The bridge between regular cost functions and omega-regular languages

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A quantitative extension of regular languages

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Regular languages over words of length ω

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A way to transfer results and constructions

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For all these questions, we do not care about precise values.

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Main theorem of regular cost functions:

B-side		S-side
B-rational expressions		S-rational expressions
=		=
B-automata	complement	S-automata
=		=
cost MSO		cost MSO*
=		=
(stabilisation monoids,up-sets)		(stabilisation monoids,down-sets)

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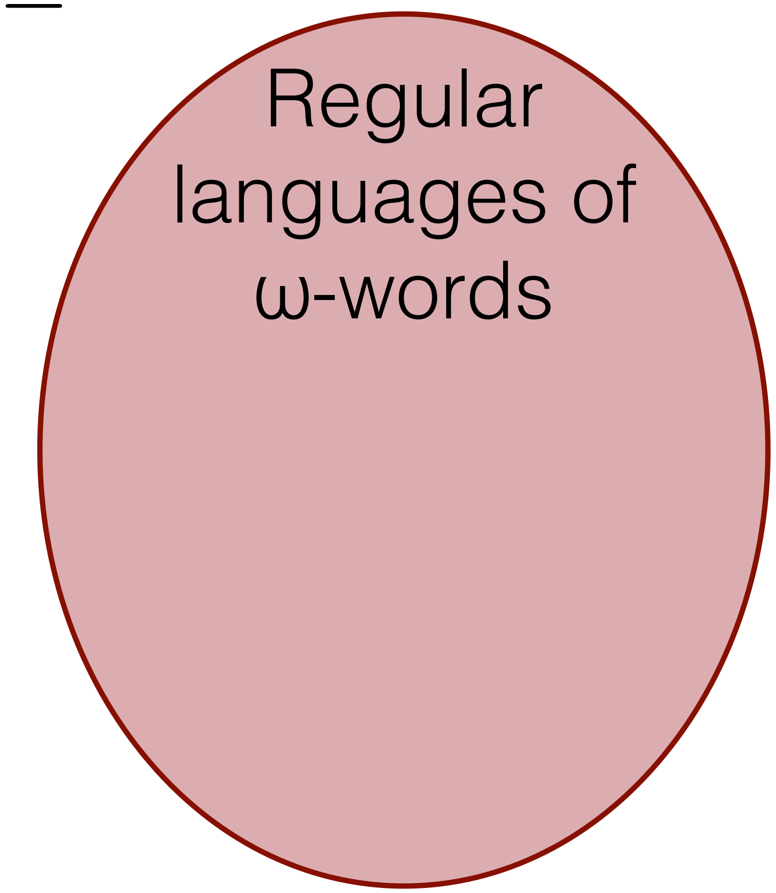
This work makes formal some similarities, and use it to factorizing proofs.

The bridge

The bridge

$L \subseteq A^\omega$



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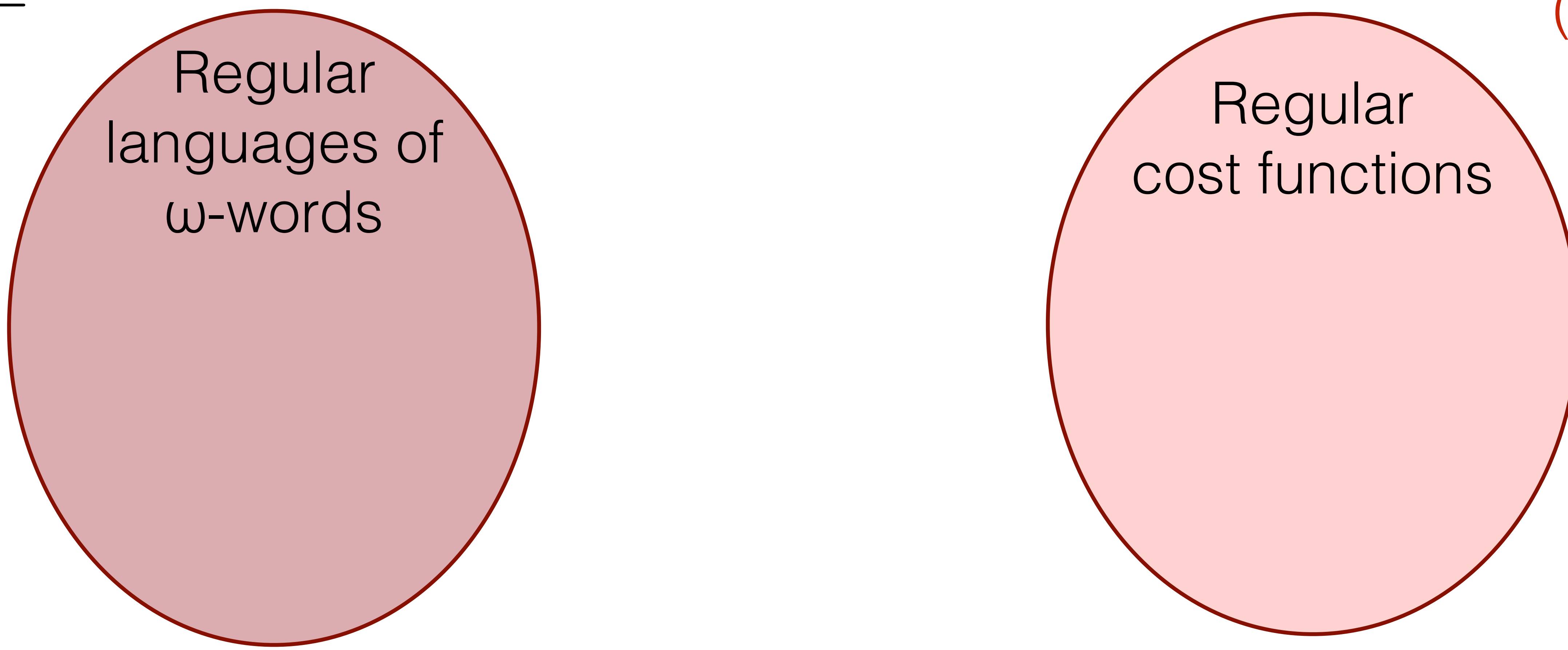

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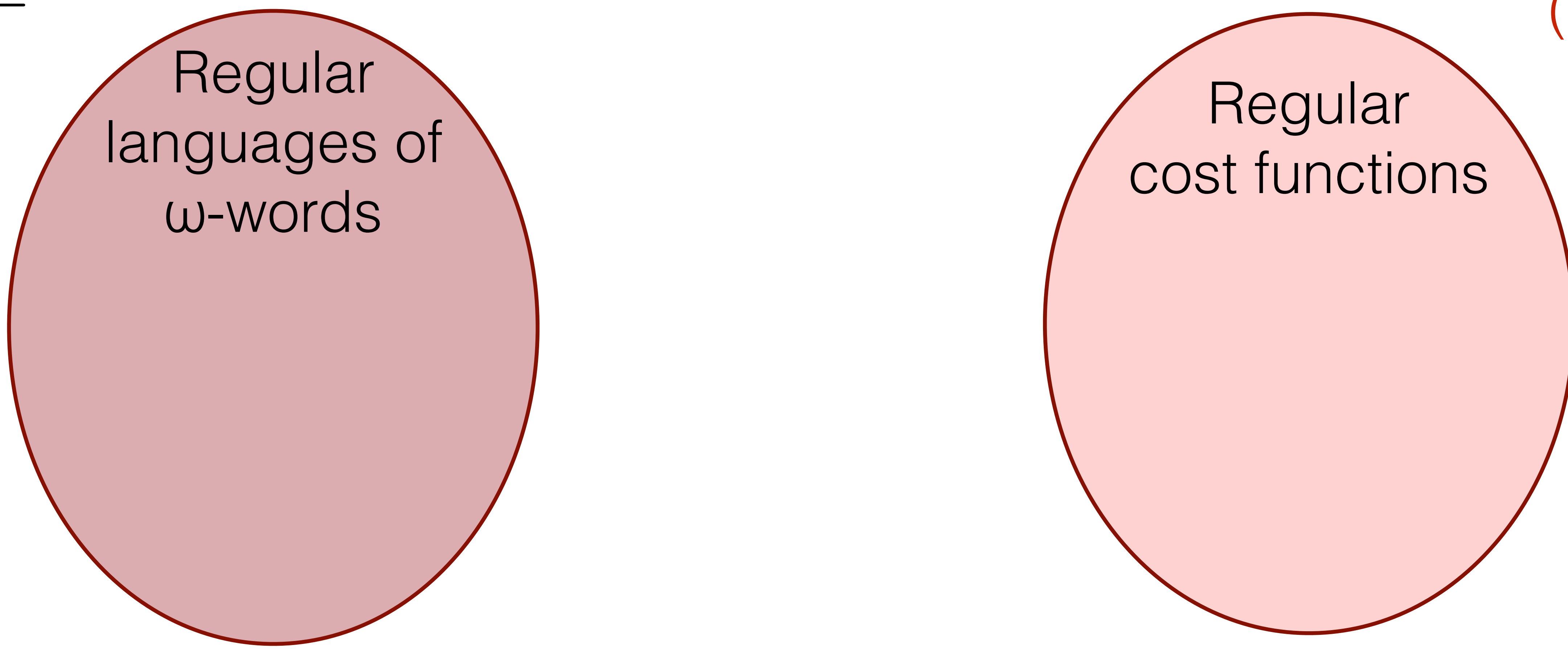

Definition: For $L \subseteq A^\omega$,

$$\begin{aligned} L^{\mathtt{ol}} : A^* &\to \mathbb{N} \cup \{\infty\} \\ u &\mapsto \sup\{n \ : \ u = v w_1 \dots w_n v' \\ &\qquad\qquad w_1, \dots, w_n \neq \varepsilon \\ &\qquad\qquad v\{w_1, \dots, w_n\}^\omega \subseteq L\} \end{aligned}$$

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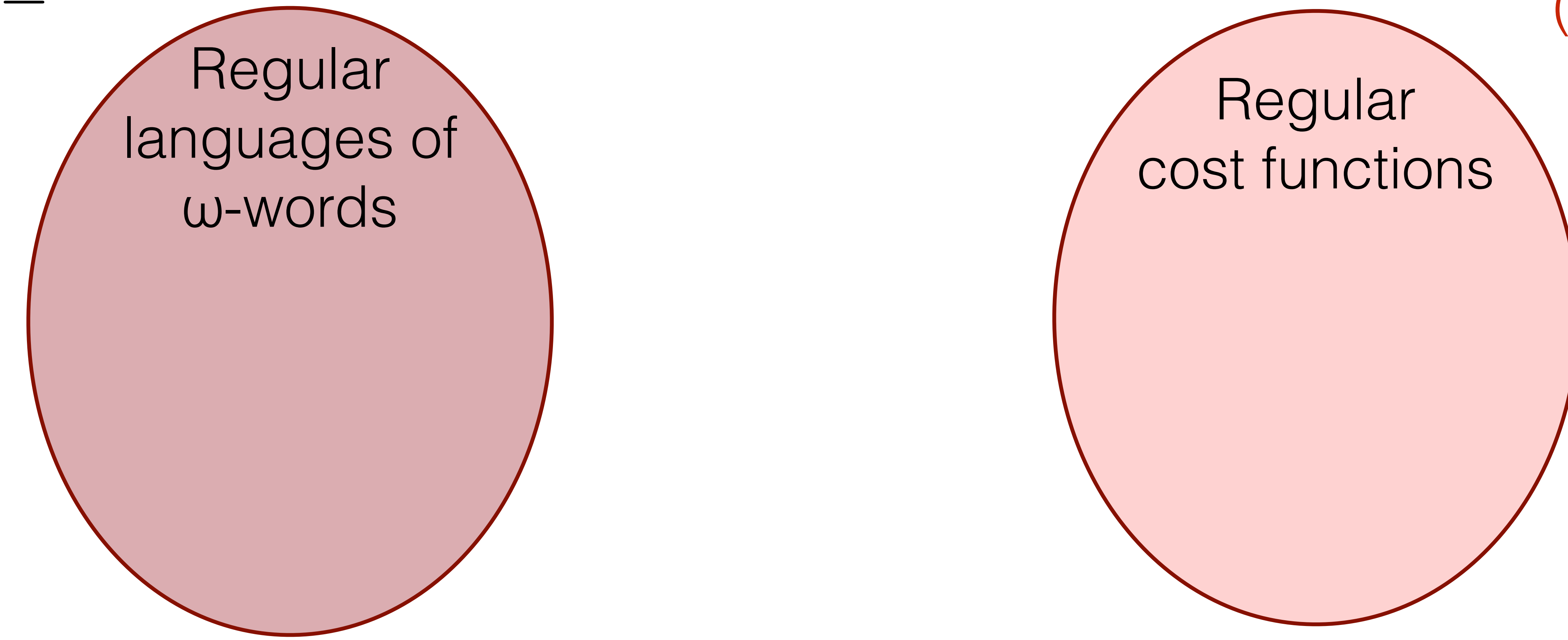

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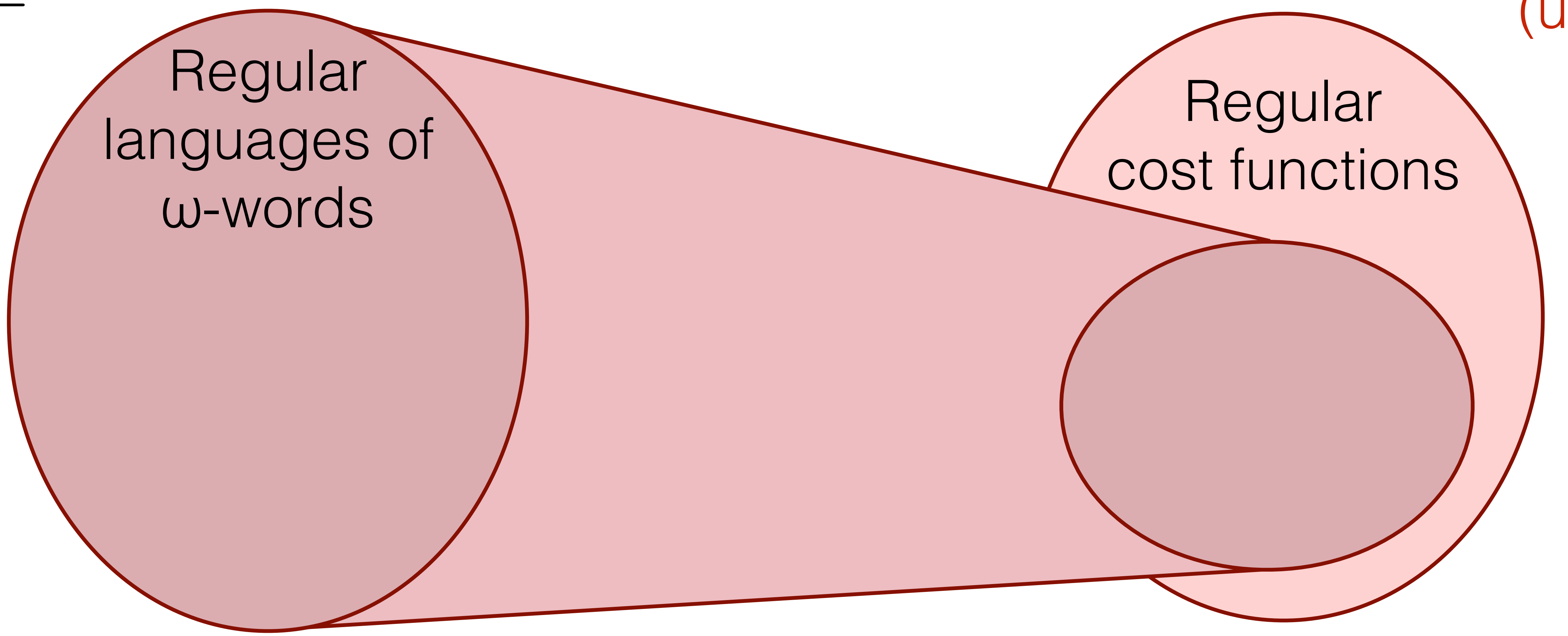

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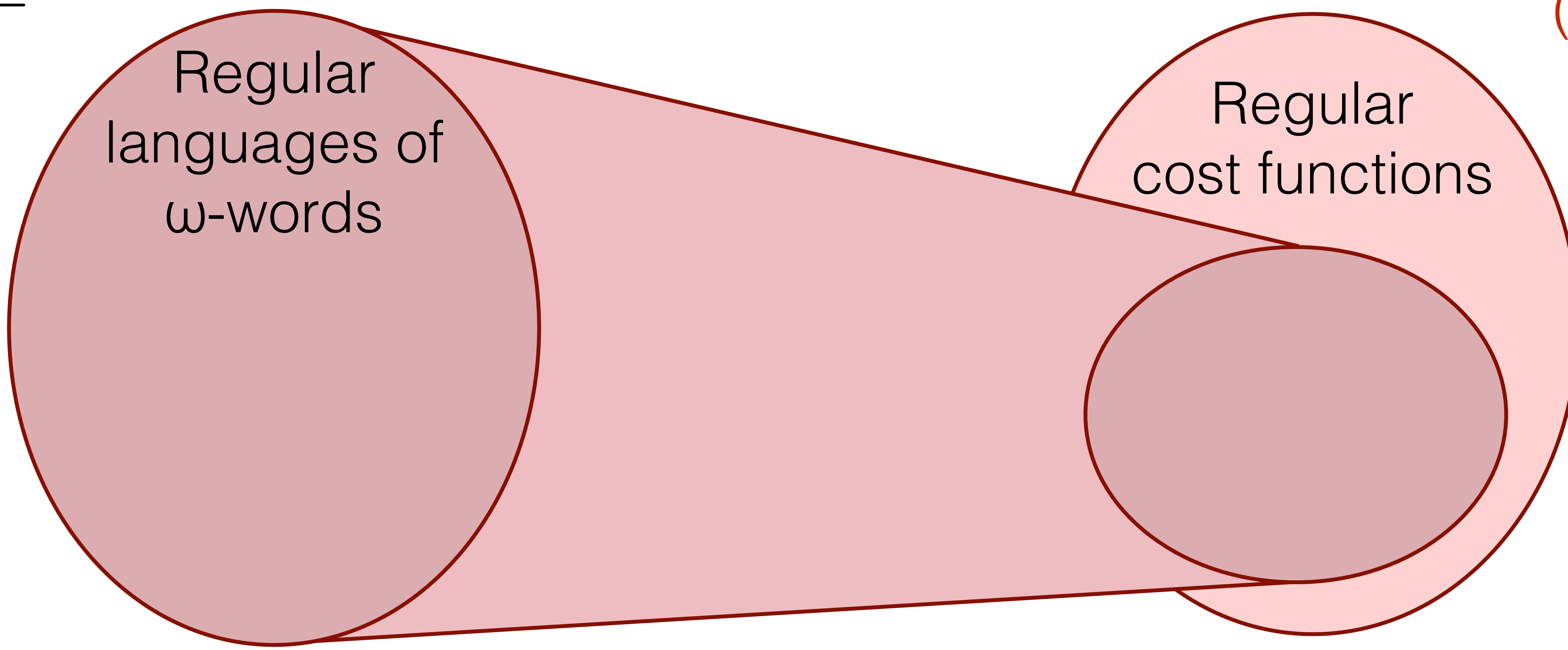

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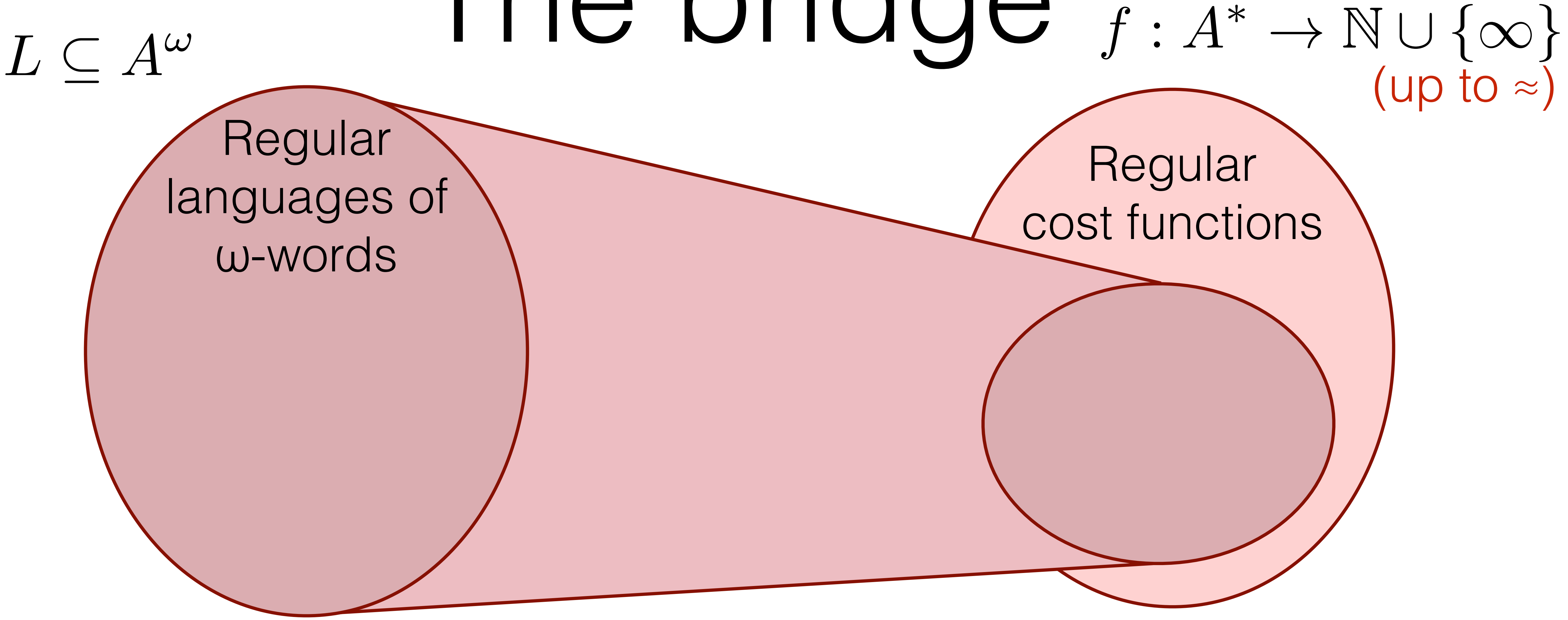

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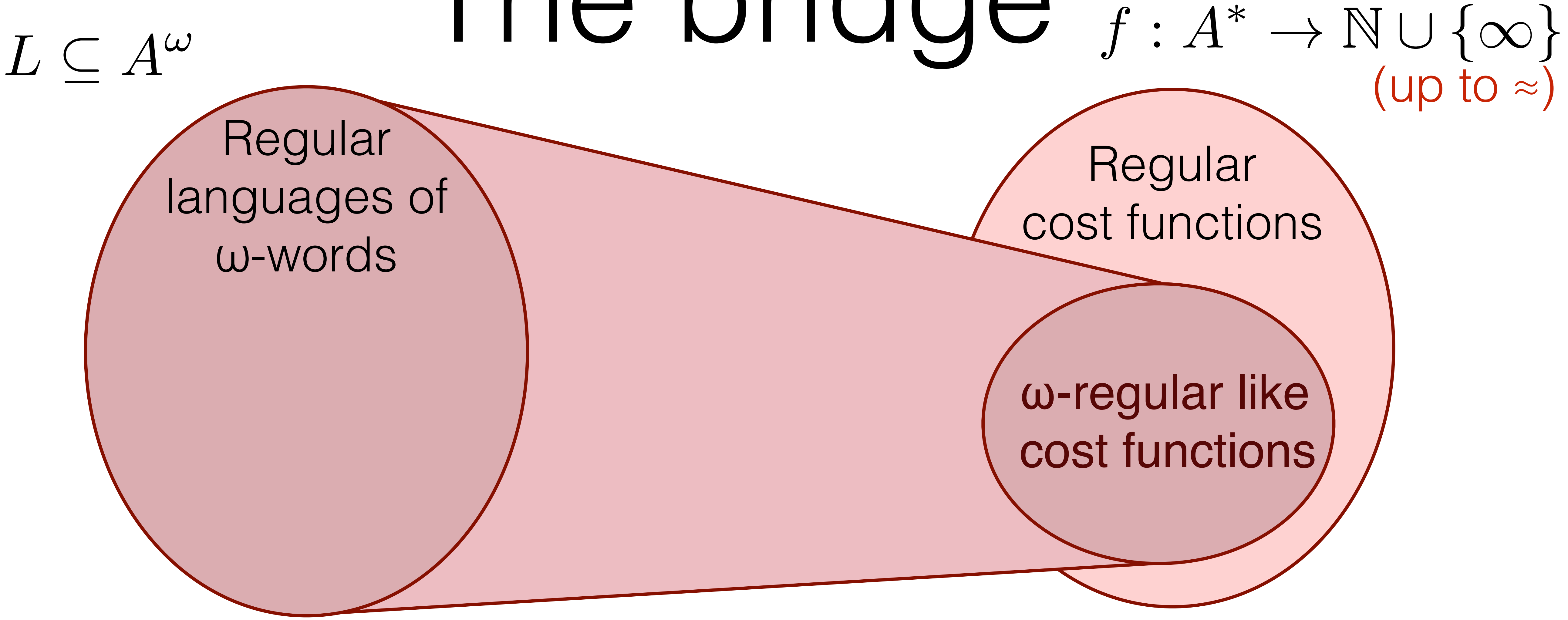

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hierarchical B-condition
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Büchi automaton	Same automaton seen as max-prefix-distance (up to ≈)

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f(u) = maximum number of Büchi transitions seen on a *prefix-run over u.*

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A sequence of transitions starting in an initial state, forming a path reading the prefix of the input.

Goal: $L^{\texttt{ol}} \approx f$

I.e. $L^{\texttt{ol}}(u)$ large iff $f(u)$ large.

Assume $f(u)$ large.

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It can be decomposed into $\alpha\beta_1 \ldots \beta_n\gamma$ with n large and each of the β's start and end in the state state, and contain at least one Büchi transition.

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This witnesses that $f(u)$ is large.

First examples of the bridge

$$L \subseteq A^\omega \quad \Rightarrow \quad \begin{aligned} L^{\mathtt{ol}} : A^* &\to \mathbb{N} \cup \{\infty\} \\ u &\mapsto \sup\{n \ : \ u = v w_1 \ldots w_n v' \\ & \qquad w_1, \ldots, w_n \neq \varepsilon \\ & \qquad u\{w_1, \ldots, w_n\}^\omega \subseteq L\} \end{aligned}$$

$Buchi = (1^* 2)^\omega$	$\lvert \cdot \rvert_2$
$coBuchi = (0+1)^* 0^\omega$	$maxblock_0$
$parity_{i,j} = \{u \in [i,j]^\omega \mid \limsup_n u_n \text{ even}\}$	hierarchical B-condition (up to ≈)
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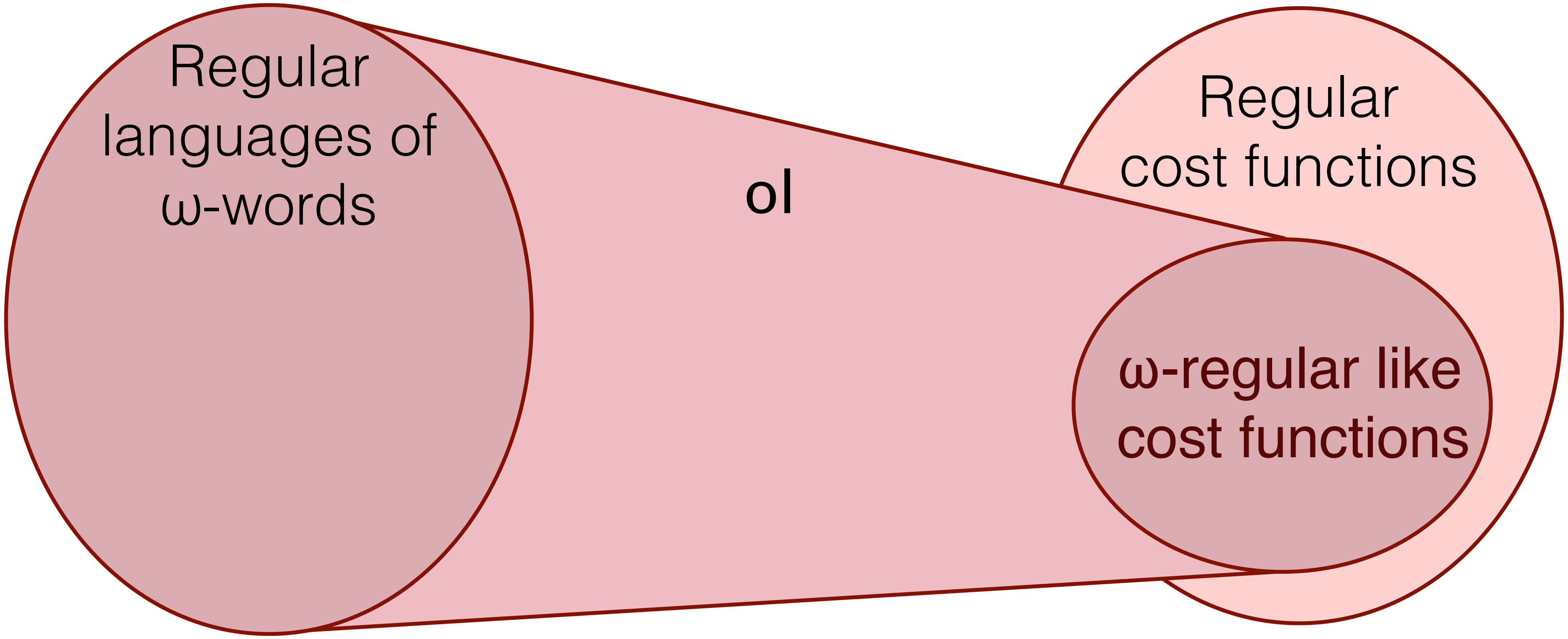
Büchi automaton — Same automaton seen as max-prefix-distance (up to ≈)

Rabin automaton — Same automaton seen as max-prefix-B automaton (up to ≈)

The bridge


Regular
languages of
ω-words
ol
Regular
cost functions
ω-regular like
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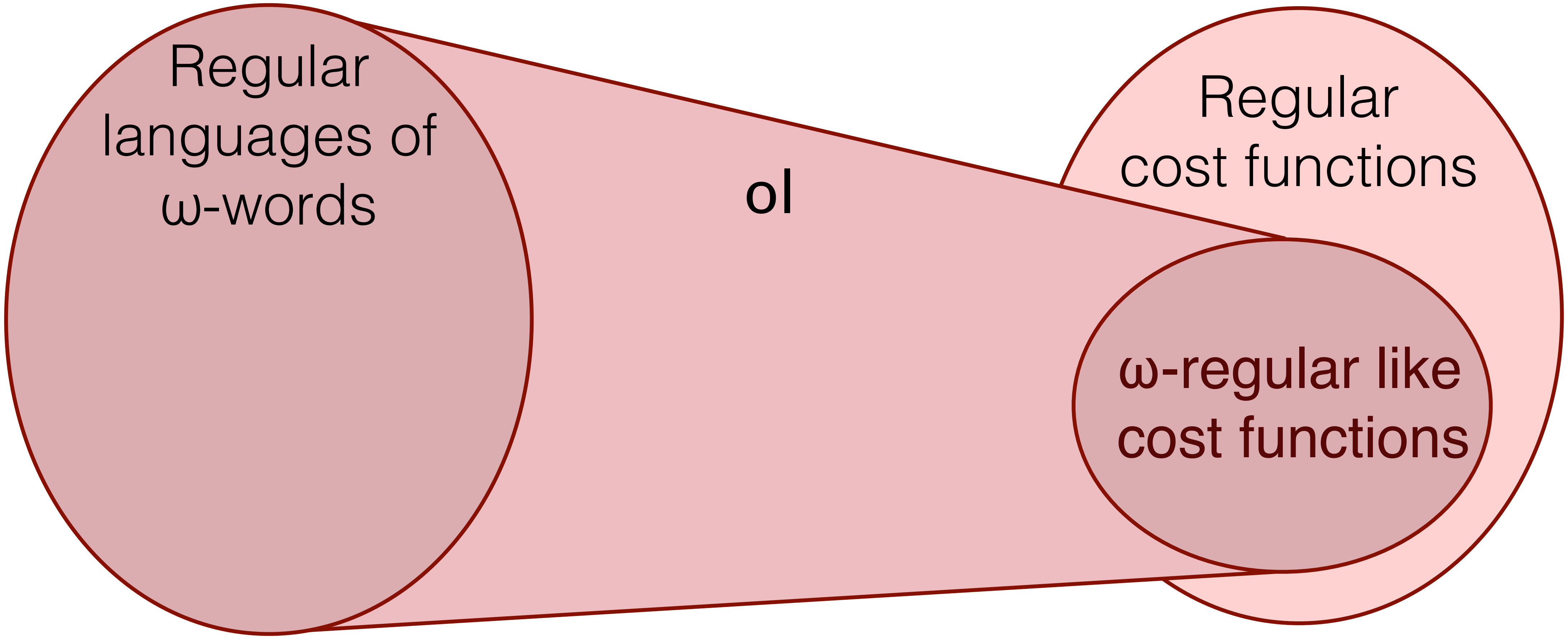

Consequence: For all ω-regular like cost function, there exists effectively a B-deterministic a that recognizes it (up to ≈).

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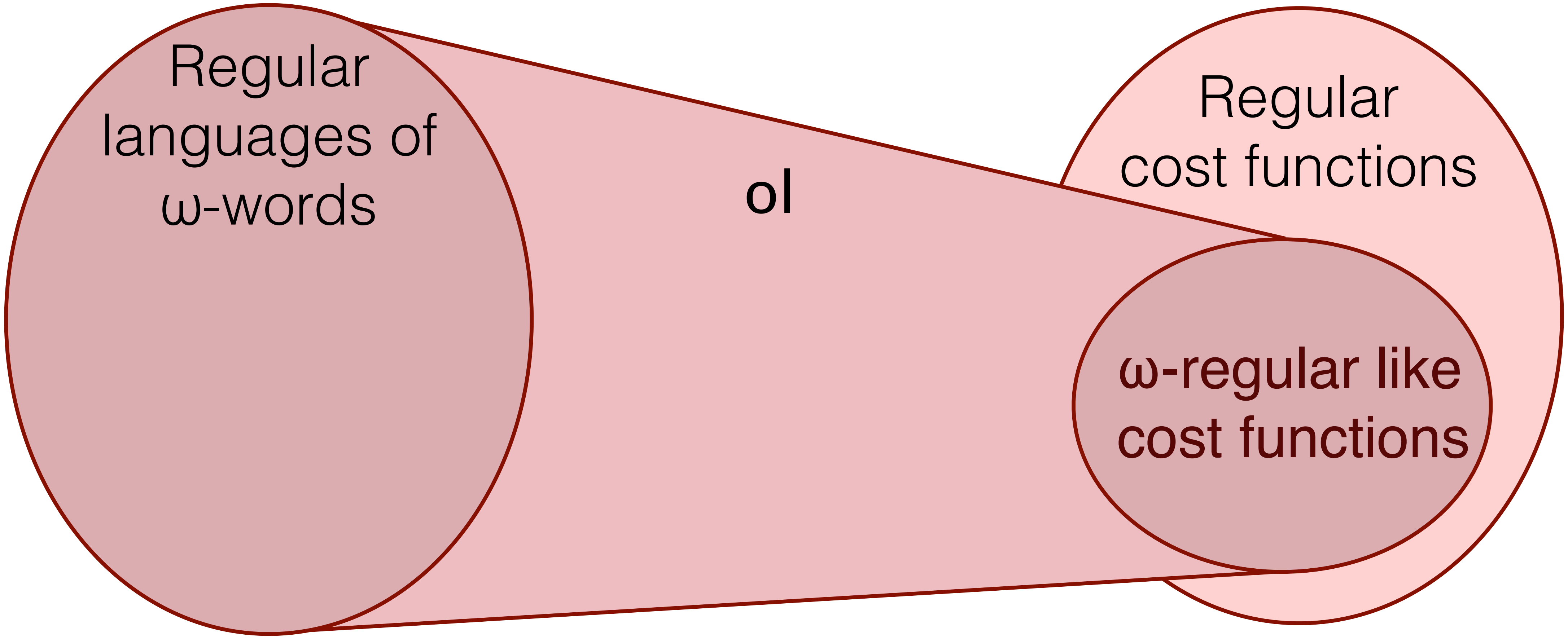

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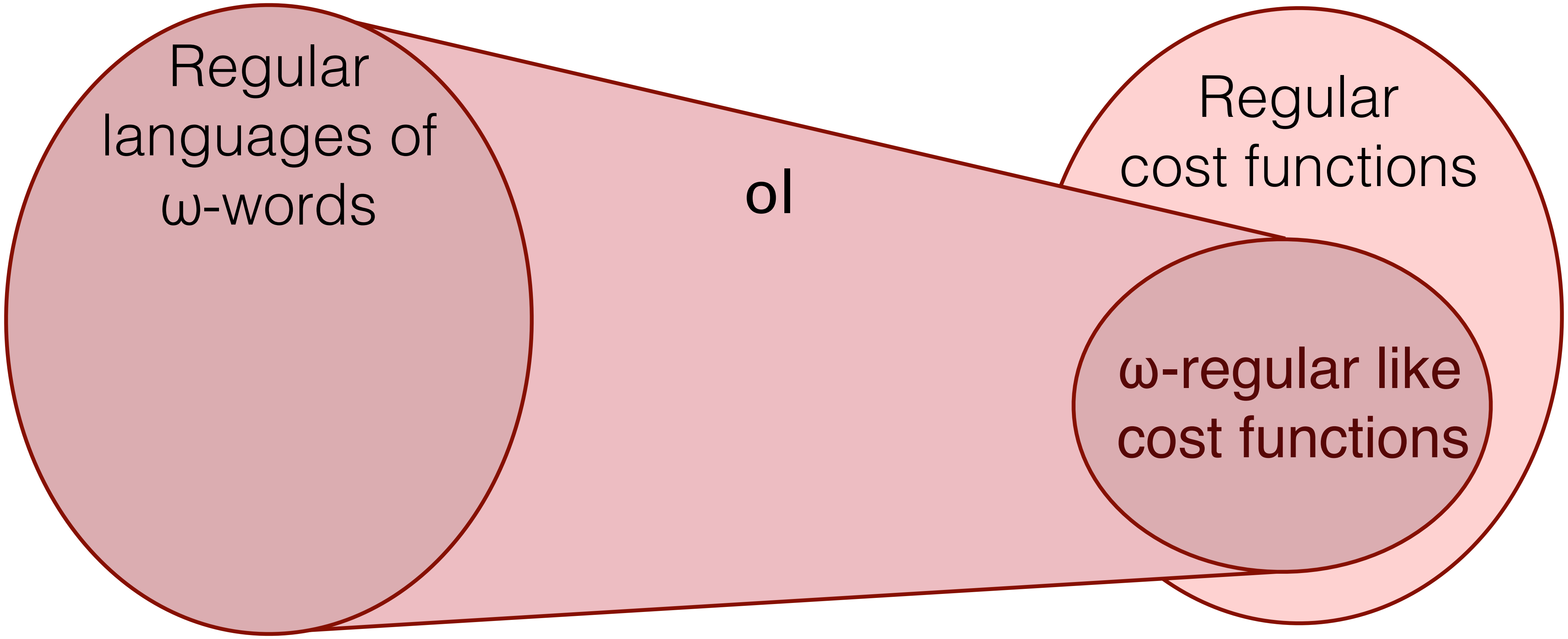

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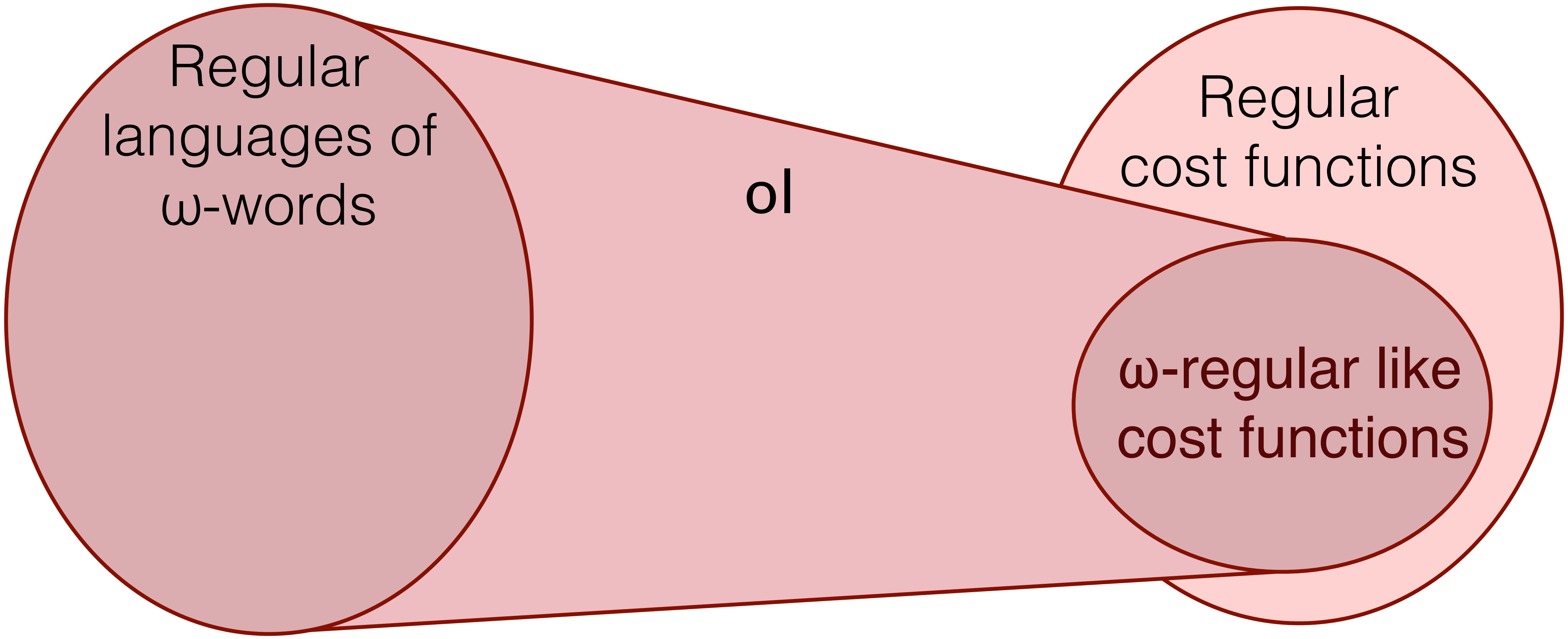

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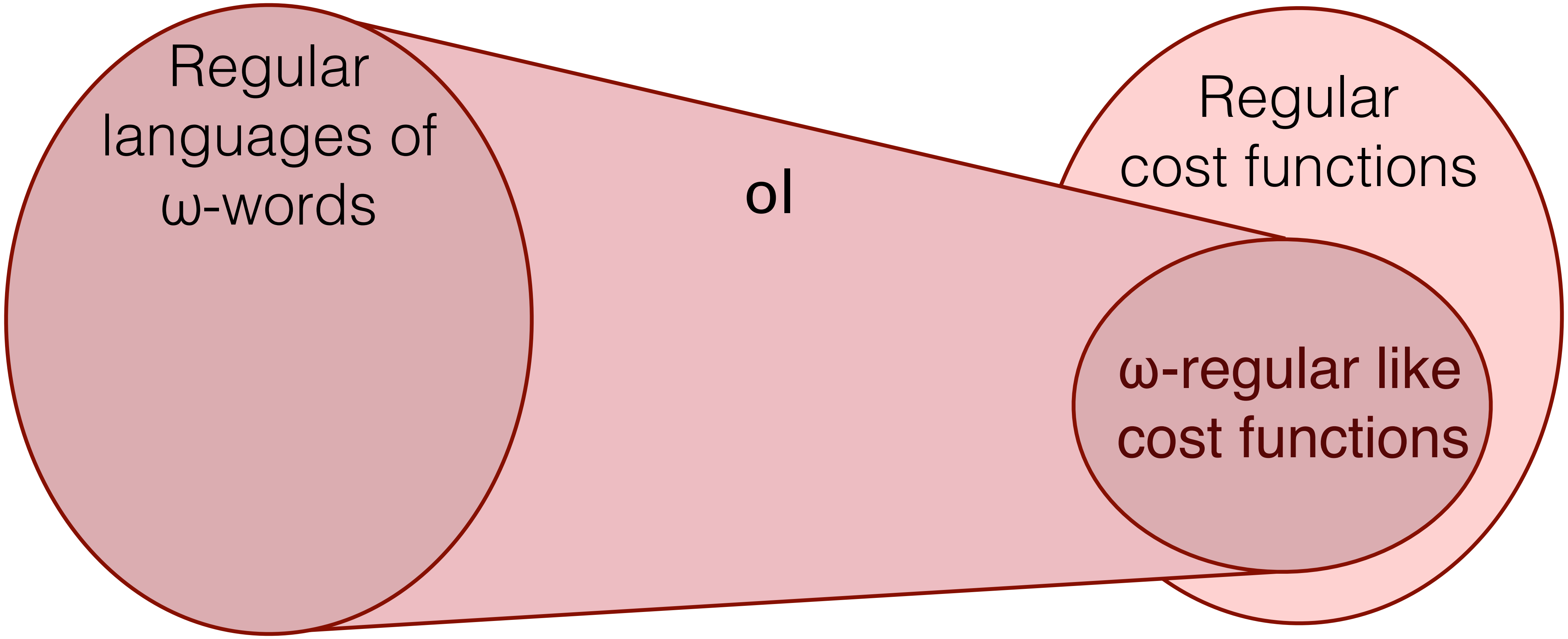

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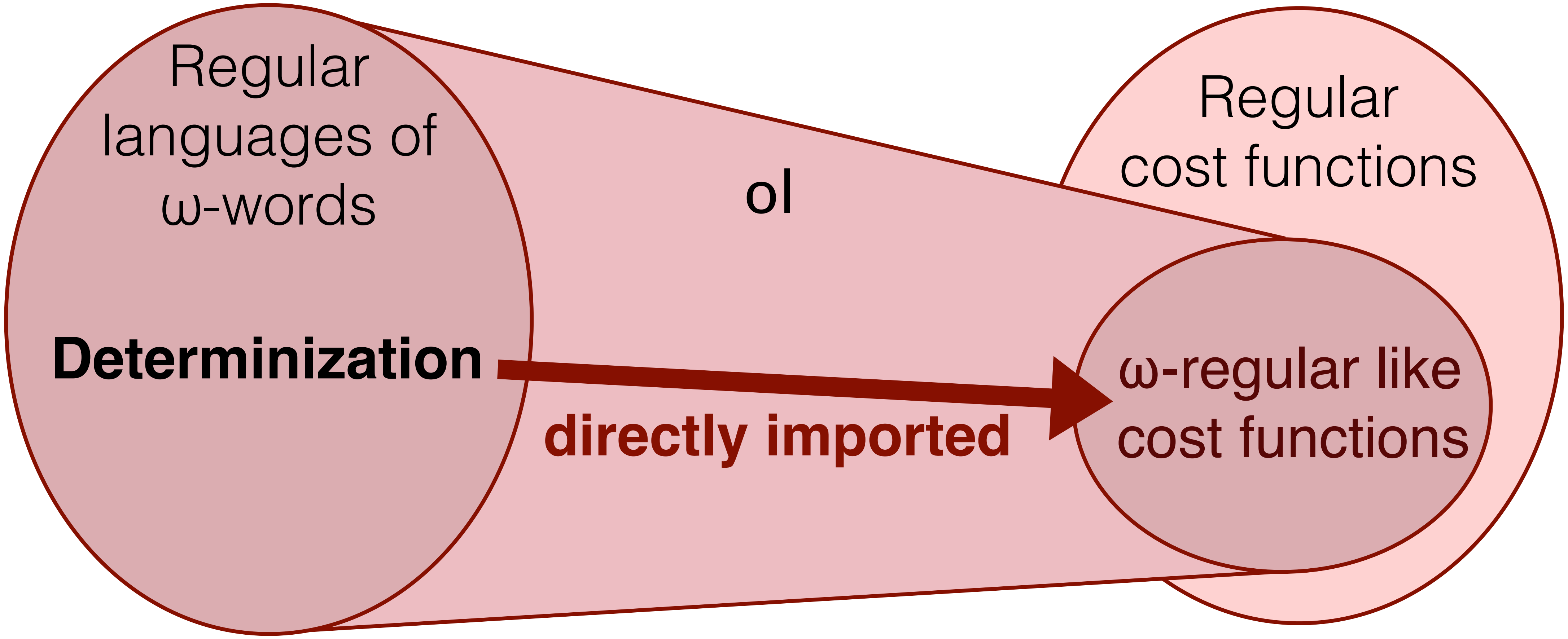

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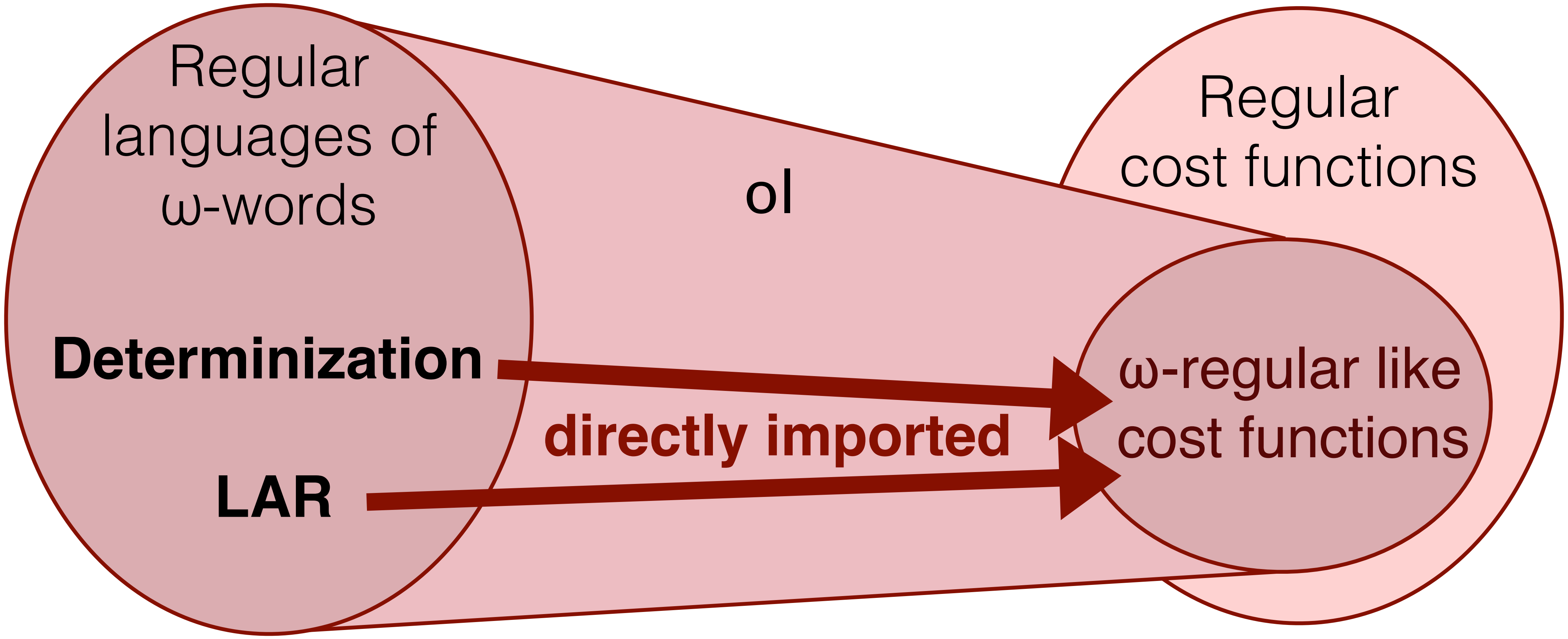

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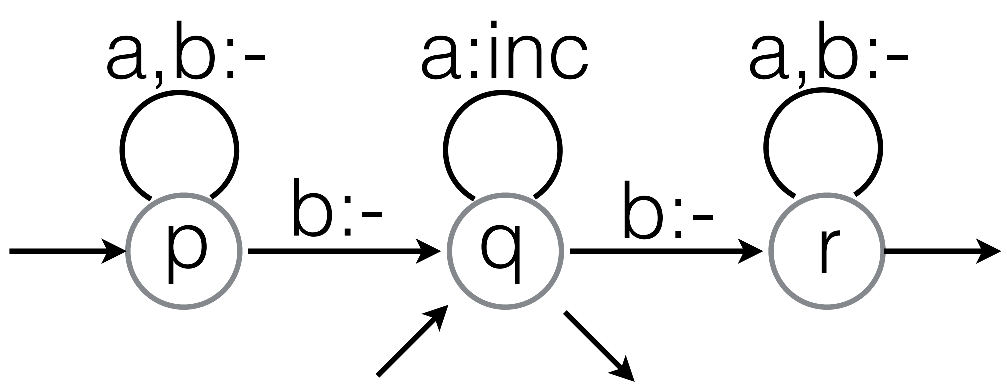

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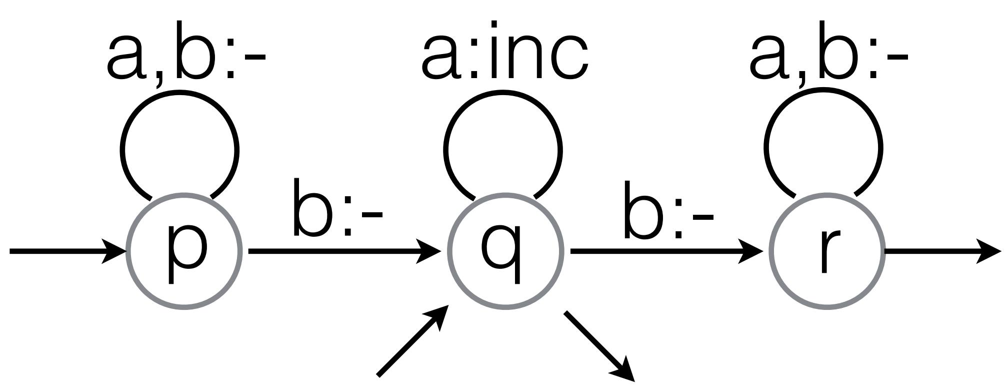

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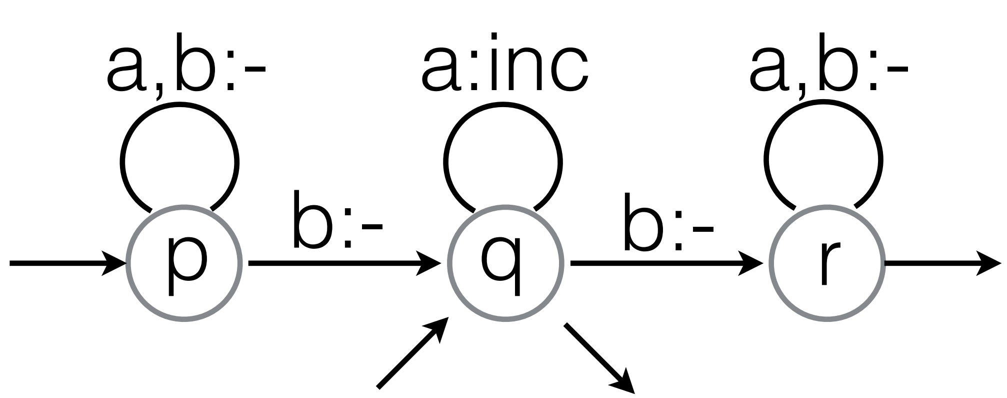

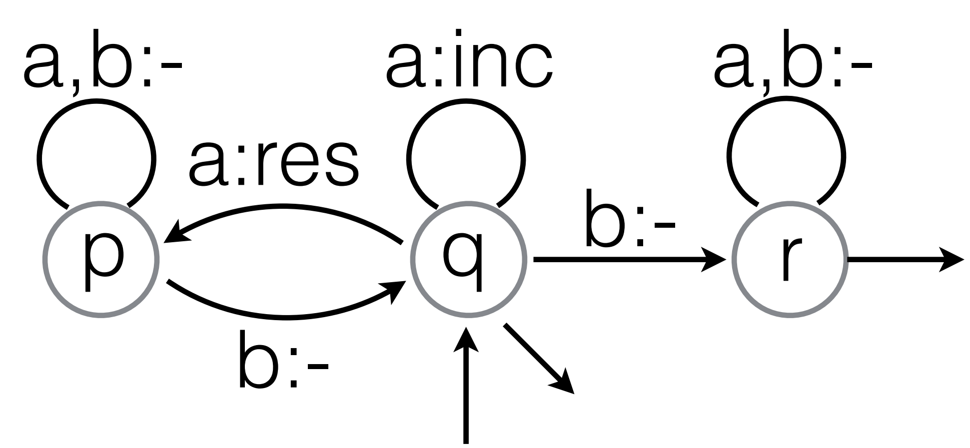

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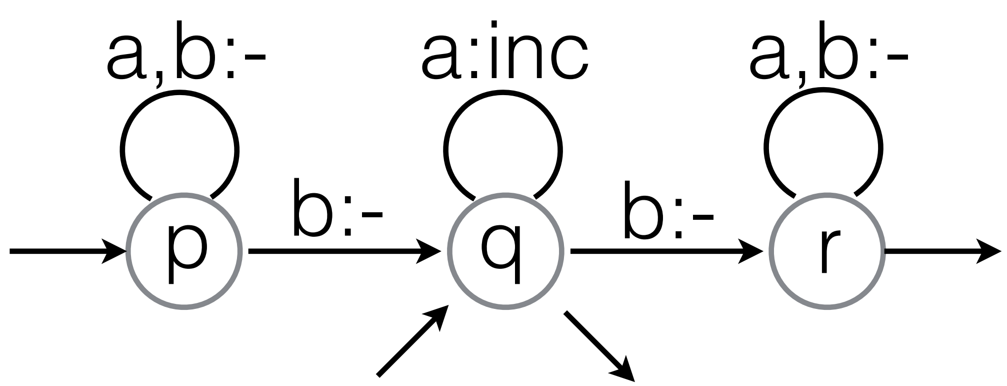

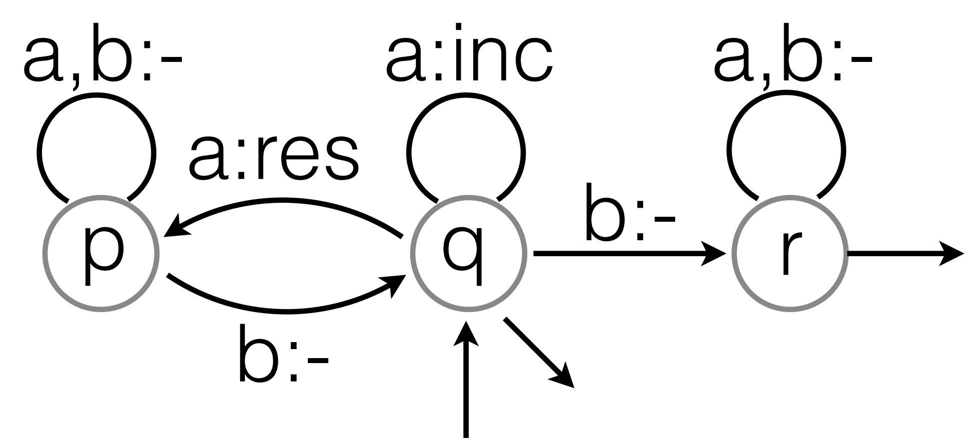

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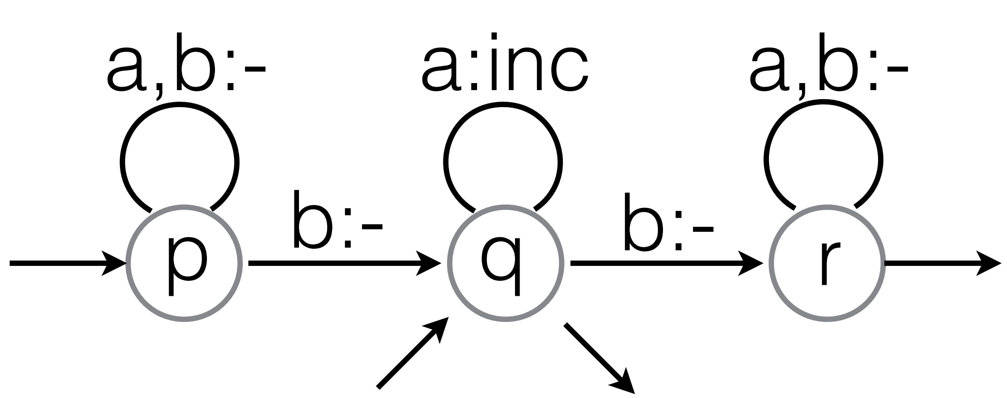

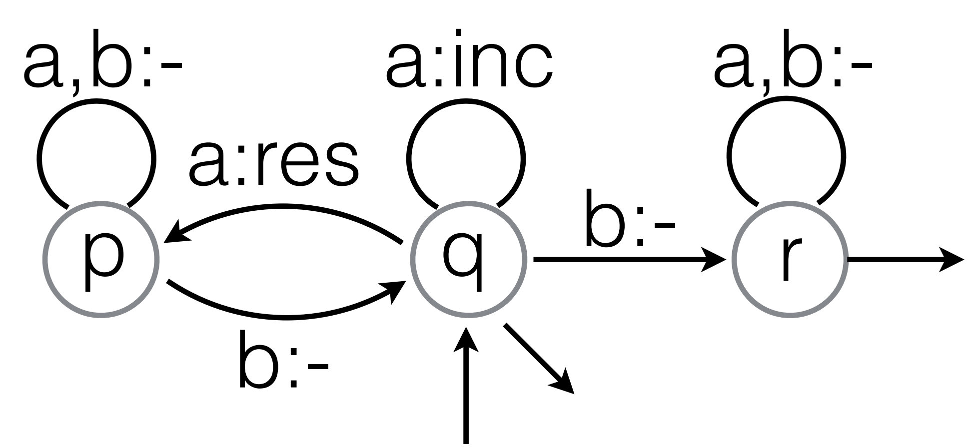

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Because (inspired from **[Bojanczyk15]**) :

determinization of ω-regular like cost functions + positional determinacy of hierarchical B-games **[C&Löding06]** = history-determinization of regular cost functions **[C09,C11 unp]**

Conclusion

We have provided a bridge from regular languages of infinite words to a subset of regular cost functions over finite words.

This allows to transport constructions and results from the well studied and simpler theory of regular languages of infinite words to cost functions over infinite trees.

In particular a new, simple and optimal proof for transforming B-automata in historic-deterministic form can be derived (a central result for working on games and trees).

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TODO:
Extend the approach to produce history deterministic S-automata.