Polynomial time approximation of entropy of shifts of finite type

Stefan Adams, Raimundo Briceno, Brian Marcus, Ronnie Pavlov

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$\mathbb{Z}^d$ Shift spaces

- **Full shift:** $\mathcal{A}^{\mathbb{Z}^d}$ over a finite alphabet $\mathcal{A}$.
- **Shift space:** for some list $\mathcal{F}$ of "forbidden" configurations on finite shapes, $X = X_{\mathcal{F}} := \{x \in \mathcal{A}^{\mathbb{Z}^d} : x \text{ contains no elements of } \mathcal{F}\}$
- **Shift of finite type (SFT):** a shift space where $\mathcal{F}$ can be chosen finite.
- **Nearest neighbor (n.n.) SFT:** a shift space where all elements of $\mathcal{F}$ are configurations on *edges* of $\mathbb{Z}^d$.

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Topological entropy

- $B_n := [0, n-1]^d$
- **globally admissible** configurations:

$$GA_n(X) = \{x(B_n) : x \in X\}$$

- **Topological entropy:**

$$h(X) := \lim_{n\to\infty} \frac{\log |GA_n(X)|}{n^d}$$

- **locally admissible** configurations:

$$LA_n(X) = \{ \text{ configs. on } B_n \text{ forbidding } \mathcal{F}\}$$

- Theorem (Ruelle, Friedland):

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SFT's, $d = 1$

- A $\mathbb{Z}$ n.n. SFT X over alphabet $\mathcal{A}$ is specified by a directed graph G with vertices indexed by $\mathcal{A}$ and an edge from a to b iff $ab \notin \mathcal{F}$.

Golden Mean Shift:

- Adjacency matrix A of G is the square matrix indexed by $\mathcal{A}$: $A_{ab} = \left\{ \begin{array}{cc} 1 & ab \notin \mathcal{F} \\ 0 & ab \in \mathcal{F} \end{array} \right\}$
- $h(X) = \log \lambda(A)$, where $\lambda(A)$ is the spectral radius of A.
- Characterization of entropies for $d = 1$ (Lind):

$$\{\log \lambda^{1/q}\}$$

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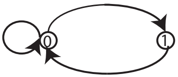

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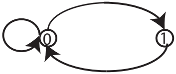

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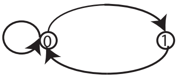

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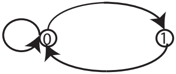

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Examples of $\mathbb{Z}^2$ SFTs: hard squares

- **hard squares** $\mathcal{A} = \{0, 1\}, \mathcal{F} = \{11, \begin{array}{c}1\\1\end{array}\}$
- h(hard squares) = ???
- h(hard hexagons) = log(λ) where λ is an algebraic integer of degree 24.

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Examples of $\mathbb{Z}^2$ SFTs: checkerboard (coloring) constraints

- ***q*-checkerboard** $\mathcal{C}_q$: $\mathcal{A} = \{1, \ldots, q\}, \mathcal{F} = \{aa, \begin{matrix} a \\ a \end{matrix}\}$

- $h(C_2) = 0$
- (Lieb): $h(C_3) = (3/2)\log(4/3)$
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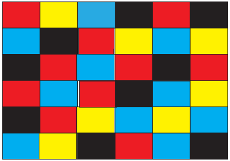

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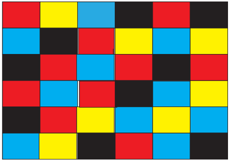

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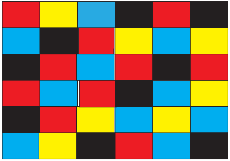

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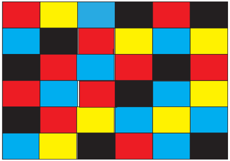

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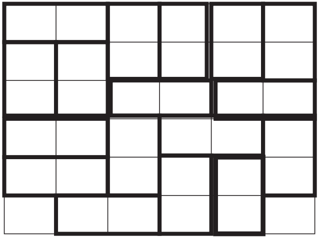

$$\mathcal{F} = \{LL, LT, LB, RR, TR, BR, \begin{matrix}T\\L\end{matrix}, \begin{matrix}T\\R\end{matrix}, \begin{matrix}T\\T\end{matrix}, \begin{matrix}B\\B\end{matrix}, \begin{matrix}L\\B\end{matrix}, \begin{matrix}R\\B\end{matrix}\}$$

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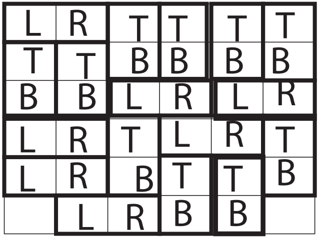

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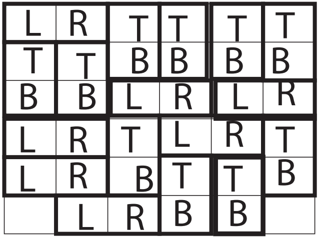

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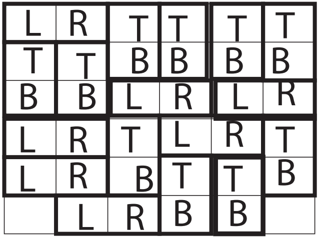

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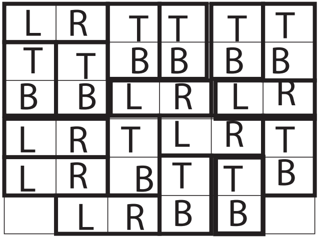

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Ledrappier 3-dot

$$X = \{x \in \{0,1\}^{\mathbb{Z}^2} : x((i,j)) + x((i+1,j)) + x((i,j+1)) = 0 \bmod 2\}$$

$$\mathcal{F} = \left\{ \begin{matrix} a & \\ b & c \end{matrix} : a + b + c \neq 0 \bmod 2 \right\}$$

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Examples of $\mathbb{Z}^2$ SFTs: iceberg model

- $\mathcal{A} = \{-M, \ldots, -1, 0, 1, \ldots M\}$
- $\mathcal{F} = \{ab, \begin{matrix} a \\ b \end{matrix} \; : \; a, b \text{ have opposite signs}\}$
- positives can sit next to positives and zeros, negatives can sit next to negatives and zeros, and zeros can sit next to anyone.
- Example: $M = 2$

$$\begin{array}{cccccccccc}
1 & 2 & 2 & 1 & 0 & 0 & -1 & -2 & 0 & 2 \\
0 & 2 & 1 & 0 & -1 & 0 & -2 & -1 & 0 & 2 \\
1 & 1 & 0 & -2 & 0 & 0 & 0 & 0 & 1 & 2 \\
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Examples of $\mathbb{Z}^2$ SFTs: iceberg model

- $\mathcal{A} = \{-M, \ldots, -1, 0, 1, \ldots M\}$
- $\mathcal{F} = \{ab, \begin{matrix} a \\ b \end{matrix} : a, b \text{ have opposite signs}\}$
- positives can sit next to positives and zeros, negatives can sit next to negatives and zeros, and zeros can sit next to anyone.
- Example: $M = 2$

1	2	2	1	0	0	−1	−2	0	2
0	2	1	0	−1	0	−2	−1	0	2
1	1	0	−2	0	0	0	0	1	2
0	2	1	0	−1	0	−2	−1	0	2
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Topological entropy, $d \geq 2$

- Exact formula known only in a few cases.
- Characterization of entropies for $d \geq 2$ (Hochman-Meyerovitch):

  {**right recursively enumerable** (RRE) numbers $h \geq 0$}

  i.e, there is an algorithm that produces a sequence $r_n \geq h$ s.t. $r_n \to h$.

  Proof:
  - Necessity: Let $r_n := \frac{\log |LA_n|}{n^d}$.
    By Ruelle/Friedland Theorem, $r_n \to h$.
    By subadditivity of $\log |LA_n|$, each $r_n \geq h$.
  - Sufficiency (hard): Emulate Turing machine with an SFT.
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- A **polynomial time approximation algorithm**: on input n, produces a sequence r_n s.t. $|r_n - h| < 1/n$ and r_n can be computed in time poly(n).
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Measure-theoretic entropy

Given a shift-invariant Borel probability measure μ on $\mathcal{A}^{\mathbb{Z}^d}$,

- For finite $S \Subset \mathbb{Z}^d$,

$$H_\mu(S) := \sum_{x \in \mathcal{A}^S} -\mu(x) \log \mu(x) = \int -\log \mu(x) d\mu(x)$$

- For finite disjoint S, T,

$$H_\mu(S \mid T) := \sum_{x \in \mathcal{A}^S, y \in \mathcal{A}^T :\ \mu(y) > 0} -\mu(x, y) \log \mu(x \mid y)$$

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- $h(\mu) := \lim_{n\to\infty} \frac{H_\mu(B_n)}{n^d}$
- $d = 1$: Theorem: $h(\mu) = H_\mu(0 \mid \{-1,-2,-3,\dots\})$
- $d = 2$: Let $\prec$ denotes lexicographic order: $(i,j) \prec (i',j')$ iff either $j < j'$ or ($j = j'$ and $i < i'$).

For $\overline{z} \in \mathbb{Z}^2$, let $\mathcal{P}^-(\overline{z}) := \{\overline{z}' \in \mathbb{Z}^2 : \ \overline{z}' \prec \overline{z}\}$ the lexicographic past of $\overline{z}$, and $\mathcal{P}^- := \mathcal{P}^-(0)$

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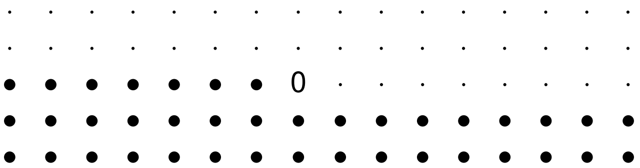

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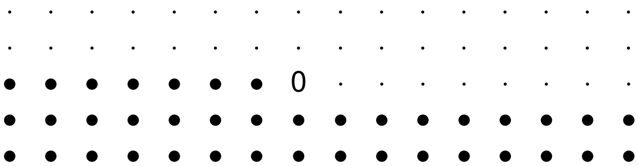

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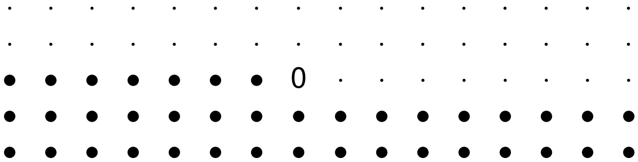

Theorem: $h(\mu) = H_{\mu}(0 \mid \mathcal{P}^{-})$.

Defn: The **information function** of μ is defined as

$$I_{\mu}(x) := -\log \mu(x(0) | \, x(\mathcal{P}^{-})) \quad (\mu - a.e.)$$

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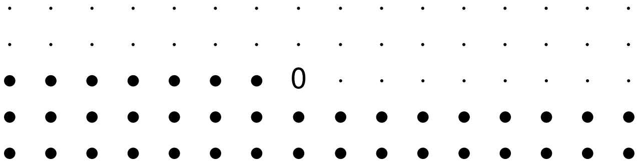

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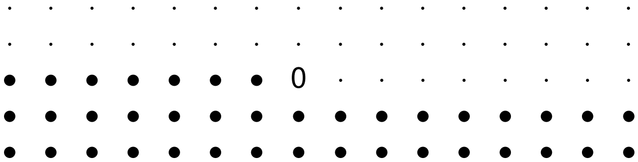

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Variational Principle for Topological Entropy

- For a shift space X,

$$h(X) = \sup_{\mu} h(\mu)$$

where the sup is taken over all shift-invariant Borel probability measures μ s.t. support$(\mu) \subseteq X$.
- Fact: The sup is always achieved. A measure which achieves the sup is called a **measure of maximal entropy (MME)**.
- So for an MME μ, $h(X) = h(\mu) = \int I_\mu(x) d\mu(x)$
- Under certain conditions, $h(X) = h(\mu) = \int I_\mu d\nu$ for some other invariant measure ν and, under stronger conditions, for *all* invariant measures ν.
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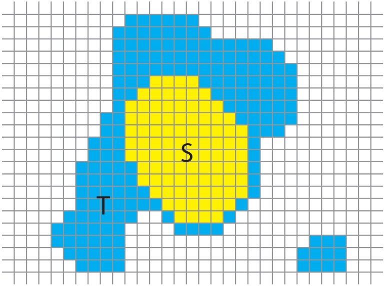

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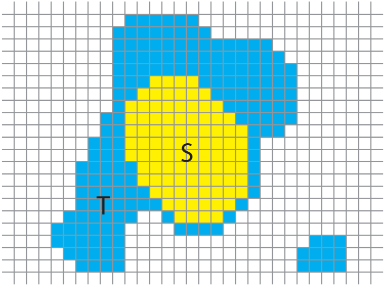

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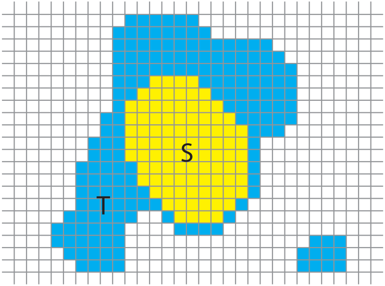

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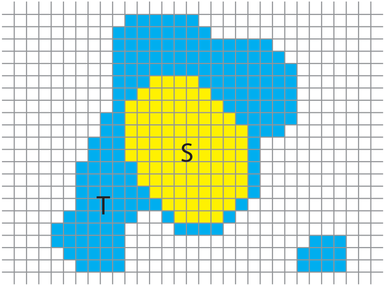

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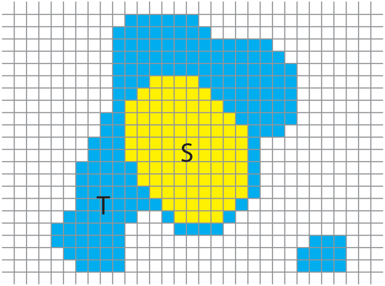

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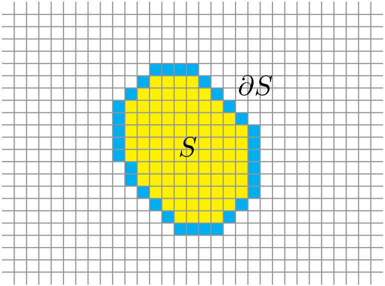

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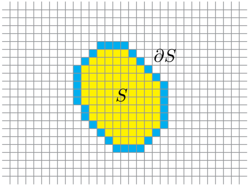

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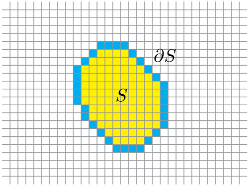

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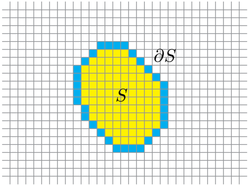

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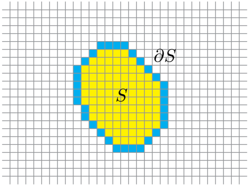

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Example: $R_{3,4,3}$:

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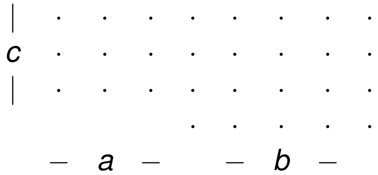

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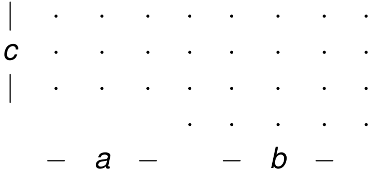

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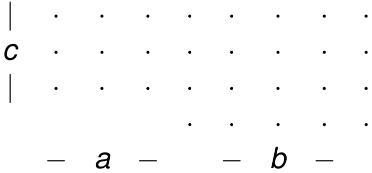

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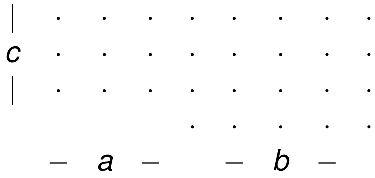

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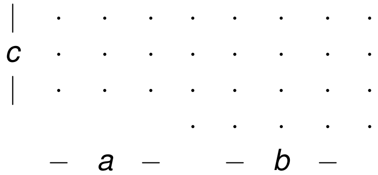

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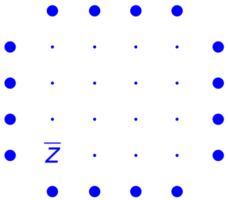

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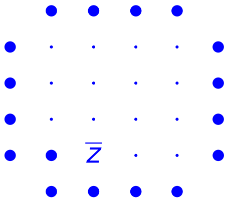

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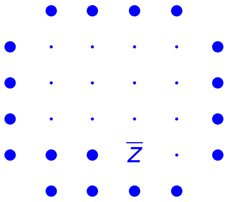

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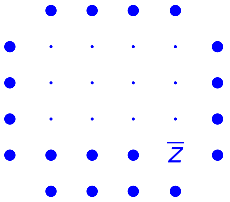

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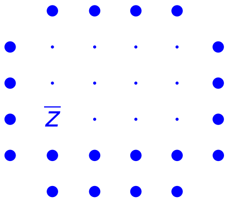

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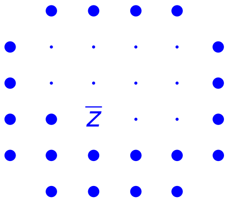

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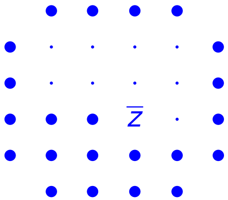

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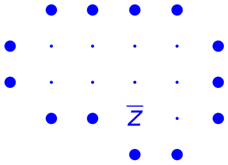

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- By the convergence assumption, for "most" $\bar{z} \in B_n$

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Algorithmic consequence

Theorem (special case of Gamarnik-Katz): Let X be a n.n. $\mathbb{Z}^2$ SFT and μ an MME on X. If

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Proof: Approximate L by $\mu(s^0 \mid s^{\partial R_{n,n,n}})$.

- Accuracy is $e^{-\Omega(n)}$
- Claim: Computation time is $e^{O(n)}$
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Proof of Claim, via transition matrices

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M_i is transition matrix from column i to column $i+1$ compatible with $s^{\partial S_{n,n,n}}$ and

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- A $\mathbb{Z}^d$ SFT X satisfies **topological strong spatial mixing (TSSM)** with gap g if

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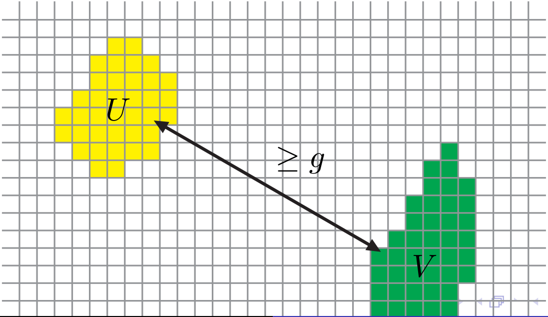

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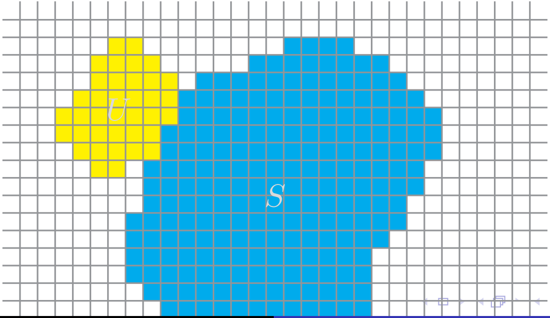

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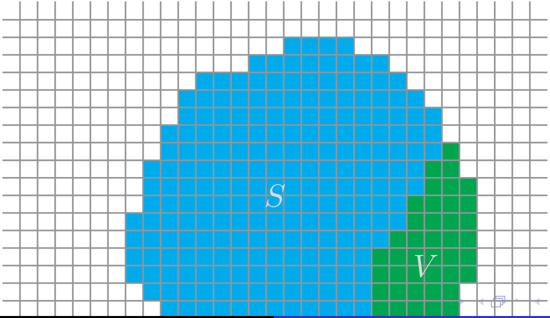

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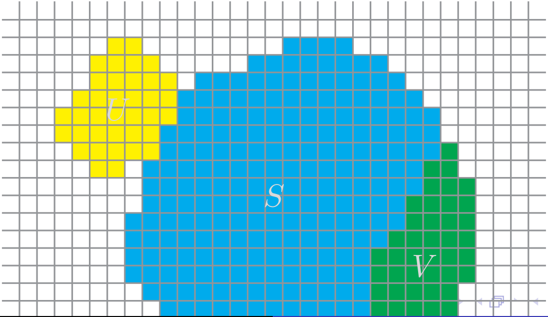

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Generalization

Theorem (Adams, Briceno, Marcus, Pavlov): Let X be a $\mathbb{Z}^d$ n.n. SFT and μ an MME on X. If

1. X satisfies TSSM
2. For some periodic orbit O in X and all $\omega \in O$

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$$h(X) = -\frac{1}{|O|} \sum_{\omega \in O} \log L(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $h(X)$.

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Applies to:

- hard squares
- q-checkerboard with $q \geq 6$
- iceberg with $M \geq 24$.

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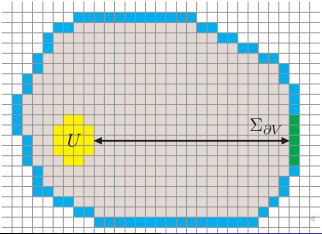

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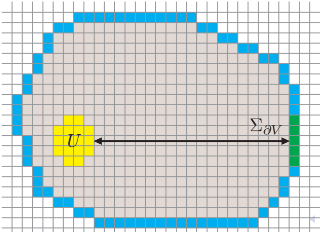

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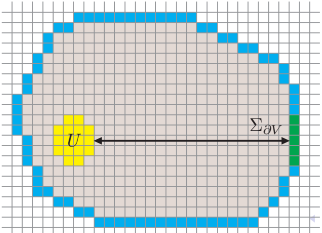

Stronger conclusion

Theorem (Briceno): Let X be a $\mathbb{Z}^d$ n.n. SFT and μ an MME on X. If

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Then for *all* invariant measures ν s.t. support$(\nu) \subseteq X$,

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$$h(X) = \int I_\mu(x)\, d\nu(x)$$

Applies to:

- hard squares
- q-checkerboard with $q \geq 6$

but *not* to iceberg.

Extension to Pressure

- Generalize results from entropy to *pressure* of nearest neighbour interactions
- Applies to large sets of parameters for classical models in statistical physics, including Hard squares, Ising, Potts, and Widom-Rowlinson.

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Topological Pressure and Variational Principle

- Let X be a shift space and $f : X \to \mathbb{R}$ a continuous function.
- **Topological Pressure** (defined by Variational Principle):

$$P_X(f) := \sup_\mu h(\mu) + \int f d\mu$$

  where the sup is taken over all shift-invariant Borel probability measures μ such that support$(\mu) \subseteq X$.
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Nearest-Neighbour interactions and Gibbs measures

- A *nearest-neighbor interaction* is a shift-invariant function Φ from a set of configurations on vertices and edges in $\mathbb{Z}^d$ to $\mathbb{R} \cup \infty$
- For a nearest-neighbor interaction Φ, the *underlying SFT*:

$$X = X_\Phi := \{x \in \mathcal{A}^{\mathbb{Z}^d} \ : \ \Phi(x(\{v, v'\})) \neq \infty, \text{ for all } v \sim v'\}.$$

- A *nearest neighbour (n.n.) Gibbs measure* μ corresponding to Φ is an MRF on X such that for $S \Subset \mathbb{Z}^d$, $\delta \in \mathcal{A}^{\partial S}$, $\mu(\delta) > 0$, $w \in \mathcal{A}^S$:

$$\mu(w|\delta) = \frac{e^{-U^\Phi(w\delta)}}{Z^{\Phi,\delta}(S)}.$$

where
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Examples of n.n. Gibbs measures

- uniform MME on n.n. SFT
- hard square model with activities
- ferromagnetic Ising model with no external field.

Equilibrium states versus n.n. Gibbs measures

- Pressure of n.n. interaction Φ:

$$P(\Phi) := \lim_{n\to\infty} \frac{\log Z^{\Phi}(B_n)}{n^d}$$

where $Z^{\Phi}(B_n)$ is the "free boundary" normalization.

- Let $A_{\Phi}(x) := -\Phi(x(0)) - \sum_{i=1}^{d} \Phi(x(0), x(e_i))$.
- Fact: $P_{X_{\Phi}}(A_{\Phi}) = P(\Phi)$.
- Lanford-Ruelle Theorem: Every equilibrium state for A_{Φ} is a Gibbs measure for Φ.
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- These theorems hold in much greater generality.

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Pressure representation and approximation

Theorem (Adams, Briceno, Marcus, Pavlov): Let μ a Gibbs measure for a n.n. interaction Φ with underlying $\mathbb{Z}^d$ n.n. SFT X. If

1. X satisfies TSSM
2. For some periodic orbit O in X and all $\omega \in O$

$$L(\omega) := \lim_{a,b,c \to \infty} \mu(\omega(0) \mid \omega(\partial R_{a,b,c})) \text{ exists}$$

Then

$$P(\Phi) = \frac{1}{|O|} \sum_{\omega \in O} -\log L(\omega) + A_\Phi(\omega)$$

Moreover, if $d = 2$ and convergence in hypothesis 2 is exponential, then there is a polynomial time algorithm to compute $P(\Phi)$.

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An SFT X satisfies the **D-condition** if

- there exist sequences of finite subsets (Λ_n), (M_n) of $\mathbb{Z}^d$ such that $\Lambda_n \nearrow \infty$, $\Lambda_n \subseteq M_n$, $\frac{|M_n|}{|\Lambda_n|} \to 1$, such that
- for any globally admissible $v \in \mathcal{A}^{\Lambda_n}$ and finite $S \subset M_n^c$ and globally admissible $w \in \mathcal{A}^S$, we have that vw is globally admissible.

Safe symbol $\Rightarrow$ TSSM $\Rightarrow$ D-condition

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MME, $d = 1$

- Assuming adjacency matrix A is irreducible and aperiodic, there is a unique MME $\mu_{\max}$, which is a Markov chain given by transition matrix

$$P_{ij} = \left\{ \begin{array}{cc} \frac{r_j}{\lambda r_i} & ij \notin \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{array} \right\}$$

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- Assuming adjacency matrix A is irreducible and aperiodic, there is a unique MME $\mu_{\max}$, which is a Markov chain given by transition matrix

$$P_{ij} = \left\{ \begin{array}{cc} \frac{r_j}{\lambda r_i} & ij \notin \mathcal{F} \\ 0 & ij \in \mathcal{F} \end{array} \right\}$$

  where $\lambda = \lambda(A)$ and r is a right eigenvector for λ, and stationary vector $r_i \ell_i$ where ℓ is a left eigenvector for λ (suitably normalized)
- Thus, if $\mu(w_1 w_2 \ldots w_{n-1} w_n) > 0$, then

$$\mu(w_1 w_2 \ldots w_{n-1} w_n) = \frac{\ell_{w_1} r_{w_n}}{\lambda^{n-1}}$$

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Entropy representation for MME, $d = 1$

- $$\begin{aligned} I_\mu(x) &= -\log \mu(x(0)|\, x(\mathcal{P}^-)) \\ &= -\log P_{x_0 x_{-1}} \\ &= \log \lambda + \log r_{x_{-1}} - \log r_{x_0} \end{aligned}$$
- So, for *all* invariant measures ν,

$$\begin{aligned} \int I_\mu(x) d\nu(x) &= \int (\log \lambda + \log r_{x_{-1}} - \log r_{x_0}) d\nu(x) \\ &= \log \lambda \\ &= h(X) \end{aligned}$$

In particular, if the SFT has a fixed point $x^* := a^{\mathbb{Z}}$ and ν is the delta measure on x^*, then on

$$h(X) = \int I_\mu(x) d\nu(x) = I_\mu(x^*) = -\log \mu(x^*)$$

and so $h(X)$ can be computed from the value of the information function at only one point.
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