The joint distribution of descent and major index over restricted sets of permutations

Sylvie Corteel
LRI, CNRS et Université Paris-Sud
Bât. 490, F-91405 Orsay, France
corteel@lri.fr

Ira M. Gessel
Brandeis University
Waltham, MA 02454-9110
gessel@brandeis.edu

Carla D. Savage*
North Carolina State University
Raleigh, NC 27695-8206
savage@csc.ncsu.edu

Herbert S. Wilf
University of Pennsylvania
Philadelphia, PA 19104-6395
wilf@math.upenn.edu

Abstract

We compute the joint distribution of descent and major index over permutations of $\{1, \ldots, n\}$ with no descents in positions $\{n-i, n-i+1, \ldots, n-1\}$ for fixed $i \geq 0$. This was motivated by the problem of enumerating symmetrically constrained compositions and generalizes Carlitz's q-Eulerian polynomial.

1 Introduction

In [9], S. Lee and the third author of this paper consider the problem of enumerating *symmetrically constrained compositions*. This study was motivated by problems in [3]. These symmetrically constrained compositions are integer sequences defined by linear constraints that are symmetric in the variables. For example, the integer sequences $(\lambda_1, \lambda_2, \lambda_3)$ satisfying

$$\lambda_{\pi(1)} + \lambda_{\pi(2)} \geq \lambda_{\pi(3)} \tag{1}$$

for every permutation π of $\{1, 2, 3\}$, are known as *integer-sided triangles* [1, 2, 8, 11]. However, in contrast to other treatments, we are counting the number of *ordered* solutions, a harder problem. Generalizing to n dimensions, one could ask for the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying the constraint (1) for every permutation π of $[n] = \{1, 2, \ldots n\}$, or, more generally, given positive integers k, ℓ, m with $k \geq \ell$, the integer sequences $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ satisfying

$$k\lambda_{\pi(1)} + \ell\lambda_{\pi(2)} \geq m\lambda_{\pi(3)} \tag{2}$$

for every permutation π of $[n]$.

*Research supported in part by NSF grants DMS-0300034 and INT-0230800

If the constraints are symmetric in the λ_i, then the generating function

$$G(x_1, x_2, \ldots, x_n) = \sum_{\lambda} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n}$$

will be a symmetric function of the x_i. The work in [9] is to show how to exploit the symmetry to compute the generating function $G_n(q) = G(q, q, \ldots, q)$.

For example, the generating function for (2) has the following form when $m = k + \ell - 1$.

Proposition 1. [9] *If $m = k + \ell - 1$, then for $n \geq 3$, the generating function for the solutions to (2) is*

$$G_n(q) \quad = \quad \frac{1}{(1-q^n)(1-q^{n\ell-1})\prod_{j=1}^{n-2}(1-q^{j+nm})} \sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nb_i}, \tag{3}$$

where $b_i = m$, $1 \leq i \leq n-2$ and $b_{n-1} = m - k$ and $D(\pi)$ is the set of descents of π:

$$D(\pi) \quad = \quad \{i \mid 1 \leq i < n \text{ and } \pi(i) > \pi(i+1)\}.$$

In order to simplify this generating function, we consider a new twist on the problem of computing the distribution of permutation statistics. For a permutation π, $\mathrm{des}(\pi) = |D(\pi)|$ is the number of descents of π and the *major index* of π is the sum of the descent positions: $\mathrm{maj}(\pi) = \sum_{i \in D(\pi)} i$. The joint distribution of $\mathrm{des}(\pi)$ and $\mathrm{maj}(\pi)$ over the set S_n of all permutations of $[n]$ is given by Carlitz's q-Eulerian polynomial [5, 6]:

$$C_n(x, q) = \sum_{\pi \in S_n} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} \;\; = \;\; \prod_{i=0}^{n} (1 - xq^i) \sum_{j=1}^{\infty} [\,j\,]_q^n x^{j-1}, \tag{4}$$

where $[\,j\,]_q = (1-q^j)/(1-q)$. (This is a special case of a result of MacMahon, who computed the distribution of permutations of a multiset by descents and major index in [10], Vol. 2, Chapter 4.)

For $i \leq n-1$, let $S_n^{(i)}$ be the set of permutations of $[n]$ that have no descent in positions $\{n-i, n-i+1, ..., n-1\}$. Let $C_n^{(i)}(x, q)$ be the joint distribution of *maj* and *des* over $S_n^{(i)}$:

$$C_n^{(i)}(x, q) \quad = \quad \sum_{\pi \in S_n^{(i)}} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)}. \tag{5}$$

Then $C_n^{(0)}(x, q) = C_n(x, q)$. Now we can express $G_n(q)$ in (3) via the following.

Proposition 2.

$$\sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nb_i} \quad = \quad C_n^{(1)}(q^{nm}, q) + q^{-nk}(C_n(q^{nm}, q) - C_n^{(1)}(q^{nm}, q)). \tag{6}$$

Proof. If $\pi \in S_n$ does not have a descent in position $n-1$, then $\pi \in S_n^{(1)}$ and

$$\sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nb_i} = \sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nm} = \sum_{\pi \in S_n^{(1)}} q^{\mathrm{maj}(\pi)} (q^{nm})^{\mathrm{des}(\pi)} = C_n^{(1)}(q^{nm}, q).$$

If $\pi \in S_n$ has a descent in position $n-1$, then $\pi \in S_n - S_n^{(1)}$ and

$$\begin{aligned}\sum_{\pi \in S_n - S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nb_i} &= \sum_{\pi \in S_n - S_n^{(1)}} q^{-nk} \prod_{i \in D(\pi)} q^{i+nm} \\ &= q^{-nk} \Bigg(\sum_{\pi \in S_n} \prod_{i \in D(\pi)} q^{i+nm} - \sum_{\pi \in S_n^{(1)}} \prod_{i \in D(\pi)} q^{i+nm} \Bigg) \\ &= q^{-nk}(C_n(q^{nm}, q) - C_n^{(1)}(q^{nm}, q)).\end{aligned}$$

Putting these two cases together gives the result. □

Further simplification of (6) requires computing $C_n^{(1)}(x,q)$. In this paper, we compute $C_n^{(i)}(x,q)$, for general i, in two ways. The first method derives a recurrence for $C_n^{(i)}(x,q)$ and solves it in terms of Carlitz polynomials. The second is a "P-partitions" approach.

In the last section, the results are applied to $G_n(q)$ of (3) to enumerate the sequences satisfying the symmetric constraints (2).

Throughout the paper, the following notation is used: $[n] = \{1, 2, \ldots, n\}$; $[\,n\,]_q = (1-q^n)/(1-q)$; $[\,n\,]_q! = \prod_{i=1}^{n} [\,i\,]_q$; and $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$.

2 The joint distribution of maj and des over $S_n^{(i)}$

2.1 Recursive approach

In this section, we give a recursive approach to the problem of computing the joint distribution of *inv* and *maj* over the permutations of $[n]$ with no descent in positions $\{n-1, n-2, \ldots, n-i\}$.

A standard technique for counting the permutations with k descents is to derive a recurrence (see, for example, Bóna [4], Theorem 1.7). What we need is a q-analog of this count, refined to consider only permutations in $S_n^{(i)}$.

Proposition 3. *Define* $e_{n,k}^{(i)}(q)$ *by*

$$\sum_{k=0}^{n-1} e_{n,k}^{(i)}(q) x^k = \sum_{\pi \in S_n^{(i)}} x^{\mathrm{des}(\pi)} q^{\mathrm{maj}(\pi)} = C_n^{(i)}(x, q).$$

Then $e^{(i)}_{n,k}(q)$ *satisfies*

$$e^{(i)}_{n,k}(q) \quad = \quad q[\,k\,]_q e^{(i)}_{n-1,k}(q) + e^{(i-1)}_{n-1,k}(q) + ([\,n-i\,]_q - [\,k\,]_q)e^{(i)}_{n-1,k-1}(q),$$

with initial conditions $e^{(-1)}_{n,k} = e^{(0)}_{n,k}$*;* $e^{(i)}_{n,0} = 1$*; and* $e^{(i)}_{n,k} = 0$ *if* $k \geq n-i$*.*

Proof. We get a permutation in $S^{(i)}_n$ with k descents by inserting n into

(i) a permutation in $S^{(i)}_{n-1}$ with k descents, immediately following a descent; or

(ii) a permutation in $S^{(i-1)}_{n-1}$ at the end of the permutation; or

(iii) a permutation in $S^{(i)}_{n-1}$ with $k-1$ descents, immediately following any of the positions $0, 1, \ldots, n-i-2$ that are not descents (leaving a total of $(n-i-1)-(k-1) = n-k-i$ positions for inserting n).

In (i) above, if n is inserted after the ith descent, *maj* increases by 1 for that descent and for every later descent, i.e., by $k+1-i$. This gives the first term in the recurrence, with its factor $q + q^2 + \cdots q^k$.

In (ii) above, if n is inserted at the end of the permutation, *maj* does not increase at all, giving the second term.

To see how (iii) gives rise to the third term in the recurrence, let $\pi = \pi(1)\pi(2)\ldots\pi(n-1)$ be a permutation in $S^{(i)}_{n-1}$ with $k-1$ descents. Let $0 = j_1 < j_2 < \cdots < j_{n-k-i} \leq n-i-2$ be the $n-k-i$ positions where n can be inserted into π to create a permutation in $S^{(i)}_n$ with k descents. We claim that inserting n immediately following any of $j_1, j_2, \ldots, j_{n-k-i}$ increases *maj* by $k, k+1, \ldots, n-i-1$, respectively. This will give the third term of the recurrence with its factor $q^k + q^{k+1} + \cdots + q^{n-i} = [\,n-i\,]_q - [\,k\,]_q$.

To prove this claim, let t_ℓ be the number of descents of π that are greater than j_ℓ. Inserting n just after $\pi(j_\ell)$ creates a descent in position $j_\ell + 1$ and increases by 1 the position of each of the t_ℓ descents in π that are greater than j_ℓ. Thus *maj* increases by

$$m = j_\ell + 1 + t_\ell.$$

Let $d = j_{\ell+1} - j_\ell$. Then all of the positions $j_\ell+1, j_\ell+2, \ldots, j_\ell+d-1$ are descents. So $t_{\ell+1} = t_\ell - (d-1)$ and inserting n just after $\pi(j_{\ell+1})$ increases *maj* by

$$(j_{\ell+1}+1) + t_{\ell+1} = (d + j_\ell + 1) + (t_\ell - (d-1)) = m+1.$$

Since inserting n at the front of π would increase *maj* by $1 + (k-1) = k$, the claim is proved. □

From Proposition 3, we can derive a recurrence for $C^{(i)}_n(x,q)$. First, the $i = 0$ case gives the following recurrence for $C_n(x,q)$. It is straighforward to verify that (4) satisfies the recurrence, giving a simple proof of Carlitz's formula.

Proposition 4.

$$C_n(x,q) = \frac{1-xq^n}{1-q} C_{n-1}(x,q) - \frac{q(1-x)}{1-q} C_{n-1}(qx,q), \tag{7}$$

with $C_0(x,q) = 1$.

Proof. Let $e_{n,k}(q) = e^{(0)}_{n,k}(q)$. Since $S^{(-1)}_n = S^{(0)}_n$, setting $i = 0$ in Proposition 3 gives

$$e_{n,k}(q) = [\,k+1\,]_q e_{n-1,k}(q) + ([\,n\,]_q - [\,k\,]_q) e_{n-1,k-1}(q) \tag{8}$$

(which also appears in [5]). Multiply (8) by x^k and sum over k. Substitute the definition of $[\,..\,]_q$ and (7) results immediately. $\square$

Similarly, from Proposition 3, we get a recurrence for general i.

Proposition 5. *For $n \geq 0$ and $0 \leq i < n$,*

$$C^{(i)}_n(x,q) = \left(\frac{q - xq^{n-i}}{1-q}\right) C^{(i)}_{n-1}(x,q) - \left(\frac{q(1-x)}{1-q}\right) C^{(i)}_{n-1}(xq,q) + C^{(i-1)}_{n-1}(x,q),$$

with $C^{(-1)}_n(x,q) = C^{(0)}_n(x,q) = C_n(x,q)$, and $C^{(i)}_n(x,q) = 1$ if $i \geq n$.

To solve the recurrence of Proposition 5, first observe that it can be simplified as follows.

Proposition 6. *For $i > 0$ and $n \geq i$*

$$C^{(i)}_n(x,q) = \frac{C^{(i-1)}_n(x,q) - \binom{n}{i} xq^{n-i} C_{n-i}(x,q)}{1 - xq^{n-i}}. \tag{9}$$

Proof. For $n = i$ the proposition is true, as both sides are equal to 1. For $n > i$, apply the recurrence of Proposition 5 and then induction:

$$\begin{aligned}
C^{(i)}_n(x,q) &= \frac{q}{1-q}\Big((1 - xq^{n-i-1}) C^{(i)}_{n-1}(x,q) - (1-x) C^{(i)}_{n-1}(xq,q)\Big) + C^{(i-1)}_{n-1}(x,q) \\
&= \frac{q(1 - xq^{n-i-1})}{1-q} \frac{C^{(i-1)}_{n-1}(x,q) - \binom{n-1}{i} xq^{n-i-1} C_{n-i-1}(x,q)}{1 - xq^{n-i-1}} \\
&\qquad - \frac{q(1-x)}{1-q} \frac{C^{(i-1)}_{n-1}(xq,q) - \binom{n-1}{i} xq^{n-i} C_{n-i-1}(xq,q)}{1 - xq^{n-i}} + C^{(i-1)}_{n-1}(x,q).
\end{aligned}$$

Rearranging terms,

$$\begin{aligned}
C^{(i)}_n(x,q) &= \frac{q}{1-q} \frac{(1 - xq^{n-i}) C^{(i-1)}_{n-1}(x,q) - (1-x) C^{(i-1)}_{n-1}(xq,q)}{1 - xq^{n-i}} + \frac{C^{(i-2)}_{n-1}(x,q)}{1 - xq^{n-i}} \\
&\qquad + C^{(i-1)}_{n-1}(x,q) - \frac{C^{(i-2)}_{n-1}(x,q)}{1 - xq^{n-i}} \\
&\qquad - \frac{\binom{n-1}{i} xq^{n-i}}{1-q} \frac{(1 - xq^{n-i}) C_{n-i-1}(x,q) - q(1-x) C_{n-i-1}(xq,q)}{1 - xq^{n-i}}.
\end{aligned}$$

Apply Proposition 5 to the first line, the induction hypothesis to the second line, and Proposition 7 to the last line to obtain

$$\begin{aligned}
C^{(i)}_n(x,q) &= \frac{C^{(i-1)}_n(x,q)}{1 - xq^{n-i}} - \frac{\binom{n-1}{i-1} xq^{n-i} C_{n-i}(x,q)}{1 - xq^{n-i}} - \frac{\binom{n-1}{i} xq^{n-i} C_{n-i}(x,q)}{1 - xq^{n-i}}. \\
&= \frac{C^{(i-1)}_n(x,q) - \binom{n}{i} xq^{n-i} C_{n-i}(x,q)}{1 - xq^{n-i}}.
\end{aligned}$$

$\square$

Finally, iterating the recurrence (9), we can solve for $C^{(i)}_n(x,q)$ in terms of Carlitz functions.

Theorem 1.

$$C_n^{(i)}(x,q) = \frac{C_n(x,q)}{(xq^{n-i};q)_i} - \sum_{k=1}^{i} \binom{n}{k} xq^{n-k} \frac{C_{n-k}(x,q)}{(xq^{n-i};q)_{i-k+1}}. \tag{10}$$

In particular, the motivating problem of computing $C_n^{(1)}(x,q)$ is solved:

Corollary 1.

$$C_n^{(1)}(x,q) \quad = \quad \frac{C_n(x,q) - nxq^{n-1}C_{n-1}(x,q)}{1-xq^{n-1}}. \tag{11}$$

Substituting (4) into (10) and simplifying gives $C_n^{(i)}(x,q)$ explicitly:

Corollary 2.

$$C_n^{(i)}(x,q) = (x;q)_{n-i} \sum_{j\geq 1} \left((1-xq^n)[\,j\,]_q^n x^{j-1} - x^j \sum_{\ell=n-i}^{n-1} \binom{n}{\ell} (q[\,j\,]_q)^\ell \right).$$

Remark. Let $C_n^{(i)}(q) = C_n^{(i)}(1,q)$. It is easy to see from Proposition 5 that $C_n^{(-1)}(q) = C_n^{(0)}(q)$ and that for $i \geq 0$, $C_n^{(i)}(q) = C_{n-1}^{(i-1)}(q) + q[\,n-i-1\,]_q C_{n-1}^{(i)}(q)$ if $n > i$ and $C_n^{(i)}(q) = 1$ otherwise. Therefore we have

Corollary 3.

$$C_n^{(i)}(q) = \sum_{\pi \in S_n^{(i)}} q^{\mathrm{maj}(\pi)} = \prod_{j=1}^{n-i-1} [\,j\,]_q \sum_{k=1}^{n-i} \binom{k+i-1}{i} q^{k-1} = [\,n-i-1\,]_q! \sum_{k=1}^{n-i} \binom{k+i-1}{i} q^{k-1}.$$

Compare this with the distribution over all permutations: $C_n^{(0)}(q) = [\,n\,]_q!$.

2.2 Direct approach

Now we give a direct approach to the problem of computing the joint distribution of *inv* and *maj* over the permutations of $[n]$ with no descent in positions $\{n-1, n-2, \ldots, n-i\}$.

Theorem 2.

$$C_n^{(i)}(x,q) = (x;q)_{n-i} \sum_{j=1}^{n-i} \binom{n-j}{i} q^{n-i-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{n-i-j} x^k.$$

Proof. Let $S_n^{(i,j)}$ be the set of permutations π of $[n]$ with no descents in positions $\{n-i, n-i+1, \ldots, n-1\}$ with the additional property that $\pi(n-i) = j$. To obtain a permutation in $S_n^{(i,j)}$ we first choose a subset $I \subseteq [n]$ of cardinality i to be the values of $\pi(n-i+1), \pi(n-i+2), \ldots, \pi(n)$.

Since $j = \pi(n-i) < \pi(n-i+1) < \ldots < \pi(n)$ every element of I must be greater than j, so I must be a subset of $\{j+1, j+2, \ldots, n\}$, and $(\pi(n-i+1), \pi(n-i+2), \ldots, \pi(n))$ must consist of the elements of I arranged in increasing order. Then $(\pi(1), \pi(2), \ldots, \pi(n-i-1))$ may be an arbitrary permutation of $[n] \setminus (I \cup \{j\})$. Since π has no descents in positions $\{n-i, n-i+1, \ldots, n-1\}$, the descent number and major index of π are the same as for $(\pi(1), \pi(2), \ldots, \pi(n-i))$, and these statistics are unchanged if we replace $(\pi(1), \pi(2), \ldots, \pi(n-i))$ with the permutation of $[n-i]$ in which the entries have the same relative order (i.e., we replace $(\pi(1), \pi(2), \ldots, \pi(n-i))$ with its "pattern"). Note that this replacement does not change $\pi(n-i) = j$, since $[j] \subseteq [n] \setminus I$ (so $1, 2, \ldots, j$ all occur in $(\pi(1), \pi(2), \ldots, \pi(n-i))$.) Now let

$$A_m^{(j)}(x,q) = \sum_{\sigma} x^{\operatorname{des}(\sigma)} q^{\operatorname{maj}(\sigma)},$$

where the sum is over all permutations σ of $[m]$ with $\sigma(m) = j$. Then the contribution to $C_n^{(i)}(x,q)$ from permutations π with $\pi(n-i) = j$ corresponding to a given i-subset $I \subseteq \{j+1, j+2, \ldots, n\}$ is $A_{n-i}^{(j)}(x,q)$ independent of the choice of I. There are $\binom{n-j}{i}$ possible choices for such a subset I, so summing on j gives

$$C_n^{(i)}(x,q) = \sum_{j=1}^{n-i} \binom{n-j}{i} A_{n-i}^{(j)}(x,q).$$

We now derive a formula for $A_m^{(j)}(x,q)$ using the "P-partition" approach [11, 7]. This relies on the observation that nonnegative integer sequences $(a_1, \ldots, a_m)$ in which $a_i \le k$, $1 \le i \le m$ have generating function:

$$\sum_{\max(a) \le k} q^{|a|} = [\,k+1\,]_q^m,$$

where $|a| = a_1 + \cdots + a_m$.

The method is to count nonnegative integer sequences $(a_1, \ldots, a_m)$ such that $a_j = 0$ and $a_i > 0$ for $i > j$ in two different ways.

First way: First count those in which $a_i \le k$, $1 \le i \le m$:

$$\sum_{\max(a) \le k} q^{|a|} = [\,k+1\,]_q^{j-1} (q[\,k\,]_q)^{m-j}.$$

Then multiply by x^k and sum over k:

$$\sum_{k=0}^{\infty} \sum_{\max(a) \le k} q^{|a|} x^k = q^{m-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{m-j} x^k.$$

If we want only those a such than $\max(a) = k$:

$$\sum_a q^{|a|} x^{\max(a)} = \sum_{k=0}^{\infty} \sum_{\max(a) = k} q^{|a|} x^k = (1-x) q^{m-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{m-j} x^k. \tag{12}$$

Second way: For a permutation π, let $D(\pi)$ be the descent set of π. Now, let π be the unique permutation of $[m]$ satisfying: (i) $a_{\pi(1)} \ge a_{\pi(2)} \ge \ldots \ge a_{\pi(m)}$ and (ii) $a_{\pi(i)} > a_{\pi(i+1)}$ when $i \in D(\pi)$. As $a_j = 0$ and $a_i > 0$ for $i > j$, this implies that $\pi(m) = j$.

Let $\lambda_i = a_{\pi(i)} - a_{\pi(i+1)}$ for $1 \le i < m$.

Then $(a_1, \ldots a_m) \leftrightarrow (\pi, \lambda)$ is a bijection between nonnegative integer sequences of length m such that $a_j = 0$ and $a_i > 0$ for $i > j$ and pairs (π, λ) where π is a permutation of $[m]$ with $\pi(m) = j$ and λ is a nonnegative integer sequence of length $m-1$ satisfying $\lambda_i > 0$ when $i \in D(\pi)$. Then $\sum_{i=1}^{m-1} \lambda_i = \max(a)$ and $|a| = \sum_{i=1}^{m-1} i\lambda_i$. So,

$$\begin{aligned}
\sum_a q^{|a|} x^{\max(a)} &= \sum_{\{\pi\in S_n\,:\,\pi(m)=j\}} \sum_{\lambda_1,\ldots,\lambda_{m-1}} q^{\sum_{i=1}^{m-1} i\lambda_i} x^{\sum_{i=1}^{m-1}\lambda_i} \\
&= \sum_{\{\pi\in S_n:\pi(m)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)} \sum_{\lambda'_1,\ldots,\lambda'_{m-1}\ge 0} \prod_{i=1}^{m-1} (xq^i)^{\lambda'_i} \\
&= \frac{\sum_{\{\pi\in S_n:\pi(m)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)}}{(1-xq)(1-xq^2)\cdots(1-xq^{m-1})}.
\end{aligned}$$

Equating this with (12) gives the desired formula for $A_m^{(j)}(x,q)$, namely,

$$A_m^{(j)}(x,q) = \sum_{\{\pi\in S_n:\pi(m)=j\}} q^{\mathrm{maj}(\pi)} x^{\mathrm{des}(\pi)} = (x;q)_m q^{m-j} \sum_{k=0}^{\infty} [\,k+1\,]_q^{j-1} [\,k\,]_q^{m-j} x^k.$$

□

We can also give a direct proof of Corollary 2 using a similar approach.

Direct proof of Corollary 2. Let T_j be the sum of $q^{|a|}$ over all nonnegative integer sequences $(a_1, \ldots, a_n)$ with $\max(a) \le j$ and at least $i+1$ entries equal to 0. We compute the generating function $\sum_{j=0}^{\infty} T_j x^j$ in two ways.

First we relate the sequences a to the permutations in $S_n^{(i)}$. Given a sequence $(a_1, \ldots, a_n)$ with at least $i+1$ entries equal to 0, we associate to it the unique permutation π of $[n]$ satisfying (i) $a_{\pi(1)} \ge a_{\pi(2)} \ge \cdots \ge a_{\pi(n)}$, and (ii) $a_{\pi(j)} > a_{\pi(j+1)}$ if $\pi(j) > \pi(j+1)$. Since $a_{\pi(n-i)} = a_{\pi(n-i+1)} = \cdots = a_{\pi(n)} = 0$, we must have $\pi(n-i) < \pi(n-i+1) < \cdots < \pi(n)$, so π has no descents in $\{n-i, n-i+1, \ldots, n-1\}$. Let $\lambda_j = a_{\pi(j)} - a_{\pi(j+1)}$ for $j = 1, 2, \ldots, n-i-1$. Then $(a_1, \ldots, a_n) \leftrightarrow (\pi, \lambda)$ is a bijection between nonnegative integer sequences of length n with at least $i+1$ zeroes and pairs (π, λ) where $\pi \in S_n^{(i)}$ and λ is a nonnegative integer sequence of length $n-i-1$ satisfying $\lambda_j > 0$ for $j \in D(\pi)$. Moreover, $\sum_{j=1}^{n-i-1} \lambda_i = \max(a)$ and $|a| = \sum_{j=1}^{n-i-1} i\lambda_i$. Now define nonnegative integers $\lambda'_1, \ldots, \lambda'_{n-i-1}$ by

$$\lambda'_j = \begin{cases} \lambda_j, & \text{if } j \notin D(\pi) \\ \lambda_j - 1, & \text{if } j \in D(\pi). \end{cases}$$

So

$$\begin{aligned}
\sum_{j=0}^{\infty} T_j x^j &= \sum_j \sum_{\max(a)\le j} q^{|a|}x^j = \frac{1}{1-x}\sum_a q^{|a|}x^{\max(a)} \\
&= \frac{1}{1-x}\sum_{(\pi,\lambda)} q^{\lambda_1+2\lambda_2+\cdots+(n-i-1)\lambda_{n-i-1}} x^{\lambda_1+\lambda_2+\cdots+\lambda_{n-i-1}} \\
&= \frac{1}{1-x}\sum_{\pi\in S_n^{(i)}} q^{\mathrm{maj}(\pi)}x^{\mathrm{des}(\pi)} \sum_{\lambda'_1,\ldots,\lambda'_{n-i-1}} \prod_{j=1}^{n-i-1}(xq^j)^{\lambda'_j} \\
&= \frac{C_n^{(i)}(x,q)}{(1-x)(1-xq)\ldots(1-xq^{n-i-1})}. \qquad (13)
\end{aligned}$$

Next we compute T_j directly. The sum of $q^{|a|}$ over all nonnegative integer sequences a with $\max(a)\le j$ is $[j+1]_q^n$, and the sum of $q^{|a|}$ over all sequences a with $\max(a)\le j$ and with exactly ℓ nonzero entries is $\binom{n}{\ell}(q[j]_q)^\ell$. Thus

$$T_j = [j+1]_q^n - \sum_{\ell=n-i}^{n}\binom{n}{\ell}(q[j]_q)^\ell,$$

so

$$\begin{aligned}
\sum_{j=0}^{\infty} T_j x^j &= \sum_{j=0}^{\infty}\left([j+1]_q^n x^j - x^j\sum_{\ell=n-i}^{n}\binom{n}{\ell}(q[j]_q)^\ell\right) \\
&= \sum_{j=1}^{\infty}[j]_q^n x^{j-1} - \sum_{j=0}^{\infty}x^j\sum_{\ell=n-i}^{n}\binom{n}{\ell}(q[j]_q)^\ell.
\end{aligned}$$

The sum on ℓ vanishes when $j=0$ since for each term, $\ell\ge n-i>0$ so $[0]_q^\ell=0$. Thus

$$\begin{aligned}
\sum_{j=0}^{\infty} T_j x^j &= \sum_{j=1}^{\infty}[j]_q^n x^{j-1} - \sum_{j=1}^{\infty}x^j\left((q[j]_q)^n + \sum_{\ell=n-i}^{n-1}\binom{n}{\ell}(q[j]_q)^\ell\right) \\
&= \sum_{j=1}^{\infty}\left((1-xq^n)[j]_q x^{j-1} - x^j\sum_{\ell=n-i}^{n-1}\binom{n}{\ell}(q[j]_q)^\ell\right). \qquad (14)
\end{aligned}$$

Then Corollary 2 follows from (13) and (14).

3 Application

We apply the results of Section 2.1 to enumerate the sequences satisfying the constraints (2).

Theorem 3.

$$G_n(q) \quad = \quad \frac{C_n(q^{nm},q)(1-q^{n\ell-1}) - C_{n-1}(q^{nm},q)nq^{nm+n-1}(1-q^{-nk})}{(1-q^n)(1-q^{n\ell-1})(q^{nm+1};q)_{n-1}}. \qquad (15)$$

Proof. From Proposition 1,

$$G_n(q) = \frac{1}{(1-q^n)(1-q^{n\ell-1})(q^{nm+1};q)_{n-2}} \sum_{\pi\in S_n} \prod_{i\in D(\pi)} q^{i+nb_i}, \tag{16}$$

where $b_i = m$, $1 \le i \le n-2$ and $b_{n-1} = m-k$, and $m = k+\ell-1$.

Start with (6), apply (11) and rearrange terms to get:

$$\begin{aligned}
\sum_{\pi\in S_n} \prod_{i\in D(\pi)} q^{i+nb_i} &= C_n^{(1)}(q^{nm},q) + q^{-nk}(C_n(q^{nm},q) - C_n^{(1)}(q^{nm},q)) \\
&= (1-q^{-nk})C_n^{(1)}(q^{nm},q) + q^{-nk}C_n(q^{nm},q) \\
&= \frac{(1-q^{-nk})(C_n(q^{nm},q) - nq^{nm+n-1}C_{n-1}(q^{nm},q))}{(1-q^{nm+n-1})} + q^{-nk}C_n(q^{nm},q) \\
&= C_n(q^{nm},q)\frac{(1-q^{-nk}) + q^{-nk}(1-q^{nm+n-1})}{(1-q^{nm+n-1})} \\
&\qquad -C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})} \\
&= C_n(q^{nm},q)\frac{(1-q^{-nk+nm+n-1})}{(1-q^{nm+n-1})} - C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})} \\
&= C_n(q^{nm},q)\frac{(1-q^{n\ell-1})}{(1-q^{nm+n-1})} - C_{n-1}(q^{nm},q)\frac{nq^{nm+n-1}(1-q^{-nk})}{(1-q^{nm+n-1})},
\end{aligned}$$

the last since $m = k+\ell-1$. Substitution into (16) gives the result. □

For example, since $C_2(x,q) = 1+xq$ and $C_3(x,q) = 1+2xq+2xq^2+x^2q^3$, solutions to

$$\{k\lambda_{\pi(1)} + \ell\lambda_{\pi(2)} \ge (k+\ell-1)\lambda_{\pi(3)}, \mid \pi\in S_3\}$$

are given by:

$$G_3(q) = \frac{1+2q^{3\ell-1}+2q^{3(k+\ell)-2}+q^{6\ell+3(k-1)}}{(1-q^3)(1-q^{3\ell-1})(1-q^{3k+3\ell-2})}.$$

When $k=\ell$, this becomes:

$$G_3(q) = \frac{1+2q^{3k-1}+2q^{6k-2}+q^{9k-3}}{(1-q^3)(1-q^{3k-1})(1-q^{6k-2})}$$

and when $k=1$, this gives the generating function for (ordered) integer-sided triangles:

$$G_3(q) = \frac{1+2q^2+2q^4+q^6}{(1-q^3)(1-q^2)(1-q^4)} = \frac{1-q+q^2}{(1-q^2)^2(1-q)}.$$

When $n=4$, solutions to

$$\{k\lambda_{\pi(1)} + \ell\lambda_{\pi(2)} \ge (k+\ell-1)\lambda_{\pi(3)}, \mid \pi\in S_4\}$$

are given by:

$$G_4(q) = \frac{1+3q^{4\ell-1}+3q^{4k+4\ell-3}+5q^{4k+4\ell-2}+5q^{8\ell+4k-4}+3q^{8\ell+4k-3}+3q^{8k+8\ell-5}+q^{12\ell+8k-6}}{(1-q^4)(1-q^{4\ell-1})(1-q^{4k+4\ell-3})(1-q^{4k+4\ell-2})}.$$

When $k = \ell$, this becomes:

$$G_4(q) = \frac{1 + 3q^{4k-1} + 3q^{8k-3} + 5q^{8k-2} + 5q^{12k-4} + 3q^{12k-3} + 3q^{16k-5} + q^{20k-6}}{(1-q^4)(1-q^{4k-1})(1-q^{8k-3})(1-q^{8k-2})}.$$

Acknowledgement. Thanks to Sunyoung Lee for her contributions to calculations in Section 3. We are grateful to the referees for their comments to improve the presentation.

References

[1] George E. Andrews. A note on partitions and triangles with integer sides. *Amer. Math. Monthly*, 86(6):477–478, 1979.

[2] George E. Andrews. MacMahon's partition analysis. II. Fundamental theorems. *Ann. Comb.*, 4(3-4):327–338, 2000. Conference on Combinatorics and Physics (Los Alamos, NM, 1998).

[3] George E. Andrews, Peter Paule, and Axel Riese. MacMahon's partition analysis. VII. Constrained compositions. In *q-series with applications to combinatorics, number theory, and physics (Urbana, IL, 2000)*, volume 291 of *Contemp. Math.*, pages 11–27. Amer. Math. Soc., Providence, RI, 2001.

[4] Miklós Bóna. *Combinatorics of permutations.* Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2004. With a foreword by Richard Stanley.

[5] L. Carlitz. q-Bernoulli and Eulerian numbers. *Trans. Amer. Math. Soc.*, 76:332–350, 1954.

[6] L. Carlitz. A combinatorial property of q-Eulerian numbers. *Amer. Math. Monthly*, 82:51–54, 1975.

[7] A. M. Garsia and I. Gessel. Permutation statistics and partitions. *Adv. in Math.*, 31(3):288–305, 1979.

[8] J. H. Jordan, Ray Walch, and R. J. Wisner. Triangles with integer sides. *Amer. Math. Monthly*, 86(8):686–689, 1979.

[9] Sunyoung Lee and Carla D. Savage. Symmetrically constrained compositions. preliminary draft.

[10] Percy A. MacMahon. *Combinatory analysis.* Two volumes (bound as one). Chelsea Publishing Co., New York, 1960.

[11] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.