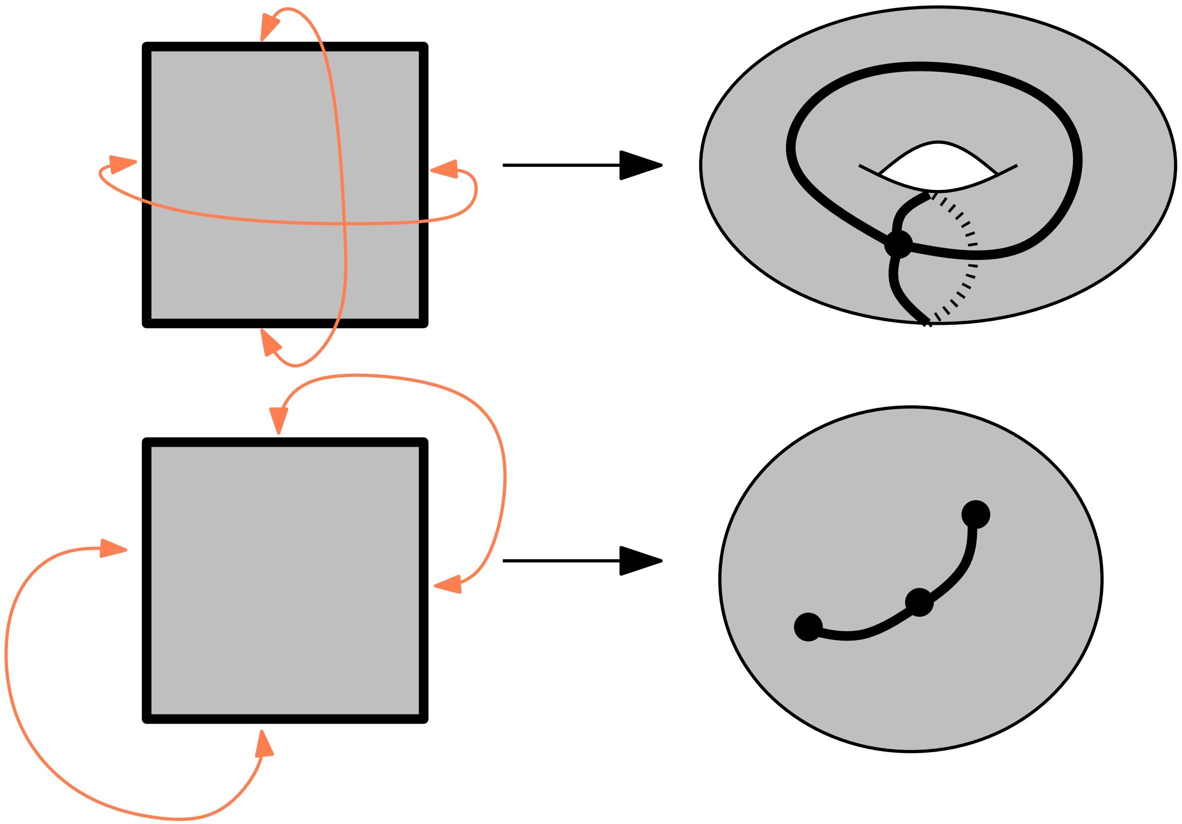

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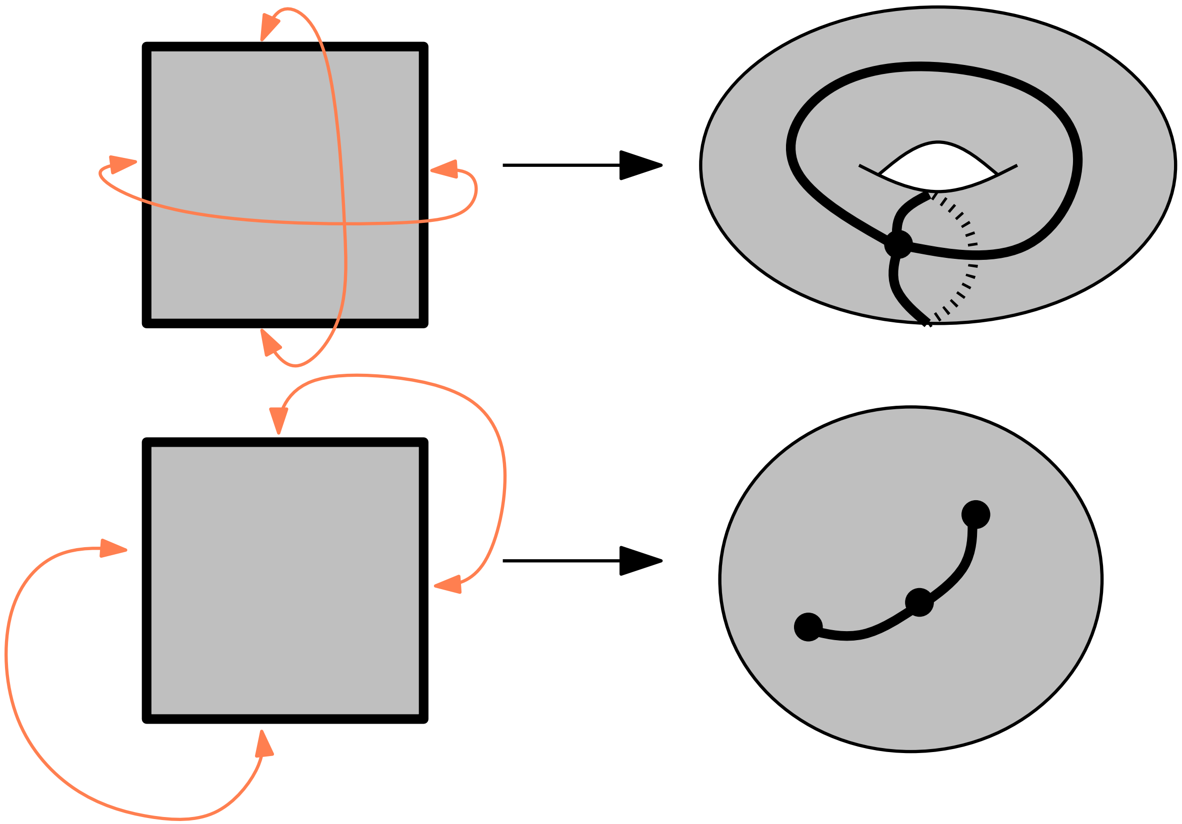

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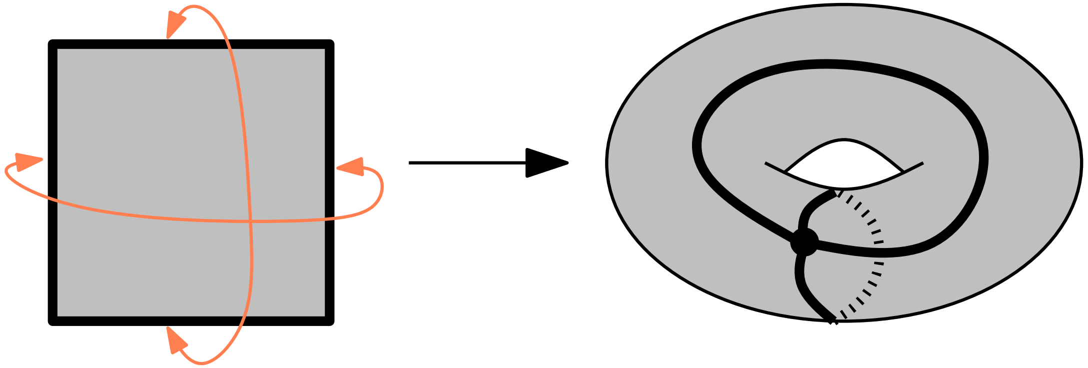
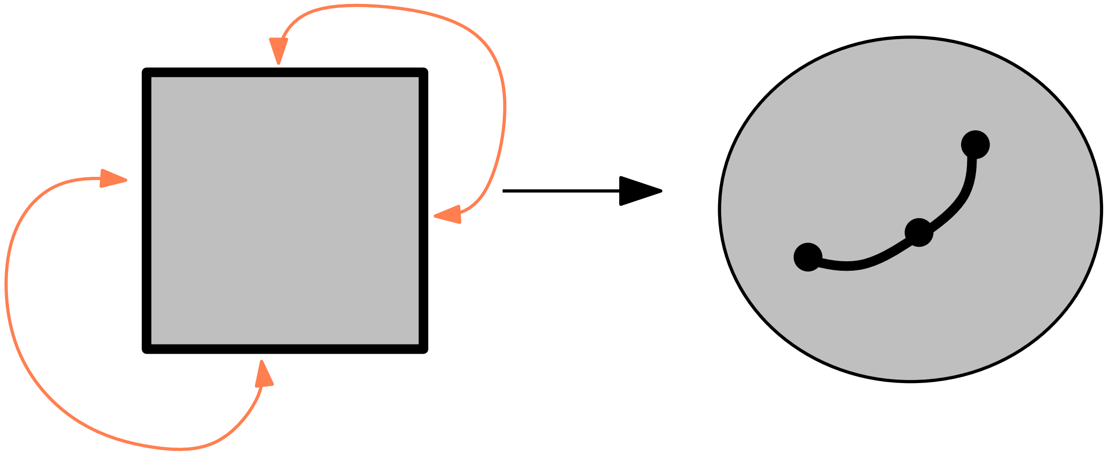


1 vertex, genus 1



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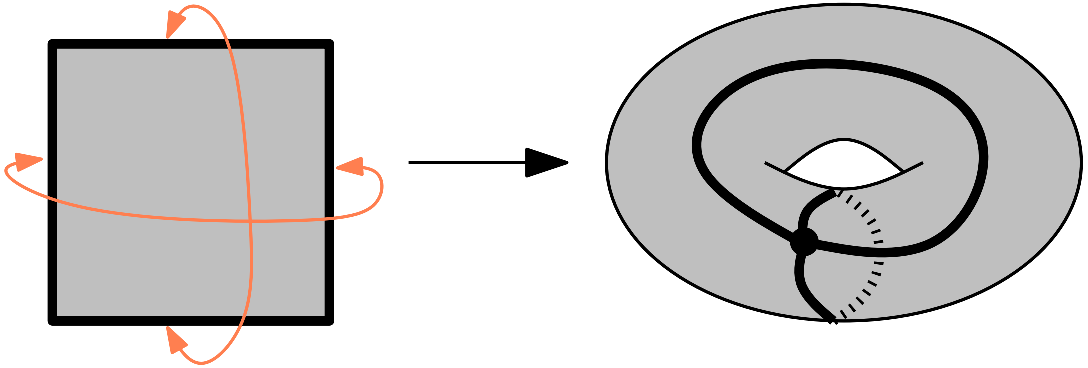

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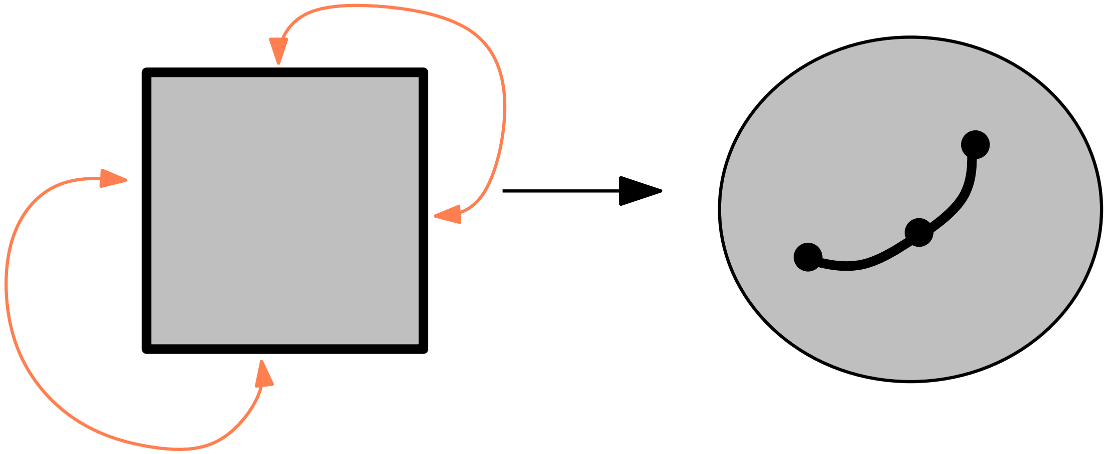

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Counting

The number of one-face maps with n edges is equal to the number of distinct matchings of the edges : $(2n-1)!! = \dfrac{(2n)!}{2^n n!}$.

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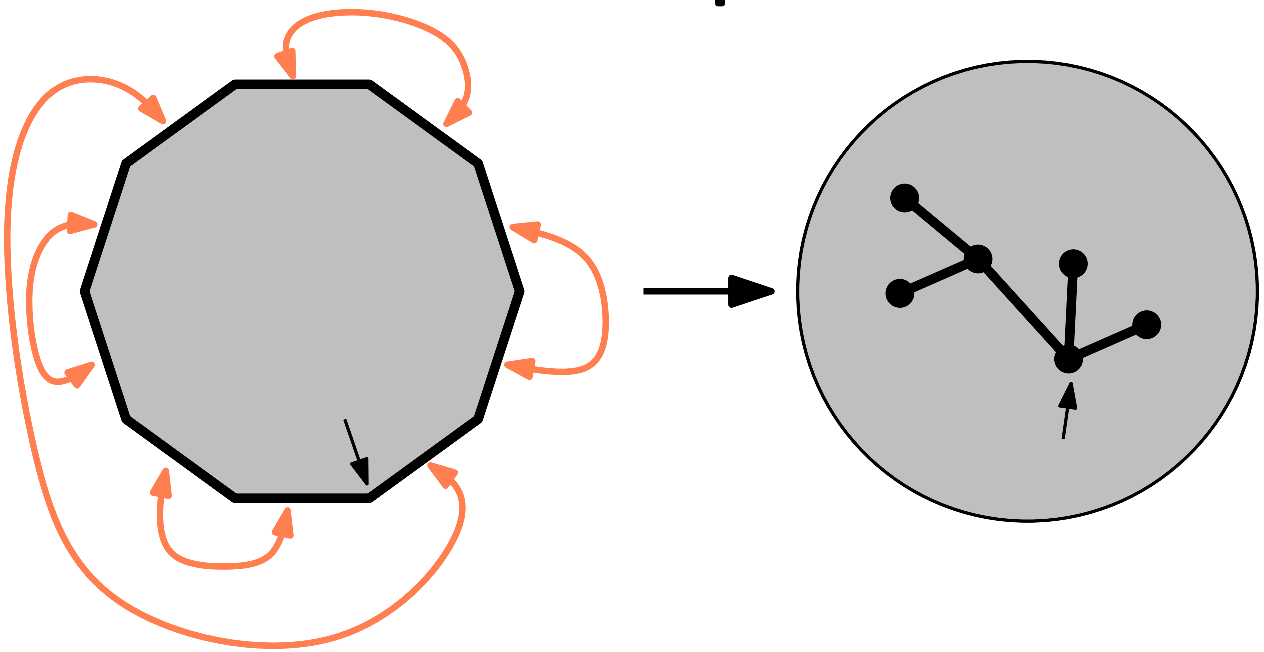
For instance, in the planar case...



One-face maps are exactly plane trees.

Therefore the number of n-edge one-face maps of genus 0 is :

$$\epsilon_0(n) = \mathrm{Cat}(n) = \frac{1}{n+1}\binom{2n}{n}$$

Higher genus surfaces ?

For each g the number of n-edge one-face maps of genus g has the (beautiful) form :

$$\epsilon_g(n) = \text{(some polynomial)} \times \mathrm{Cat}(n)$$

For instance :

$$\epsilon_1(n) = \frac{(n+1)n(n-1)}{12}\mathrm{Cat}(n)$$

$$\epsilon_2(n) = \frac{(n+1)n(n-1)(n-2)(n-3)(5n-2)}{1440}\mathrm{Cat}(n)$$

References : Lehman and Walsh 72 (formal power series), Harer and Zagier 86 (matrix integrals).

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No combinatorial interpretation !

For years people have tried to give an interpretation of the Harer-Zagier formula:

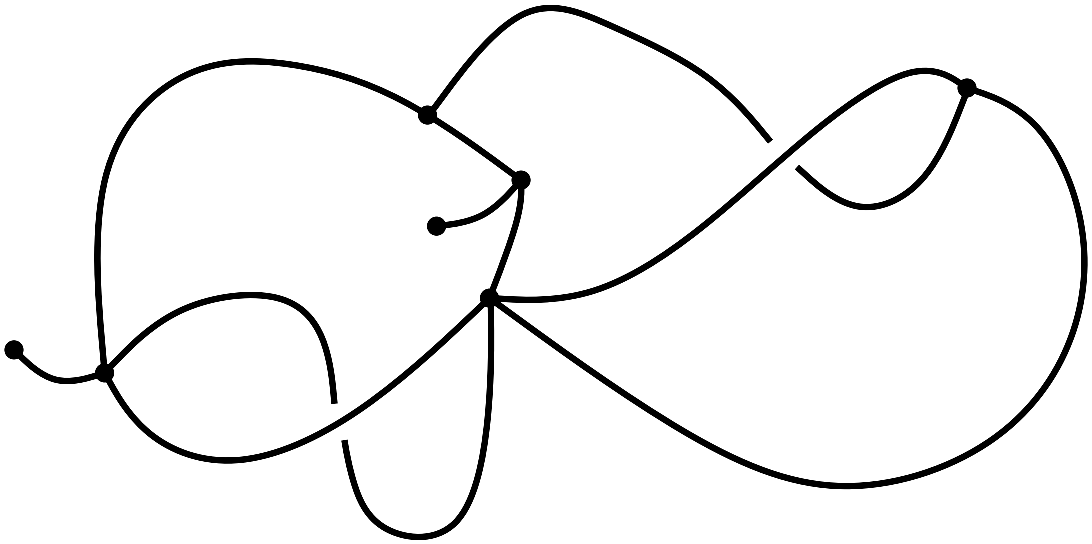
$$(n+1)\epsilon_g(n) = 2(2n-1)\epsilon_g(n-1) + (2n-1)(n-1)(2n-3)\epsilon_{g-1}(n-2)$$

Aim of the talk: discover and prove, with bijections, other kind of identities.

Trisections, and a bijection.

Numbering the corners.

We follow the border of the map starting from the root, and we number the corners from 1 to $2n$.



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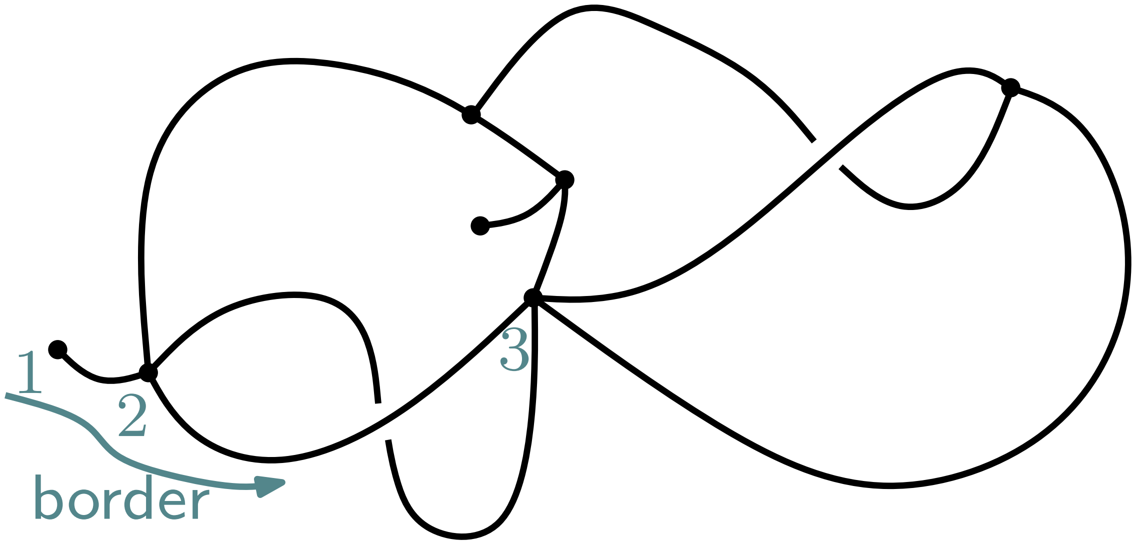

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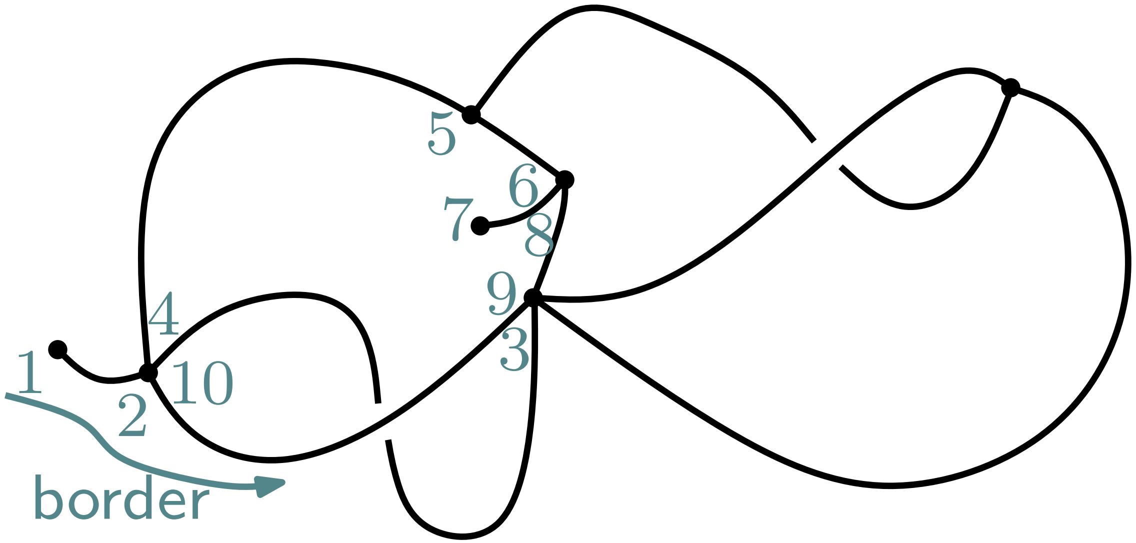

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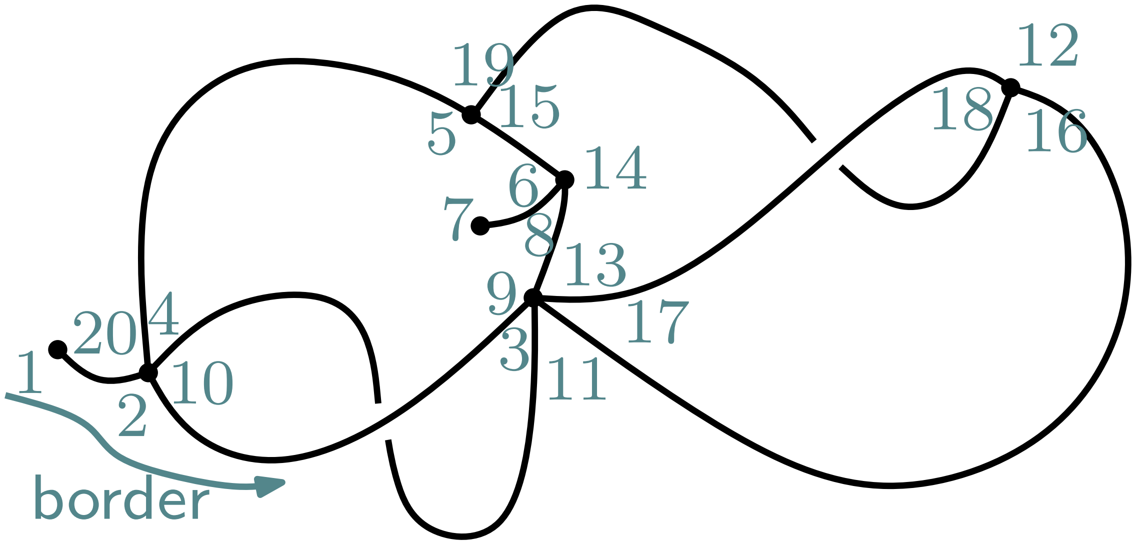

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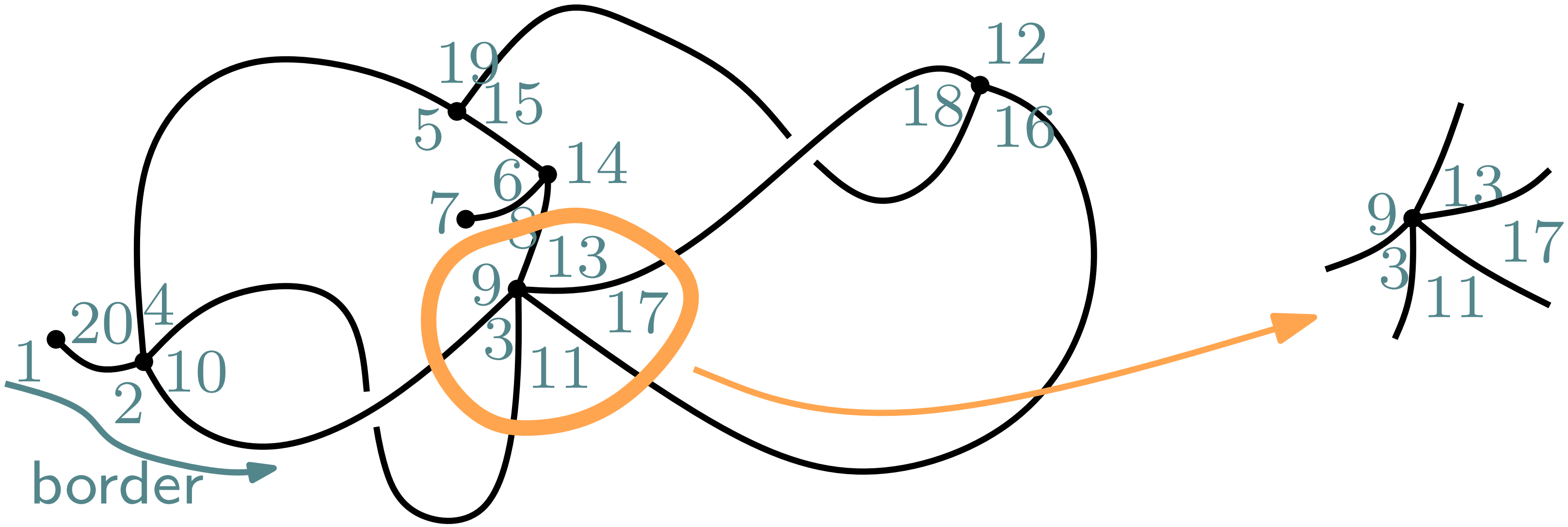

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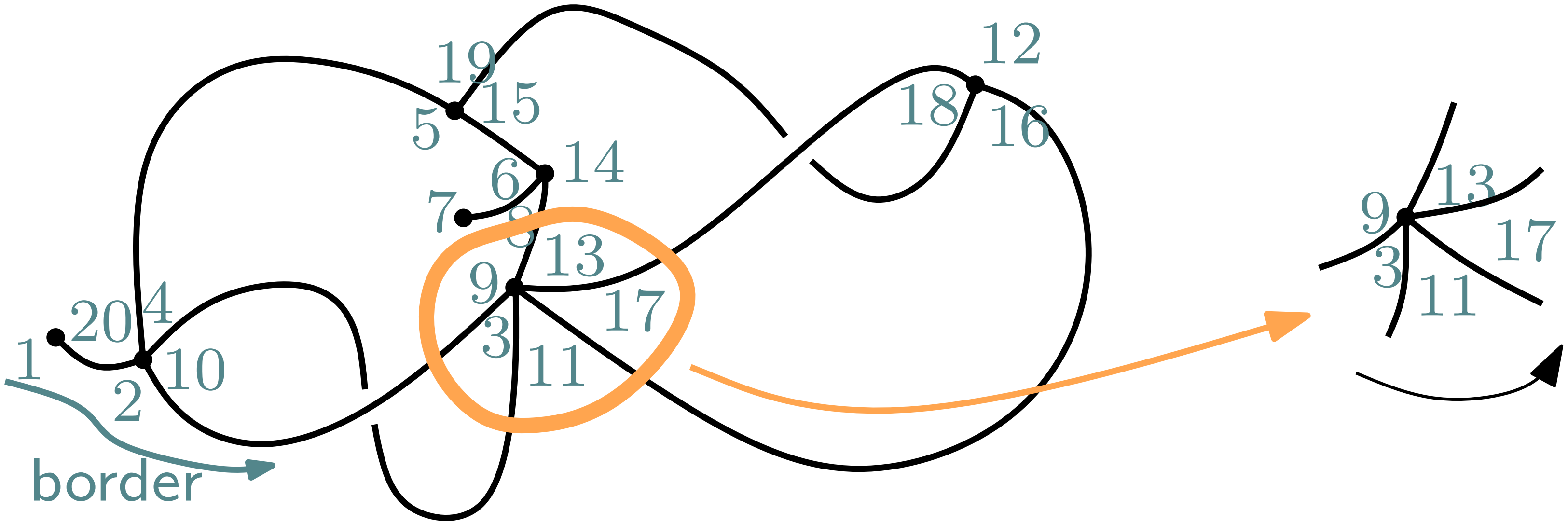

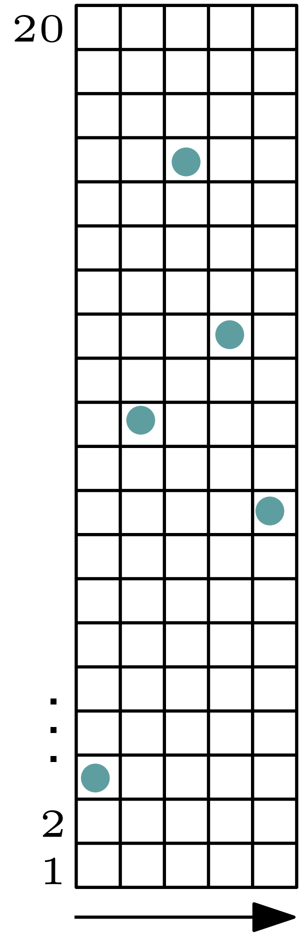

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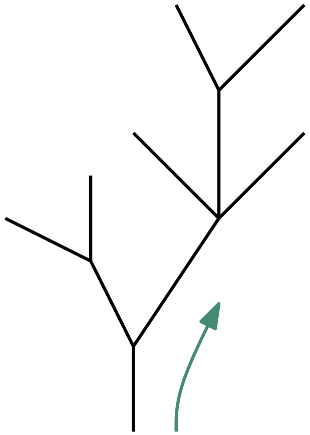
Planar case

In the planar case, the border-numbering and the cyclic ordering always coincide:



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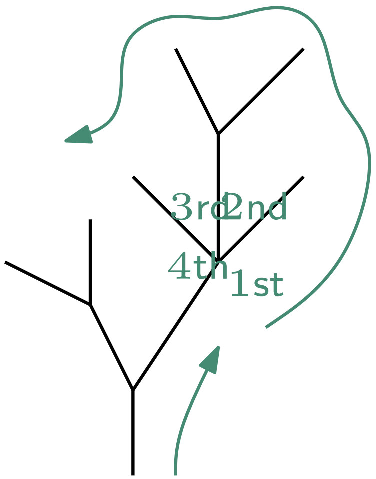

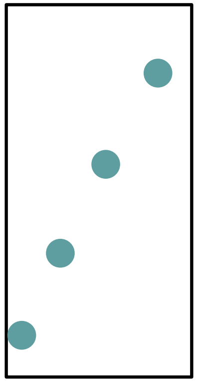

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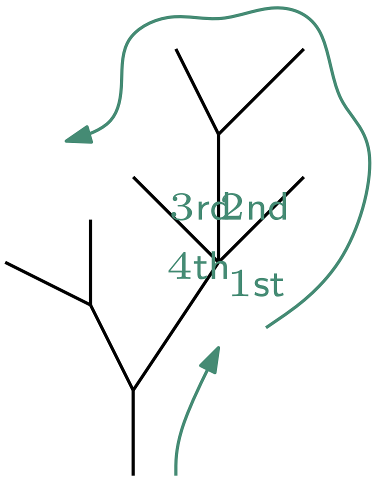
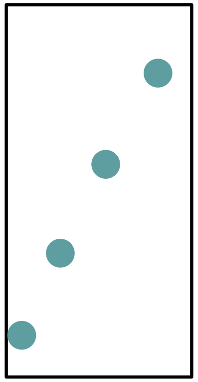
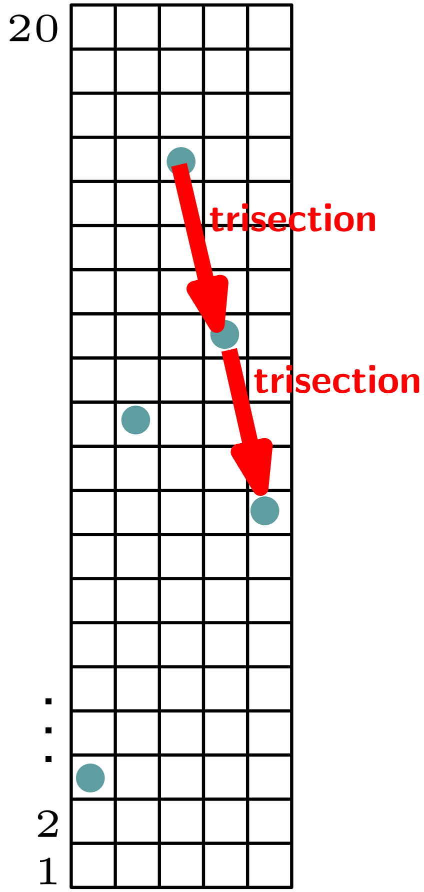
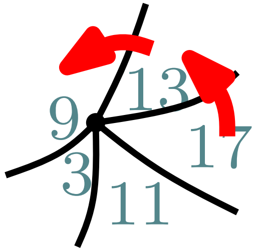






Higher genus

Around each vertex, a decrease in the diagram is called a trisection.



The trisection lemma

A one-face map of genus g always has exactly $2g$ trisections.

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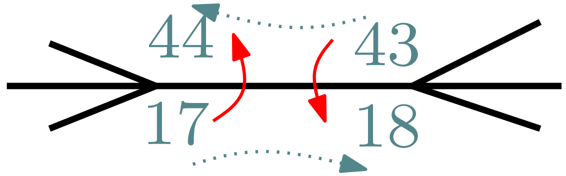
- each non-root edge contains exaclty one descent and one ascent.



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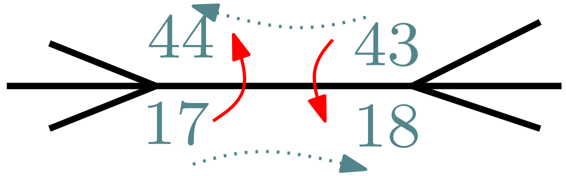

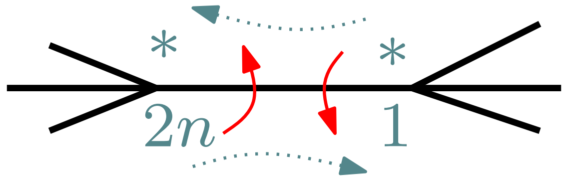

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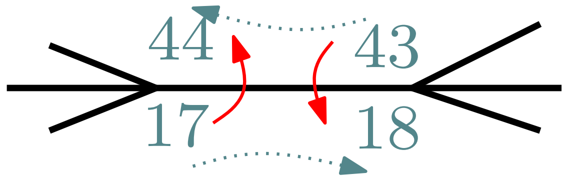

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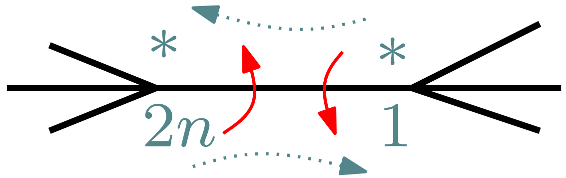

- hence there are $(n-1)+2 = n+1$ descents in total.
- but each vertex contains one descent which is not a trisection:

$$\# \text{ trisections} = (\# \text{ descents}) - (\# \text{ vertices})$$
$$= (n+1) - (n+1-2g) \quad \text{QED.}$$

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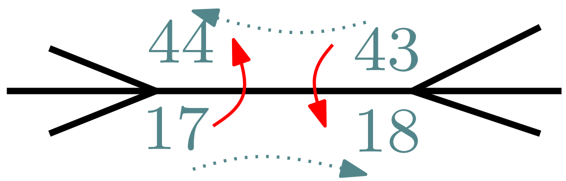

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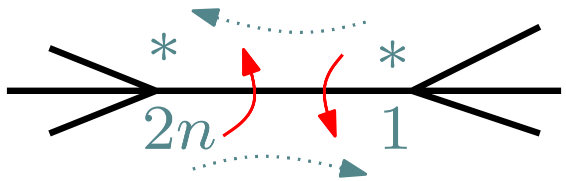

- but each vertex contains one descent which is not a trisection:

$$\# \text{ trisections} = (\# \text{ descents}) - (\# \text{ vertices})$$
$$= (n+1) - (n+1-2g) \quad \text{QED.}$$

→ It is an equivalent problem to count one-face maps with a distinguished trisection.

How to build a trisection : first method.

- Start with a map of genus $(g-1)$ with three marked vertices.
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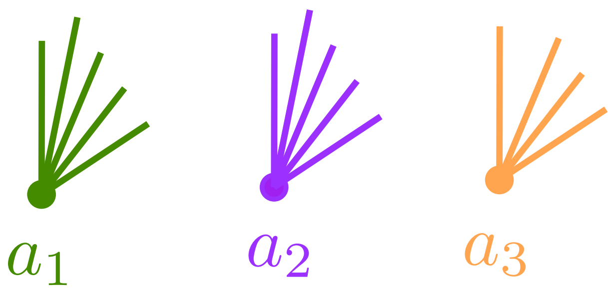

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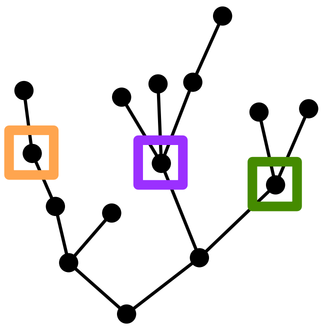

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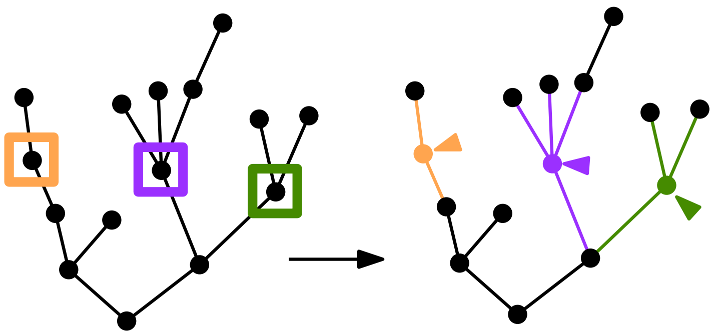

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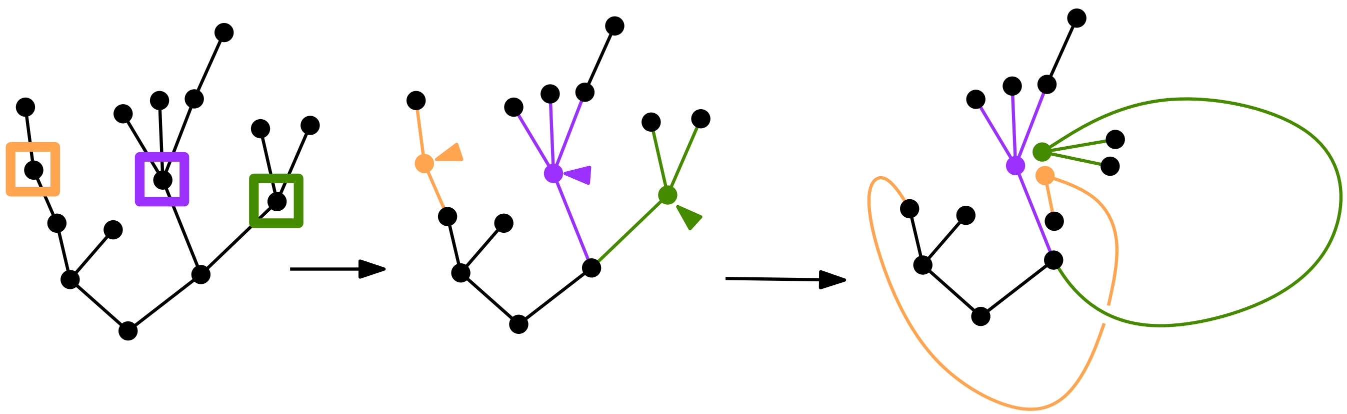

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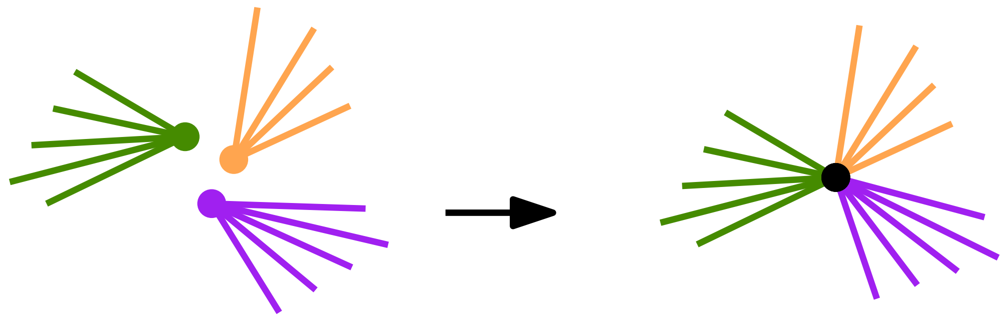

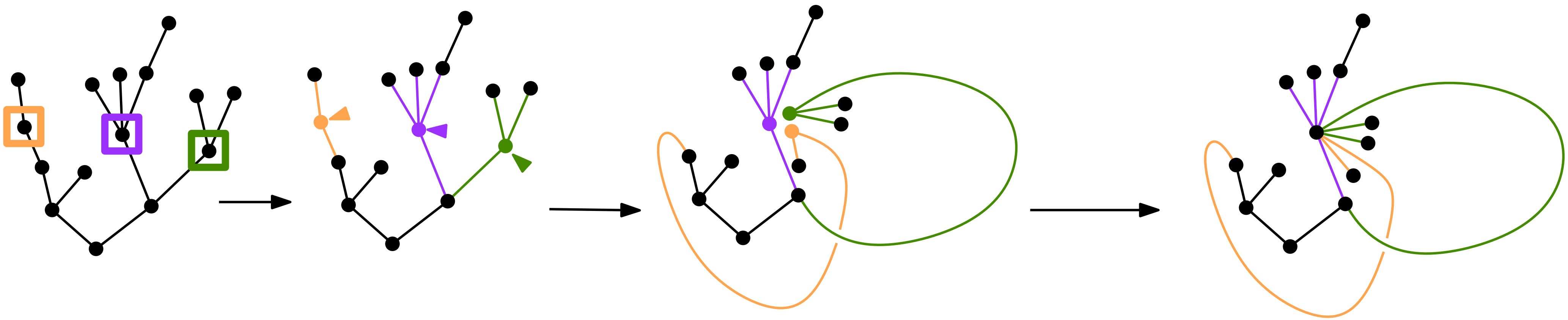

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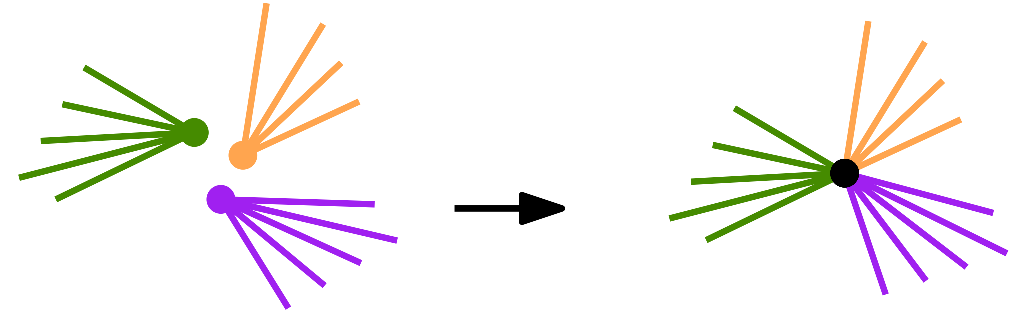
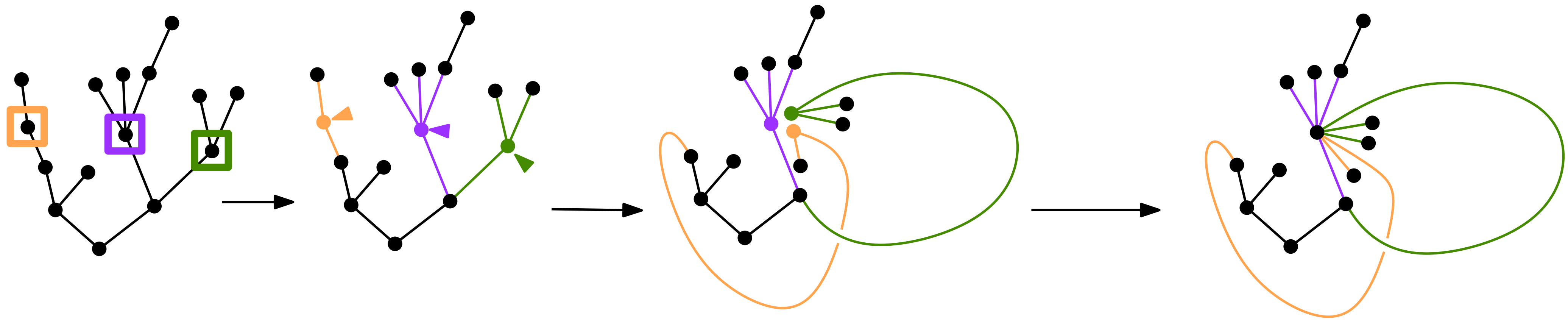


- The resulting map has only one border :

$$1 \to 2 \to \dots \to a_1 \to \dots \to a_2 \to \dots \to a_3 \to \dots \to 2n$$



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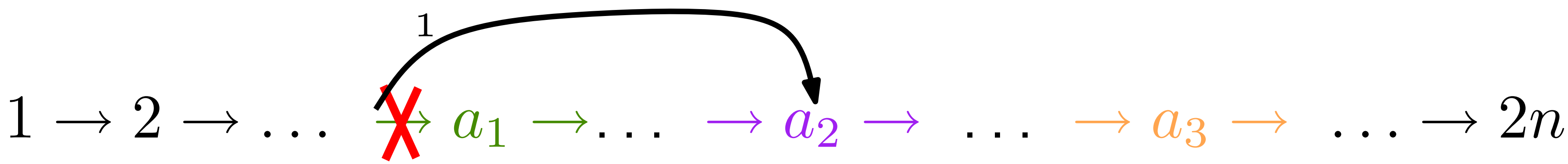

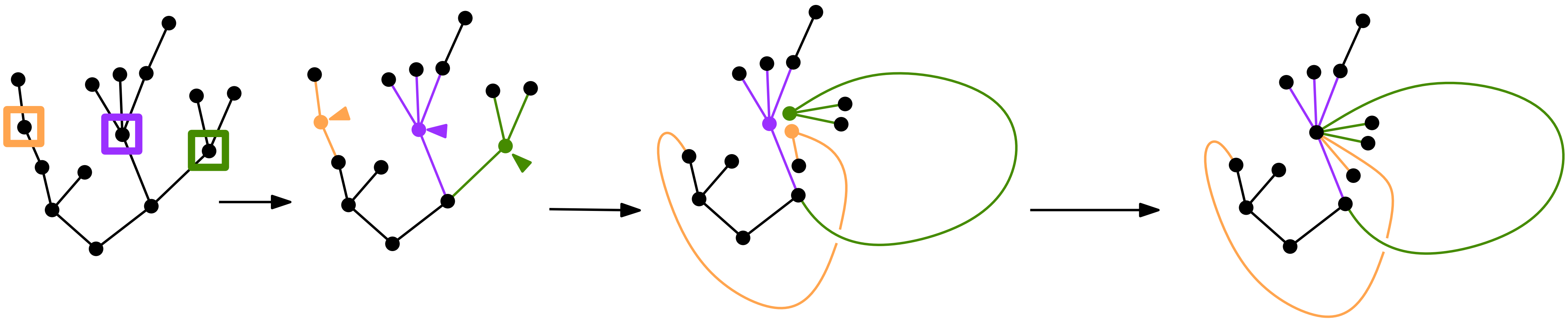

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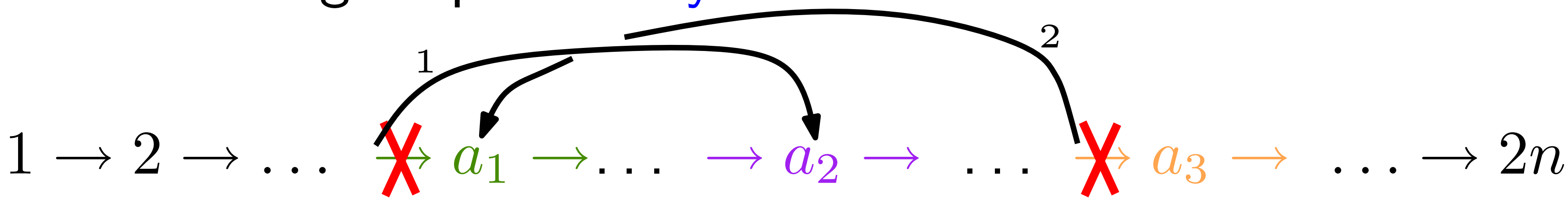

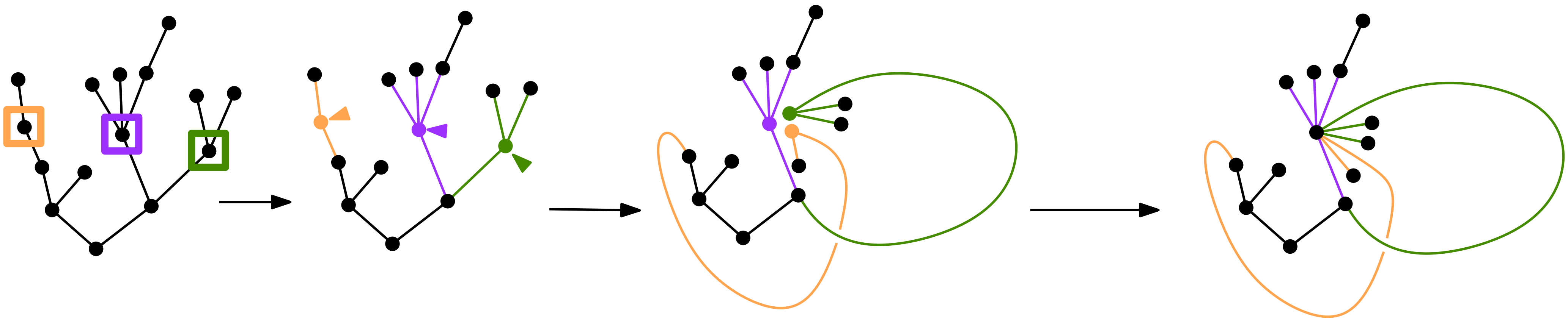

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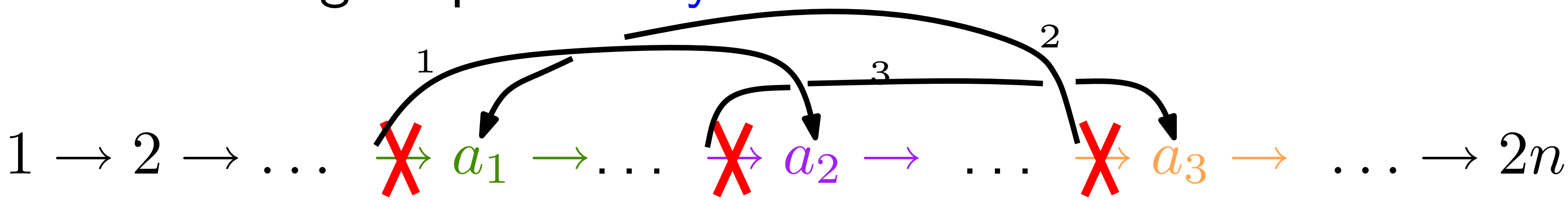

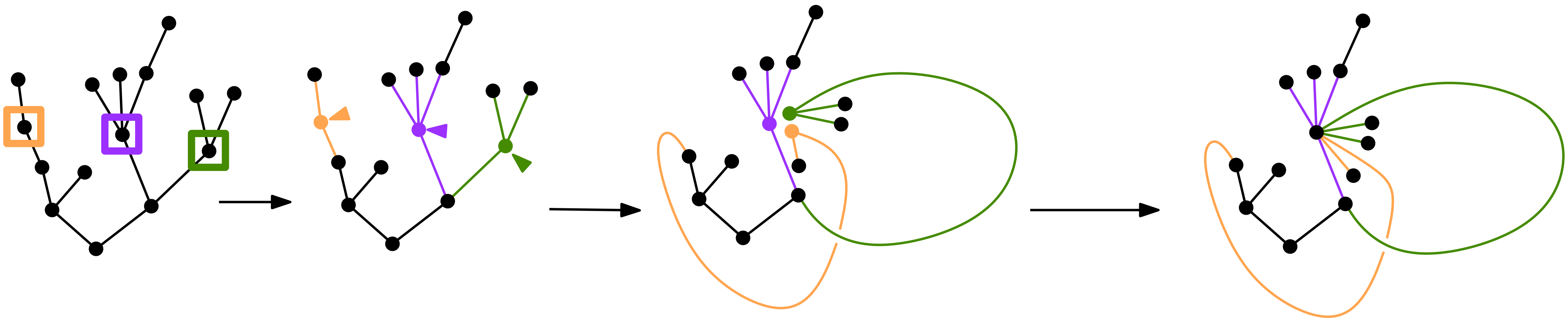

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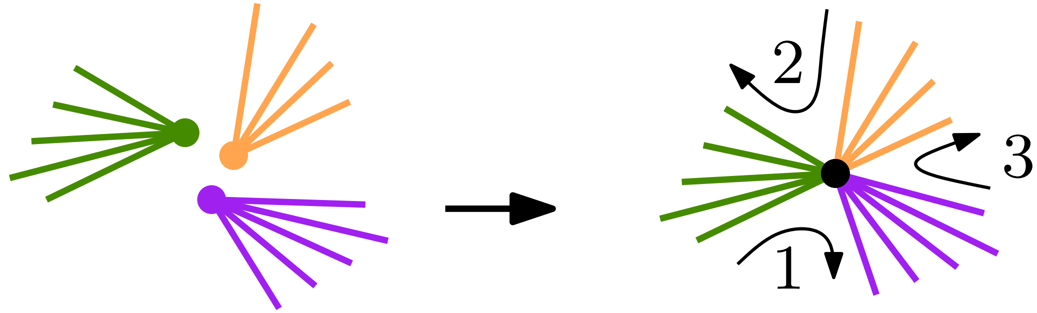

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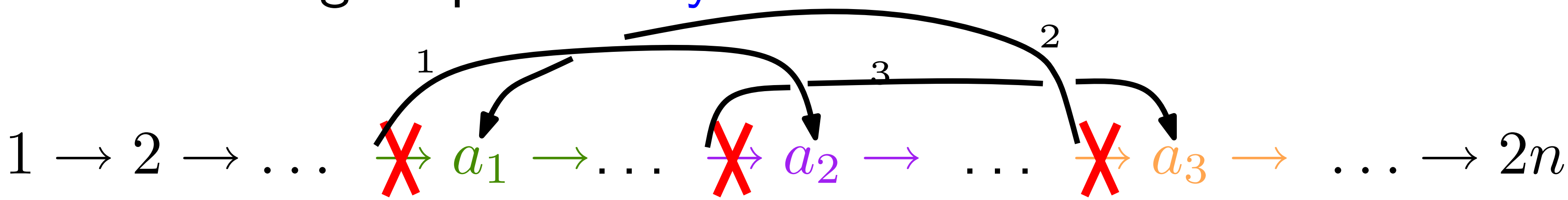

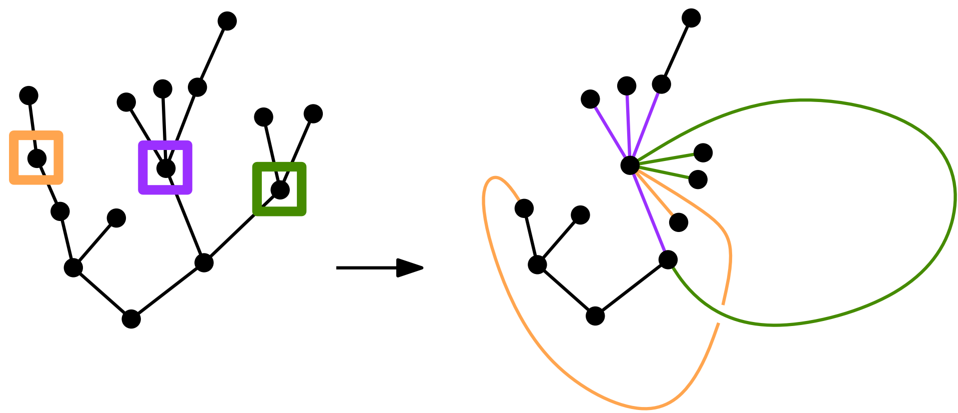

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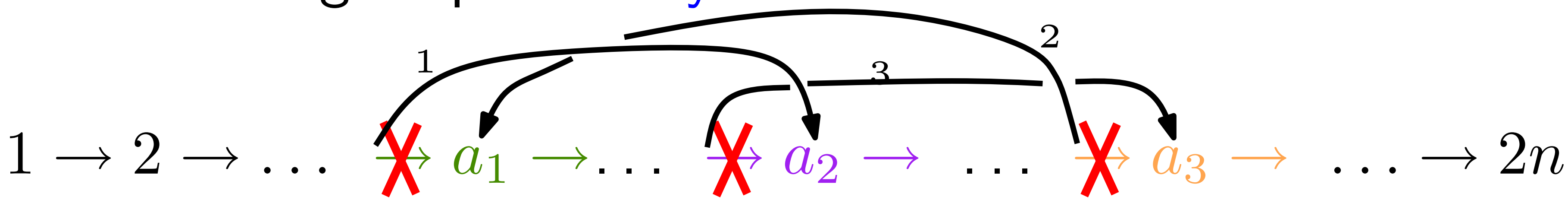
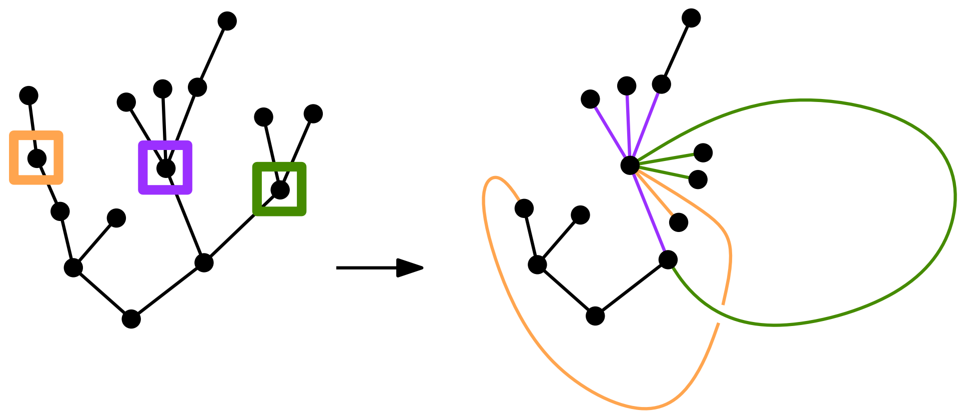


- By Euler's formula, it has genus g.



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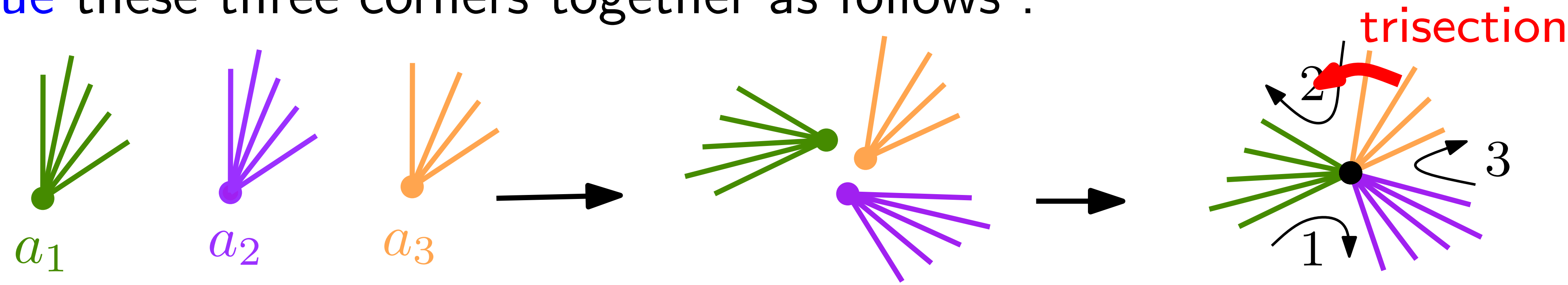

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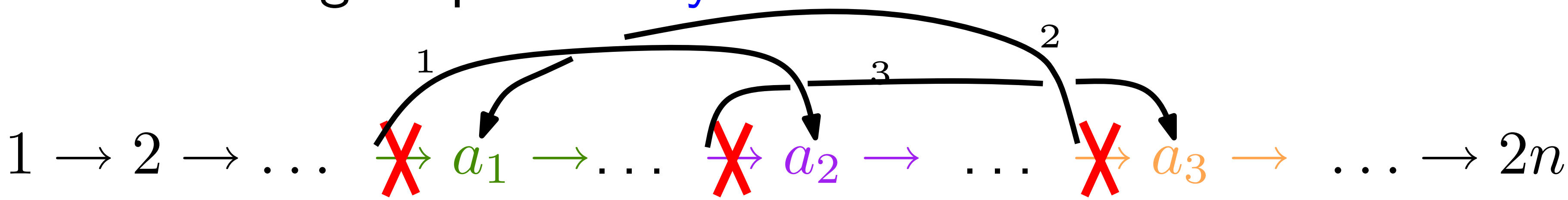
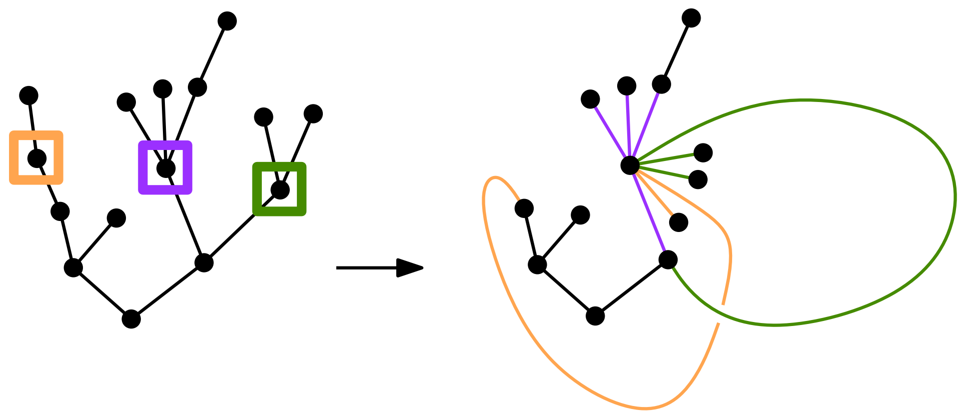
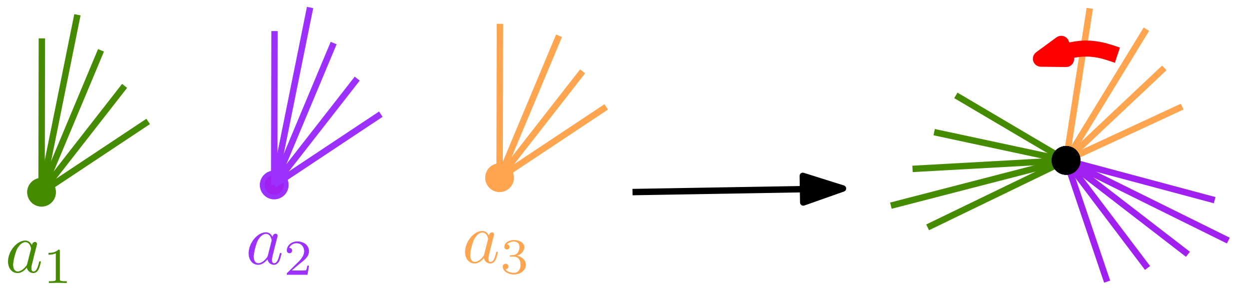


- By Euler's formula, it has genus g.
- Moreover we have built a trisection.



Therefore we have a mapping :



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genus g, one marked trisection

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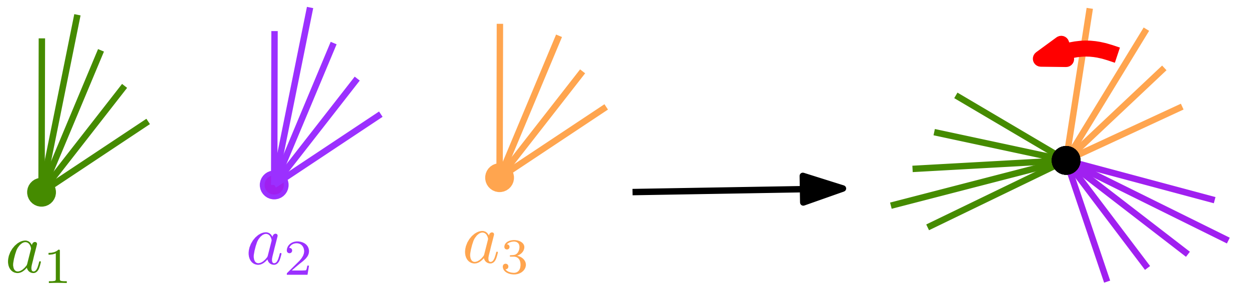

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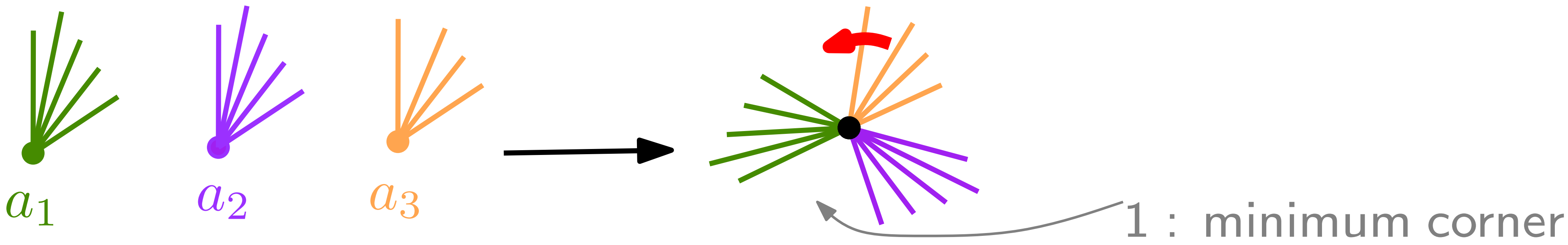

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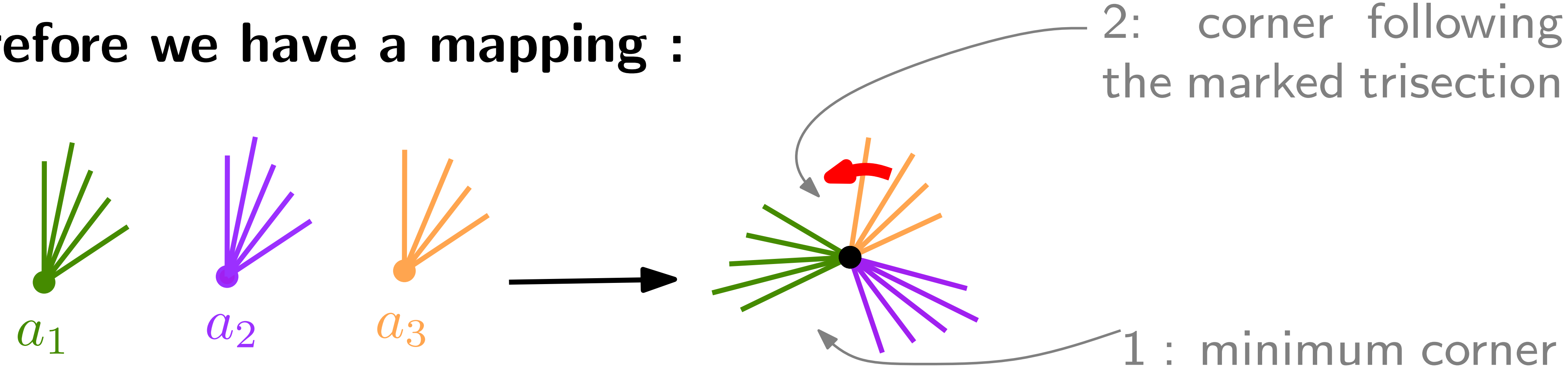

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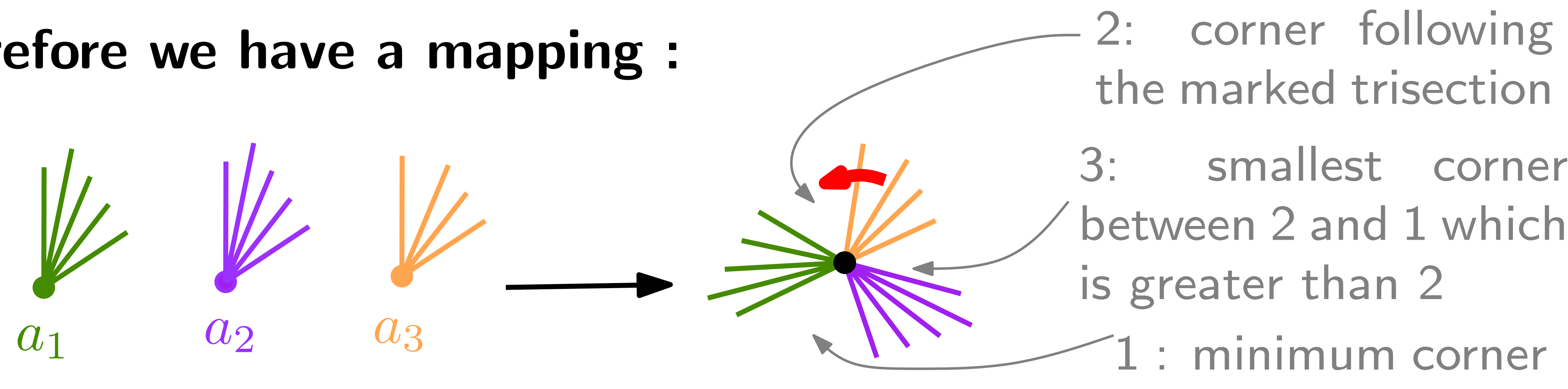

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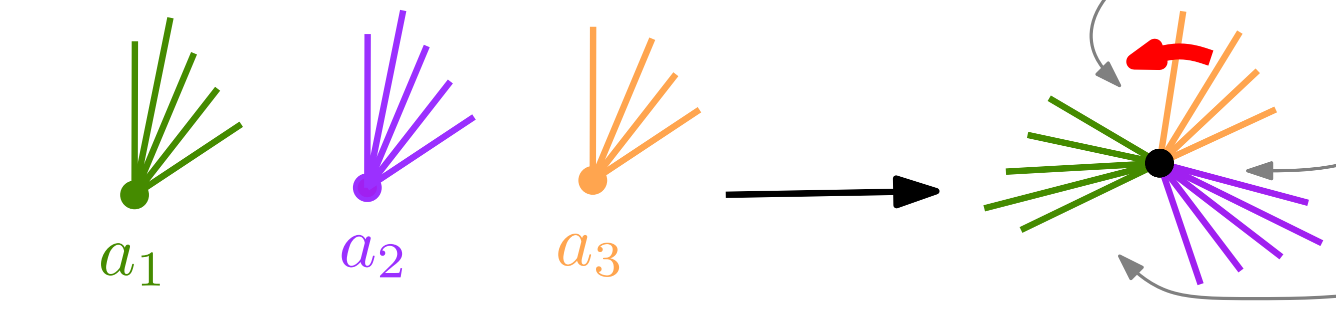

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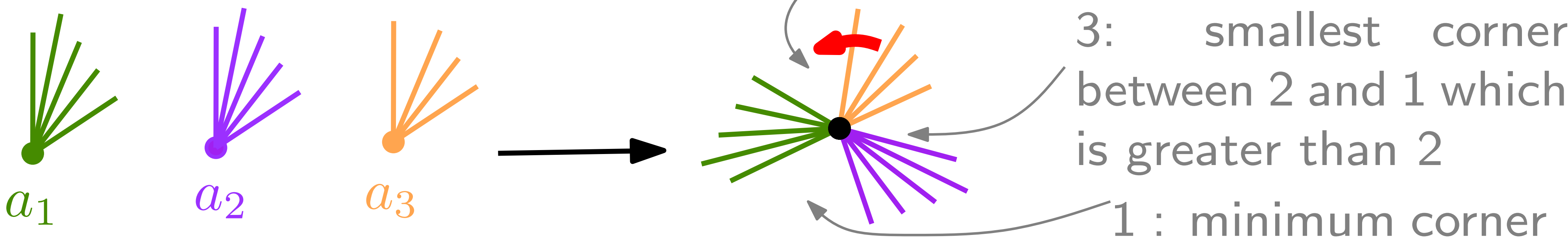

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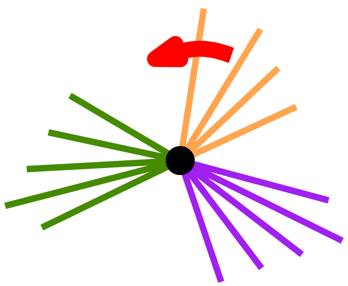

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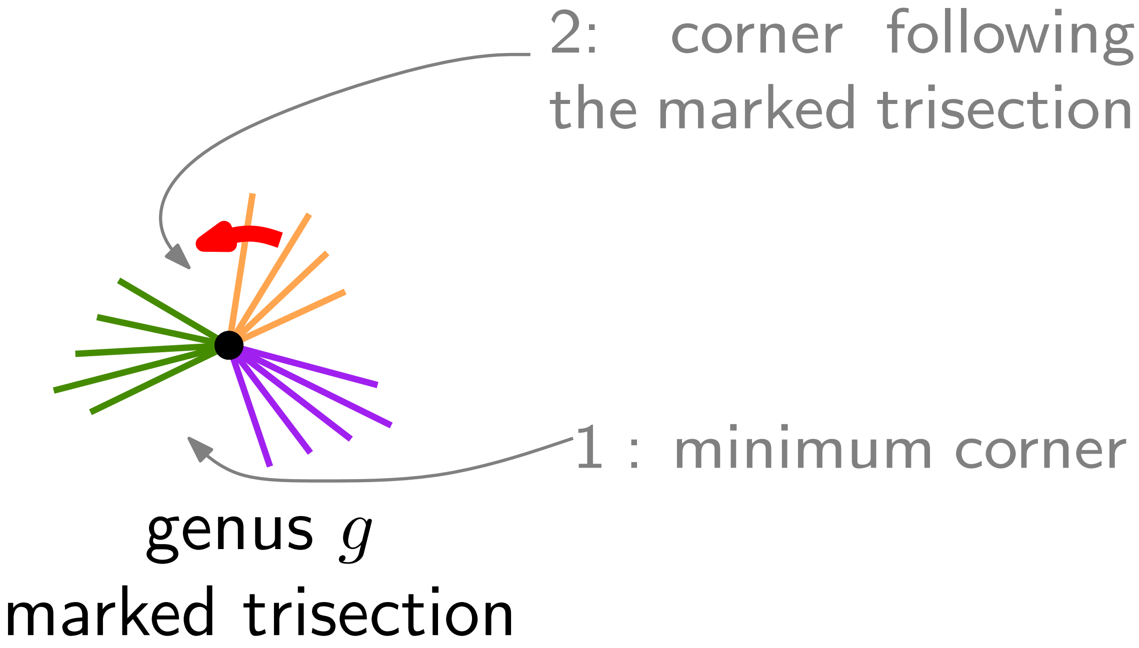

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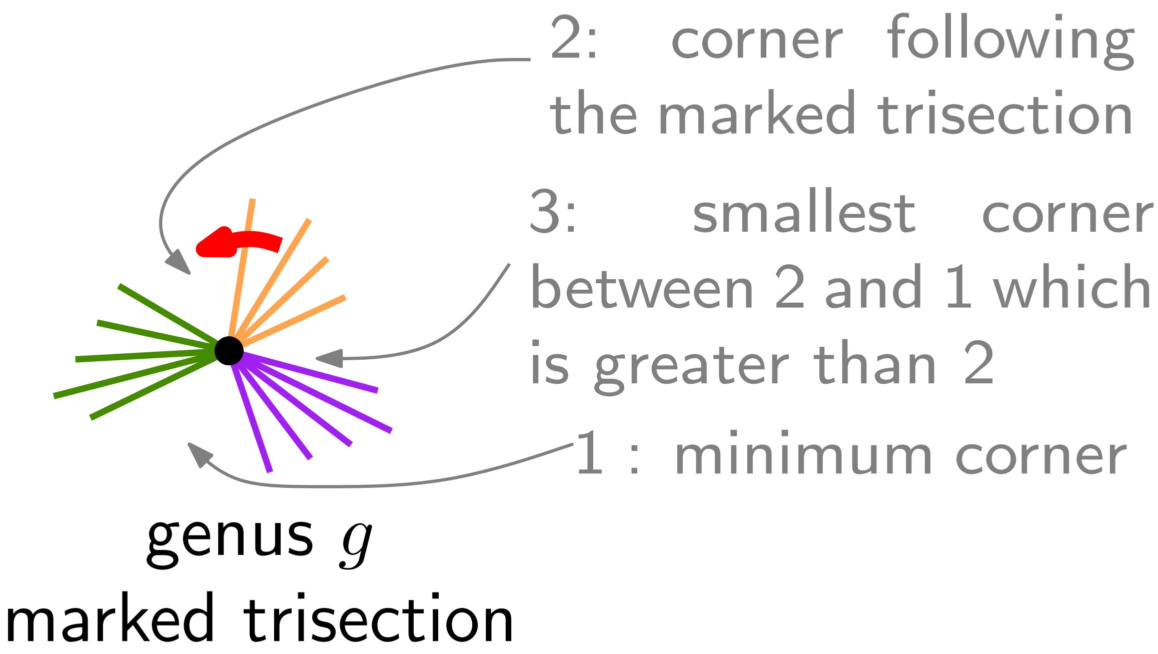

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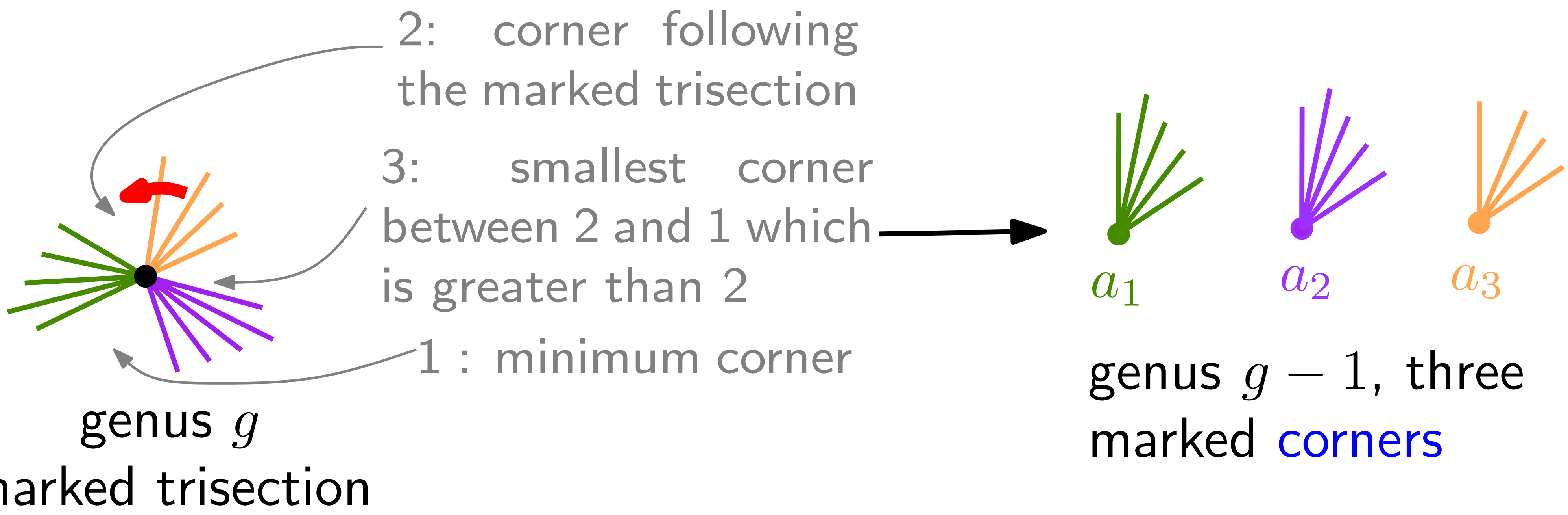

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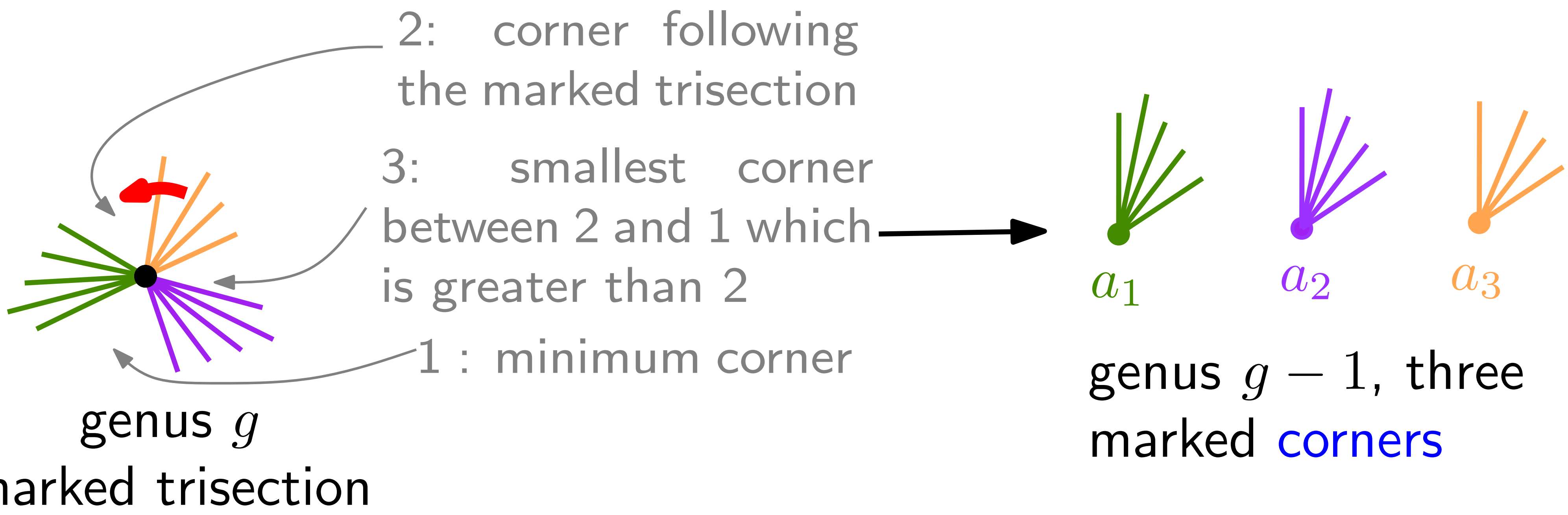

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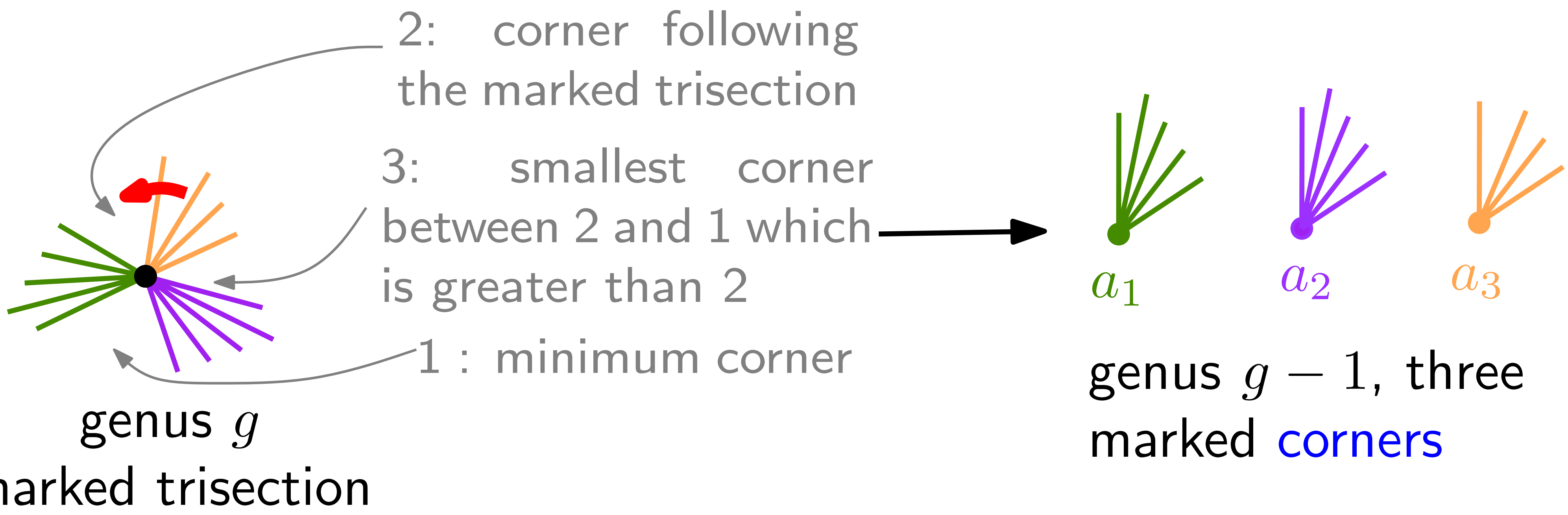

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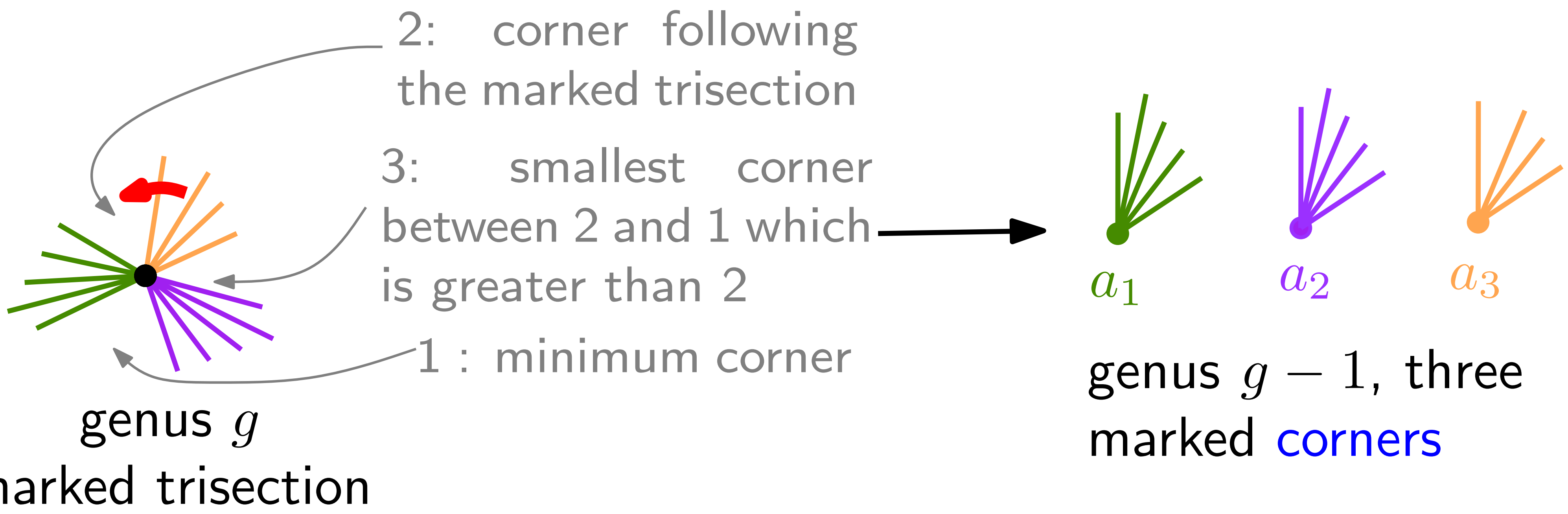

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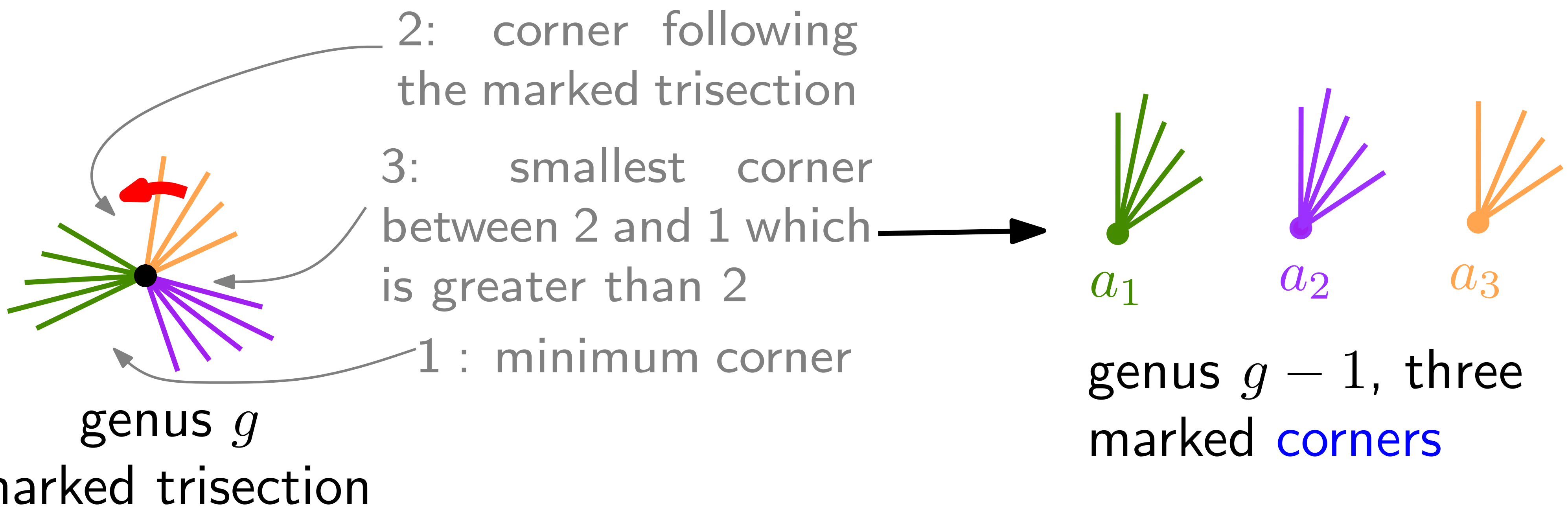

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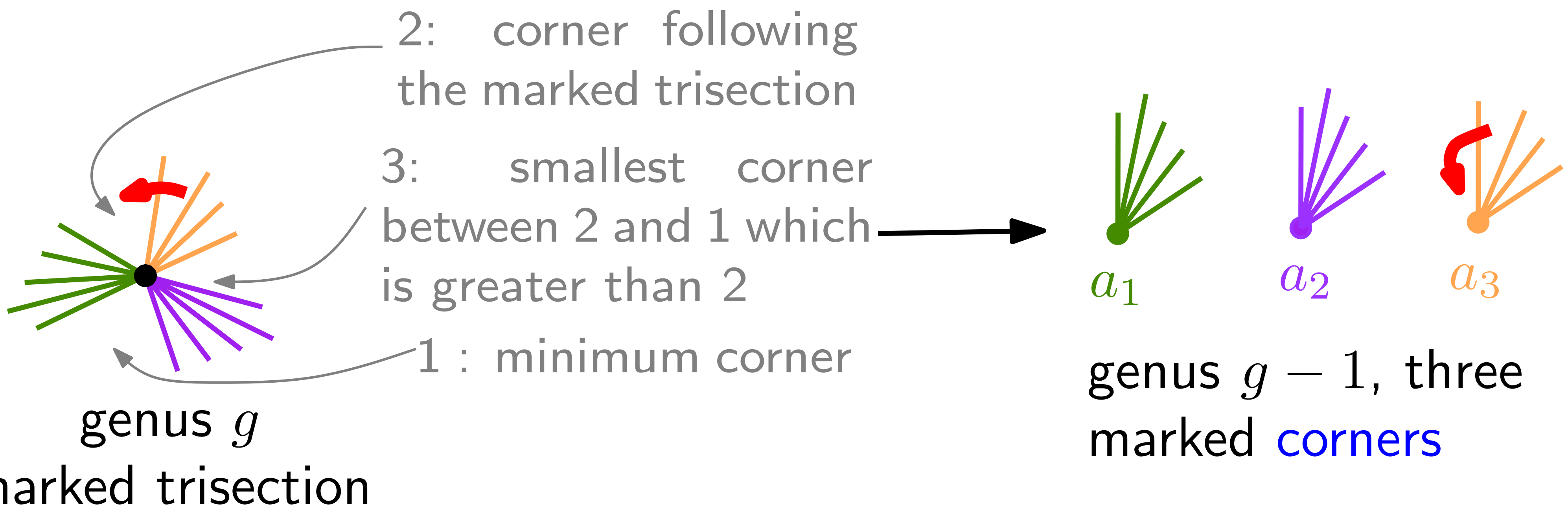

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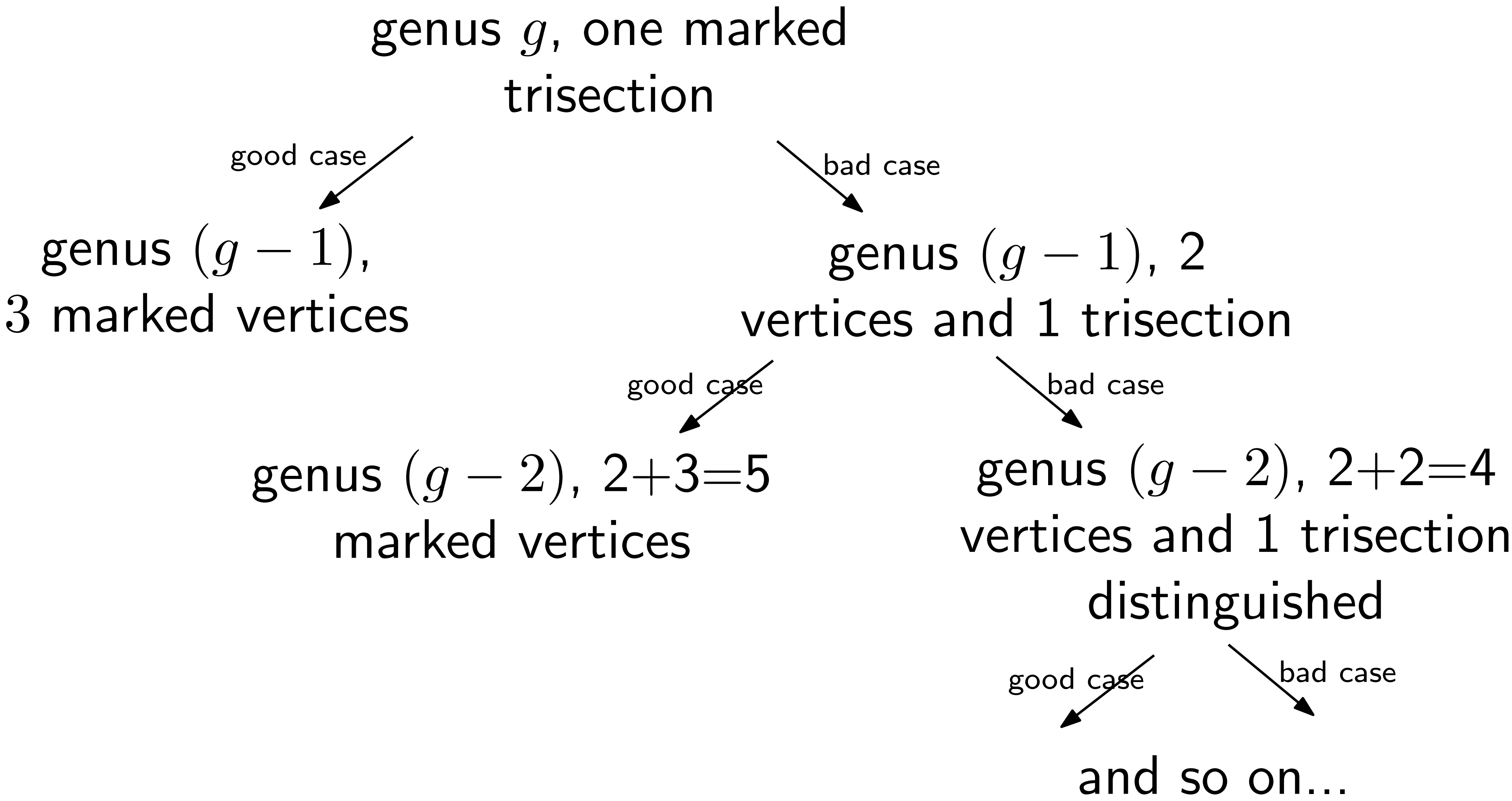
- This is not always the case for a_3 :
  - If a_3 is the minimum of its vertex : we are in the image of the previous construction.
  - Else a_3 is incident to a trisection of the map of genus $(g-1)$.

Therefore :

genus g, one marked trisection

- good case → genus $(g-1)$, 3 marked vertices
- bad case → genus $(g-1)$, 2 vertices and 1 trisection

Therefore :


genus g, one marked trisection
good case
bad case
genus $(g-1)$, 3 marked vertices
genus $(g-1)$, 2 vertices and 1 trisection
good case
bad case
genus $(g-2)$, 2+3=5 marked vertices
genus $(g-2)$, 2+2=4 vertices and 1 trisection distinguished
good case
bad case
and so on...

Hence we have a bijection:

$$\begin{matrix}\text{genus } g, \\ \text{one marked trisection}\end{matrix} \overset{\text{bij.}}{=\!=} \bigcup_{i>0} \begin{pmatrix}\text{genus } g-i \text{ and } 2i+1 \\ \text{marked vertices.}\end{pmatrix}$$

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Everything boils down to plane trees:

$$\epsilon_g(n) = \underbrace{(\text{some polynomial})}\times \mathrm{Cat}(n)$$

= "number" of possibilities for the successive choices of vertices.

$$= \sum_{0=g_0<g_1<\ldots<g_r=g} \prod_{i=1}^{r} \frac{1}{2g_i}\binom{n+1-2g_{i-1}}{2(g_i-g_{i-1})+1}$$

A special case:

A map is precubic if all its vertices have degree 1 or 3.
(always rooted at a vertex of degree one).

In the planar case, precubic maps are planted binary trees, and the number of precubic maps with $n = 2m+1$ edges is given by the Catalan number $\mathrm{Cat}(m)$.

Here:

The number of precubic maps of genus g with $n = 2m+1$ edges is:

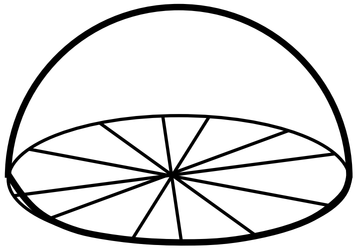
$$\xi_g(m) = \frac{1}{2^g g!} \binom{m+1}{3, 3, \ldots, 3, m+1-3g} \mathrm{Cat}(m)$$
$$= \frac{(2m)!}{12^g g! m! (m+1-3g)!}$$

Non-orientable case.

...work in progress with Olivier Bernardi (MIT).

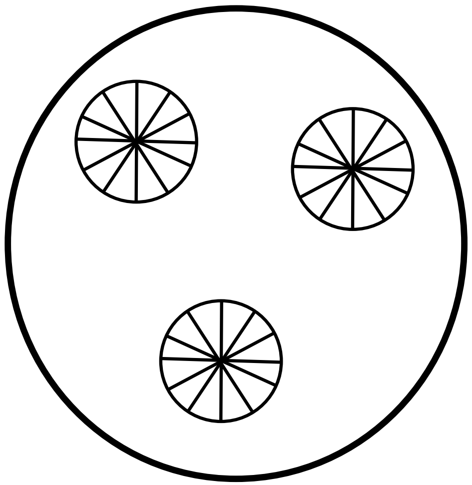
Projective plane

= upper hemisphere with antipodal points identified on the equator.



Non-orientable surface $\mathbb{N}_h$

= connected sum of the sphere and h projective planes.



What about maps on $\mathbb{N}_h$?

Maps become more complicated combinatorial objects...

Maps $\neq$ graph $+$ rotation system

In order to define the rotation system at each vertex, one must first choose arbitrarily the clockwise orientation around each vertex.

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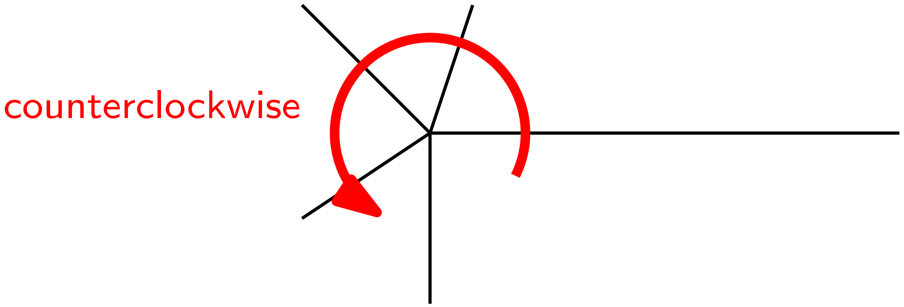
When the two orientations disagree along an edge, this edge is called a twist:



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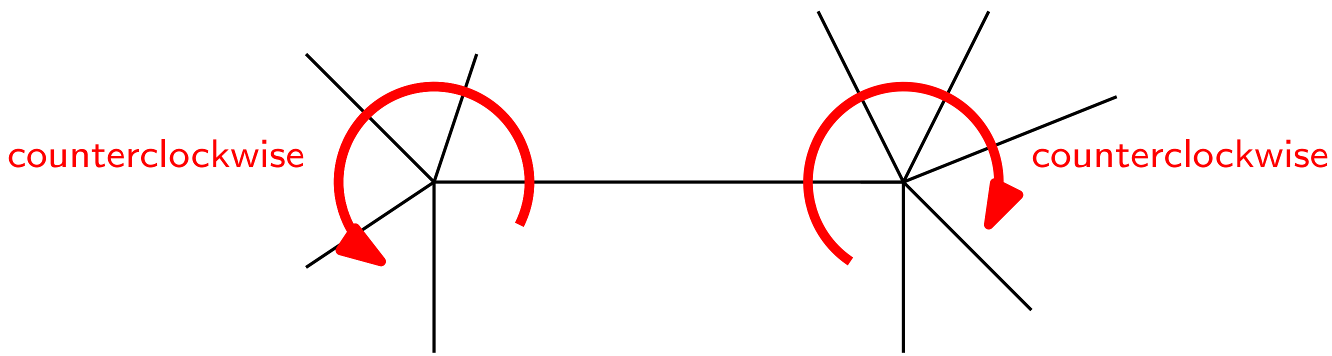
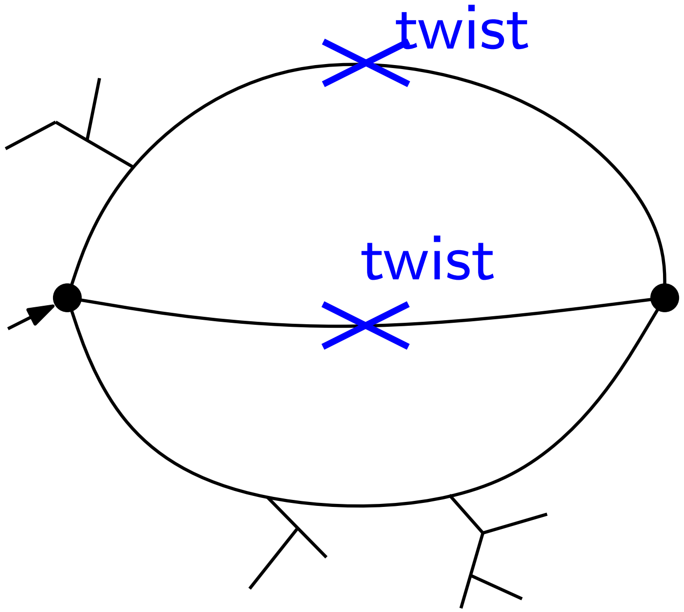


Drawing maps on the plane

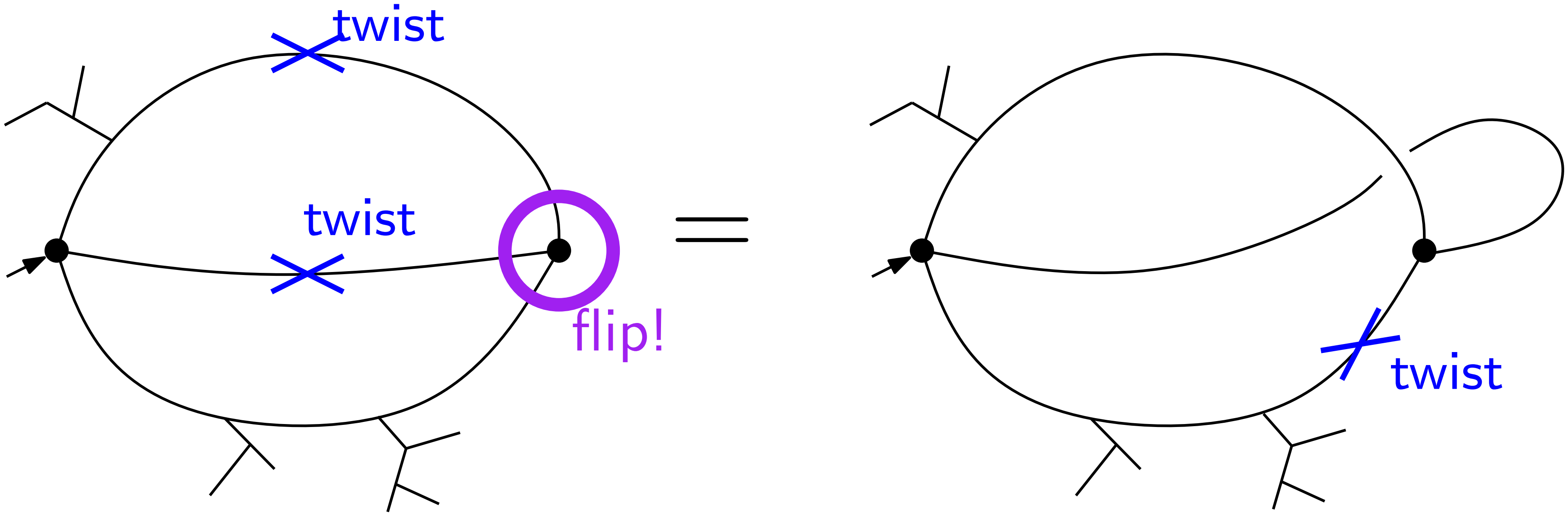
Once an orientation convention is fixed, one can draw the map on the plane as before. But now one has to remember the position of the twists.



Drawing maps on the plane

Once an orientation convention is fixed, one can draw the map on the plane as before. But now one has to remember the position of the twists.

This representation is not unique: it is defined up to flips of the vertices.



Hence: map = (graph + rotation system + set of twists), considered up to flips of the vertices.

Euler's formula: $s + f = e + 2 - h$

Hard to count with such a definition.

We weed to define a canonical orientation.

For the moment, we only know how to do that (well) in the case of precubic maps (all vertices have degree 1 or 3).

During the tour of the map, certain corners are visited on the left of the tour, and others on the right.

The canonical orientation of a precubic one-face map is the only one such that around each vertex, there are more left-corners than right corners.

Good news

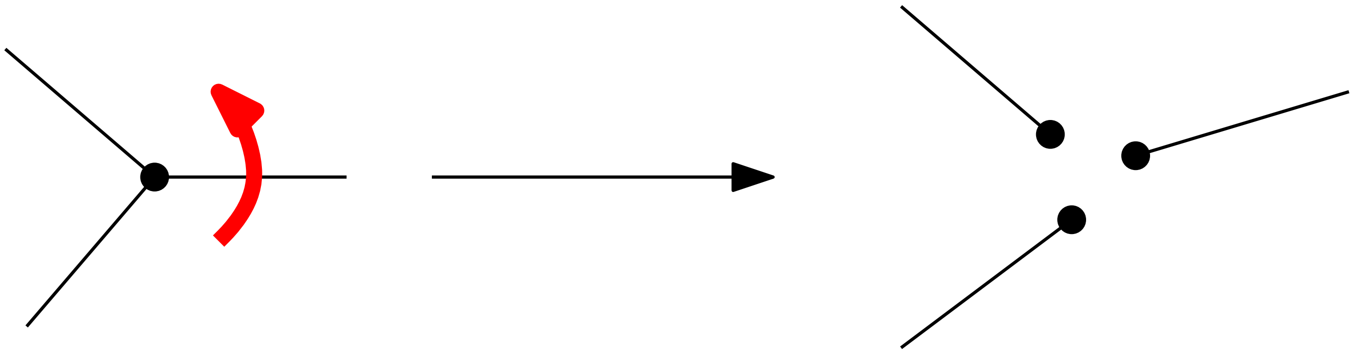
In the canonical orientation, the notion of trisection still makes sense.

We have a mapping

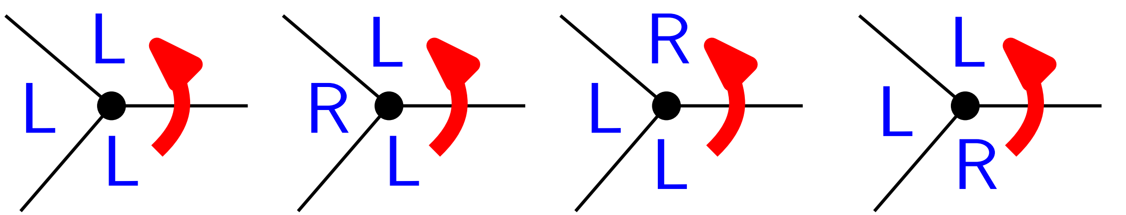
Precubic maps of type h with a distinguished trisection

↓

Precubic maps of type $(h-2)$ with 3 distinsguished leaves



This mapping is one to four:

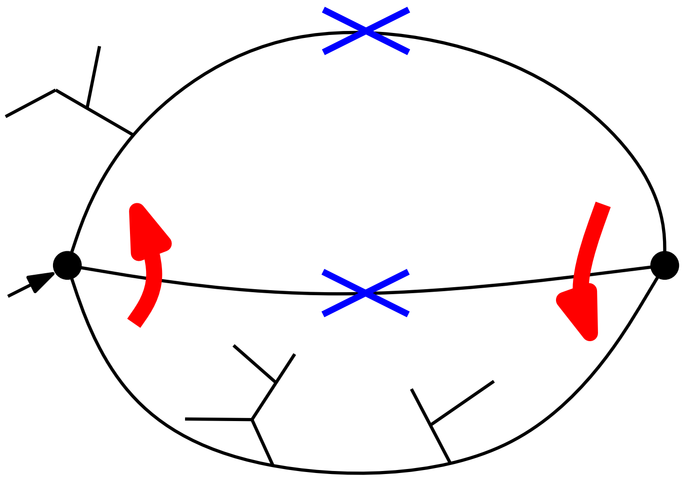


Bad news

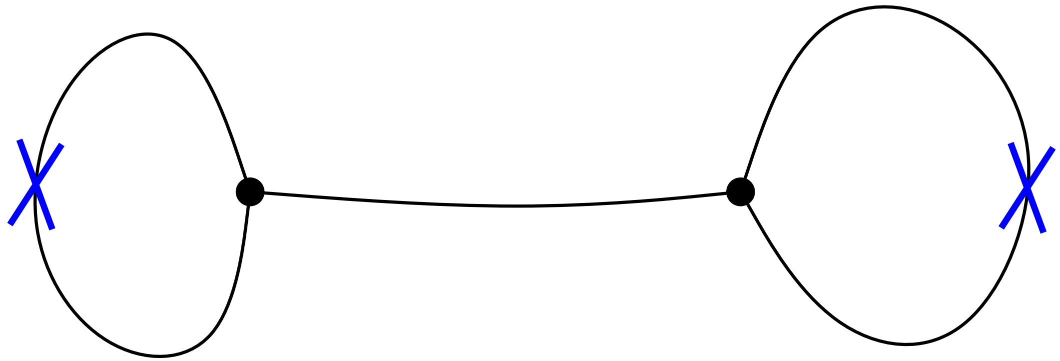
The trisection lemma does not work !

For example

These two maps have type $h = 2$ (Klein bottle):



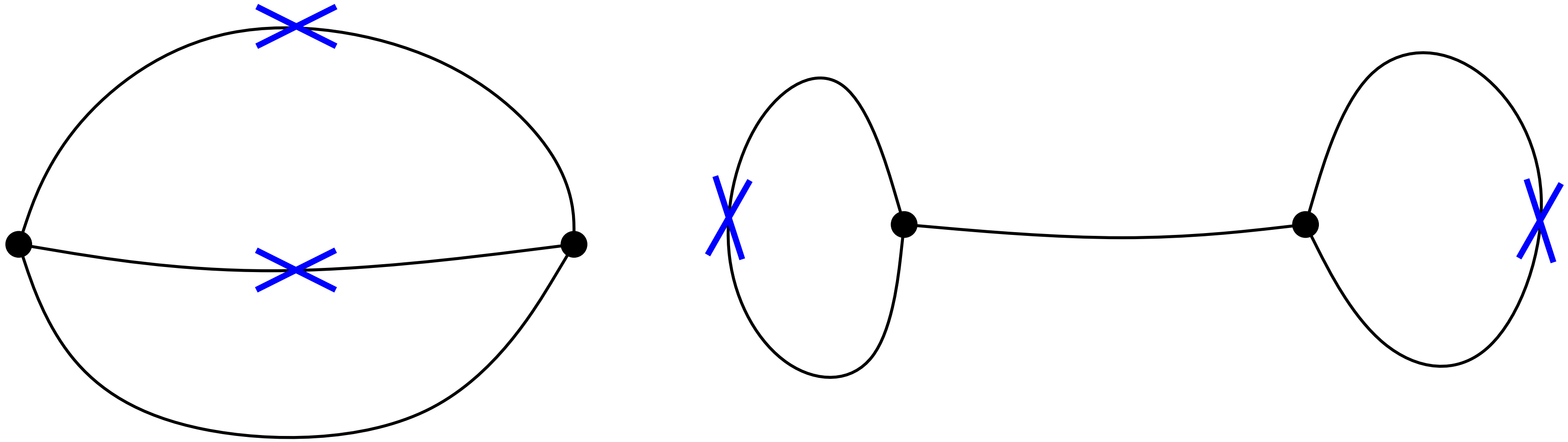
2 trisections



0 trisections

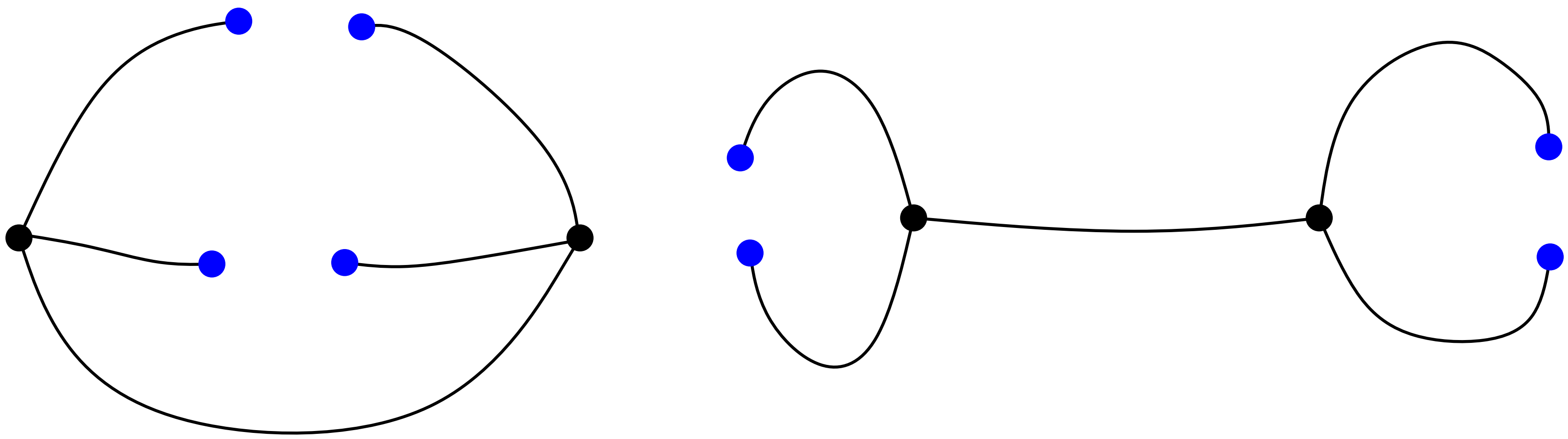
What to do then? ...the trisection lemma is the key of our approach.

A very strange, global, and still mysterious involution



Cut all the twists...

A very strange, global, and still mysterious involution



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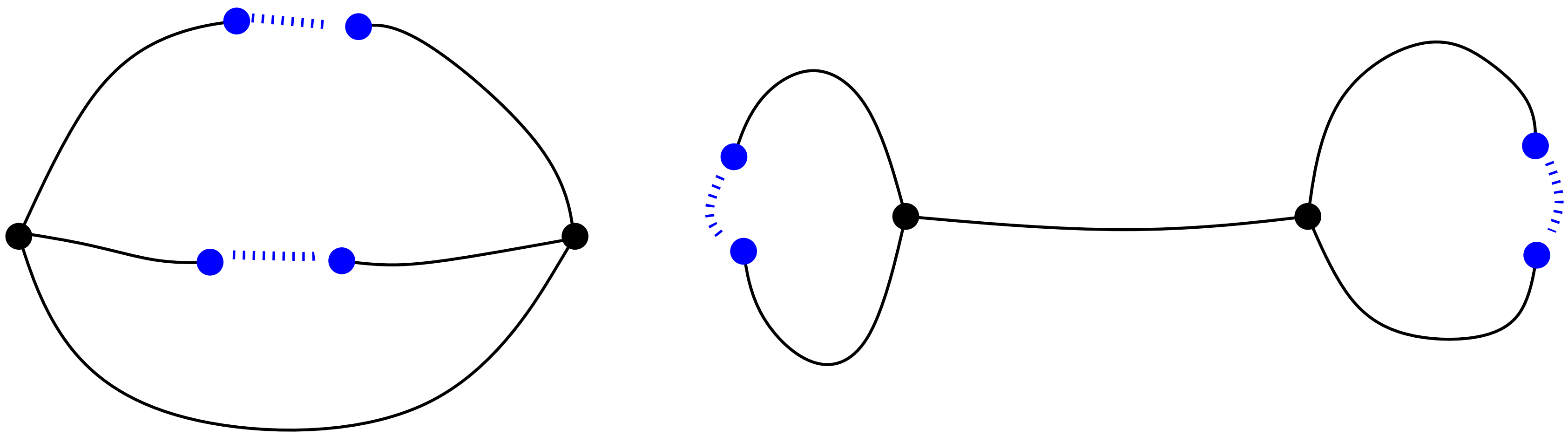
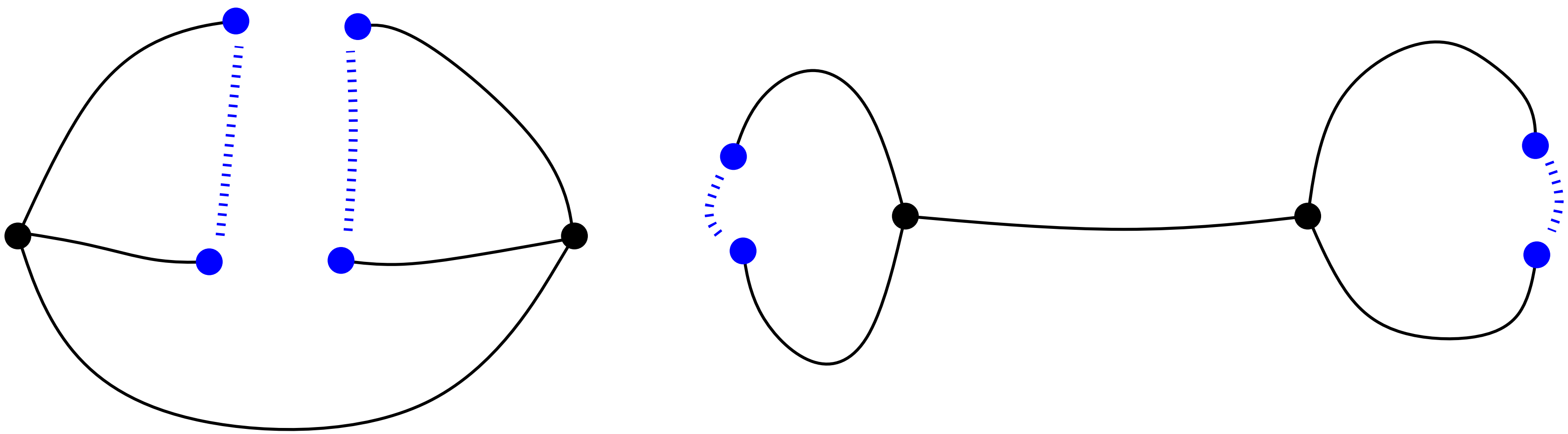


Cut all the twists...

Make a rotation of the matching system of the twists.

A very strange, global, and still mysterious involution



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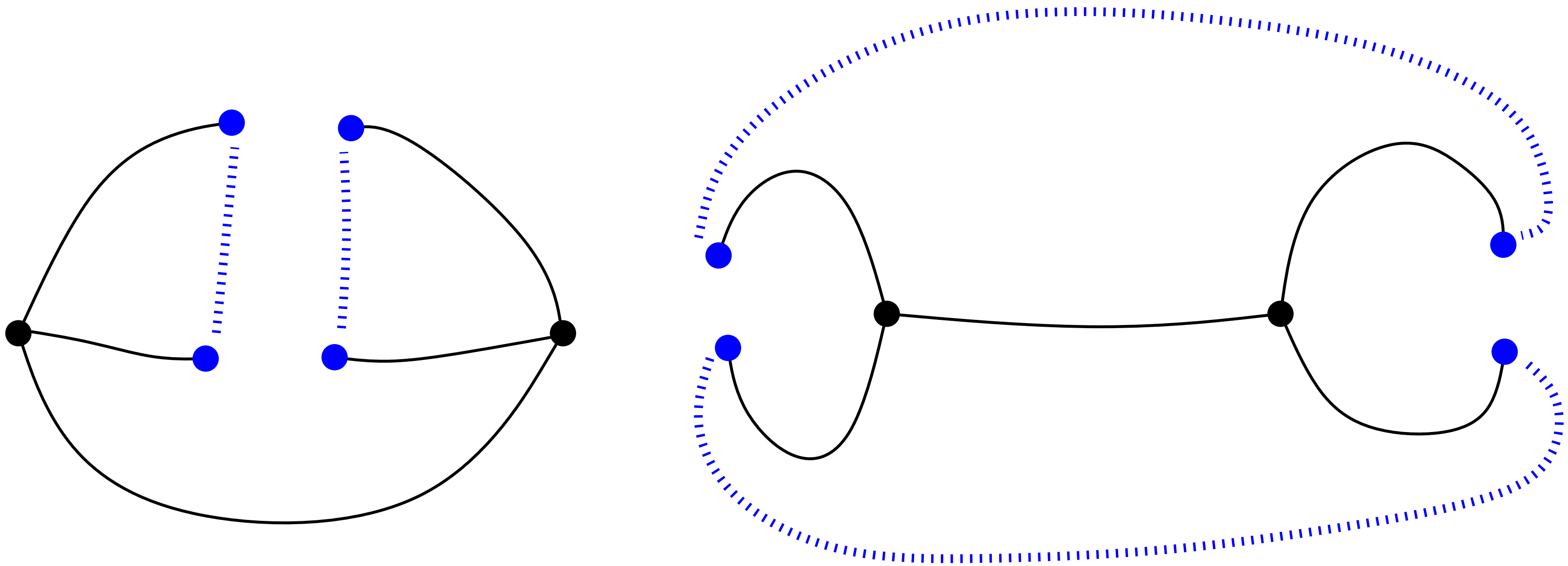

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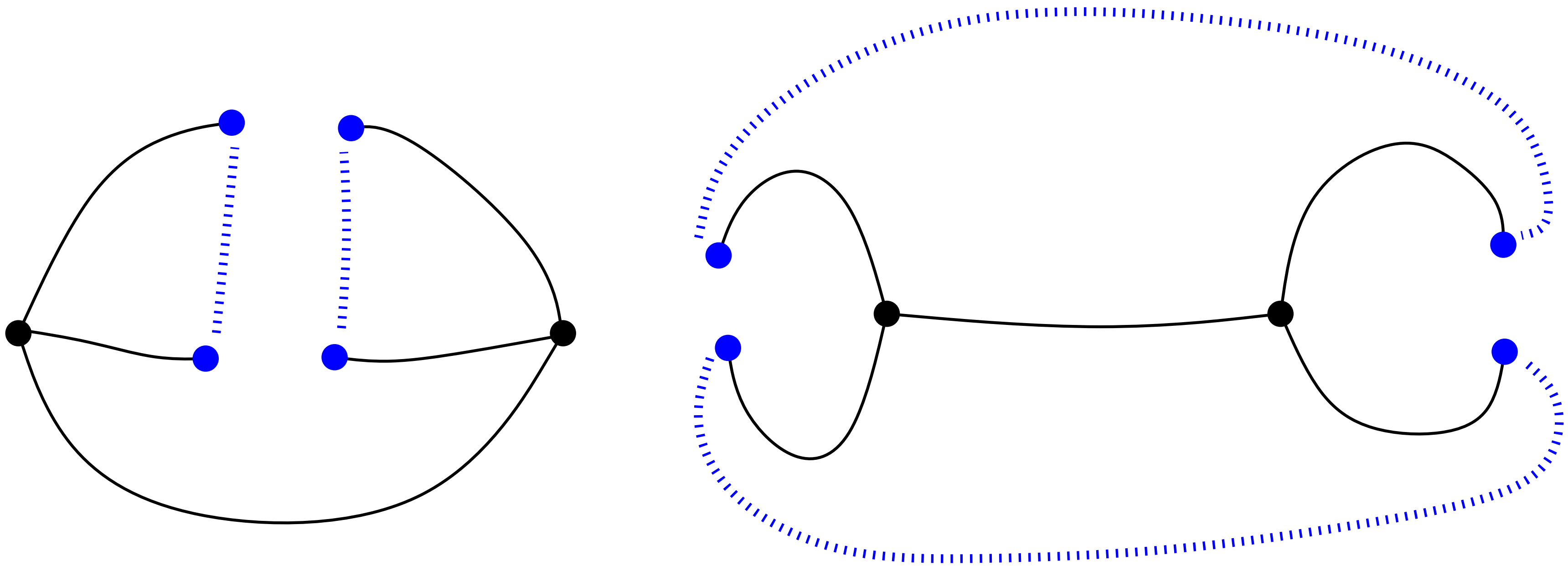

Cut all the twists...

Make a rotation of the matching system of the twists.

Believe me

The involution exchanges maps with $h+k$ trisections and maps with $h-2-k$ trisections. (here $k \geq 0$)

The averaged trisection lemma

For each $h \geq 0$ the average number of trisections among non-orientable precubic one-face maps of type h with n edges is $(h-1)$.

In other words, the average excess of trisections is:

- 0 for orientable maps
- -1 for non-orientable maps.

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In other words, the average excess of trisections is:

- 0 for orientable maps
- -1 for non-orientable maps.

Therefore we can count !

orientable ones

$$h \cdot \eta_h(m) = \underbrace{4\binom{m+1-2h}{3}\eta_{h-2}(m)}_{\text{distinguished trisection}} + \underbrace{\eta_h(m) - \xi_h(m)}_{\text{to compensate the excess}}$$

all maps of type h (orientable + non-orientable)

from which closed formulas follow...

Thank you !