Size-Change Abstraction and Max-Plus Automata

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Abstract. Max-plus automata (over $\mathbb{N} \cup \{-\infty\}$) are finite devices that map input words to non-negative integers or $-\infty$. In this paper we present (a) an algorithm allowing to compute the asymptotic behaviour of max-plus automata, and (b) an application of this technique to the evaluation of the computational time complexity of programs.

1 Introduction

The contributions of this paper are two-fold. First, we provide an algorithm that given a function computed by a max-plus automaton over $\mathbb{N} \cup \{-\infty\}$ computes the asymptotic minimal behaviour of the automaton as a function of the length of the input. We then apply this result for characterizing the asymptotic complexity bounds that can be obtained by the size-change abstraction, which is a widely used technique in automated termination analysis. These two contributions are of independent interest. Let us introduce them successively.

Weighted automata, and the main theorem

Max-plus automata belong to the wider family of weighted automata, as introduced by Schützenberger [7]. The principle of weighted automata is to consider non-deterministic automata that take values in a semiring $(S, \oplus, \otimes, 0, 1)$ (i.e., a ring in which the addition is not required to have an inverse). Weighted automata interpret the non-determinism of the automaton as the sum in the semiring and the sequence as the product. Standard non deterministic automata correspond to the case of the Boolean semiring $(\{0,1\}, \vee, \wedge, 0, 1)$. Probabilistic automata correspond to the case $([0,1], +, \times, 0, 1)$ (with a stochasticity restriction). Distance automata (or min-plus automata) correspond to the case $(\mathbb{N}\cup\{\infty\}, \min, +, 0, \infty\}$.

In this paper, we concentrate our attention to max-plus automata, which correspond to the semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, 0, -\infty)$. Such automata have transition with weights in $\mathbb{N}$. Over a given input, they output the maximum over all accepting runs of the sum of the weights of transitions (and $-\infty$ if there is no accepting run). Such automata are natural candidates for modelling worst case behaviours of systems, as shown in the subsequent application. Remark that max-plus automata share a lot of common points with min-plus automata, and indeed, many results for max-plus automata can be converted into results for min-plus automata and vice-versa[1].

[1] Indeed, if we allow negative weights, then negating all weights turns max-plus automata into min-plus automata and vice-versa, while preserving the semantics. How-

We seek to analyse the asymptotic behaviour of such automata. More precisely, fix a max-plus automaton computing a function f from the words in $\mathbb{A}^*$ to $\mathbb{N} \cup \{-\infty\}$. We study the asymptotic evolution of $c(n)$ defined for $n \in \mathbb{N}$ as:

$$c(n) = \inf\{f(w) \ : \ w \in \mathbb{A}^*,\ |w| \geq n\} \ .$$

We show that this quantity either is $-\infty$ for all n, or it is in $\Theta(n^\beta)$ for a computable rational $\beta \in [0, 1]$. Our main theorem, Theorem 2, expresses this property in a dual, yet equivalent, way as the asymptotic behaviour of the longest word that happen to have a value smaller than n.

From a logical perspective, it has to do with a quantifier alternation since the quantity studied is computed as a minimum (inf) of a function which, itself, is defined as a maximum (as a max-plus-automaton). In particular, in our case, it is immediately PSPACE hard (using reduction of the universality problem for non-deterministic automata). Such quantifier alternations are often even more complex when weighted automata are considered. For instance, a natural question involving such an alternation is to test whether $f(u) < |u|$ for some u, and it turns out to be undecidable [4]. On the other side, the boundedness question for min-plus automata (determining if there exists n such that $f(u) \leq n$ for all words u), which also has a similar quantifier alternation flavour, turns out to be decidable [3]. The work of Simon [9] has the most similarities with our contribution. It shows that, for a min-plus automaton computing a function g, the dual quantity $d(n) = \sup\{g(w) \ : \ |w| \leq n\}$ has a behaviour that is asymptotically between $n^{1/(k+1)}$ and $n^{1/k}$ for some non-negative integer k. Our result differs in two ways. First, the results for min-plus automata and for max-plus automata cannot be converted directly into results over the other form of automata. Second, our main result is significantly more precise since it provides the exact asymptotic coefficient. The proof of this theorem is the subject of the first part of this paper.

Program Analysis and Size Change Abstraction

The second contribution in this work consists in applying Theorem 2 for characterizing the asymptotic complexity bounds that can be obtained by the size-change abstraction, which is a popular program abstraction for automated termination analysis (e.g. [5, 6]). This question was the primary reason for this investigation.

We start with definitions needed to precisely state our contribution. We fix some finite set of *size-change variables* Var. We denote by Var' the set of primed versions of the variables Var. A *size-change predicate* (SCP) is a formula $x \triangleright y'$ with $x, y \in Var$, where $\triangleright$ is either $>$ or $\geq$. A *size-change transition* (SCT) T is a set of SCPs. A *size-change system* (SCS) $\mathcal{S}$ is a set of SCTs.

We define the semantics of size-change systems by *valuations* $\sigma : Var \to [0..N]$ of the size-change variables to natural numbers in the interval $[0..N]$, where

ever, such kind of reductions can get more complicated, if not impossible, when negative values are forbidden, as it is in our case.

N is a (symbolic) natural number. We write $\sigma, \tau' \models x \triangleright y'$ for two valuations σ, τ, if $\sigma(x) \triangleright \tau(y)$ holds over the natural numbers. We write $\sigma, \tau' \models T$, if $\sigma, \tau' \models x \triangleright y'$ holds for all $x \triangleright y' \in T$. A *trace* of an SCS $\mathcal{S}$ is a sequence $\sigma_1 \xrightarrow{T_1} \sigma_2 \xrightarrow{T_2} \cdots$ such that $T_i \in \mathcal{S}$ and $\sigma_i, \sigma'_{i+1} \models T_i$ for all i. The *length* of a trace is the number of SCTs that the trace uses, counting multiple SCTs multiple times. An SCS $\mathcal{S}$ is *terminating*, if $\mathcal{S}$ does not have a trace of infinite length.

We note that in earlier papers, e.g. [5], the definition of a size-change system includes a control flow graph that restricts the set of possible traces. For the ease of development we restrain from adding control structure. We discuss in Appendix I that our result also holds when we add control structure. Moreover, earlier papers, e.g. [5], consider SCSs semantics over the natural numbers, i.e., valuations $\sigma : \mathit{Var} \to \mathbb{N}$. In contrast, we restrict values to the interval $[0, N]$ in order to guarantee that the length of traces is bounded for terminating SCSs: no valuation $\sigma \in \mathit{Var} \to [0..N]$ can appear twice in a trace (otherwise we would have a cycle, which could be pumped to an infinite trace); thus the length of traces is bounded by $(N+1)^k$ for SCSs with k variables.

Problem Statement: Our goal is to determine a function $h_{\mathcal{S}} : \mathbb{N} \to \mathbb{N}$ such that the length of the longest trace of a terminating SCS $\mathcal{S}$ is of asymptotic order $\Theta(h_{\mathcal{S}}(N))$. This question has also been of interest in a recent report [1], which claims that SCSs always have a polynomial bound, i.e., a bound $\Theta(N^k)$ for some $k \in \mathbb{N}$. However, this is not the case (see example below). We believe that the development in [1] either contains a gap or that the results of [1] have to be stated differently.

Example 1. The length of the longest trace of the SCS $\mathcal{S} = \{T_1, T_2, T_3\}$ with
$T_1 = \{x_1 > x'_1, x_2 \geq x'_2, x_3 > x'_3, x_4 \geq x'_4\}$,
$T_2 = \{x_1 > x'_1, x_2 \geq x'_2, x_2 \geq x'_3, x_2 > x'_4, x_3 > x'_4, x_4 > x'_4\}$ and
$T_3 = \{x_2 > x'_2, x_2 > x'_3, x_2 > x'_4, x_3 > x'_2, x_3 > x'_3, x_3 > x'_4, x_4 > x'_2, x_4 > x'_3, x_4 > x'_4\}$ is of asymptotic order $\Theta(N^{\frac{3}{2}})$. For comparison, [1] considers SCSs bounded in terms of the initial state; we can make $\mathcal{S}$ bounded in terms of the initial state by adding a new variable x_N to $\mathcal{S}$, and adding the constraints $\{x_N \geq x'_N, x_N \geq x'_1, x_N \geq x'_2, x_N \geq x'_3, x_N \geq x'_4\}$ to each of T_1, T_2, T_3.

The asymptotic order $\Theta(N^{\frac{3}{2}})$ of $\mathcal{S}$ can be established by Theorem 1 stated below (a corresponding max-plus automaton is stated in Example 2 and its asymptotic behavior is analyzed in Appendix J)). For illustration purposes, we sketch here an elementary proof. For the lower bound we consider the sequence $s_N = ((T_1^{\frac{\sqrt{N}}{2}-1} T_2)^{\frac{\sqrt{N}}{2}-1} T_3)^{\frac{\sqrt{N}}{2}-1}$. For example, for $N = 36$ we have $s_N = T_1 T_1 T_2 T_1 T_1 T_2 T_3 T_1 T_1 T_2 T_1 T_1 T_2 T_3$. Note that s_N is of length $l_N = \frac{\sqrt{N}}{2} \cdot \frac{\sqrt{N}}{2} \cdot (\frac{\sqrt{N}}{2} - 1) = \Omega(N^{\frac{3}{2}})$. We define valuations σ_i, with $0 \leq i \leq l_N$, that demonstrate that s_N belongs to a trace of $\mathcal{S}$: given some index $0 \leq i \leq l_N$, let t_3 denote the number of T_3 before index i in the sequence s_N, let t_2 denote the number of T_2 before index i since the last T_3, and let t_1 denote the number of T_1 before index i since the last T_2 (note that we have $0 \leq t_1, t_2, t_3 < \frac{\sqrt{N}}{2}$ by the shape of s_N); we set $\sigma_i(x_1) = N - t_2 \cdot \frac{\sqrt{N}}{2} - t_1, \sigma_i(x_2) = N - t_3 \cdot \sqrt{N}, \sigma_i(x_3) =$

$N - t_3 \cdot \sqrt{N} - t_1, \sigma_i(x_4) = N - t_3 \cdot \sqrt{N} - \frac{\sqrt{N}}{2} - t_2$. It is easy to verify that the valuations σ_i satisfy all constraints of s_N.

We move to the upper bound. Let S be a sequence of SCTs that belongs to a trace of $\mathcal{S}$. We decompose $S = S_1 T_3 S_2 T_3 \cdots$ into subsequences S_i that do not contain any occurrence of T_3. We define a_i to be the maximal number of consecutive T_1 in S_i, and b_i to be the total number of T_2 in S_i. We set $c_i = \max\{a_i, b_i\}$. We start with some observations: We have $|S_i| \leq c_i(c_i+1)+c_i = c_i(c_i+2)$ (i) by the definition of the c_i. We have $|S_i| \leq N$ (ii) because the inequality $x_1 > x_1'$ is contained in T_1 as well as in T_2 and the value of x_1 can only decrease N times in S_i. Combining (i) and (ii) we get $|S_i| \leq \min\{c_i(c_i+2), N\}$ (iii). We have $\sum_i c_i \leq N$ (iv); this holds because there is a chain of inequalities from the beginning to the end of S that for every i either uses all inequalities $x_3 > x_3'$ of the consecutive T_1 or all inequalities $x_4 > x_4'$ of the T_2 in S_i, and this chain can only contain N strict inequalities. Finally, by the definition of the S_i we have $|S| \leq \sum_i |S_i| + 1$. With (iii) we get $|S| \leq \sum_i \min\{c_i(c_i+2), N\} + 1 \leq 5 \sum_i \min\{c_i^2, N\}$ (v). Using associativity and commutativity we rearrange the sum $\sum_i c_i = \sum_i d_i + \sum_i e_i + r$, where the d_i are summands $c_i > \sqrt{N}$ and the e_i and r are the sum of summands $c_i \leq \sqrt{N}$ with $\frac{\sqrt{N}}{2} \leq e_i \leq \sqrt{N}$ and $r < \frac{\sqrt{N}}{2}$; we denote $e_i = \sum_j c_{ij}$ for some c_{ij}. By (iv) there are at most $\sqrt{N}$ of the d_i and at most $2\sqrt{N}$ of the e_i. Using these definitions in (v) we get $|S| \leq 5(\sum_i \min\{d_i^2, N\} + \sum_{i,j} \min\{c_{ij}^2, N\} + \min\{r^2, N\}) \leq 5(\sqrt{N} \cdot N + \sum_{i,j} c_{ij}^2 + N) \leq 5(\sqrt{N} \cdot N + \sum_i e_i^2 + N) \leq 5(\sqrt{N} \cdot N + 2\sqrt{N} \cdot N + N) = O(N^{\frac{3}{2}})$.

In this paper we establish the fundamental result that the *computational time complexity* of terminating SCA instances is decidable:

Theorem 1. *Let $\mathcal{S}$ be a terminating SCS. The length of the longest trace of $\mathcal{S}$ is of order $\Theta(N^\alpha)$, where $\alpha \geq 1$ is a rational number; moreover, α is computable.*

We highlight that our result provides a *complete characterization* of the complexity bounds arising from SCA and gives means for determining the exact asymptotic bound of a given abstract program. Our investigation was motivated by previous work [10], where we introduced a practical program analysis based on SCA for computing resource bounds of imperative programs; in contrast to this paper, [10] does not study the completeness of the proposed algorithms and does not contain any result on the expressivity of SCA.

Organization of the Paper In Section 2, we give the automata definitions and sketch the proof of Theorem 2. In Section 3 we provide a reduction from size-change systems to max-plus automata that allows to prove Theorem 1 from Theorem 2.

2 Max-Plus Automata

In this section, we first define max-plus automata (section 2.1), and then sketch the proof of Theorem 2 (section 2.2).

2.1 Definition of max-plus automata

A **semigroup** $(S, \cdot)$ is a set S equipped with an associative binary operation '$\cdot$'. If the product has furthermore a neutral element 1, $(S, \cdot, 1)$ is called a **monoid**. The monoid is said to be **commutative** if $\cdot$ is commutative. An **idempotent** in a semigroup is an element e such that $e \cdot e = e$. Given a subset A of a semigroup, $\langle A \rangle$ denotes the closure of A under product, *i.e.,* the least sub-semigroup that contains A. Given $X, Y \subseteq S$, $X \cdot Y$ denotes $\{a \cdot b \ : \ a \in X,\ b \in Y\}$.

A **semiring** $(S, \oplus, \otimes, 0_S, 1_S)$ is a set S equipped with two binary operations $\oplus$ and $\otimes$ such that $(S, \oplus, 0_S)$ is a commutative monoid, $(S, \otimes, 1_S)$ is a monoid, 0_S is absorbing for $\otimes$ (for all $x \in S$, $x \otimes 0_S = 0_S \otimes x = 0_S$) and $\otimes$ distributes over $\oplus$. We shall use the **max-plus semiring** $(\{-\infty\} \cup \mathbb{N}, \max, +, -\infty, 0)$, denoted $\overline{\mathbb{N}}$, and its extension $\overline{\mathbb{R}^+} = \{-\infty, 0\} \cup \{x \ : \ x \in \mathbb{R},\ x \geq 1\}$, that we name the **real semiring**. This semiring will be used instead of $\overline{\mathbb{N}}$ during the computations. The operation over matrices induced by this semiring is denoted $\otimes$. Remark that $0_{\overline{\mathbb{N}}} = -\infty$, and $1_{\overline{\mathbb{N}}} = 0$.

Let S be a semiring. The set of matrices with m rows and n columns over S is denoted $\mathcal{M}_{m,n}(S)$, or simply $\mathcal{M}_n(S)$ if $m = n$. As usual, $A \otimes B$ for two matrices A, B (provided the width of A and the height of B coincide) is defined as:

$$(A \otimes B)_{i,j} = \bigoplus_{0<k\leq n} (A_{i,k} \otimes B_{k,j}) \quad \left(= \max_{0<k\leq n} (A_{i,k} + B_{k,j}) \text{ for } S = \overline{\mathbb{N}} \text{ or } \overline{\mathbb{R}^+} \right) .$$

It is standard that $(\mathcal{M}_n(S), \otimes, I_n)$ is a monoid, whose neutral element is the diagonal matrix I_n with 1_S (i.e., 0 for $\overline{\mathbb{N}}$) on the diagonal, and 0_S (i.e., $-\infty$ for $\overline{\mathbb{N}}$) elsewhere. For a positive integer k, we set $M^0 = I_n$, and $M^k = M^{k-1} \otimes M$. For $\lambda \in \mathbb{R}^+$, we denote by λA the matrix such that $(\lambda A)_{i,j} = \lambda A_{i,j}$ for all i, j (this matrix has non-negative real coefficients, which might not be over $\overline{\mathbb{R}^+}$ if $\lambda \leq 1$). Finally, we write $A \leq B$ if for all i, j, $A_{i,j} \leq B_{i,j}$.

A **max-plus automaton** over the alphabet $\mathbb{A}$ (with k states) is a map δ from $\mathbb{A}$ to $\mathcal{M}_k(\overline{\mathbb{N}})$ together with initial and final vectors $I, F \in \mathcal{M}_{1,k}(\{0, -\infty\})$. The map δ is uniquely extended into a morphism from $\mathbb{A}^*$ to $\mathcal{M}_k(\overline{\mathbb{N}})$, that we also denote δ. The **function computed by the automaton** maps each word $u \in \mathbb{A}^*$ to ${}^tI \otimes \delta(u) \otimes F \in \overline{\mathbb{N}}$ where ${}^tI$ denotes the transpose of I.

Example 2. We consider the following automaton, over the alphabet $\{a, b, c\}$, for $k = 6$ and defined by (where $-\infty$ is not written for readability):

$$\delta(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & & & & 0 \\ & & 0 & & & 0 \\ & & & 1 & & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \qquad \delta(b) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & 1 & & & & 0 \\ & & 0 & 0 & 1 & 0 \\ & & & & 1 & 0 \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix},$$

$$\delta(c) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & 0 \\ & & 1 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 0 \\ & & 1 & 1 & 1 & 0 \\ & & & & & 0 \end{pmatrix}, \qquad \text{and } I = F = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is sometimes convenient to see such matrices as a weighted automaton [7]. Such a presentation is provided in Figure 1. The states of the automaton are $q_1, \ldots, q_6$ and correspond respectively to the lines and the columns 1 to 6 of the matrices. There is a transition from q_i to q_j corresponding to letter $x = a, b, c$ if the entry i, j of the matrix $\delta(x)$ is $z \neq -\infty$. In this case, the transition is weighted by z. The initial states are the states q_i such that $I_i = 0$. The final states are the states q_j such that $F_j = 0$. A run over the word w is a path (a sequence of compatible transitions) in the graph labelled by w. Its weight is the sum of the weights of the transitions. Finally the weight of a given word w is the maximum of the weights of the runs labelled by w and going from an initial state to a final state. The weight of w, given by the graph representation is exactly the value ${}^tI \otimes \delta(w) \otimes F$, given by the matrix presentation.

2.2 Main theorem

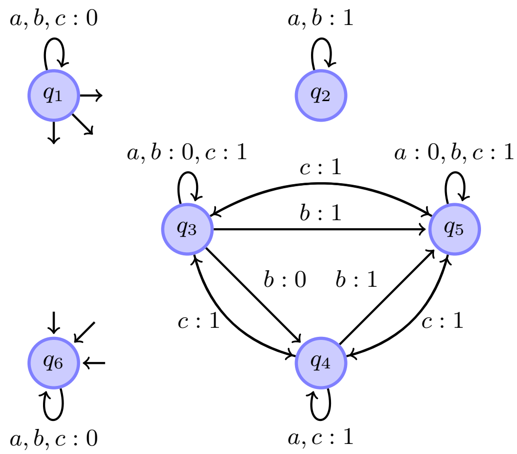
Theorem 2. *Given a max-plus automaton computing* $f : \mathbb{A}^* \to \mathbb{N} \cup \{-\infty\}$, *there exists an algorithm that computes the value* $\alpha \in \{+\infty\} \cup \{\beta \in \mathbb{Q} \ : \ \beta \geq 1\}$ *such that*

$$g(n) = \Theta(n^\alpha)$$

where $g(n) = \sup\{|w| \ : \ f(w) \leq n\}$, *with the convention that* $n^{+\infty} = +\infty$.

Example 3. The algorithm applied on the automaton given in exemple 2 outputs value $2/3$. A sequence of words that witness this growth is $(((a^n b)^n) c^n)_{n \in \mathbb{N}}$.

The semigroup of weighted matrices Our goal is to analyse the relationship between the output of the automaton and the length of the input. Thus we use weighted matrices that are pairs of a matrix representing the behaviour of



where:
- there are edges from state q_1 to every state labelled by every letter with weight 0,
- there are edges from every state to state q_6 labelled by every letter with weight 0,
- every state is initial and final.

Fig. 1. A weighted automaton over the semiring $(\overline{\mathbb{N}}, \max, +)$.

the automaton with a value standing for the length of the input. Formally, a **weighted matrix** is an ordered pair (M, x) where $M \in \mathcal{M}_k(\overline{\mathbb{R}^+})$ and $x \geq 1$ is a real number called the **weight** of the weighted matrix. They are usefull to represent pairs $(\delta(w), |w|)$. The set of weighted matrices is denoted by $\mathcal{W}_k$. Weighted matrices have a semigroup structure $(\mathcal{W}_k, \otimes)$, where $(M, x) \otimes (N, y)$ stands for $(M \otimes N, x + y)$. By definition, the function $w \mapsto (\delta(w), |w|)$ is a morphism of semigroups. As in the general case, we use $\otimes$ over subsets of $\mathcal{W}_k$. Given $A \subseteq \mathcal{W}_k$, $\langle A \rangle$ is the closure under $\otimes$ of A. Our goal is to study the set

$$\{(\delta(w), |w|) \mid w \in \mathbb{A}^*\} = \langle \{(\delta(a), 1) \mid a \in \mathbb{A}\} \rangle$$

and more precisely to give a finite representation of it up to some approximation. The key to our algorithm is the ability to (a) finitely represent infinite sets of weighted matrices and (b) define a notion of approximation between such sets. Then our algorithm computes using such sets, and guarantees that, up to the approximation, it is consistent with the behaviour of the automaton. We present these notions below. From now we fix a max-plus automaton with k states computing a function f and defined by the morphism δ. Let us first introduce another semiring usefull for defining finite representation.

The $\overline{\mathbb{R}^+_\odot}$ and small semirings, and the semigroup of weighted matrices We have seen the semirings $\overline{\mathbb{N}}$ and $\overline{\mathbb{R}^+}$. We use another semiring over the same ground set $\overline{\mathbb{R}^+}$ but with a different product, $\odot$. For all $x, y \in \overline{\mathbb{R}^+}$ set $x \odot y$ to be:

$$x \odot y = \begin{cases} -\infty & \text{if either } x = -\infty \text{ or } y = -\infty, \\ \max(x, y) & \text{otherwise.} \end{cases}$$

Again, $(\overline{\mathbb{R}^+}, \max, \odot, -\infty, 0)$ is a semiring, denoted $\overline{\mathbb{R}^+_\odot}$. As before, this induces a product operation $\odot$ for matrices. The product operation $\odot$ is a good approximation of $\otimes$ as shown by the following key lemma that follows from the similar property for real number and monotonicity of max and plus (proof in Appendix A).

Lemma 1. *Given matrices $M_1, \ldots, M_q$, $q \geq 1$ over $\overline{\mathbb{R}^+}$, then*

$$M_1 \odot \cdots \odot M_q \leq M_1 \otimes \cdots \otimes M_q \leq q(M_1 \odot \cdots \odot M_q) \ .$$

The last semiring we use is the **small semiring** $(\mathbb{S}, \max, \odot, -\infty, 0)$, simply denoted $\mathbb{S}$, which is the restriction of $\overline{\mathbb{R}^+_\odot}$ to $\{-\infty, 0, 1\}$. There is a natural map φ from $\overline{\mathbb{R}^+}$ to $\mathbb{S}$ obtained by collapsing all elements above or equal to 1 to 1. It happens that φ is at the same time a **morphism** of semirings from $\overline{\mathbb{R}^+}$ to $\mathbb{S}$ and from $\overline{\mathbb{R}^+_\odot}$ to $\mathbb{S}$. Matrices over the small semiring are called **small matrices**.

The morphism φ is also extended to weighted matrices by $\varphi((M, x)) = \varphi(M)$.

Our goal is, given a finite set of weighted matrices A, to compute a presentation of $\langle A \rangle$ up to approximation (Lemma 7) — the notion of presentation of sets of weighted matrices and the notion of approximation are the subject of the two subsequent sections.

Presentable Sets of Weighted Matrices We introduce now the notion of presentable sets of matrices, i.e., sets of matrices that we can manipulate via their finite presentation. Our sets of weighted matrices are presented in 'exponential form', i.e., given a weight $x \geq 1$, an entry of the matrix will be of the form x^α. In fact, some special cases have to be treated, that results in the use of $\alpha = \perp$ or $-\infty$.

Exponents and exponentiations The semiring of **exponents** (the choice of this name will be explained when defining exponentiation in the next paragraph) is $(\mathit{Exps}, \max, \max_\odot, \perp, -\infty)$ where

$$\mathit{Exps} = \{\perp, -\infty\} \cup [0, 1] \ ,$$

where max is defined with respect to the order $\perp < -\infty < x < y$ for all $x < y \in [0, 1]$, and where $\max_\odot(\alpha, \beta)$ for $\alpha, \beta \in \mathit{Exps}$ is defined by:

$$\max_\odot(\alpha, \beta) = \begin{cases} \perp & \text{if } \alpha = \perp \text{ or } \beta = \perp, \\ \max(\alpha, \beta) & \text{otherwise.} \end{cases}$$

This semiring will be simply denoted *Exps*, and the induced operation over matrices $\odot$ (we will see that this notation is not ambiguous). We take the convention to denote by α, β exponents, and by X, Y, Z vectors and matrices of exponents.

We define now the exponentiation operation. For $x \geq 1$ and $\alpha \in \mathit{Exps}$, set

$$x^\alpha = \begin{cases} -\infty & \text{if } \alpha = \perp, \\ 0 & \text{if } \alpha = -\infty, \\ x^\alpha & \text{otherwise, i.e., if } \alpha \in [0, 1], \text{ for the usual exponent.} \end{cases}$$

Lemma 2. *For all $x \geq 1$, $\alpha \mapsto x^\alpha$ is a semiring morphism from Exps to $\overline{\mathbb{R}^+_\odot}$.*

Note that this morphism can be applied to vectors (or matrices). In this case, given a matrix $Y \subseteq Exps^{k\times k}$, and some $x \geq 1$, we denote by $Y[x] \in \overline{\mathbb{R}^+}^{k\times k}$ the matrix such that $(Y[x])_{i,j} = x^{Y_{i,j}}$ for all $i, j = 1 \dots k$. According to the previous lemma, the map $Y \mapsto Y[x]$ is a morphism from matrices over *Exps* to matrices over $\overline{\mathbb{R}^+_\odot}$.

It is also sometimes convenient to send the small semiring to the exponent semiring. It is done using the following straightforward lemma.

Lemma 3. *The function γ that maps $-\infty$ to $\perp$, 0 to $-\infty$, and 1 to 0 is a semiring morphism from $\mathbb{S}$ to Exps such that $x^{\gamma(a)} = a$ for all $a \in \mathbb{S}$ and $x \geq 1$.*

Polytopes and presentable sets. Our goal it to describe finitely some infinite sets of matrices over $\overline{\mathbb{R}^+}$. We start from the notion of polytope. For this, we rely on the definition of polytopes in $\mathbb{R}^k$: a **polytope** (in $\mathbb{R}^k$) is a convex hull of finitely many points of $\mathbb{R}^k$. We would like to use this definition for subsets of $Exps^k$. For that we send *Exps* to $\mathbb{R}$ by $t(\perp) = -2$, $t(-\infty) = -1$ and $t(s) = s$ if s is real.

A subset of $Exps^k$ is called a polytope if its image under t is a polytope in $\mathbb{R}^k$. In particular, we can use this definition for matrices of exponents, yielding polytopes of matrices.

We can now define presentable sets of matrices over $\overline{\mathbb{R}^+}$. Essentially, a set of matrices over $\overline{\mathbb{R}^+}$ is presentable if it is the image under exponentiation of a finite union of polytopes of exponent matrices. Let us define precisely how this is defined. A set of weighted matrices $A \subseteq \mathcal{W}_k$ is **presentable** if it is of the form:

$$A = \{(M, 1) \ : \ M \in S\} \cup \{(Y[x], x) \ : \ Y \in P, \ x \geq 1\} \ ,$$

where S is a set of small matrices of dimension $k \times k$, and P is a finite union of polytopes of $Exps^{k\times k}$. The pair (S, P) is called the **presentation** of A. A presentation is said **small** if $P = \emptyset$. It is said **asymptotic** if $S = \emptyset$. Obviously, any presentable set is the union of a set of small presentation with a set of asymptotic presentation. Of course presentable sets are closed under union.

The approximation and simulation scheme We describe now the notion of approximation that we use. Indeed, our goal is to compute the set of weighted matrices $\{(\delta(w), |w|)\}$. We cannot expect to do it in general, and, at any rate, presentable sets of matrices cannot capture exactly the behaviour of the automaton. That is why we reason about sets of matrices up to some approximation relation that is sufficiently precise for our purpose, and at the same time is sufficiently relaxed for allowing to approximate the behaviour of the automaton by a presentable set of weighted matrices.

Given some $a \geq 1$ and two weighted matrices (M, x) and (N, y), we write

$$(M, x) \preccurlyeq_a (N, y) \quad \text{if} \quad M \leq aN, \ y \leq ax \ \text{ and } \varphi(M) = \varphi(N) \ .$$

This definition extends to sets of weighted matrices as follows. Given two such sets A, B, $A \preccurlyeq_a B$ if for all $(N, y) \in B$, there exists $(M, x) \in A$ such that

$(M,x) \preccurlyeq_a (N,y)$. We write $A \approx_a B$ if $A \preccurlyeq_a B$ and $B \preccurlyeq_a A$ and say that A is **a-equivalent** to B. We drop the a parameter when not necessary, and simply write $A \approx B$ if $A \approx_a B$ for some a.

A first consequence of this definition is that every weighted matrix (M,x) is a-equivalent to the weighted matrix $(\varphi(M),1)$ where a is the maximum of the entries of M and x. This justifies that, in the definition of a presentable set, the weighted matrices of the finite part are of this form.

Let us give some intuition why this approximation may help. For instance consider some exponent matrix M, and let us show:

$$\{(M[x],x) \ : \ x \geq 1\} \approx_2 \{(M[y],y) \ : \ y \in \mathbb{N},\ y \geq 1\} .$$

Indeed, one inclusion is obvious, yielding $\preccurlyeq_1$. For the other direction, consider some $x \geq 1$, and take $y = \lfloor x \rfloor$, then $2y \geq x$ and $M[y] \leq M[x]$, thus $(M[y],y) \preccurlyeq_2 (M[x],x)$. More generally imagine the y's would be further constrained to be multiples of some value, say 2, then the same arguments would work. Hence this equivalence relation allows to absorb a certain number of phenomena that can occur in an automaton and are irrelevant for our specific problem. In particular, if the least growing rate is achieved for words of length n for n even only, then this 'computing modulo 2' can be 'hidden' thanks to the $\approx$-approximation.

The following lemma establishes some essential properties of the $\preccurlyeq_a$ relations (as a consequence, the same properties hold for $\approx_a$). (Proofs in Appendix B).

Lemma 4. *Given A, A', B, B', C sets of weighted matrices and $a, b \geq 1$,*

1. *if $A \preccurlyeq_a B$ and $b \geq a$, then $A \preccurlyeq_b B$,*
2. *if $A \preccurlyeq_a A'$ and $B \preccurlyeq_a B'$, then $A \cup B \preccurlyeq_a A' \cup B'$,*
3. *if $A \preccurlyeq_a B$ and $B \preccurlyeq_b C$ then $A \preccurlyeq_{ab} C$,*
4. *if $A \preccurlyeq_a A'$ and $B \preccurlyeq_a B'$ then $A \otimes B \preccurlyeq_a A' \otimes B'$,*
5. *if $A \preccurlyeq_a B$ then $\langle A \rangle \preccurlyeq_a \langle B \rangle$.*

The main induction: the forest factorization theorem of Simon The forest factorization theorem of Simon [8] is a powerful combinatorial tool for understanding the structure of finite semigroups. In this short abstract, we will not describe the original statement of this theorem, in terms of trees of factorizations, but rather a direct consequence of it which is central in our proof (the presentation of the theorem was used in a similar way in [2]).

Theorem 3 (equivalent to the forest factorization theorem [8]). *Given a semigroup morphism φ from $(S, \otimes)$ (possibly infinite) to a finite semigroup $(T, \odot)$, and some $A \subseteq S$, set $B_0 = A$ and for all $n \geq 0$,*

$$B_{n+1} = B_n \cup B_n \otimes B_n \cup \bigcup_{\substack{e \in T \\ \text{is idempotent}}} \langle B_n \cap \varphi^{-1}(e) \rangle ,$$

then $\langle A \rangle = B_N$ for $N = 3|T| - 1$.

This theorem teaches us that, for computing the closure under product in the semigroup S, it is sufficient to be able to know how to compute (a) the union of sets, (b) the product of sets, and (c) the restriction of a set to the inverse image of an idempotent by φ, and (d) the closure under product of sets of elements that all have the same idempotent image under φ. Of course, this proposition is only interesting when the semigroup T is cleverly chosen.

In our case, we are going to use the above proposition with $(S, \otimes) = (\mathcal{W}_k, \otimes)$, and $(T, \odot) = (\mathcal{M}_k(\mathbb{S}), \odot)$, and φ the morphism which maps each weighted matrix (M, x) to $\varphi(M)$. Our algorithm will compute, given a presentation of a set of weighted matrices A, an approximation of $\langle A \rangle$ using the inductive principle of the factorization forest theorem. This is justified by the two following lemmas.

Lemma 5. *For all presentable sets of weighted matrices A, A', there exists effectively a presentable set of weighted matrices* $\texttt{product}(A, A')$ *such that*

$$A \otimes A' \approx \texttt{product}(A, A') \ .$$

Lemma 6. *For all presentable sets A such that $\varphi(A) = \{E\}$ for E an idempotent, there is effectively a presentable set* $\texttt{idempotent}(A)$ *such that*

$$\langle A \rangle \approx \texttt{idempotent}(A) \ .$$

Assuming that Lemmas 5 and 6 hold, it is easy to provide an algorithm which, given a presentable set A computes a presentable set $\texttt{closure}(A)$ as follows:

– Set $A_0 = A$ and for all $n = 0 \dots N-1$ (N taken from Theorem 3), set

$$A_{n+1} = A_n \cup \texttt{product}(A_n, A_n) \cup \bigcup_{\substack{E \in \mathcal{M}_k(\mathbb{S}) \\ \text{idempotent}}} \texttt{idempotent}(A_n \cap \varphi^{-1}(E)) \ .$$

– Output $\texttt{closure}(A) = A_N$.

The correctness of this algorithm is given by the following lemma. It derives from the good properties of $\approx$ given in Lemma 4. (See Appendix C.)

Lemma 7. *For all presentable sets of weighted matrices* $\texttt{closure}(A) \approx \langle A \rangle$.

This allows us to conclude the proof of Theorem 2 (see Appendix D for details). The algorithm takes an automaton δ, I, F as input, then it computes thanks to the above Lemma 7 a presentable set B that is $\approx$-equivalent to $\langle A \rangle$ where A is the set of weighted matrices corresponding to basic letters (i.e., $\{(\delta(a), 1) \ : \ a \text{ letter}\}$). Set (S, P) a presentation of B. Then the algorithm outputs $\inf\{{}^t I \odot X \odot F \mid X \in P\}$ that is computable since P is a finite union of polytopes. This cofficient is the answer of the algorithm: the minimal exponent such that the presentable set witnesses the existence of a behaviour of the automaton that has this growth-rate.

3 From Size-Change Systems to Max-Plus Automata

For proving Theorem 1, we define a translation of SCSs to max-plus automata. Let $\mathcal{S}$ be an SCS with k variables, which we assume to be numbered $x_1, \ldots, x_k$. We define an max-plus automaton $\phi(\mathcal{S})$ with $k+2$ states as follows: The alphabet $A_{\mathcal{S}}$ of $\phi(\mathcal{S})$ contains a letter a_T for every SCT $T \in \mathcal{S}$. We define the mapping δ of $A_{\mathcal{S}}$ to $\mathcal{M}_{k+2}(\overline{\mathbb{N}})$ as follows:

$$\delta(a_T)_{i,j} = \begin{cases} 0, & i = 1 \text{ or } j = k+2 \\ 1, & x_{i-1} > x'_{j-1} \in T \\ 0, & x_{i-1} \geq x'_{j-1} \in T \\ -\infty, & \text{otherwise} \end{cases}$$

Further, $\phi(\mathcal{S})$ has the initial and final vector $I = F = \mathbf{0} \in \mathcal{M}_{1,k+2}(\overline{\mathbb{N}})$. For example, the SCS from Example 1 is translated to the max-plus-automaton in Example 2.

The following lemmata relate SCSs and their translations; they allow us to derive Theorem 1 from Theorem 2 (all proofs are rather straight-forward and can be found in the appendix).

Lemma 8. *Let u be a word of $\phi(\mathcal{S})$ with ${}^tI \otimes \delta(u) \otimes F = N$. Then $\mathcal{S}$ has a trace with valuations over $[0, N]$ of length $|u|$.*

Lemma 9. *Assume $\mathcal{S}$ has a trace with valuations over $[0, N]$ of length l. Then there is a word u of $\phi(\mathcal{S})$ with ${}^tI \otimes \delta(u) \otimes F \leq N$ and $|u| = l$.*

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Appendix

A Proof of Lemma 1

Proof (of lemma 1). Let $x_1, \ldots, x_q \in \overline{\mathbb{R}^+}$. If one of $x_1, \ldots, x_q$ is $-\infty$, then $x_1 \odot \cdots \odot x_q = -\infty = x_1 + \cdots + x_q$. Otherwise, $x_1 \odot \cdots \odot x_q = \max(x_1, \ldots, x_q) \leq x_1 + \cdots + x_q \leq q \max(x_1, \ldots, x_q) = q(x_1 \odot \cdots \odot x_q)$. Since furthermore both $+$ and $\odot$ are non-decreasing with respect to their arguments, this relation between $+$ and $\odot$ over $\overline{\mathbb{R}^+}$ can be raised to matrices. □

B Proof of Lemma 4

Proof (of Lemma 4). (1) Direct from the definition.

(2) Direct from the definition of $\preccurlyeq$ over sets.

(3) Assume $(L, x) \preccurlyeq_a (M, y) \preccurlyeq_b (N, z)$, then $ax \geq y$ and $by \geq z$, thus $abx \geq z$. Furthermore $\varphi(L) = \varphi(M) = \varphi(N)$. Finally $L \leq aM$ and $M \leq bN$, thus $L \leq abN$. Overall $(L, x) \preccurlyeq_{ab} (N, z)$. This easily extends to sets.

(4) Assume $(M, x) \preccurlyeq_a (M', x')$ and $(N, y) \preccurlyeq_a (N', y')$. Then, $ax \geq x'$ and $ay \geq y'$ implies $a(x+y) \geq x'+y'$. Furthermore, since $\varphi(M) = \varphi(M')$ and $\varphi(N) = \varphi(N')$, we have $\varphi(M \otimes N) = \varphi(M) \odot \varphi(N) = \varphi(M') \odot \varphi(N') = \varphi(M' \otimes N')$. Finally, since $M \leq aM'$ and $N \leq aN'$, $M \otimes N \leq aM' \otimes aN' = a(M' \otimes N')$. Once more, this is easily extended to sets of weighted matrices.

(5) By induction, applying the second and fourth items. □

C Proof of Lemma 7

Proof (Lemma 7). We use Theorem 3 and prove by induction on n:

$$B_n \approx A_n$$

where B_n is as in Theorem 3. This is true if $n = 0$ ($B_0 = A = A_0$). Let $n \geq 0$, by lemmas 5 and 6, $\texttt{product}(A_n, A_n) \approx A_n \times A_n$ and $\texttt{idempotent}(A_n) \approx \langle A_n \rangle$. Then by lemma 4:

$$\begin{aligned} A_{n+1} = \quad & A_n \cup \texttt{product}(A_n, A_n) \\ & \cup \bigcup_{E \in \mathcal{M}_n(\mathbb{S}) \text{ idempotent}} \texttt{idempotent}(A_n \cap \varphi^{-1}(E)) \\ \approx \quad & B_n \cup B_n \times B_n \\ & \cup \bigcup_{E \in \mathcal{M}_n(\mathbb{S}) \text{ idempotent}} \langle B_n \cap \varphi^{-1}(E) \rangle \\ \approx \quad & B_{n+1} \end{aligned}$$

Then $\texttt{closure}(A) = A_N \approx B_N = \langle A \rangle$. □

D Proof of the main theorem, Theorem 2

Proof (Theorem 2). Given a max-plus automaton (δ, I, F), set $A = \{(\delta(a), 1) \mid a \in \mathbb{A}\}$ the set of small weighted matrices corresponding to letters. By lemma 7, one can compute $\texttt{closure}(A)$ that approximates $\langle A \rangle = \{(\delta(u), |u|) \mid u \in \mathbb{A}^*\}$. Since $\texttt{closure}(A)$ is presentable, let us note $\texttt{closure}(A) = \{(M, 1) \ : \ M \in S\} \cup \{(K[x], x) \ : \ K \in P, \ x \geq 1\}$. Then, by presentability, one can compute β that is the minimum of the values $K_{i,j}$ for all K matrices of exponents in P, i initial and j final. Then set $\alpha = +\infty$ if $\beta = \perp, \ -\infty$ or 0 and $\alpha = \frac{1}{\beta}$ otherwise. For $\alpha \neq +\infty$, let us prove the two following statements:

- There is an infinite sequence of words $(u_n)_n$ and an integer a such that for all n, $|u_n| \geq af(u_n)^\alpha$. Hence, denote by K the witness matrix such that $K_{i,j} = \beta$ for some i initial and j final. Since $\{(\delta(u), |u|) \mid u \in \mathbb{A}^*\} \preccurlyeq \texttt{closure}(A)$, then there is an integer b such that for all positive integer n there is a word u_n such that $(\delta(u_n), |u_n|) \preccurlyeq_b (K[n], n)$. Thus the sequence $(u_n)_{n \in \mathbb{N}-\{0\}}$ is infinite (since $n \leq b|u_n|$), and

$$\begin{aligned} (b|u_n|)^\beta &\geq n^\beta \\ &\geq^t I \otimes K[n] \otimes F \\ &\geq b({}^t I \otimes \delta(u_n) \otimes F) \\ &\geq bf(u_n) \end{aligned}$$

  Set $a = b^{\alpha - 1}$.
- There is an integer a such that for all words u of length greater than a, $|u| \leq af(u)^\alpha$. Hence, let b be the integer such that $\texttt{closure}(A) \preccurlyeq_b \{(\delta(u), |u|) \mid u \in \mathbb{A}^*\}$, there is (M, y) in $\texttt{closure}(A)$ such that $(M, y) \preccurlyeq_b (\delta(u), |u|)$. Moreover, since $|u| > b$, then $y > 1$ and thus (M, y) belongs to the asymptotic part of $\texttt{closure}(A)$. Then

$$\begin{aligned} |u|^\beta &\leq (by)^\beta \\ &\leq b^\beta({}^t I \otimes M \otimes F) \\ &\leq b^\beta bf(u) \end{aligned}$$

  Set $a = b^{1+\alpha}$.

It is similar to prove that $\alpha = +\infty$ if and only if there is a sequence of words that is bounded. $\square$

E Proofs of Lemma 5 and 6

E.1 Product of presentable sets

In this section, we establish Lemma 5. We need to compute the product of two presentable sets of weighted matrices A, B. We do this in several step. We first treat the case of the product of a set of asymptotic matrices with a set of small matrices.

Lemma 10. *Let A be a presentable set of weighted matrices, and S be a set of small matrices, then the set $\{(M \odot N, x) \ : \ (M,x) \in A,\ N \in S\}$ is effectively presentable. This set is denoted $A \odot S$.*

Proof. The results obviously holds if A has a small presentation. Thus, we just have to show it for the case when A has an asymptotic presentation $(\emptyset, P)$ (the general case being obtained by union). Let S' be $\gamma(S)$, i.e., obtained by application of the morphism from Lemma 3. Let us consider the set $Q = P \odot S' = \{Y \odot Z \ : \ Y \in P,\ Z \in S'\}$. Since P and S' as well as max and $\max_\odot$ (as subsets of Exps^3) are polytopes, and that by definition

$$Q = \left\{ X \in \mathit{Exps}^{k\times k} \ : \ \exists Y \in P \exists Z \in S' \bigwedge_{i,j \in [k]} X_{i,j} = \max_\ell(\max_\odot(Y_{i,\ell}, Z_{\ell,j})) \right\} ,$$

it follows that Q is effectively polytope.

Let us show now that $C = \{(M \odot N, x) \ : \ (M,x) \in A,\ N \in S\}$ equals $Q[*] = \{(X[n], n) \mid n \geq 1, X \in Q\}$. Consider a weighted matrix (K,x). We have $(K,x) \in C$ if and only if $K = M \odot N$ for $(M,x) \in A$ and $N \in S$, if and only if $M = Y[x]$ for some Y in P and $N = Z[x]$ for some $Z \in S'$, if and only if $K = (Y \odot Z)[x]$ for some $Y \in P$ and $Z \in S'$, if and only if $(K,x) \in Q[*]$. $\square$

Using the product with a set of small matrices as defined above, we can approximate the general product (which means, prove Lemma 5).

Lemma 11. *For all presentable sets of weighted matrices A, B, let* $\texttt{product}(A,B) = A \odot \varphi(B) \cup \varphi(A) \odot B$*, then*

$$A \otimes B \approx_2 \texttt{product}(A,B) \ .$$

Proof. First, $A \odot \varphi(B) \approx_2 \cup_{N \in \varphi(B)} A \otimes \{(N,1)\}$, that is included in $A \otimes B$. The case $\varphi(A) \odot B$ is symmetric. Hence $A \otimes B \preccurlyeq_2 \texttt{product}(A,B)$.

Conversely, consider $(M,x) \in A$ and $(N,y) \in B$. Suppose first that $x \geq y$, then, since $\varphi(N) \leq N$ and $2x \geq (x+y)$ we have:

$$\begin{aligned}(M,x) \otimes (N,y) &= (M \otimes N, x+y) \\ &\succcurlyeq_2 (M \otimes \varphi(N), x) \in A \odot \varphi(B) \ .\end{aligned}$$

If $x \leq y$, we similarly get $(M,x) \otimes (N,y) \succcurlyeq_2 (\varphi(M) \otimes N, y) \in \varphi(A) \odot B$. Hence we have $\texttt{product}(A,B) \preccurlyeq_2 A \otimes B$.

E.2 Iteration of Idempotents: Uniformization

Our goal from now is to establish Lemma 6, which means to compute, given a presentable set of weighted matrices A that is sent by φ to a single idempotent E, a set $\texttt{idempotent}(A)$ such that

$$\texttt{idempotent}(A) \approx \langle A \rangle \ .$$

We show in this first section that we can reduce this problem to sets of weighted matrices that have the further property of being uniform. We introduce here the notion of uniform matrices and show this reduction. From now a small idempotent matrix E is fixed, i.e., such that $E \odot E = E$. All matrices that we consider from now are mapped by φ to E.

Given a matrix M such that $\varphi(M) = E$, its **uniformization** $\texttt{uniform}(M)$ is the matrix

$$\texttt{uniform}(M) = E \odot M \odot E \ .$$

A matrix such that $\texttt{uniform}(M) = M$ is said **uniform**. This notation is extended to weighted matrices by $\texttt{uniform}(M, x) = (\texttt{uniform}(M), x)$. It is also extended to sets of weighted matrices as usual with:

$$\texttt{uniform}(A) = \{\texttt{uniform}(M, x) \ : \ (M, x) \in A\}.$$

Lemma 12. *If M, N are uniform then $M \leq M \otimes N$ and $N \leq M \otimes N$.*

Proof. Indeed, since $E \leq N$ and using Lemma 1, $M = M \odot E \leq M \odot N \leq M \otimes N$. The other case is symmetric.

A consequence is that we can 'eliminate' any term in a product of uniform matrices.

Corollary 1. *If $M_1, \dots, M_q$ are uniform, and $1 \leq i_1 < i_2 < \dots i_n \leq q$ with $n \geq 1$, then $M_{i_1} \otimes \cdots \otimes M_{i_n} \leq M_1 \otimes \cdots \otimes M_q$.*

To formalize this, we say that two indices i, j are **connected***, written $i \to j$ if $E_{i,j} \neq -\infty$, i.e., $E_{i,j} \in \{0, 1\}$. Remark that the usual notion of connectedness would refer to paths rather than a direct connection from i to j. In fact in our case, because E is idempotent, this is equivalent. Indeed, we have that whenever $i \to j$ and $j \to k$ then $i \to k$. If $i \to j$ and $j \to i$ then we simply write $i \leftrightarrow j$. A matrix M (such that $\varphi(M) = E$) is* **uniform** *if whenever $i \leftrightarrow i'$ and $j \leftrightarrow j'$, then $M_{i,j} = M_{i',j'}$. A weighted matrix (M, x) is* **uniform** *if M is uniform.*

Let us immediately explain why we chose this terminology:

Lemma 13. *For a matrix M such that $\varphi(M) = E$, $\texttt{uniform}(M)$ is uniform. Furthermore $\texttt{uniform}(\texttt{uniform}(M)) = \texttt{uniform}(M)$.*

Proof. Let us first show that E is uniform. This comes from the fact that E is an idempotent. Let $i \leftrightarrow i'$ and j be indices. Then $E_{i,j} = \max_\ell E_{i,\ell} \odot E_{\ell,j}$. So, in particular, $E_{i,j} \geq E_{i',j}$. By symmetry, we get that $E_{i',j} \geq E_{i,j}$ and thus $E_{i,j} = E_{i',j}$. Again by symmetry (this time by transposing the matrix E), we get that if $j \leftrightarrow j'$ then $E_{i',j} = E_{i',j'}$. Hence, if both $i \leftrightarrow i'$ and $j \leftrightarrow j'$ we have $E_{i,j} = E_{i',j'}$.

Let us now consider the indices $i \leftrightarrow i'$ and $j \leftrightarrow j'$ in the matrix $E \odot M \odot E$. Since E is uniform, $E_{i,h} = E_{i',h}$ for all h and $E_{\ell,j} = E_{\ell,j'}$ for all ℓ. Hence,

$$\begin{aligned}
\texttt{uniform}(M)_{i,j} &= \max_{h,\ell}(E_{i,h} \odot M_{h,\ell} \odot E_{\ell,j}) \\
&= \max_{h,\ell}(E_{i',h} \odot M_{h,\ell} \odot E_{\ell,j'}) \\
&= \texttt{uniform}(M)_{i',j'} \ ,
\end{aligned}$$

which means that $\mathtt{uniform}(M)$ *is uniform.*

Lemma 14. *For all presentable sets of weighted matrices A,* $\mathtt{uniform}(A)$ *is effectively presentable.*

Proof. Direct from Lemma 10.

We shall now conclude this section by proving that for computing $\langle A\rangle$ it is sufficient to be able to compute $\langle \mathtt{uniform}(A)\rangle$.

Lemma 15. *Given a set of weighted matrices A mapped to E by φ such that $(E,1)\in A$ then*

$$\langle A\rangle \approx_9 A\cup(A\otimes A)\cup(A\otimes\langle\mathtt{uniform}(A)\rangle\otimes A)\ .$$

We shall use the rest of this section for establishing Lemma 15.

For the first direction, remark that if $(M,x)\in A$, then since $(E,1)\in A$, this means that $\mathtt{uniform}(M,x)=(E\odot M\odot E,x)\approx_3 (E,1)\otimes(M,x)\otimes(E,1)\in\langle A\rangle$. Hence, $\langle A\rangle\preccurlyeq_3 A\cup(A\otimes A)\cup(A\otimes\langle\mathtt{uniform}(A)\rangle\otimes A)$.

The converse direction is slightly more involved. Consider a weighted matrix (M,x) in $\langle A\rangle$. It can be decomposed as a product

$$(M,x)=(M_1,x_1)\otimes\cdots\otimes(M_q,x_q)$$

for some $q\geq 1$, and (M_r,x_r) in A for $r=1\ldots q$. Of course, if $q=1$ or $q=2$, this means $(M,x)\in A\cup(A\otimes A)$. Thus, from now we assume that $q\geq 3$. Two positions r,s in $[1,q]$ are said **neighbors** if either $r=s+1$ or $s=r+1$.

Lemma 16. *There exists a subset $I\subseteq\{1,\ldots,q\}$ that does not contain neighbor positions, and such that $3\sum_{s\in I}x_s\geq x$.*

Proof. I is constructed inductively as follows. Start with $I_0=\emptyset$. Then at each step n, pick the position s such that (a) s is not in I_n and not a neighbour of a position in I_n, and (b) is such that x_s is maximal among the positions that satisfy (a). Then construct $I_{n+1}=I_n\cup\{s\}$. At some point the processes stops because no more position satisfy (a). We define I to be this set.

Of course, no two neighbours can be in I since this is forbidden by (a). Now, from the construction process, for all positions s, then either s is in I, and we set $g(s)=s$, or s has a neighbour t in I such that $x_t\geq x_s$ (indeed, by contraposition, if all the neighbours u of s are such that $x_u<x_s$, then s would have been chosen in the construction process before its neightbours, and thus would be in I), and we set $g(s)=t$. Hence, we have $x_s\leq x_{g(s)}$ for all $s=1\ldots q$. Furthermore, $g(s)$ is eigher s or a neighbor of s. As a consequence no more than three s can have the same value under g. It follows that $x=\sum_{s=1\ldots q}x_s\leq\sum_{s=1\ldots q}x_{g(s)}\leq 3\sum_{s\in I}x_s$.

Consider now $I=\{1\}\cup I'\cup\{q\}$ with I' the subset constructed in the previous lemma. Let us write $I=\{1=i_1<\cdots<i_\ell=q\}$. Remark that $E\otimes E\preccurlyeq_2 E\preccurlyeq_1$

M_s for all $s = 1 \dots q$. Thus, we can substitute each M_s that does not belong to I with either $(E, 1)$ or $(E, 1) \otimes (E, 1)$ and get:

$$\begin{aligned}
&(M_1, x_1) \otimes \texttt{uniform}(M_{i_2}, x_{i_2}) \otimes \dots \\
&\qquad\qquad \dots \otimes \texttt{uniform}(M_{i_{\ell-1}}, x_{i_{\ell-1}}) \otimes (M_q, x_q) \\
&\qquad \preccurlyeq_3 (M_1, x_1) \otimes ((E, 1) \otimes (M_{i_2}, x_{i_2}) \otimes (E, 1)) \otimes \dots \\
&\qquad \dots \otimes ((E, 1) \otimes (M_{i_{\ell-1}}, x_{i_{\ell-1}}) \otimes (E, 1)) \otimes (M_q, x_q) \\
&\qquad\qquad\qquad\qquad \preccurlyeq_3 (M, x) \ .
\end{aligned}$$

The last approximation is a combination od the previous lemma and of corollary 1. In particular, the weight of the left matrix is at least equal to $\sum_{s \in I} x_s$ and $3(\sum_{s \in I} x_s) \geq x$. Thus, $A \otimes \langle \texttt{uniform}(A) \rangle \otimes A \preccurlyeq_9 (M, x)$.

E.3 Iteration of Idempotents: Iteration of Uniform Matrices

In the previous section we have seen that it is sufficient to be able to compute $\langle A \rangle$ for A a presentable set of uniform matrices. In this section, we reduce this problem to iterate a single matrix at a time. From now, we fix A a presentable set of uniform matrices. As before, φ maps A to a single idempotent element E.

We first start by describing a property of iteration of uniform weighted matrices, namely that removing terms from the iteration (but not too many of them) can only decrease the result of the product (up to approximation). This result is elementary, but will prove useful.

Lemma 17. *Let $(M, x) = (M_1, x_1) \otimes \dots \otimes (M_q, x_q)$ for $(M_1, x_1), \dots, (M_q, x_q) \in A$, and let $1 \leq i_1 < \dots < i_j \leq q$ be integers such that $a(x_{i_1} + \dots + x_{i_j}) \geq x_1 + \dots + x_q$ for some $a \geq 2$. Then*

$$(M_{i_1}, x_{i_1}) \otimes \dots \otimes (M_{i_j}, x_{i_j}) \preccurlyeq_a (M, x) \ .$$

Proof. From 1, $M_{i_1} \otimes \dots \otimes M_{i_j} \leq M$. Since furthermore $a(x_{i_1} + \dots + x_{i_j}) \geq x_1 + \dots + x_q$, the result follows.

The following lemma shows how that in a product of uniform weighted matrices, it is possible to take one of the weighted matrices, and iterate only this one. The following remark is useful beforehand for understanding:

Lemma 18. *Let $M = M_1 \otimes \dots \otimes M_q$ be a product of uniform matrices with $\varphi(M_s) = E$ for all s, then for all indices i, $M_{i,i} = \sum_{s=1}^{q} (M_s)_{i,i}$.*

Furthermore, for all indices i, j,

$$M_{i,j} \leq k \max(\max_{s=2 \dots q-1} (M_1 \otimes M_s \otimes M_q)_{i,j}, \max_{i \to \ell, \ell \to j} \sum_{s=2 \dots q-1} (M_s)_{\ell,\ell}) \ .$$

Proof. For the first statement, let h, j be indices such that $i \to h \to j \to i$, then since the matrices are uniform, for all s, $(M_s)_{h,j} = (M_s)_{i,i}$. That is why $M_{i,i} = \sum_{s=1}^{q} (M_s)_{i,i}$.

For the second statement, let ℓ such that $i \to \ell \to j$. Then $M_{i,\ell}$ and $M_{\ell,j}$ are different from $-\infty$. So $M_{i,j} \geq (M_1)_{i,\ell} + \sum_{p=2\dots q-1}(M_s)_{\ell,\ell} + (M_q)_{\ell,j} \geq \sum_{s=2\dots q-1}(M_s)_{\ell,\ell}$. Moreover, for all s, $(M_1 \otimes M_s \otimes M_q)_{i,j} \leq M_{i,j}$ by Corollary 1. Conversely, consider the maximal sum involved in the computation of $M_{i,j}$. Suppose that you use for some s, a coefficient $(M_s)_{h,\ell}$ greater than all the diagonal coefficients in M_s. Then $E_{\ell,h} = -\infty$ (otherwise by uniformity, $(M_p)_{h,\ell} = (M_s)_{h,h} = (M_s)_{\ell,\ell} = (M_s)_{\ell,h}$, that contradicts the hypothesis). Thus the index h will be no longer be used in the sum. So at most k coefficients greater than the diagonal coefficients can be used. Moreover, by using the first part of the lemma, $(M_p \otimes \cdots \otimes M_t)_{\ell,\ell} = \sum_{s=p}^{t}(M_s)_{\ell,\ell}$. Therefore

$$M_{i,j} \leq k \max_{\substack{i \to h \\ \ell \to j \\ 1 \leq s \leq q}} (M_s)_{h,\ell} + k \max_{\substack{i \to \ell \\ \ell \to j}} \sum_{s=1\dots q-1} (M_s)_{\ell,\ell}.$$

Lemma 19. *There exist $a \geq 1$ such that for all products $(M,x) = (M_1,x_1) \otimes \dots \otimes (M_q,x_q)$, $q \geq 3$ of uniform weighted matrices, then for some $p = 2 \dots q-1$ and some positive integer n,*

$$(M_1,x_1) \otimes (M_p,x_p)^n \otimes (M_q,x_q) \preccurlyeq_a (M,x) \ .$$

Proof. Let $(N,y) = (M_2,x_2) \otimes \cdots \otimes (M_{q-1},x_{q-1})$. For an index i such that $E_{i,i} = 1$, and some $p = 2 \dots q-1$, we say that p is i-**bad** if $(M_p)_{i,i} > \frac{kx_p}{y} N_{i,i}$. If there is some i such that p is i-bad, then p is simply said **bad**. Otherwise p is **good**.

We claim that there exists a good p. The argument proceeds by counting the weight of bad p's. For all i such that $E_{i,i} = 1$, set $z_i = \sum_{p\ i\text{-bad}} x_p$. We have

$$N_{i,i} \geq \sum_{p\ i\text{-bad}} (M_p)_{i,i} > \sum_{p\ i\text{-bad}} \frac{kx_p}{y} N_{i,i} > \frac{kz_i}{y} N_{i,i} \ .$$

Thus, $z_i < \frac{y}{k}$ (since $N_{i,i} \geq E_{i,i} = 1$). Hence we get $\sum_{p\ \text{bad}} x_p \leq \sum_i z_i < y$. This means that there exists a good p.

Let now $n = \lceil \frac{y}{x_p} \rceil$. Let $(R,z) = (M_1,x_1) \otimes (M_p,x_p)^n \otimes (M_q,x_q)$. Let us remark first that $z = x_1 + nx_p + x_q \geq x$ by definition of n. Let now i,j be indices such that $E_{i,j} \geq 1$. Remark that $M_1 \otimes M_p \otimes M_q \leq M$ by Lemma 1. Furthermore, consider ℓ such that $E_{\ell,\ell} = 1$. Then (since p is good, $x_p \geq 1$ and $y \geq x_p$) $(M_p^n)_{\ell,\ell} = n(M_p)_{\ell,\ell} \leq \lceil \frac{y}{x_p} \rceil \frac{kx_p}{y} N_{\ell,\ell} \leq 2kN_{\ell,\ell}$. In combination with Lemma 18, we get that

$$(M_1,x_1) \otimes (M_p,x_p)^n \otimes (M_q,x_q) \preccurlyeq_{2kb} (M,x) \ ,$$

where b is the constant from Lemma 18.

E.4 Iteration of Idempotents: the Single Matrix Case

We shall now prove that we can compute the effect of iterating a single matrix taken from a presentable set. Since presentable sets of weighted matrices contain a finite part and an asymptotic part, we successively treat the two situations. Again, E is a fixed small idempotent matrix. Furthermore A is a presentable set of uniform weighted matrices that are all mapped to E by φ. We aim at computing a presentable set $B \approx \{(M,x)^m \ : \ m \in \mathbb{N}^+,\ (M,x) \in A\}$. From this we will be able to conclude.

The case of small matrices. Let $(M,1)$ be a small matrix from A (in fact, there is exactly one such matrix, namely $(E,1)$). Let us describe a presentable set and show that it is equivalent to $\langle (M,1)\rangle = \{(M^n,n) \ : \ n \in \mathbb{N}^+\}$.

Let K be the exponent matrix defined by:

$$K_{i,j} = \begin{cases} \perp & \text{if } E_{i,j} = -\infty \\ -\infty & \text{if } E_{i,j} = 0 \\ 1 & \text{if } E_{\ell,\ell} = 1 \text{ for some } \ell \text{ such that } i \to \ell \to j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove that there exists a such that $(M^n,n) \approx_a (K[n],n)$ for all positive integers n. We do it by case distinction depending on $K_{i,j}$.

Subcase 0. If $K_{i,j} = \perp$ then $K_{i,j}[n] = -\infty = M^n_{i,j}$. If $K_{i,j} = -\infty$ then $K_{i,j}[n] = 0 = M^n_{i,j}$.

Subcase 1. If $K_{i,j} = 0$, $K[n]_{i,j} = n^0 = 1 \leq E_{i,j} \leq M^n_{i,j}$. For the sake of contradiction, assume now that $M^n_{i,j} \geq k+2$ for some n. This would mean that there are indices $i = i_0, i_1, i_2, \ldots, i_\ell = j$ such that $M_{i,i_1} + M_{i_1,i_2} + M_{i_2,i_3} + \ldots + M_{i_{\ell-1},j} \geq k+2$. By pigeon hole principle, there are $0 < s < q < \ell$ such that $i_s = i_q$ and $M_{i_s,i_{s+1}} + \cdots + M_{i_{q-1},i_q} \geq 1$. Using the fact that E is idempotent, this means that $i \to i_s$, $E_{i_s,i_s} = 1$, and $i_s \to j$. This proves that $K_{i,j} = 1$ by definition. A contradiction. Hence $M^n_{i,j} \leq k+1 \preccurlyeq_{k+1} K[n]$.

Subcase 2. If $K_{i,j} = 1$, clearly $K[n]_{i,j} = n \geq M^n_{i,j}$. Conversely, by definition of $K_{i,j}$, there exists p such that $i \to p$, $p \to j$ and $E_{p,p} = 1$. If $n = 1$ or $n = 2$, since E is idempotent, $E_{i,j} = E^2_{i,j} = E_{p,p} = 1$, then $2(M^n_{i,j}) \geq 2 \geq K[n]$. Otherwise, the path $i, p, \ldots, p, j$ witnesses that $M^k_{i,j} \geq n-2$. It follows that $3M^n \geq K[n]$.

If we combine all the above cases, we get that:

$$\langle (M,1)\rangle \approx_{k+2} \{K[\ell] \ : \ \ell \in \mathbb{N}^+\} \approx_2 \{K[x] \ : \ x \geq 1\} \ .$$

Case of asymptotic presentations. In this case, we have a set of weighted uniform matrices

$$A = \{K[x] \ : \ x \geq 1,\ A \in P\},$$

where P is a polytope of exponents matrices, and our goal is to compute an approximation of the set:

$$B = \{(K[x],x)^m \ : \ x \geq 1,\ m \in \mathbb{N}^+,\ K \in P\} \ .$$

Let K be in P. The new weighted matrix $(K[x], x)^m$ depends on two parameters, namely x and m. We shall now introduce another parameter $\lambda \in [0,1]$ that informally describes the balance between the two parameters x and m. The case $\lambda = 0$ corresponds to x small (i.e., 'bounded'), and m large, while $\lambda = 1$ corresponds to x large, and m small (i.e., 'bounded').

Set $\lambda \in [0,1]$, and let $K^{(\lambda)}$ be the matrix of exponents defined for all indices i, j by

$$K^{(\lambda)}_{i,j} = \begin{cases} \perp & \text{if } E_{i,j} = -\infty \\ -\infty & \text{if } E_{i,j} = 0 \\ \alpha^{(\lambda)}_{i,j} & \text{otherwise,} \end{cases}$$

where

$$\alpha^{(\lambda)}_{i,j} = \max\Big(\max_{i \to \ell \to j, E_{\ell,\ell}=1} \big(1 - \lambda + \lambda K_{\ell,\ell}\big), \max_{i \to h, \ell \to j, E_{h,\ell}=1} \lambda K_{h,\ell} \Big) \ .$$

Define now

$$A^{(*)} = \{K^{(\lambda)}[x] \ : \ x \geq 1,\ K \in P,\ \lambda \in [0,1]\} \ .$$

Let us first treat the effectivity question.

Lemma 20. *The set of weighted matrices $A^{(*)}$ is presentable.*

Proof. We have $A^{(*)} = \big\{K[x] \ : \ x \geq 1,\ K \in \{L^{(\lambda)} \ : \ L \in P,\ \lambda \in [0,1]\}\big\}$. So we just have to show that the set of exponents matrices $\{L^{(\lambda)} \ : \ L \in P,\ \lambda \in [0,1]\}$ is presentable. This is obvious since it is obtained from P by affine interpolations according to λ, and by maxima, which maintain polytopes. $\square$

What remains to be done is to prove that $A^{(*)} \approx_a B$ for a suitable $a \geq 1$.

We start by a matrix of the form $(K[x], x)^m \in B$. If $m = 1$, $(K[x], x) = (K^{(0)}[x], x)$, and if $m = 2$, $(K[x], x)^2 \approx_2 (K^{(0)}[2x], 2x)$. Let us assume $x \geq 1$ and $m \geq 3$. Let $\gamma \geq 0$ be such that $x = m^\gamma$ (this is possible since $m > 1$), and set $\lambda = \frac{\gamma}{1+\gamma} \in [0,1)$. First let us prove that $K^{(\lambda)}[m^{\gamma+1}]_{i,j} \leq 3K[x]^m_{i,j}$. Since $\varphi(K[x]^m) = E = \varphi(K^{(\lambda)}[m^{\gamma+1}])$, we just have to compare the entries for i, j such that $E_{i,j} \geq 1$.

We have to compare the entry $(K[x]^m)_{i,j}$ with the term witnessing the maximum in the definition $\alpha^{(\lambda)}_{i,j}$. The *first subcase* is when this term corresponds to ℓ such that $i \to \ell \to j$, and the term is $1-\lambda+\lambda K_{\ell,\ell}$. Then, unfolding the definition of $K[x]^m$, we get

$$\begin{aligned} 3(K[x]^m)_{i,j} &\geq 3(K[x]_{i,\ell} + (m-2)K[x]_{\ell,\ell} + K[x]_{\ell,j}) \\ &\geq m x^{K_{\ell,\ell}} && \text{(since } m \geq 3\text{)} \\ &\geq (m^{\gamma+1})^{1-\lambda+\lambda K_{\ell,\ell}} \ . \end{aligned}$$

The *second subcase* is when the maximum involved in the computation of $\alpha_{i,j}^{(\lambda)}$ is witnessed by indices h and ℓ such that $i \to h$ and $\ell \to j$, and the corresponding term is $\lambda K_{h,\ell}$. In this case,

$$K[x]_{i,j}^m \geq K[x]_{h,\ell} \geq x^{K_{h,\ell}} \geq (m^{\gamma+1})^{\lambda K_{h,\ell}} \ .$$

Summarizing, we have $3K[x]_{i,j}^m \geq K^{(\lambda)}[m^{\gamma+1}]_{i,j}$, and hence

$$A^{(*)} \ni (K^{(\lambda)}[m^{\gamma+1}], m^{\gamma+1}) \preccurlyeq_3 (K[x]^m, mx) \ .$$

Then $A^{(*)} \preccurlyeq B$.

Conversely, let us consider some weighted matrix of the form $(K^{(\lambda)}[y], y)$ in $A^{(*)}$, i.e., for some $\lambda \in [0,1]$ and $y \geq 1$. Set $m = \lfloor y^{1-\lambda} \rfloor$, m is a positive integer, and $x = \frac{y}{m} \geq 1$.

Let us prove that $(K[x]^m, mx) \preccurlyeq_{k+2} (K^{(\lambda)}[y], y)$. First $mx = y$ and $\varphi(K[x]^m) = E = \varphi(K^{(\lambda)}[y])$. Then, if $m = 1$ and thus $y = x$ and $1 \leq y^{1-\lambda} < 2$, we have $x^{K_{i,j}} \leq (x^{1-\lambda})^{K_{i,j}} x^{\lambda K_{i,j}} \leq 2y^{K_{i,j}^{(\lambda)}}$. Otherwise if $m \geq 2$, then

$$K[x]_{i,j}^m \leq kx^{\max_{i\to k, \ell \to j, E_{k,\ell} \neq -\infty} K_{k,\ell}} + mx^{\max_{i \to \ell, \ell \to j, E_{\ell,\ell} \neq -\infty} K_{\ell,\ell}}$$

Besides,

$$mx^{\max_{i\to \ell, \ell \to j, E_{\ell,\ell} \neq -\infty} K_{\ell,\ell}} \leq \\ y^{(1-\lambda)(1-\max_{i\to \ell, \ell\to j, E_{\ell,\ell}\neq -\infty} K_{\ell,\ell})} y^{\max_{i\to\ell,\ell\to j, E_{\ell,\ell}\neq-\infty} K_{\ell,\ell}}$$

Thus,

$$K[x]_{i,j}^m \leq (k+2) y^{K_{i,j}^{(\lambda)}} \ .$$

Then $B \preccurlyeq A^{(*)}$.

If we put all the previous reasoning together, we obtain:

Lemma 21. *There exists $a \geq 1$ such that for all presentable sets of uniform weighted matrices A,*

$$A^{(*)} \approx_a \{(M,x)^m \ : \ m \in \mathbb{N}^+, \ (M,x) \in A\} \ .$$

F Proof of Lemma 8

Let $u = a_{T_1} \cdots a_{T_l}$ be a word of $\phi(\mathcal{S})$ with ${}^tI \otimes \delta(u) \otimes F = N$. We define vectors $A_i = {}^tI \otimes \delta(a_{T_1} \cdots a_{T_i})$ for all $0 \leq i \leq l$. We show $0 \leq (A_i)_j \leq N$ for all $0 \leq i \leq l$ and $1 \leq j \leq k+2$ (*): The inequality $0 \leq (A_i)_j$ holds because the first row in the matrix $\delta(a_{T_1} \cdots a_{T_i})$ consists of entries greater equal to zero and I is the $k+2$ dimensional zero-vector. The inequality $(A_i)_j \leq N$ follows from ${}^tI \otimes \delta(u) \otimes F = {}^tA_i \otimes \delta(a_{T_{i+1}} \cdots a_{T_l}) \otimes F = N$ because the column $k+2$ in $\delta(a_{T_{i+1}} \cdots a_{T_l})$ consists of entries greater equal to zero and F is the $k+2$ dimensional zero-vector.

We define valuations σ_i over $[0, N]$ as follows: We set $\sigma_i(x_j) = N - (A_i)_{j+1}$ for all $0 \leq i \leq l$ and $1 \leq j \leq k$. By (*) these valuations are well-defined. We show $\sigma_{i-1}, \sigma_i' \models T_i$ for any $1 \leq i \leq l$: We fix some size-change predicate $x_s \triangleright x_r' \in T_i$. Then $(a_{T_i})_{s,t}$ is 0 or 1 depending on whether $\triangleright$ is $\geq$ or $>$. Thus, we have $\sigma_{i-1}(x_s) = N - (A_{i-1})_{s+1} \triangleright N - (A_{i-1} \otimes \delta(a_{T_i}))_{r+1} = N - (A_i)_{r+1} = \sigma_i(x_r)$.

We have proved that $\sigma_0 \xrightarrow{T_1} \sigma_1 \cdots \xrightarrow{T_l} \sigma_l$ is a trace of $\mathcal{S}$.

G Proof of Lemma 9

Let $\sigma_0 \xrightarrow{T_1} \sigma_1 \cdots \xrightarrow{T_l} \sigma_l$ be a trace of $\mathcal{S}$ with valuations over $[0, N]$. Let $u = a_{T_1} \cdots a_{T_l}$ be the corresponding word of $\phi(\mathcal{S})$. In the following we show that ${}^tI \otimes \delta(u) \otimes F \leq N$: We set $A_0 = {}^tI$. We inductively define vectors $A_i = A_{i-1} \otimes \delta(a_{T_i})$ for all $1 \leq i \leq l$. It is easy to see that $(A_i)_1 = 0$ for all i by definition of I and the matrices $\delta(a_T)$ (i). Moreover, it is easy to prove that A_i consists only of entries greater equal than zero using the fact that every matrix $\delta(a_T)$ consists of zeros in its first row and that I is the $k+2$ dimensional zero-vector (ii).

We show by induction on i that $(A_i)_{j+1} \leq N - \sigma_i(x_j)$ holds for all $1 \leq j \leq k$ and $0 \leq i \leq l$ (iii). Clearly, this holds for $i = 0$. We consider some $1 \leq i \leq l$. We fix some j with $1 \leq j \leq k$. We have $(A_i)_{j+1} = \max_{h \in [1,k+2]}((A_{i-1})_h + \delta(a_{T_i})_{h,j+1})$. Let $h \in [1, k+2]$ be an index that gives the maximum. We proceed by a case distinction on h.

$h = 1$: then we have $(A_{i-1})_1 = 0$ from (i) and further $\delta(a_{T_i})_{h,j+1} = 0$ by definition; thus $(A_i)_{j+1} = (A_{i-1})_1 + \delta(a_{T_i})_{h,j+1} = 0$ and (iii) holds trivially.

$h = k+2$: by definition we have $\delta(a_{T_i})_{h,j+1} = -\infty$; this is impossible, because $(A_i)_{j+1} = (A_{i-1})_h + \delta(a_{T_i})_{h,j+1} = -\infty$ contradicts (ii).

$h \in [2, k+1]$: $\delta(a_{T_i})_{h,j+1} \neq -\infty$ is not possible (see previous case). Thus by the definition of $\delta(a_{T_i})$ there is a size-change predicate $x_{h-1} \triangleright x_j' \in T_i$. By the definition of a trace we have $\sigma_{i-1}, \sigma_i' \models T_i$. Thus $\sigma_{i-1}(x_{h-1}) \triangleright \sigma_i(x_j)$. By induction assumption we have $(A_{i-1})_h \leq N - \sigma_{i-1}(x_{h-1})$. Thus $(A_i)_{j+1} = (A_{i-1})_h + \delta(a_{T_i})_{j+1,h} \leq N - \sigma_{i-1}(x_{h-1}) + \delta(a_{T_i})_{j+1,h} \leq N - \sigma_i(x_j)$. This proves (iii).

Next, we prove $(A_i)_{k+2} \leq N$ for all $0 \leq i \leq l$ by induction on i (iv). Clearly, this holds for $i = 0$. We consider some $1 \leq i \leq l$. We get $(A_i)_{k+2} = \max_{h \in [1,k+2]}((A_{i-1})_h + \delta(a_{T_i})_{h,k+2}) \leq N$ from the following facts: We have $(A_{i-1})_1 = 0$ by (i). By (iii) we have $(A_{i-1})_{j+1} \leq N - \sigma_{i-1}(x_j) \leq N$ for all $1 \leq j \leq k$. By induction assumption we have $(A_{i-1})_{k+2} \leq N$. By definition we have $\delta(a_{T_{i-1}})_{h,k+2} = 0$ for all $h \in [1, k+2]$.

By (i),(iii) and (iv) we have $(A_l)_j \leq N$ for all $1 \leq j \leq k+2$. Thus we have ${}^tI \otimes \delta(u) \otimes F = A_l \otimes F \leq N$.

H Proof of Theorem 1

Let $\mathcal{S}$ be a terminating SCS. Let f be the function computed by $\phi(\mathcal{S})$, i.e., the function that takes a word u over $\mathbb{A}_{\mathcal{S}}$ and returns the natural number ${}^tI \otimes \delta(u) \otimes$

F. By Theorem 2, there effectively is a number $\alpha \in \{+\infty\} \cup \{\alpha \in \mathbb{Q} \,:\, \alpha \geq 1\}$ such that $g(n) = \Theta(n^\alpha)$ (*), where $g(n) = \sup\{|w| \mid w \in \mathbb{A}_\mathcal{S}^*, f(w) \leq n\}$.

First, we establish the lower bound. From (*) we have that there is a $c \in \mathbb{Q}^+$ and there are infinitely many words u_N with $|u_N| \geq c \cdot N^\alpha$. From Lemma 8 we get that for each u_N there is a trace with valuations over $[0, N]$ with length $|u_N|$. Thus the longest trace of $\mathcal{S}$ with valuations over $[0, N]$ is of order $\Omega(N^\alpha)$.

Next, we establish the upper bound. From (*) we have that there is a $c \in \mathbb{Q}^+$ and there is a $N \in \mathbb{N}$ such that $g(n) \leq c \cdot n^\alpha$ for all $n \geq N$ (**). Consider now the longest trace with valuations over $[0, N]$. Let us assume that this trace is of length l. By Lemma 9 there is a word u_N of $\phi(\mathcal{S})$ with $f(u_N) \leq N$ and $|u_N| = l$. From (**) we get $l = |u_N| \leq g(N) \leq c \cdot N^\alpha$. Thus the longest trace of $\mathcal{S}$ with valuations over $[0, N]$ is of order $O(N^\alpha)$.

Finally, we show that $\alpha \neq +\infty$. We will assume that $\alpha = +\infty$ and show that $\mathcal{S}$ has an infinite trace. This contradicts the assumption that $\mathcal{S}$ is terminating. Assume $\alpha = +\infty$. Then by Theorem 2 there is an infinite set of words U such that $\{f(u) \mid u \in U\}$ is bounded. Let $N = \max\{f(s) \mid u \in U\}$. Let $w = a_{T_1} a_{T_2} \cdots$ be some infinite word with $\lim_i u_i = w$ for some words $u_i \in U$. We define vectors $A_i = {}^tI \otimes \delta(a_{T_1} \cdots a_{T_i})$ for all $i \geq 0$. We define valuations σ_n as follows: We set $\sigma_i(x_j) = N - (A_i)_{j+1}$ for all $0 \leq i$ and $1 \leq j \leq k$.

We fix some $i \geq 0$. Then $a_{T_1} \cdots a_{T_i} a_{T_i+1}$ is a prefix of some u_m (because of $\lim_i u_i = w$). Using this word u_m one can show $0 \leq (A_i)_j \leq N$ for all $1 \leq j \leq k+2$ (as in the proof of Lemma 8; recall that $f(u_m) \leq N$). Thus σ_i is a valuation over $[0, N]$. Further, as in the proof of Lemma 8 one can show that $\sigma_i, \sigma'_{i+1} \models T_{i+1}$.

Thus $\sigma_0 \xrightarrow{T_1} \sigma_1 \xrightarrow{T_2} \cdots$ is an infinite trace of $\mathcal{S}$ over $[0, N]$.

I Adding Control Structure to Size-change Systems

In the following, we discuss how to add control structure to our analysis. Let L be a regular languages of SCTs. We restrict the traces of an SCS $\mathcal{S}$ as follows: For every trace $\sigma_1 \xrightarrow{T_1} \sigma_2 \xrightarrow{T_2} \cdots$ of $\mathcal{S}$ we require the word $T_1 T_2 \cdots$ to be an element of L. Such a regular language L, for example, allows to model control flow graphs with initial and final states.

It remains to show that Theorem 2 holds true, when words are restricted to some regular language. However, the following lemma shows that this question can be reduced to the asymptotic behaviour of another max-plus-automaton, where words are not restricted:

Lemma 22. *Given a max-plus automaton computing a function f over $\mathbb{A}^*$ and a regular language $L \subseteq \mathbb{A}^*$, there exists effectively a max-plus automaton computing the function g such that for all words w,*

$$g(w) = \begin{cases} f(w) & \text{if } w \in L, \\ \max(|w|, f(w)) & \text{otherwise.} \end{cases}$$

Proof. Consider a non-deterministic automaton for $\mathbb{A}^* - L$, it can be transformed into a max-plus automaton computing a function h that maps a word w to $|w|$ if $w \notin L$, and $-\infty$ otherwise. For this it is sufficient to give weight 1 to all the transitions of the automaton. The function g of the conclusion of the lemma is then simply the max of f and h, which is naturally obtained by taking the disjoint union of the two automata.

J Example

Consider the automaton given in example 2. We will apply the algorithm to this example.

First, let us give the matrices over the words $((a^pb)^qc)^r$ and $((a^nb)^nc)^n)$.

$$\delta(((a^pb)^qc)^r) = \begin{pmatrix} 0 & (p+1)q & s-1 & s-1 & s-1 & max((p+1)q, s-2) \\ & & & & & (p+1)q \\ & & s-1 & s-1 & s-1 & s-2 \\ & & s & s & s & s-1 \\ & & s-p & s-p & s-p & s-p-1 \\ & & & & & 0 \end{pmatrix}$$

where $s = r(p+q+1)$.

$$(\delta(((a^nb)^nc)^n), n^3) \approx (\begin{pmatrix} 0 & n^2 & n^2 & n^2 & n^2 & n^2 \\ & & & & & n^2 \\ & & n^2 & n^2 & n^2 & n^2 \\ & & n^2 & n^2 & n^2 & n^2 \\ & & n^2 & n^2 & n^2 & n^2 \\ & & & & & 0 \end{pmatrix}, n^3)$$

Now, let us apply the algorithm (a simplified version for this example, in fact).

$$\texttt{uniform}(\delta(a)) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ & 1 & & & & \\ & & 0 & & & 0 \\ & & & 1 & & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

$$\texttt{uniform}(\delta(a))^* = \begin{pmatrix} 0 & x & 0 & x & 0 & x \\ & x & & & & \\ & & 0 & & & 0 \\ & & & x & & x \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}$$

$$\texttt{uniform}(\delta(a))^*b = \begin{pmatrix} 0 & x & 0 & 0 & x & x \\ & x & & & & x \\ & & 0 & 0 & 1 & 0 \\ & & & & x & x \\ & & & & 1 & 0 \\ & & & & & 0 \end{pmatrix}$$

$$\texttt{uniform}(\texttt{uniform}(\delta(a))^*b) = \begin{pmatrix} 0 & x & 0 & 0 & x & x \\ & x & & & & x \\ & & 0 & 0 & x & x \\ & & & & x & x \\ & & & & 1 & 1 \\ & & & & & 0 \end{pmatrix}$$

From now on, $0 \leq \lambda \leq 1$, and we note $M = \max(\lambda, 1-\lambda)$.

$$\texttt{uniform}(\texttt{uniform}(\delta(a))^*b)^* = \begin{pmatrix} 0 & x & 0 & 0 & x^M & x^M \\ & x & & & & x \\ & & 0 & 0 & x^M & x^M \\ & & & & x^M & x^M \\ & & & & x^{1-\lambda} & x^{1-\lambda} \\ & & & & & 0 \end{pmatrix}$$

$$\texttt{uniform}(\texttt{uniform}(\delta(a))^*b)^*c = \begin{pmatrix} 0 & x & 0 & 0 & x^M & x^M \\ & & & & & x \\ & & x^M & x^M & x^M & x^M \\ & & x^M & x^M & x^M & x^M \\ & & x^{1-\lambda} & x^{1-\lambda} & x^{1-\lambda} & x^{1-\lambda} \\ & & & & & 0 \end{pmatrix}$$

$$\texttt{uniform}(\texttt{uniform}(\texttt{uniform}(\delta(a))^*b)^*c) = \begin{pmatrix} 0 & x & x^M & x^M & x^M & x^M \\ & & & & & x \\ & & x^M & x^M & x^M & x^M \\ & & x^M & x^M & x^M & x^M \\ & & x^M & x^M & x^M & x^M \\ & & & & & 0 \end{pmatrix}$$

From now on, $0 \leq \mu \leq 1$ and we note $L = \mu M + (1-\mu)$.

$$\texttt{uniform}(\texttt{uniform}(\texttt{uniform}(\delta(a))^*b)^*c)^* = \begin{pmatrix} 0 & x^\mu & x^L & x^L & x^L & x^{max(\mu,L)} \\ & & & & & x^\mu \\ & & x^L & x^L & x^L & x^L \\ & & x^L & x^L & x^L & x^L \\ & & x^L & x^L & x^L & x^L \\ & & & & & 0 \end{pmatrix}$$

Let us show that:

$$\min_{0\leq\lambda,\mu\leq 1} \max(L,\mu) = \min_{0\leq\lambda,\mu\leq 1} \max(\mu, 1-\mu\lambda, 1-\mu+\mu\lambda) = \frac{2}{3}$$

First for $\mu = \frac{2}{3}$ and $\lambda = \frac{1}{2}$, $\max(\mu, 1-\mu\lambda, 1-\mu+\mu\lambda) = \frac{2}{3}$. Besides for all $0 \leq \lambda, \mu \leq 1$, $\mu + 1 - \mu\lambda + 1 - \mu + \mu\lambda = 2$ so at least one of the elements is greater than $\frac{2}{3}$ so

$$\min_{0\leq\lambda,\mu\leq 1} \max(\mu, 1-\mu\lambda, 1-\mu+\mu\lambda) \geq \frac{2}{3}.$$