# Subtyping

# Outline

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# Simply Typed λ-calculus

## Syntax

| | | | | |
|---|---|---|---|---|
| *Types* | $T$ | $::=$ | $T \to T$ | function types |
| | | | $\texttt{Bool} \mid \texttt{Int} \mid \texttt{Real} \mid \ldots$ | basic types |
| *Terms* | $a, b$ | $::=$ | $\texttt{true} \mid \texttt{false} \mid 1 \mid 2 \mid \ldots$ | constants |
| | | $\mid$ | $x$ | variable |
| | | $\mid$ | $a\,b$ | application |
| | | $\mid$ | $\lambda x{:}T.a$ | abstraction |

## Reduction

$$\textit{Contexts} \quad C[\,] \quad ::= \quad [\,] \quad \mid \quad a[\,] \quad \mid \quad [\,]a \quad \mid \quad \lambda x{:}T.[\,]$$

BETA

$$(\lambda x{:}T.a)b \longrightarrow a[b/x]$$

CONTEXT

$$\frac{a \longrightarrow b}{C[a] \longrightarrow C[b]}$$

# Type system

## Typing

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**Theorem (Subject Reduction)**

*If $\Gamma \vdash a : T$ and $a \longrightarrow^* b$, then $\Gamma \vdash b : T$.*

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**Theorem (Subject Reduction)**

*If $\Gamma \vdash a : T$ and $a \longrightarrow^* b$, then $\Gamma \vdash b : T$.*

We will essentially focus on the subject reduction property (a.k.a. *type preservation*), though well-typed programs also satisfy *progress*:

**Theorem (Progress)**

*If $\varnothing \vdash a : T$ and $a \not\longrightarrow$, then a is a value*

where a value is either a constant or a lambda abstraction

$$v ::= \lambda x{:}T.a \mid \texttt{true} \mid \texttt{false} \mid 1 \mid 2 \mid \ldots$$

# Type checking algorithm

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```
let rec typecheck gamma = function
  | x -> gamma(x)                                    (* Var rule   *)
  | λx:T.a -> T → (typecheck (gamma, x:T) a)         (* Intro rule *)
  | ab -> let T1→T2 = typecheck gamma a in           (* Elim rule  *)
          let T3 = typecheck gamma b in
            if T1==T3 then T2 else fail
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**Exercise.** *Write the* `typecheck` *function for the following definitions:*

```
type stype = Int | Bool | Arrow of stype * stype

type term =
   Num of int | BVal of bool | Var of string
 | Lam of string * stype * term | App of term * term

exception Error
```

Use `List.assoc` for environments.

# Subtyping

The rule for application requires the argument of the function to be *exactly of the same type* as the domain of the function:

$$\frac{\Gamma \vdash a : S \to T \qquad \Gamma \vdash b : S}{\Gamma \vdash a\,b : T}\ (\to\textsc{Elim})$$

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## Subtyping polymorphism

We need a kind of polymorphism different from the ML one (parametric polymorphism).

# Subtyping relation

- Define a pre-order (*ie*, a reflexive and transitive binary relation) $\leqslant$ on types: $\leqslant \subset \mathit{Types} \times \mathit{Types}$ (some literature uses the notation `<:`)

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**Containment:** If $S \leqslant T$, then every value of type $S$ *is also* of type $T$.
For instance an odd number *is also* an integer, a student *is also* a person.
Sometimes called a "**is_a**" relation.

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- We'll see how each interpretation has a formal counterpart.

# Subtyping for simply typed λ-calculus

- We suppose to have a predefined preorder $\mathcal{B} \subset \mathit{Basic} \times \mathit{Basic}$ for basic types (given by the language designer).

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  $\{(\mathtt{Odd}, \mathtt{Int}), (\mathtt{Even}, \mathtt{Int}), (\mathtt{Int}, \mathtt{Real})\}$

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- To extend it to function types, we resort to the sustitutability interpretation. We will try to deduce when we can safely replace a function of some type by a term of a different type

# Subtyping of arrows: intuition

**Problem**

Determine for which type $S$ we have $S \leqslant T_1 \to T_2$

Let $g : S$ and $f : T_1 \to T_2$. Let us follow the **substitutability interpretation:**

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1. If $a : T_1$, then we can apply $f$ to $a$. If $S \leqslant T_1 \to T_2$, then we can apply $g$ to $a$, as well.
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2. If $a : T_1$, then $f(a)$ is well typed. If $S_1 \to S_2 \leqslant T_1 \to T_2$, then also $g(a)$ is well-typed. $g$ expects arguments of type $S_1$ but $a$ is of type $T_1$
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3. $f(a) : T_2$, but since $g$ returns results in $S_2$, then $g(a) : S_2$. If I use $g$ where $f$ is expected, then it must be safe to use $S_2$ results where $T_2$ results are expected
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## Solution

$$S_1 \to S_2 \leqslant T_1 \to T_2 \quad \Leftrightarrow \quad T_1 \leqslant S_1 \wedge S_2 \leqslant T_2$$

# Covariance and contravariance

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Notice the different orientation of containment on domains and co-domains.

We say that the type constructor $\to$ is

- *covariant* on codomains, since it preserves the direction of the relation;
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- *is also* a function that maps integers to reals: it returns results in `Int` so they will be also in `Real`.

  `Int→Int`$\leqslant$ `Int→Real` (covariance of the codomains)

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  `Int→Int≤ Int→Real` (covariance of the codomains)
- *is also* a function that maps odds to integers: when fed with integers it returns integers, so will do the same when fed with odd numbers.

  `Int→Int≤ Odd→Int` (contravariance of the codomains)

# Subtyping deduction system

$$\textsc{Basic}\ \frac{(B_1, B_2) \in \mathcal{B}}{B_1 \leqslant B_2} \qquad \textsc{Arrow}\ \frac{T_1 \leqslant S_1 \qquad S_2 \leqslant T_2}{S_1 \to S_2 \leqslant T_1 \to T_2}$$

$$\textsc{Refl}\ \frac{}{T \leqslant T} \qquad \textsc{Trans}\ \frac{T_1 \leqslant T_2 \qquad T_2 \leqslant T_3}{T_1 \leqslant T_3}$$

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**Theorem (Admissibility of Refl and Trans)**

*In the system composed just by the rules Arrow and Basic:*
*1) $T \leqslant T$ is provable for all types $T$*
*2) If $T_1 \leqslant T_2$ and $T_2 \leqslant T_3$ are provable, so is $T_1 \leqslant T_3$.*

The rules Refl and Trans are *admissible*

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This corresponds to the *containment relation*:

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Subject reduction: If $\Gamma \vdash a : T$ and $a \longrightarrow^* b$, then $\Gamma \vdash b : T$.
Progress property: If $\varnothing \vdash a : T$ and $a \not\longrightarrow$, then $a$ is a value

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$$\frac{\Gamma \vdash a : S \qquad S \leqslant T}{\Gamma \vdash a : T}$$

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Subsumption makes the type system non-algorithmic:

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$\to$ELIM$_{\leqslant}$
$$\frac{\Gamma \vdash_{\mathcal{A}} a : S {\to} T \qquad \Gamma \vdash_{\mathcal{A}} b : U \qquad U {\leqslant} S}{\Gamma \vdash_{\mathcal{A}} ab : T}$$

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$$\to\text{ELIM}_{\leqslant}\quad \frac{\Gamma \vdash_{\mathcal{A}} a : S \to T \qquad \Gamma \vdash_{\mathcal{A}} b : U \qquad U \leqslant S}{\Gamma \vdash_{\mathcal{A}} ab : T}$$

1. The system is algorithmic: it describes a typing algorithm (exercise: program `typecheck` and `subtype` by using the previous structures)
2. The system conforms the substitutability interpretation: we *use* an expression of a subtype $U$ where a supertype $S$ is expected (note “use” = elimination rule).

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1. The system is algorithmic: it describes a typing algorithm (exercise: program `typecheck` and `subtype` by using the previous structures)
2. The system conforms the substitutability interpretation: we *use* an expression of a subtype $U$ where a supertype $S$ is expected (note “use” = elimination rule).

How do we relate the two systems?

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For subtyping, admissibility ensured that the system and the algorithm prove the same judgements. Here it is no longer true. For instance:

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**This is expected:** Algorithm = one type returned for each typable term.

# Soundness and completeness of the typing algorithm

$a$ is typable by $\vdash \quad \Leftrightarrow \quad a$ is typable by $\vdash_{\mathcal{A}}$

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**Theorem (Soundness)**

*If $\Gamma \vdash_{\mathcal{A}} a : T$, then $\Gamma \vdash a : T$*

**Theorem (Completeness)**

*If $\Gamma \vdash a : T$, then $\Gamma \vdash_{\mathcal{A}} a : S$ with $S \leqslant T$*

# Minimum type and soundness

## Corollary (Minimum type)

*If* $\Gamma \vdash_{\mathcal{A}} a : T$ *then* $T = \min\{S \mid \Gamma \vdash a : S\}$

Proof. Let $\mathcal{S} = \{S \mid \Gamma \vdash a : S\}$. Soundness ensures that $\mathcal{S}$ is not empty. Completeness states that $T$ is a lower bound of $\mathcal{S}$. Minimality follows by using soundness once more.

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## Theorem (Algorithmic subject reduction)

*If* $\Gamma \vdash_{\mathcal{A}} a : T$ *and* $a \longrightarrow^* b$, *then* $\Gamma \vdash_{\mathcal{A}} b : S$ *with* $S \leqslant T$.

The theorem above explains that the computation reduces the minimum type of a program. As such it increases the type information about it.

# Summary for simply-typed $\lambda$-calculs + $\leqslant$

- The *containment* interpretation of the subtyping relation corresponds to the “logical” view of the type system embodied by subsumption.
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- The obtained algorithm works on the *minimum types* of the logical system
- Computation reduces the (algorithmic) type thus increasing type information (the result of a computation represents the best possible type information: it is the *singleton type* containing the result).
- The last point makes *dynamic dispatch* (aka, dynamic binding) meaningful.

# Products I

## Syntax

$$\begin{array}{llcll} \textit{Types} & T & ::= & \ldots \mid T \times T & \text{product types} \\ \textit{Terms} & a, b & ::= & \ldots & \\ & & \mid & (a, a) & \text{pair} \\ & & \mid & \pi_i(a) \quad (i{=}1{,}2) & \text{projection} \end{array}$$

## Reduction

$$\pi_i((a_1, a_2)) \longrightarrow a_i \qquad (i{=}1{,}2)$$

## Typing

$$\times\textsc{Intro}\ \frac{\Gamma \vdash a_1 : T_1 \qquad \Gamma \vdash a_2 : T_2}{\Gamma \vdash (a_1, a_2) : T_1 \times T_2} \qquad\qquad \times\textsc{Elim}_i\ \frac{\Gamma \vdash a : T_1 \times T_2}{\Gamma \vdash \pi_i(a) : T_i}\ (i{=}1{,}2)$$

# Products II

## Subtyping

$$\frac{\textsc{Prod}\quad S_1 \leqslant T_1 \qquad S_2 \leqslant T_2}{S_1 \times S_2 \leqslant T_1 \times T_2}$$

**Exercise:** *Check whether the above rule is compatible with the containement and/or the substitutability interpretation of the subtyping relation.*

The subtyping rule above is also algorithmic. Similarly, for the typing rules there is no need to embed subtyping in the elimination rules since $\pi_i$ is an operator that works on all products, not a particular one (*cf.* with the application of a function, which requires a particular domain).

Of course subject reduction and progress still hold.

**Exercise:** *Define values and reduction contexts for this extension.*

# Records

Up to now subtyping rules « lift » the subtyping relation $\mathcal{B}$ on basic types to constructed types. But if $\mathcal{B}$ is the identity relation, so is the whole subtyping relation. Record subtyping is non-trivial even when $\mathcal{B}$ is the identity relation.

## Syntax

$$\begin{array}{llcll} \textit{Types} & T & ::= & \ldots \mid \{\ell : T, \ldots, \ell : T\} & \text{record types} \\ \textit{Terms} & a, b & ::= & \ldots & \\ & & \mid & \{\ell = a, \ldots, \ell = a\} & \text{record} \\ & & \mid & a.\ell & \text{field selection} \end{array}$$

## Reduction

$$\{\ldots, \ell = a, \ldots\}.\ell \longrightarrow a$$

## Typing

{}INTRO

$$\frac{\Gamma \vdash a_1 : T_1 \ \ldots \ \Gamma \vdash a_n : T_n}{\Gamma \vdash \{\ell_1 = a_1, \ldots, \ell_n = a_n\} : \{\ell_1 : T_1, \ldots, \ell_n : T_n\}}$$

{}ELIM

$$\frac{\Gamma \vdash a : \{\ldots, \ell : T, \ldots\}}{\Gamma \vdash a.\ell : T}$$

# Record Subtyping

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We can replace some record by a record of different type if in the latter we can select the same fields as in the former and their contents can substitute the respective contents in the former.

## Subtyping

RECORD

$$\frac{S_1 \leqslant T_1 \ \dots\ S_n \leqslant T_n}{\{\ell_1{:}S_1,\dots,\ell_n{:}S_n,\dots,\ell_{n+k}{:}S_{n+k}\} \leqslant \{\ell_1{:}T_1,\dots,\ell_n{:}T_n\}}$$

**Exercise.** *Which are the algorithmic typing rules?*

# Outline

# Iso-recursive and Equi-recursive types

Lists are a classic example of recursive types:

$$X \quad \approx \quad (\mathtt{Int} \times X) \vee \mathtt{Nil}$$

also written as $\mu X.((\mathtt{Int} \times X) \vee \mathtt{Nil})$

Two different approaches according to whether $\approx$ is interpreted as an isomorphism or an equality:

Iso-recursive types: $\mu X.((\mathtt{Int} \times X) \vee \mathtt{Nil})$ is considered *isomorphic* to its one-step unfolding $(\mathtt{Int} \times \mu X.((\mathtt{Int} \times X) \vee \mathtt{Nil})) \vee \mathtt{Nil})$. Terms include a pair of built-in coercion functions for each recursive type $\mu X.T$:

$$\mathtt{unfold} : \mu X.T \to T[\mu X.T/X] \qquad \mathtt{fold} : T[\mu X.T/X] \to \mu X.T$$

Equi-recursive types: $\mu X.((\mathtt{Int} \times X) \vee \mathtt{Nil})$ is considered *equal* to its one-step unfolding $(\mathtt{Int} \times \mu X.((\mathtt{Int} \times X) \vee \mathtt{Nil})) \vee \mathtt{Nil})$. The two types are completely interchangeable. No support needed from terms.

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Equi-recursive types: $\mu X.((\texttt{Int} \times X) \vee \texttt{Nil})$ is considered *equal* to its one-step unfolding $(\texttt{Int} \times \mu X.((\texttt{Int} \times X) \vee \texttt{Nil})) \vee \texttt{Nil})$. The two types are completely interchangeable. No support needed from terms.

Subtyping for recursive types generalizes the equi-recursive approach.
The $\approx$ relation corresponds to subtyping in both directions:

$$\mu X.T \leqslant T[\mu X.T/X] \qquad\qquad T[\mu X.T/X] \leqslant \mu X.T$$

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- Actually if you have recursive terms and allow infinite values you can easily jeopardize decidability of the subtyping relation (which resorts to checking type emptiness)
- This contrasts with their intuition which looks simple: we always informally applied a rule such as:

$$\frac{A, X \leqslant Y \vdash S \leqslant T}{A \vdash \mu X.S \leqslant \mu Y.T}$$

# Subtyping recursive types

## Syntax

| | | | | |
|---|---|---|---|---|
| *Types* | $T$ | $::=$ | $\texttt{Any}$ | top type |
| | | $\mid$ | $T \to T$ | function types |
| | | $\mid$ | $T \times T$ | product types |
| | | $\mid$ | $X$ | type variables |
| | | $\mid$ | $\mu X.T$ | recursive types |

where $T$ is *contractive*, that is (two equivalent definitions):

1. $T$ is contractive iff for every subexpression $\mu X.\mu X_1.\ldots.\mu X_n.S$ it holds $S \neq X$.
2. $T$ is contractive iff every type variable $X$ occurring in it is separated from its binder by a $\to$ or a $\times$.

# Subtyping recursive types

The subtyping relation is defined *COINDUCTIVELY* by the rules

$$\textsc{Top}\ \frac{}{T \leqslant \texttt{Any}} \qquad \textsc{Prod}\ \frac{S_1 \leqslant T_1 \qquad S_2 \leqslant T_2}{S_1 \times S_2 \leqslant T_1 \times T_2} \qquad \textsc{Arrow}\ \frac{T_1 \leqslant S_1 \qquad S_2 \leqslant T_2}{S_1 \to S_2 \leqslant T_1 \to T_2}$$

$$\textsc{Unfold Left}\ \frac{S[\mu X.S/X] \leqslant T}{\mu X.S \leqslant T} \qquad \textsc{Unfold Right}\ \frac{S \leqslant T[\mu X.T/X]}{S \leqslant \mu X.T}$$

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3. Why no rule to compare two $\mu$-types?

**Short answers (more detailed answers to come):**

1. Because we compare infinite expansions
2. Because it would be unsound
3. Useless since obtained by coinduction and unfold

# Example of coinductive derivation

$$\textsc{Unfold Left}\ \dfrac{\textsc{Unfold Right}\ \dfrac{\textsc{Arrow}\ \dfrac{\texttt{Even} \leqslant \texttt{Int} \qquad \mu X.\texttt{Int} \to X \leqslant \mu Y.\texttt{Even} \to Y}{\texttt{Int} \to (\mu X.\texttt{Int} \to X) \leqslant \texttt{Even} \to (\mu Y.\texttt{Even} \to Y)}}{\texttt{Int} \to (\mu X.\texttt{Int} \to X) \leqslant \mu Y.\texttt{Even} \to Y}}{\mu X.\texttt{Int} \to X \leqslant \mu Y.\texttt{Even} \to Y}$$

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**Notice the use of coinduction**

# Amadio and Cardelli's subtyping algorithm

Let $A \subset \mathit{Types} \times \mathit{Types}$

$$\frac{}{A \vdash S \leqslant T}\ (S,T) \in A$$

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### The rest is similar

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# Properties

**Theorem (Soundness and Completeness)**

*Let $S$ and $T$ be closed types. $S \leqslant T$ belongs the relation coinductively defined by the rules in slide 374 if and only if $\varnothing \vdash S \leqslant T$ is provable*

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## Theorem (Soundness and Completeness)

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To see the proof of the above theorem you can refer to the following reference Pierce et al. Recursive types revealed, Journal of Functional Programming, 12(6):511-548, 2002.

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Notice that the algorithm above is exponential. We will show how to define an $O(n^2)$ algorithm to decide $S \leqslant T$, where $n$ is the total number of different subexpressions of $S \leqslant T$.

# Induction and coinduction

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Given a deduction system, it characterizes two possible distinct sets (of provable judgements) according to whether an inductive or a coinductive approach is used.

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$\mathcal{F}$-closed if it contains all the elements that can be deduced by $\mathcal{F}$ with hypothesis in $X$.

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A deduction system

- *inductively* defines the least $\mathcal{F}$-closed set
- *coinductively* defines the greatest $\mathcal{F}$-consistent set

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$\{a, b, c, d, e, f, g\} = \mathcal{U}$

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Inductively: $\{d, e\}$

Coinductively: $\{a, b, c, d, e\}$

Self-justifying set: $\{a, b, c\}$

# Exercises

1. Let $\mathcal{U} = \mathbb{Z}$ and take as deduction system all the instances of the rule

$$\frac{n}{n+1}$$

   for $n \in \mathbb{Z}$. Which are the sets inductively and coinductively defined by it?

2. Same question but with $\mathcal{U} = \mathbb{N}$.
3. Same question but with $\mathcal{U} = \mathbb{N}^2$ and as deduction system all the rules instance of

$$\frac{(m,n) \qquad (n,o)}{(m,o)}$$

   for $m, n, o \in \mathbb{N}$

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Now consider $f : S$, then $f$:

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> $S$ and $T$ are in subtyping relation because their infinite expansions are in subtyping relation.

$$S \leqslant T \quad \Longrightarrow \quad \mathtt{Int} \to S \leqslant \mathtt{Even} \to T \quad \Longrightarrow \quad S \leqslant T \wedge \mathtt{Even} \leqslant \mathtt{Int}$$

This is exactly the proof we saw at the beginning:

$$\textsc{Unfold Left}\ \dfrac{\textsc{Unfold Right}\ \dfrac{\textsc{Arrow}\ \dfrac{\texttt{Even} \leqslant \texttt{Int} \qquad \overbrace{\mu X.\texttt{Int} \to X}^{S} \leqslant \overbrace{\mu Y.\texttt{Even} \to Y}^{T}}{\texttt{Int} \to (\mu X.\texttt{Int} \to X) \leqslant \texttt{Even} \to (\mu Y.\texttt{Even} \to Y)}}{\texttt{Int} \to (\mu X.\texttt{Int} \to X) \leqslant \mu Y.\texttt{Even} \to Y}}{\underbrace{\mu X.\texttt{Int} \to X}_{S} \leqslant \underbrace{\mu Y.\texttt{Even} \to Y}_{T}}$$

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## Observation:

1. The deduction above shows why a specific rule for $\mu$ is useless (apply consecutively the two unfold rules).
2. If we added reflexivity and/or transitivity rules, then $\mathcal{U}$ would be $\mathcal{F}$-consistent (*cf.* the third exercise few slides before).

A naive implementation of the Amadio-Cardelli algorithm is exponential (why?). If we “thread” the computation of the memoization environments we obtain a quadratic complexity. This is done as follows:

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& & \textbf{else } \mathtt{fail}
\end{array}
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## Compare the previous algorithm with the Amadio-Cardelli algorithm:

$$\frac{}{A \vdash S \leqslant T}\ (S,T) \in A$$

$$\frac{}{A \vdash S \leqslant \texttt{Any}}\ (S,\texttt{Any}) \notin A$$

$$\frac{A' \vdash S_1 \leqslant T_1 \qquad A' \vdash S_2 \leqslant T_2}{A \vdash S_1 \times S_2 \leqslant T_1 \times T_2}\ A' = A \cup (S_1 \times S_2, T_1 \times T_2); A \mathrel{+\!\!\!\!|} A$$

$$\frac{A' \vdash T_1 \leqslant S_1 \qquad A' \vdash S_2 \leqslant T_2}{A \vdash S_1 \to S_2 \leqslant T_1 \to T_2}\ A' = A \cup (S_1 \to S_2, T_1 \to T_2); A \mathrel{+\!\!\!\!|}$$

$$\frac{A' \vdash S[\mu X.S/X] \leqslant T}{A \vdash \mu X.S \leqslant T}\ A' = A \cup (\mu X.S, T); A \mathrel{+\!\!\!\!|} A'; T \mathrel{+\!\!\!\!|} \texttt{Any}$$

$$\frac{A' \vdash S \leqslant T[\mu X.T/X]}{A \vdash S \leqslant \mu X.T}\ A' = A \cup (S, \mu X.T); A \mathrel{+\!\!\!\!|} A'; S \mathrel{+\!\!\!\!|} \mu Y.U$$

**They both check containment in the relation coinductively defined by:**

$$\textsc{Top}\ \frac{}{T \leqslant \texttt{Any}} \qquad \textsc{Prod}\ \frac{S_1 \leqslant T_1 \qquad S_2 \leqslant T_2}{S_1 \times S_2 \leqslant T_1 \times T_2} \qquad \textsc{Arrow}\ \frac{T_1 \leqslant S_1 \qquad S_2 \leqslant T_2}{S_1 \to S_2 \leqslant T_1 \to T_2}$$

$$\textsc{Unfold Left}\ \frac{S[\mu X.S/X] \leqslant T}{\mu X.S \leqslant T} \qquad \textsc{Unfold Right}\ \frac{S \leqslant T[\mu X.T/X]}{S \leqslant \mu X.T}$$

But the former is far more efficient.

# Outline

# References


- R. Amadio and L. Cardelli. Subtyping recursive types. ACM Transactions on Programming Languages and Systems, 14(4):575-631, 1993.
- Pierce et al. Recursive types revealed, Journal of Functional Programming, 12(6):511-548, 2002.