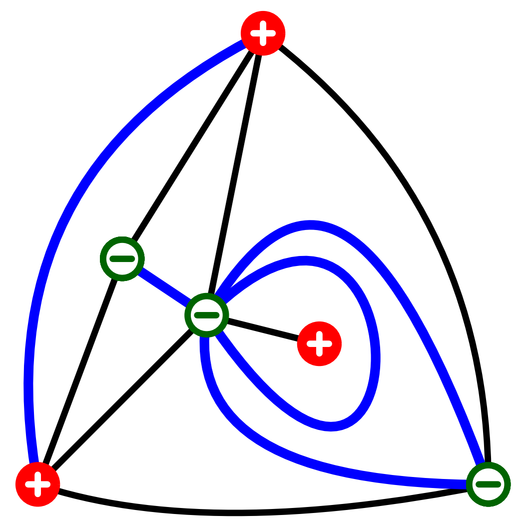
Géométrie des clusters de spins dans les triangulations munies d'un modèle d'Ising

Marie Albenque (CNRS, LIX, École Polytechnique)

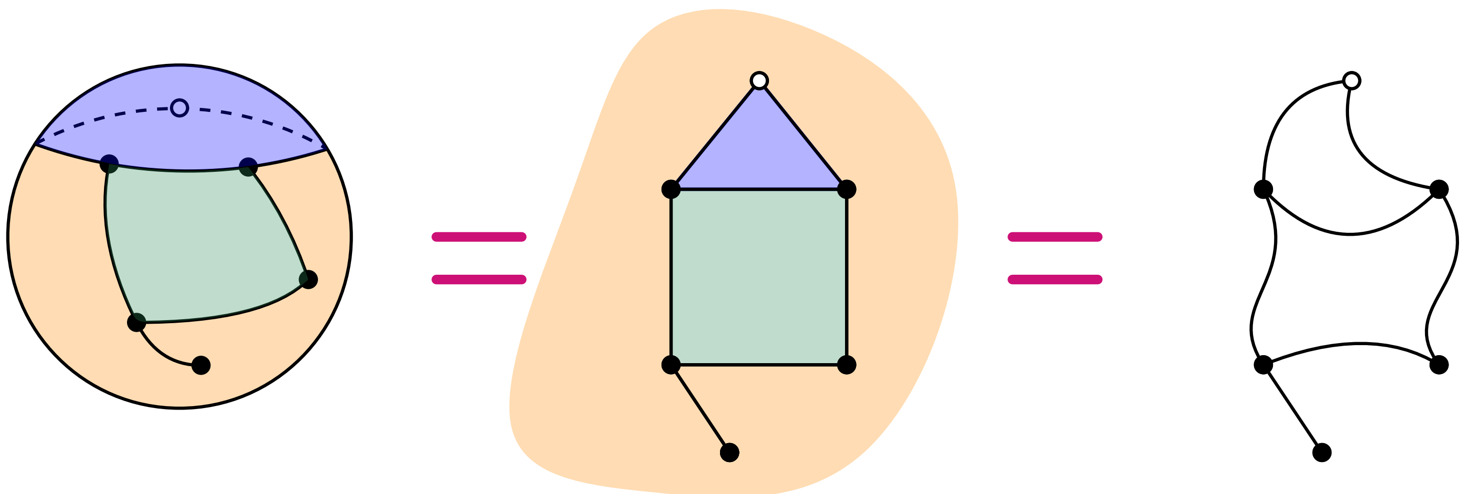
joint works with **Laurent Ménard** (Univ. Paris Nanterre – NYU Shanghai)
and **Gilles Schaeffer** (CNRS, LIX – École Polytechnique)



Séminaire Flajolet, Février 2022

Maps – Definition(s)

A **planar map** is a proper embedding of a planar connected graph in the 2-dimensional sphere (considered up to orientation-preserving homeomorphisms).



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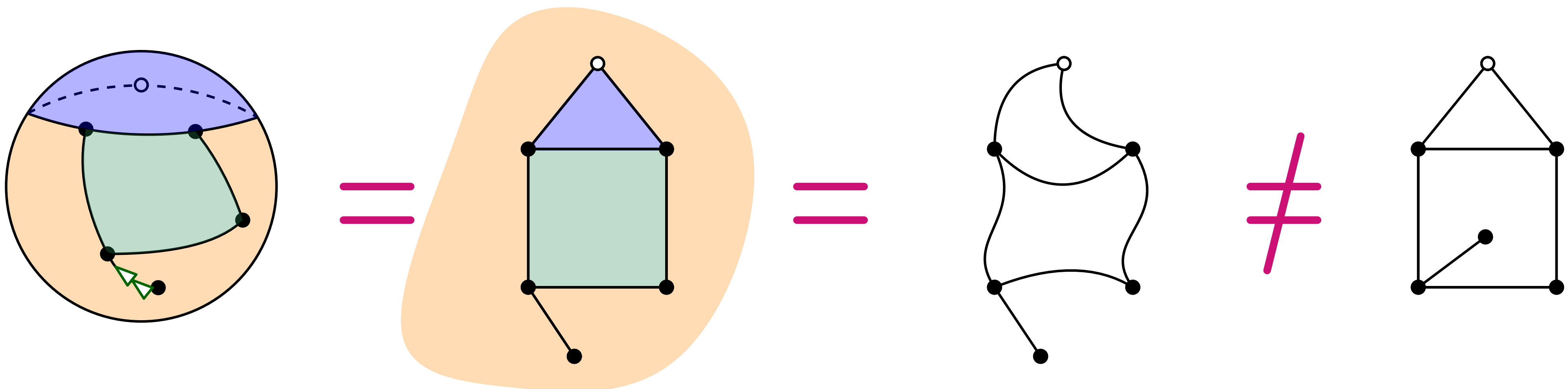

Planar map = planar graph + cyclic order of edges around each vertex.

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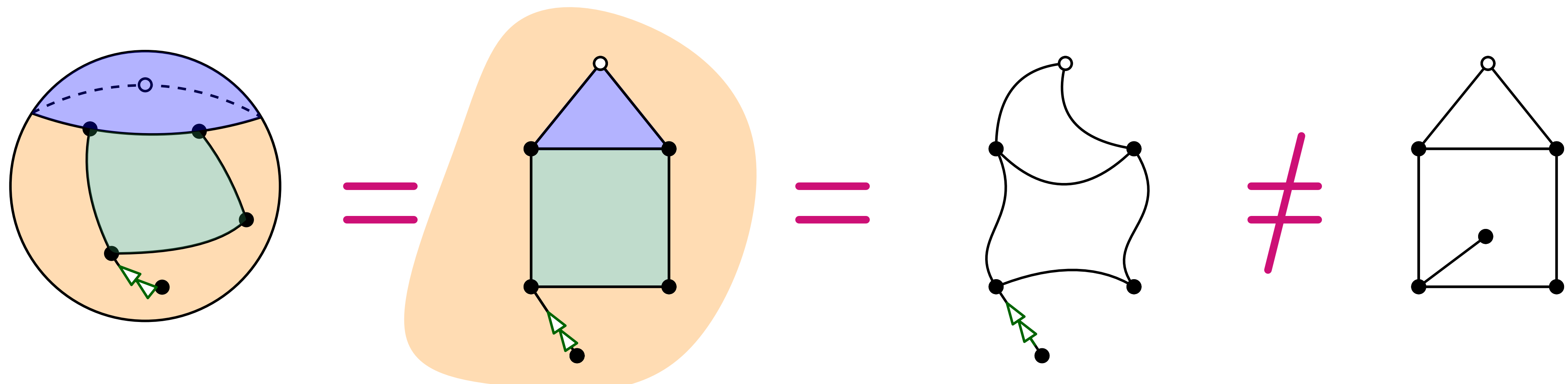

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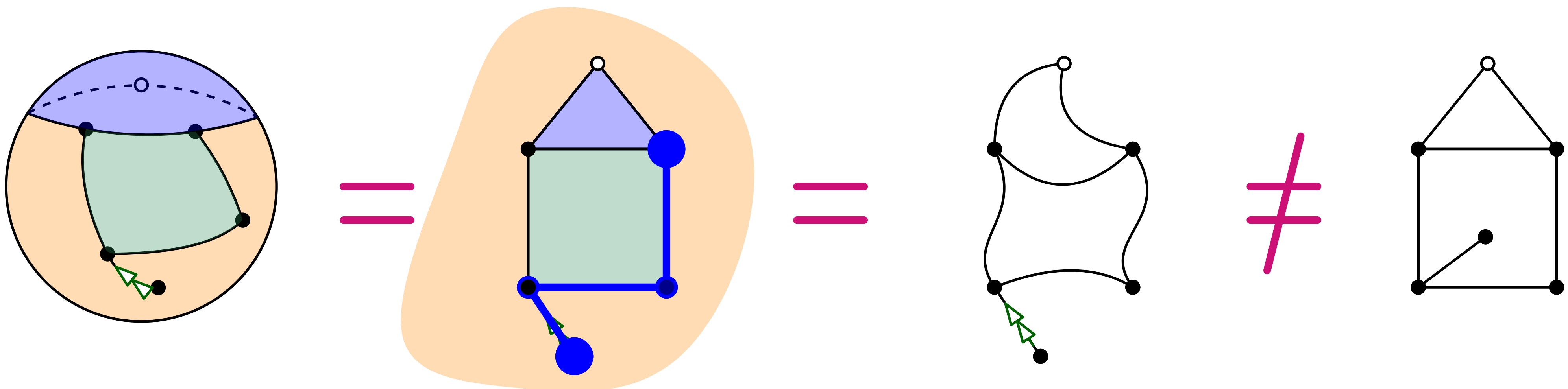

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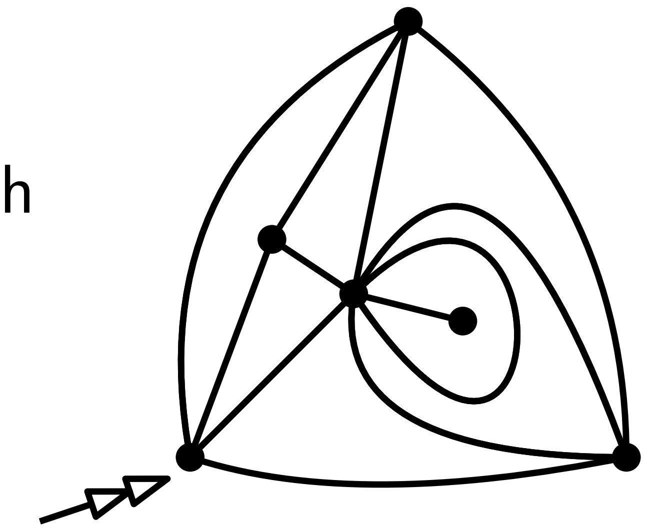
A map M defines a discrete **metric space**:

- points: set of vertices of $M = V(M)$.
- distance: graph distance $= d_{gr}$.

Triangulations

A **triangulation** is a planar map in which all faces have degree 3.

Triangulation of **size** n has $3n$ edges
(or equivalently $n+2$ vertices, $2n$ faces).



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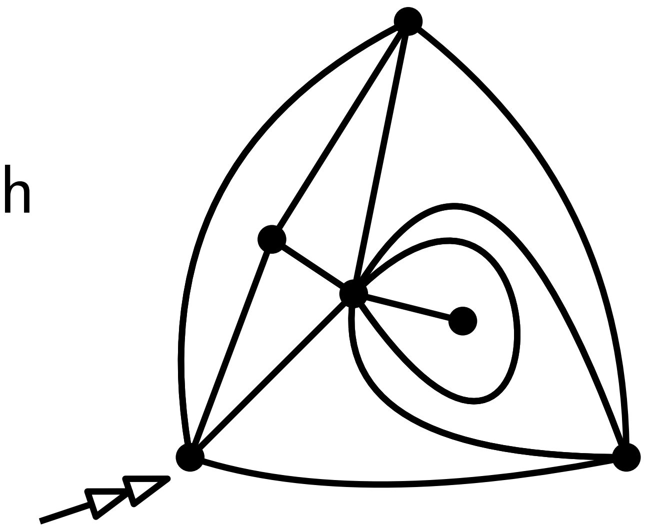
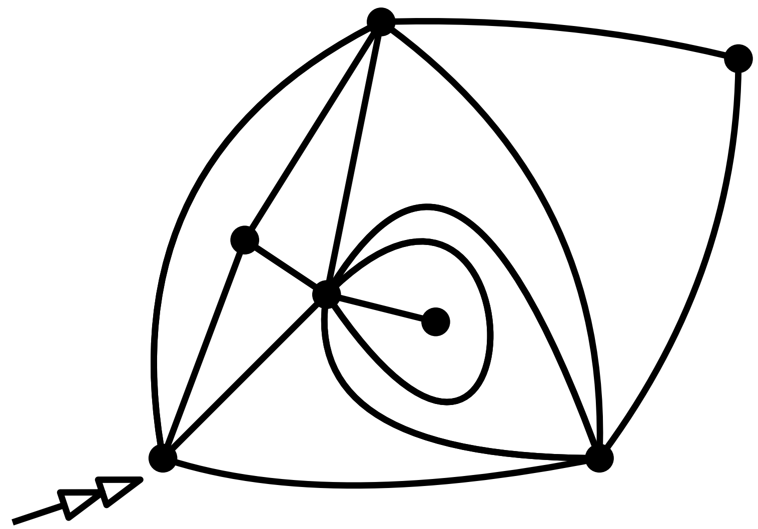


A **triangulation with a boundary** is a planar map in which all faces have degree 3, except possibly the root face.



Triangulation with a boundary of length 4.

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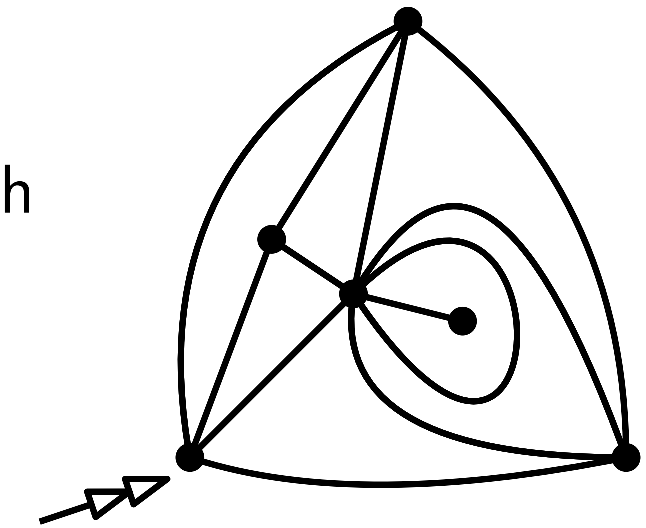
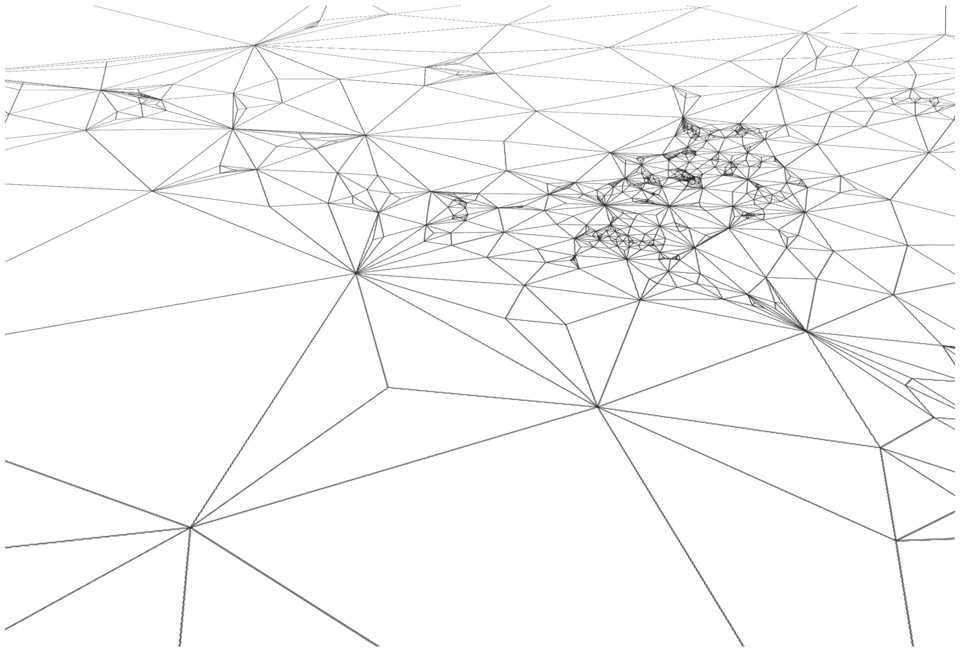
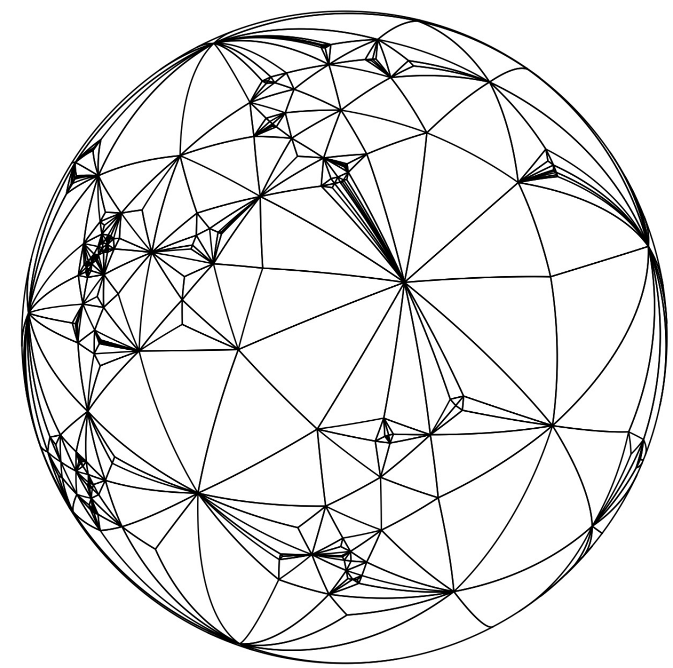


What does a **random triangulation** of size n look like (as n tends to ∞)?



Simulation by I.Kortchemski

Local limit point of view



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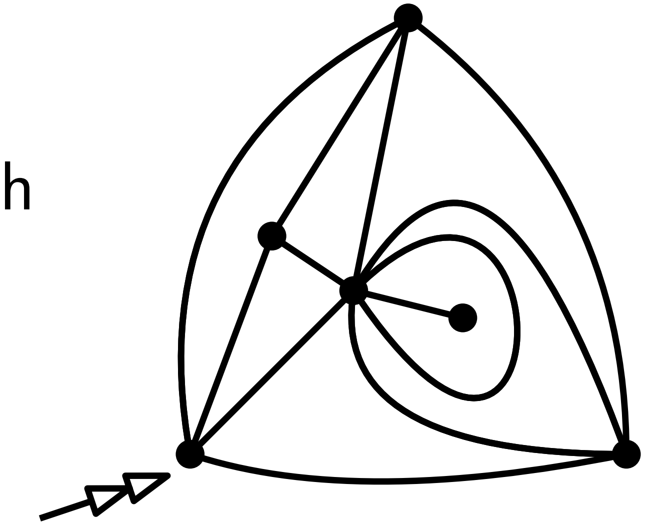

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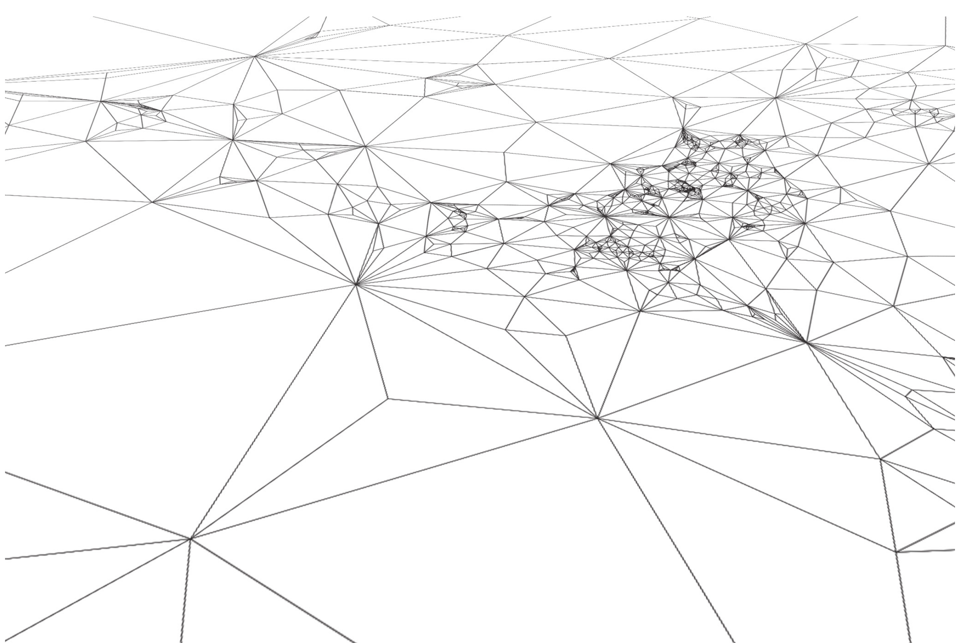

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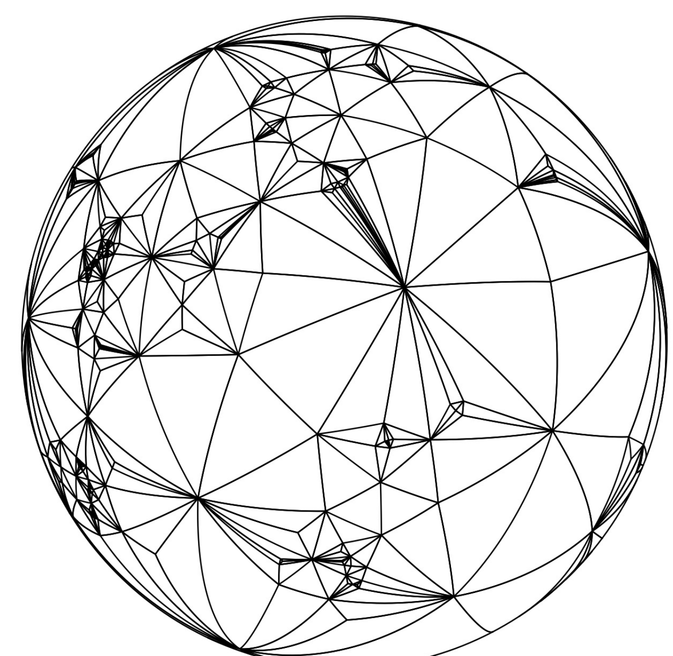
Today: local limit point of view



Scaling limit point of view

Local topology ($\sim$ Benjamini–Schramm convergence)

For m a rooted planar map and $R \in \mathbb{N}^\star$,

$$B_R(\mathrm{m}) = \text{ball of radius } R \text{ around the root vertex of } \mathrm{m}$$

Definition:
The **local topology** on $\mathcal{G}$ is induced by the distance:

$$d_{loc}(\mathrm{m}, \mathrm{m}') := \frac{1}{1 + \max\{R \geq 0 : B_R(\mathrm{m}) = B_R(\mathrm{m}}$$

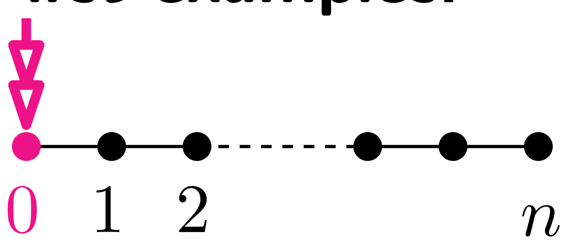
$\mathrm{m}_n \to \mathrm{m}$ for the local topology

$\Leftrightarrow$

For all **fixed** R, there exists n_0 s.t.:

$$B_R(\mathrm{m}_n) = B_R(\mathrm{m}) \quad \text{for } n \geq n_0$$

First examples:



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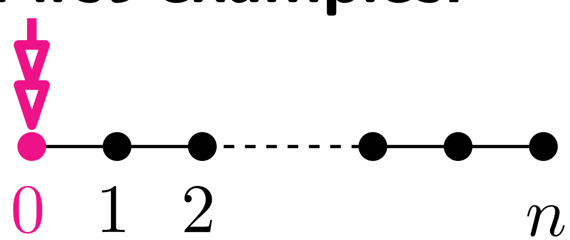

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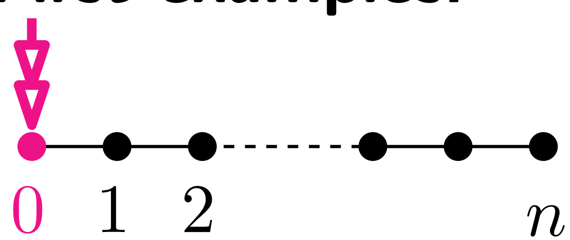

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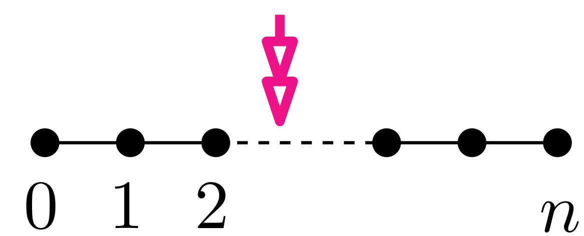

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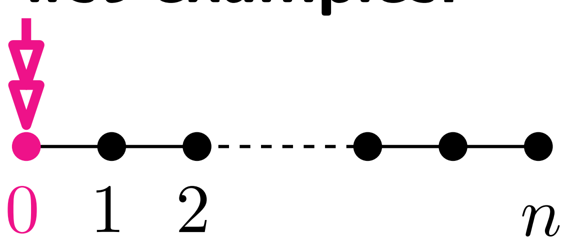

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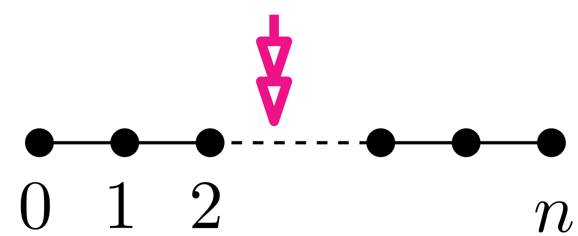
Root $=$ **0**



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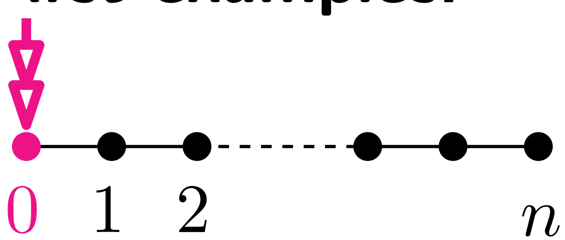

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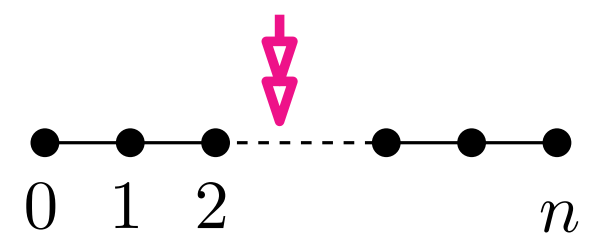

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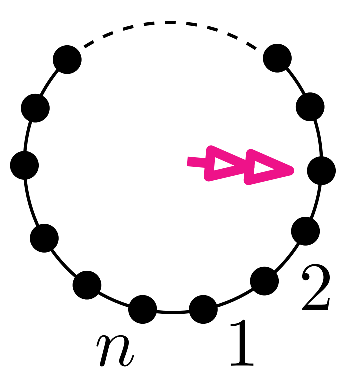
Uniformly chosen root

Root does not matter



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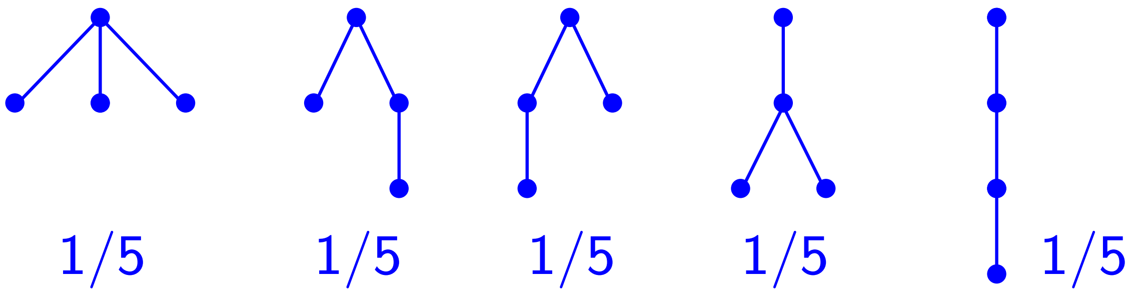
Local convergence: more complicated examples

μ_n = uniform measure on plane trees with n vertices:

$\mu_4 =$



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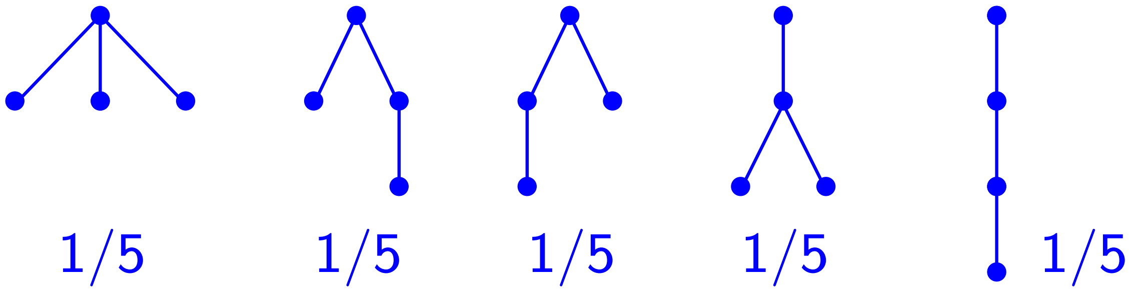

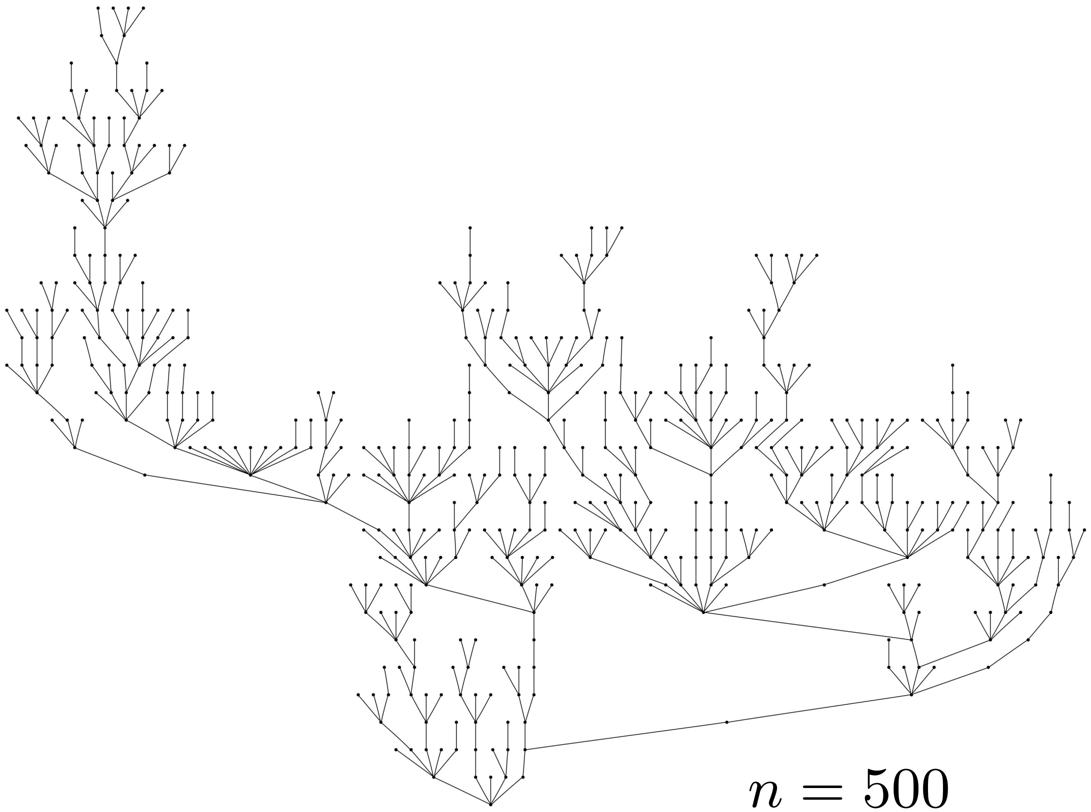

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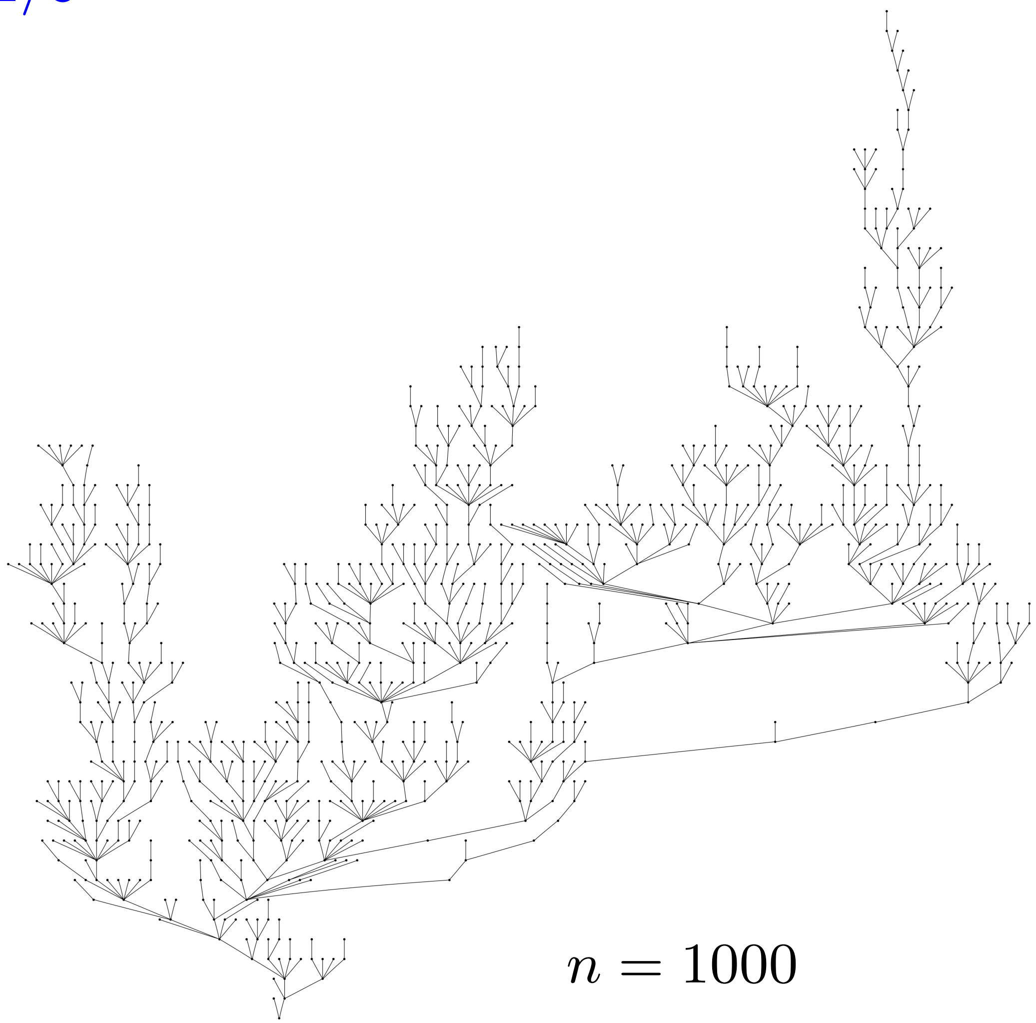

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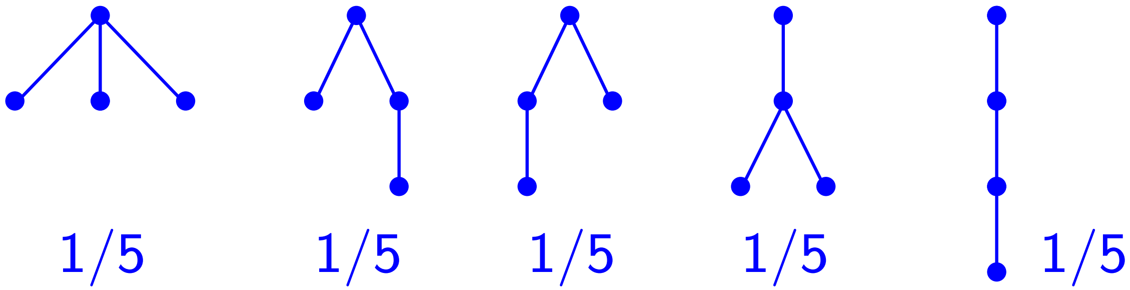
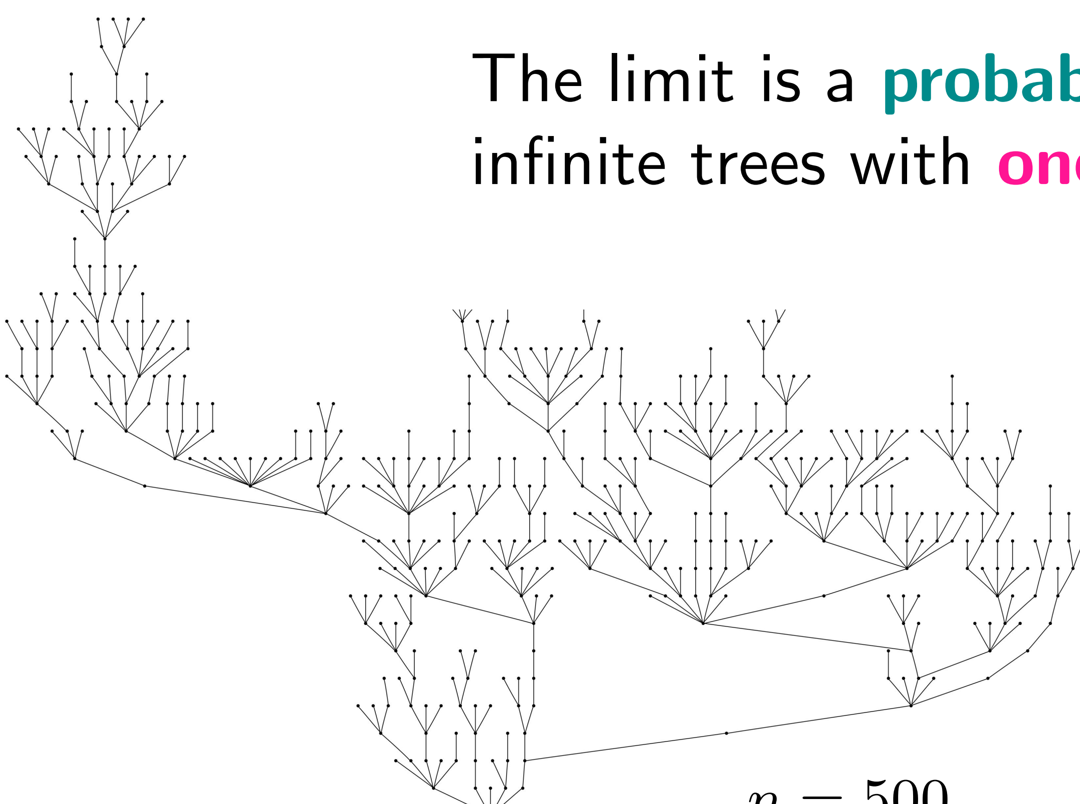


The limit is a **probability distribution** on infinite trees with **one** infinite branch [Kesten].



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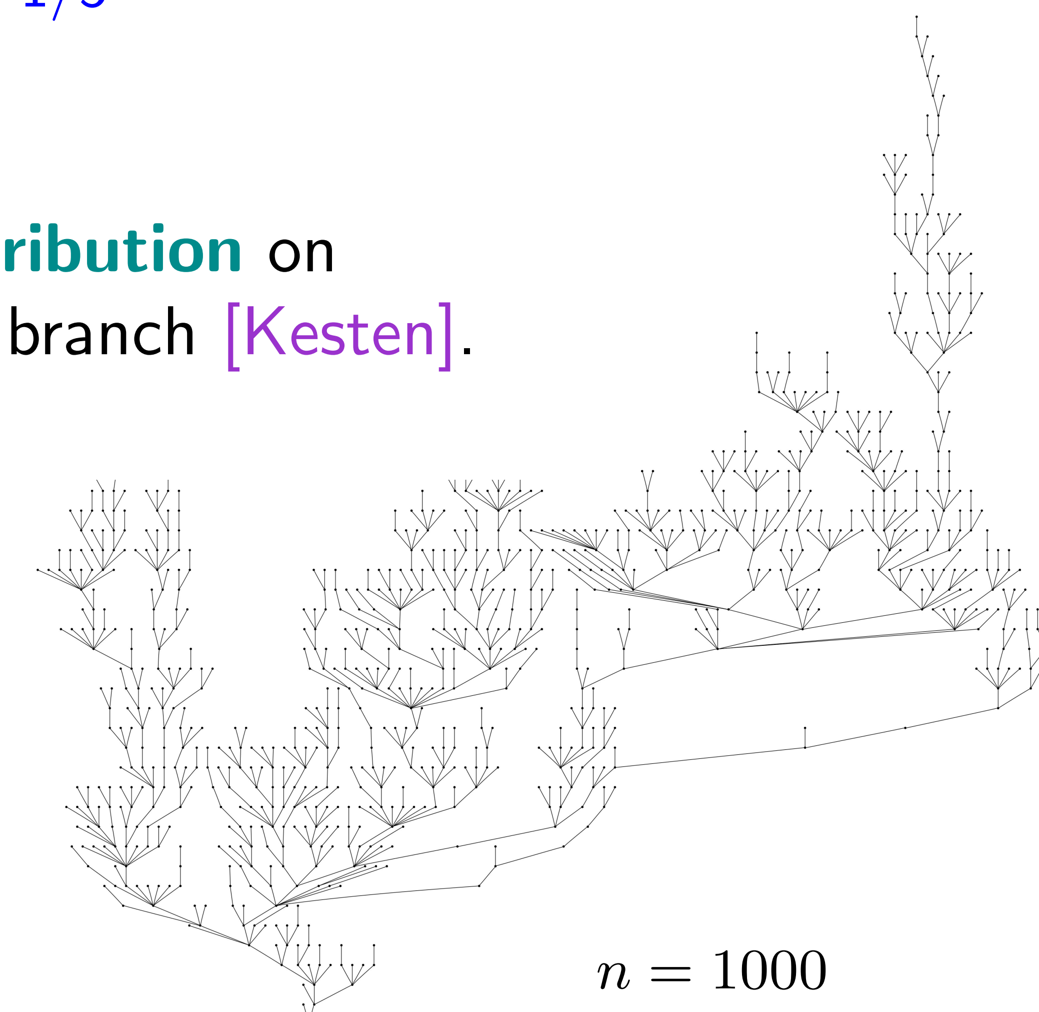

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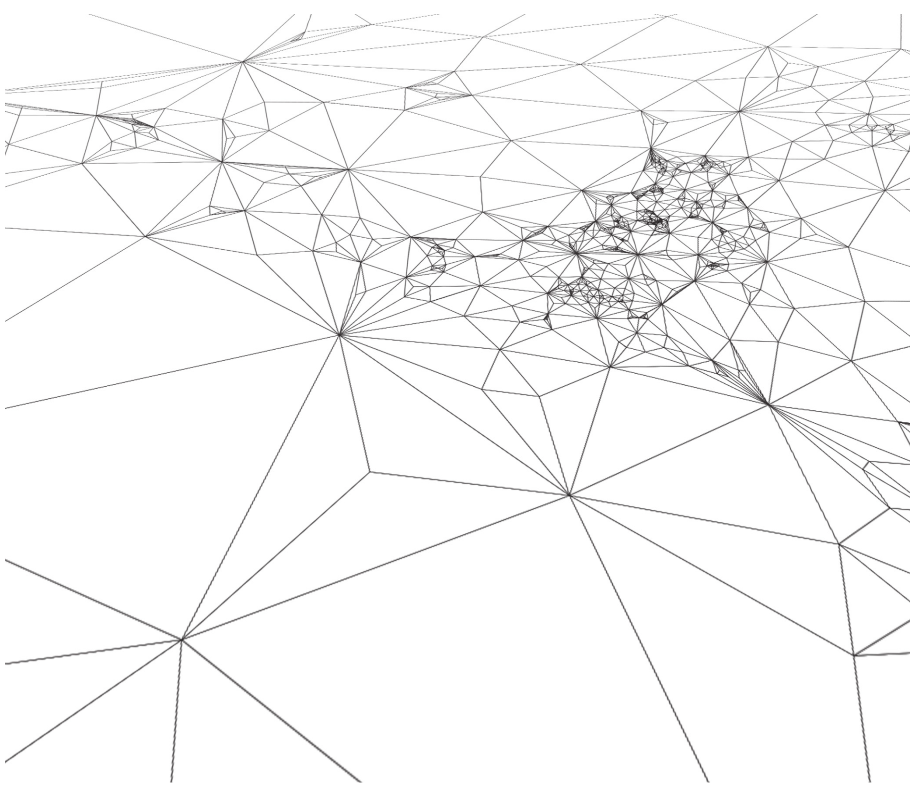
Local limit of large uniformly random triangulations

Theorem [Angel – Schramm, '03]

Let $\mathbb{P}_n$ = uniform distribution on triangulations of size n.

$$\mathbb{P}_n \xrightarrow{(d)} \text{UIPT}, \qquad \text{for the local topology}$$

UIPT = Uniform Infinite Planar Triangulation
= measure supported on infinite planar triangulations.



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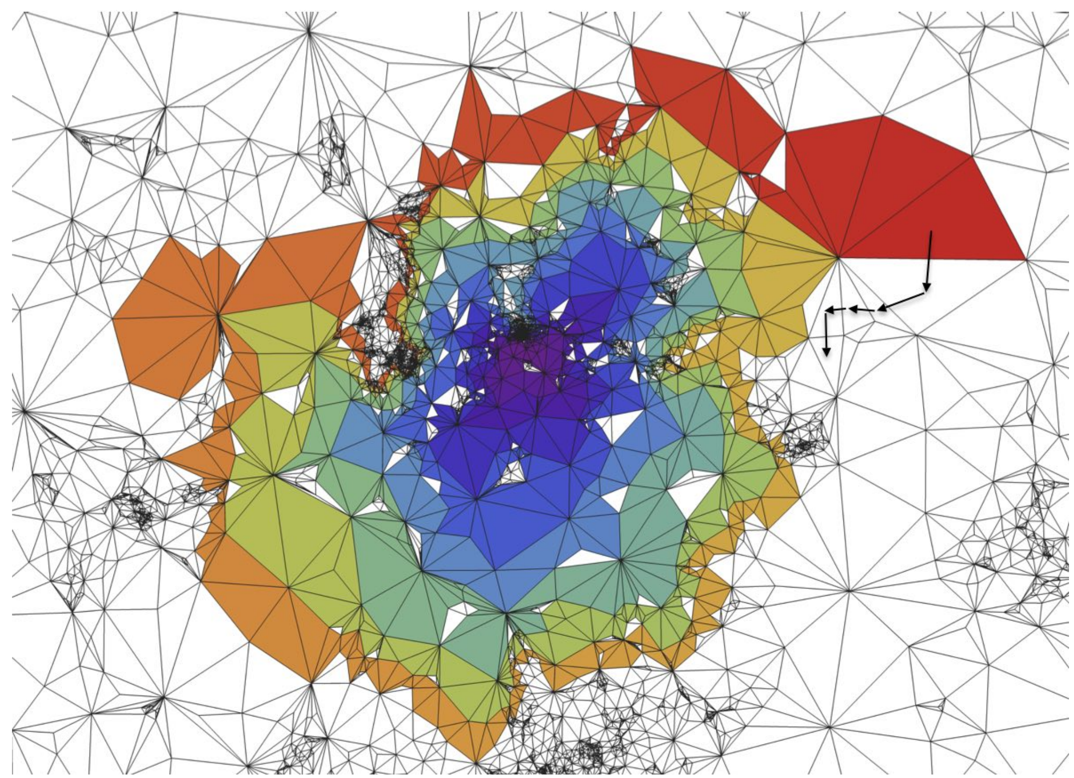

Simulation by T.Budd

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Some properties of the UIPT:

- The UIPT has almost surely one end [Angel – Schramm, 03]
- Volume (nb. of vertices) and perimeters of balls known to some extent.

$$\mathbb{E}\left[|B_R(\mathbf{T}_\infty)|\right] \sim \frac{2}{7} R^4 \qquad \text{[Angel 04, Curien – Le Gall 12]}$$

- The simple random walk is recurrent [Gurel-Gurevich + Nachmias, 13]

Universality: we expect the **same behavior** for other "reasonable" models of maps.
In particular, we expect the volume growth to be 4.
(proved for quadrangulations [Krikun 05], simple triangulations [Angel 04])

Intermezzo: why should we care about local limits ?

Suppose that a sequence of random graphs G_n admits a local weak limit G_∞,

$$\text{Then,} \quad f(G_n) \xrightarrow{proba} f(G_\infty) \quad \text{for any } f \text{ which is continuous for } d_{loc}.$$

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Main idea: The limiting object is often "nicer".

Hence, it is easier to compute $f(G_\infty)$, from which we can deduce the behavior of $f(G_n)$.

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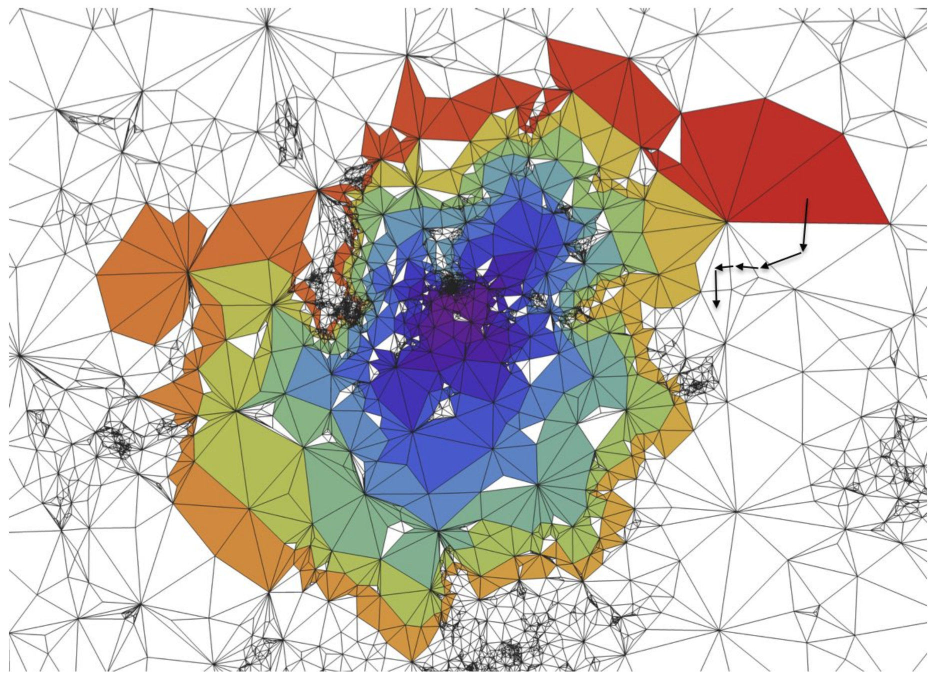
Two example for maps:

- one-endedness in the UIPT:

Allows to give an explicit description of what can happen when the map gets disconnected.

- spatial Markov property

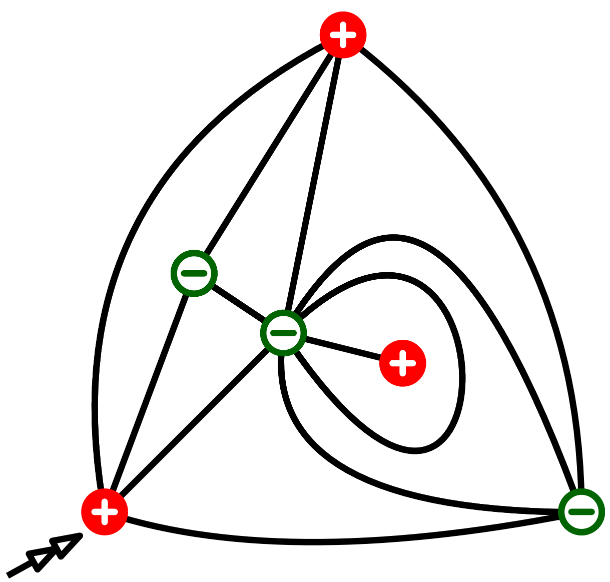


Simulation by T.Budd

II - Local limits of Ising-weighted triangulations

Escaping universality: adding matter

First, **Ising model** on a finite deterministic planar triangulation T:



Spin configuration on T:

$$\sigma : V(T) \to \{-1, +1\} = \{ ⊖ , ⊕ \}.$$

Ising model on T: take a random spin configuration with probability:

$$P(\sigma) \propto e^{\beta J \sum_{v \sim v'} \mathbf{1}_{\{\sigma(v) = \sigma(v')\}}}$$

$\beta > 0$: inverse temperature.
$J = \pm 1$: coupling constant.
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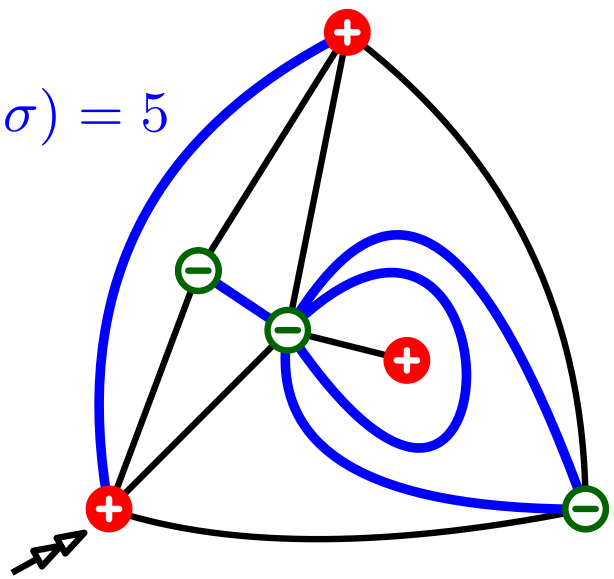

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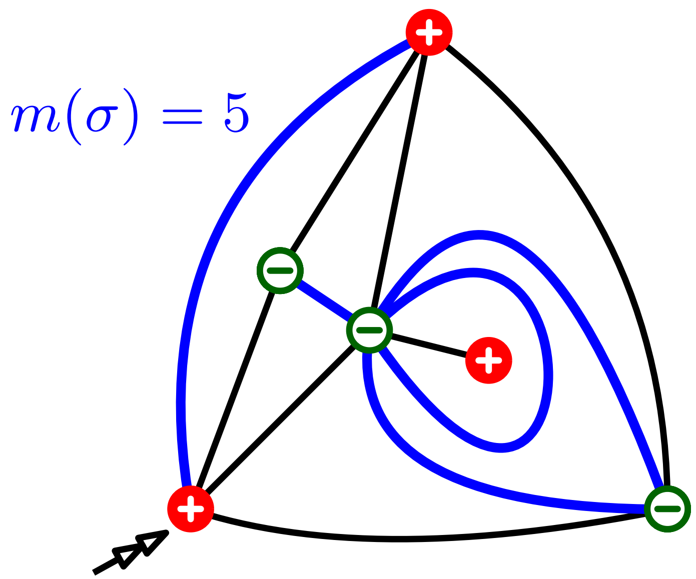

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Next step: Sample a triangulation of size n **together** with a spin configuration, with probability $\propto \nu^{m(T,\sigma)}$.

$$\mathbb{P}_n^{\nu}\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{\mathcal{Z}_n}.$$

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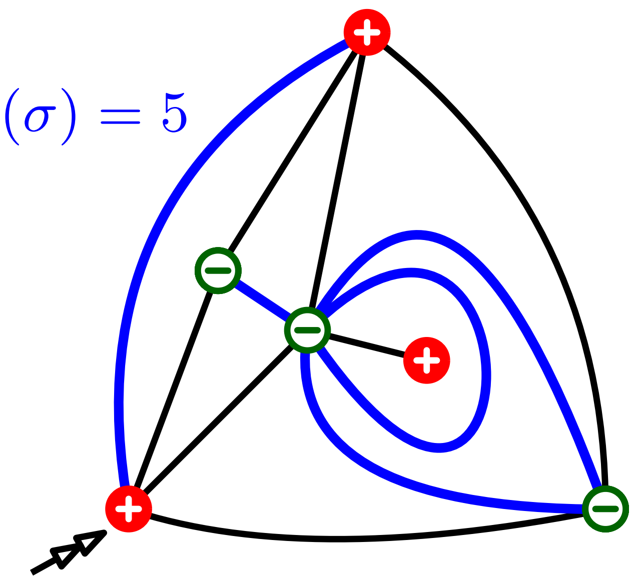

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Remark: This is a probability distribution on triangulations **with** spins. But, forgetting the spins gives a probability a distribution on triangulations **without** spins **different from the uniform distribution**.

Escaping universality: new asymptotic behavior

Counting exponent for undecorated maps:

number of (undecorated) maps of size $n \sim \kappa \rho^{-n} n^{-\mathbf{5/2}}$

(e.g.: triangulations, quadrangulations, general maps, simple maps,...)

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Generating series of Ising-weighted triangulations:

$$Z(\nu, t) = \sum_{T \text{ triangulation}} \sum_{\sigma : V(T) \to \{-1,+1\}} \nu^{m(T,\sigma)} t^{e(T)}.$$

Theorem [Bernardi – Bousquet-Mélou 11]
For every $\nu > 0$, $Z(\nu, t)$ is algebraic and satisfies

$$[t^{3n}]Z(\nu, t) \underset{n\to\infty}{\sim} \begin{cases} \kappa\, \rho_{\nu_c}^{-n}\, n^{\mathbf{-7/3}} & \text{if } \nu = \nu_c = 1 + \frac{1}{\sqrt{7}}, \\ \kappa\, \rho_{\nu}^{-n}\, n^{\mathbf{-5/2}} & \text{if } \nu \neq \nu_c. \end{cases}$$

See also [Boulatov – Kazakov 1987], [Bousquet-Melou – Schaeffer 03] and [Bouttier – Di Francesco – Guitter 04].

This suggests a **different behavior** of the underlying maps for $\nu = \nu_c$.

Local convergence of triangulations with spins

Theorem [A. – Ménard – Schaeffer, 21]

Let $\mathbb{P}_n^\nu = \nu$–Ising weighted probability distribution for triangulations of size n:

$$\mathbb{P}_n^\nu \xrightarrow{(d)} \nu\text{-IIPT}, \qquad \text{for the local topology } \textbf{with spins}$$

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Some properties of the ν-IIPT:

- One-ended a.s.
- Simple random walk is recurrent.
- Geometry of the clusters ?
- Volume (nb. of vertices) and perimeters of balls ???

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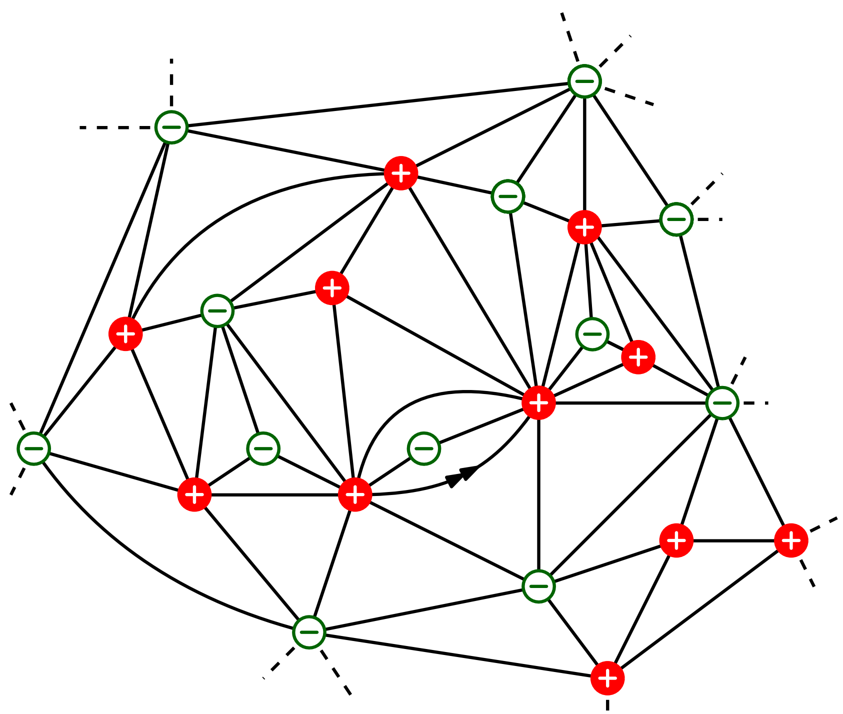
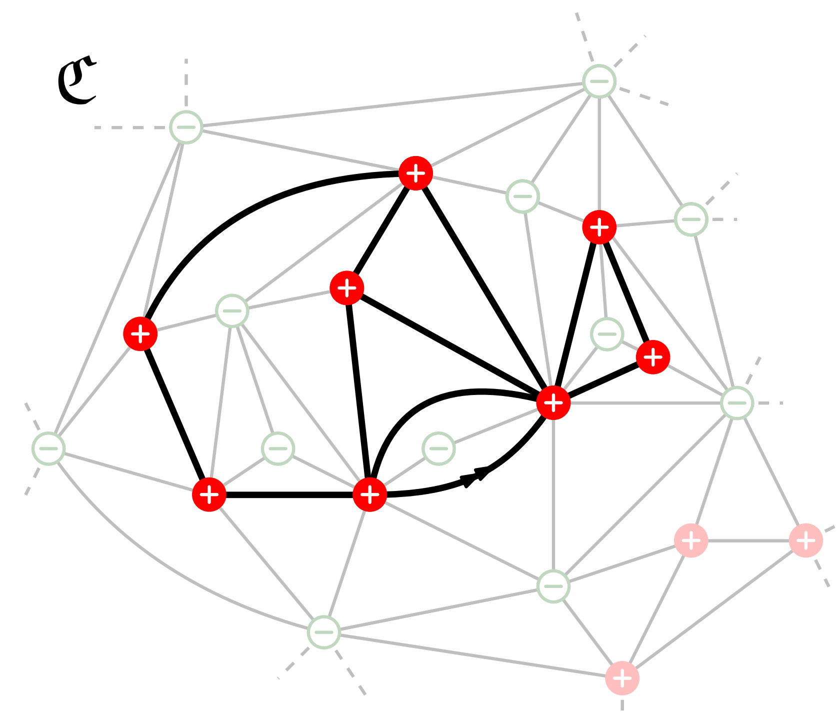
- Volume (nb. of vertices) and perimeters of balls ???

Non-universality: we expect a **different** behavior for $\nu = \nu_c$

In particular, we expect the volume growth to be different from 4.
Watabiki's conjecture: $\frac{7+\sqrt{97}}{4} \sim 4.21...$

III - Clusters in the ν-IIPT

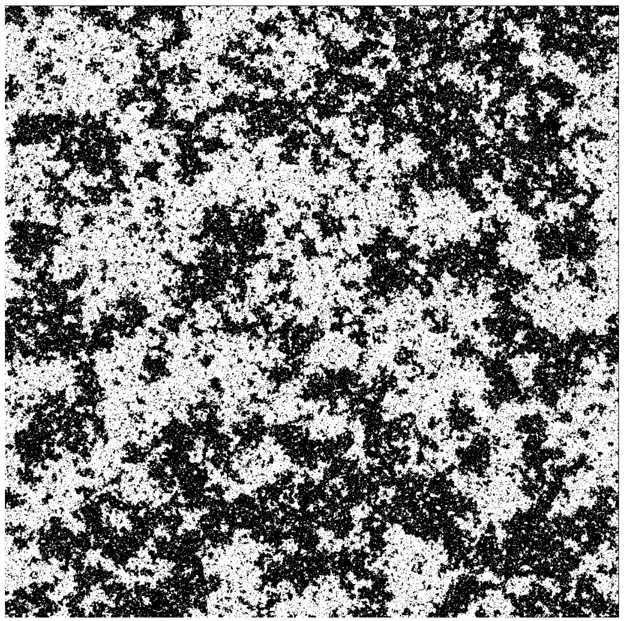




Ferromagnetic Ising model on $\mathbb{Z}^2$: clusters

Simulations by R.Cerf:

$\nu < \nu_c$



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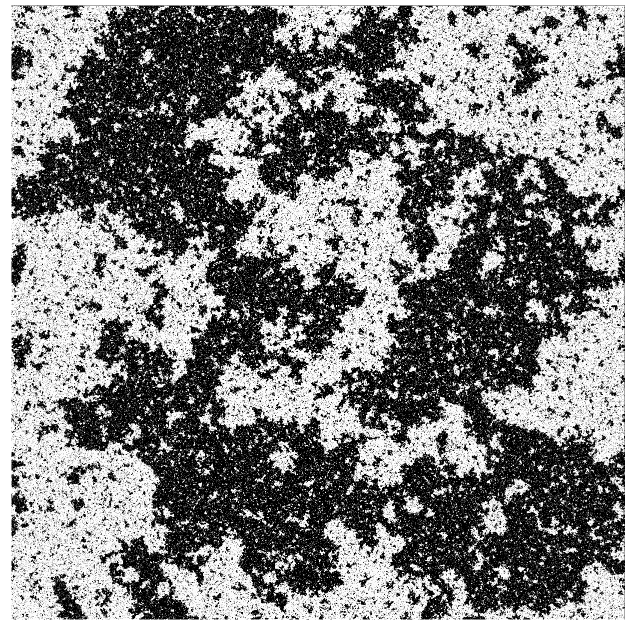

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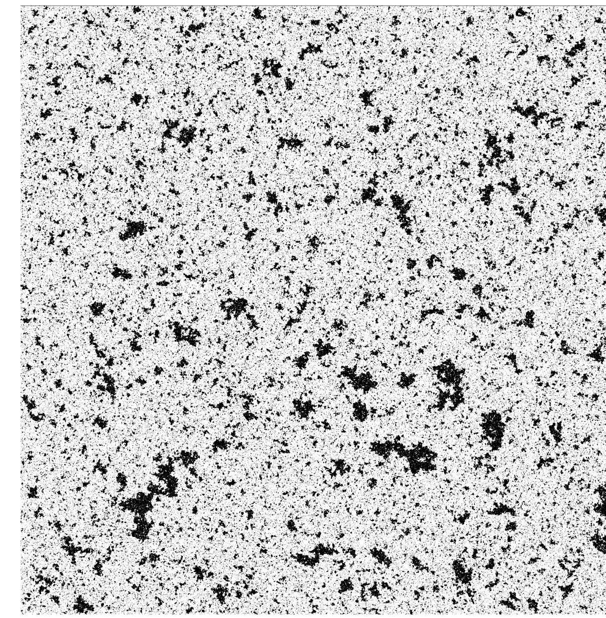

One infinite cluster

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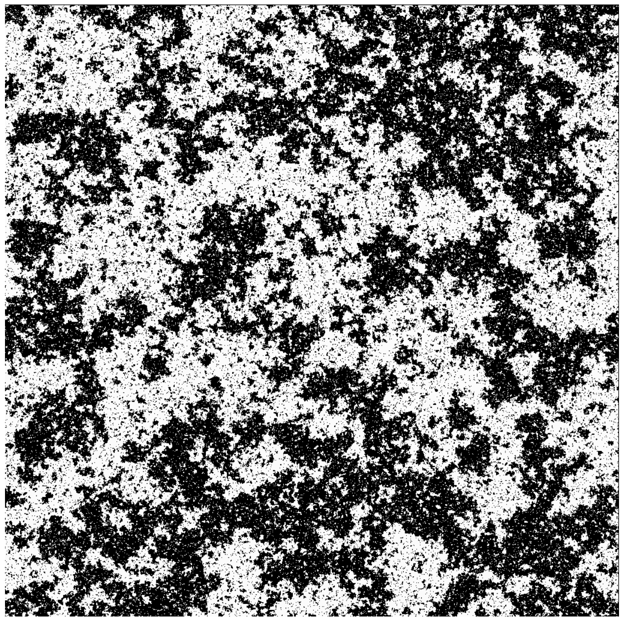

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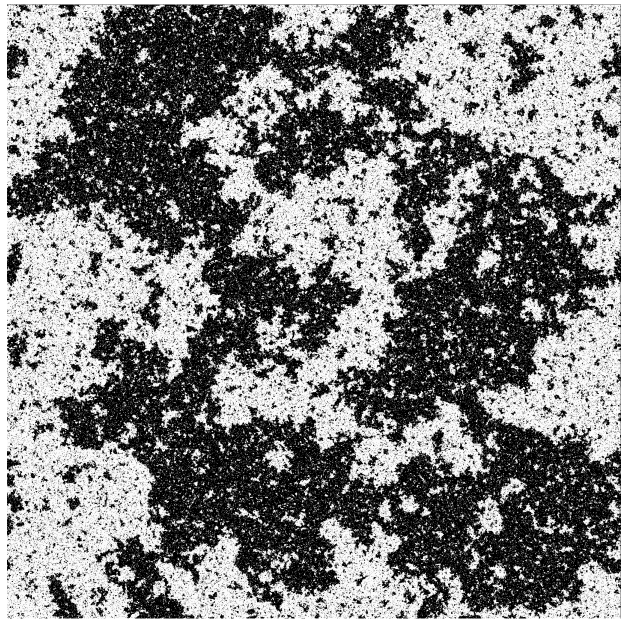

CLE$_3$

[Chelkak+Smirnov 2012]

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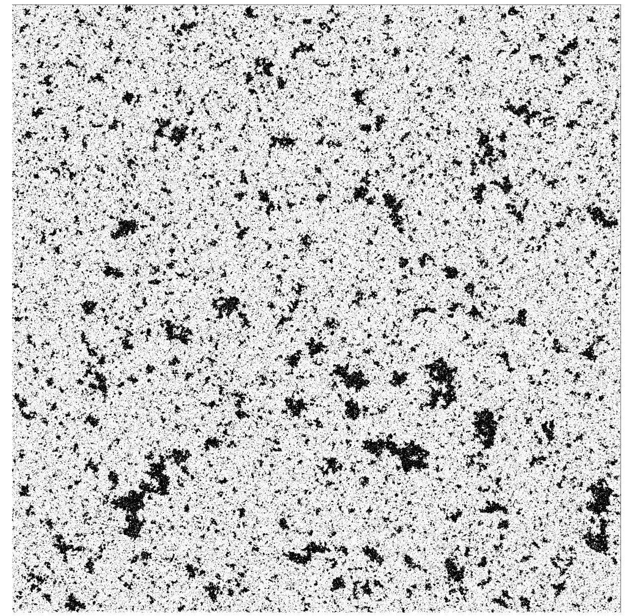

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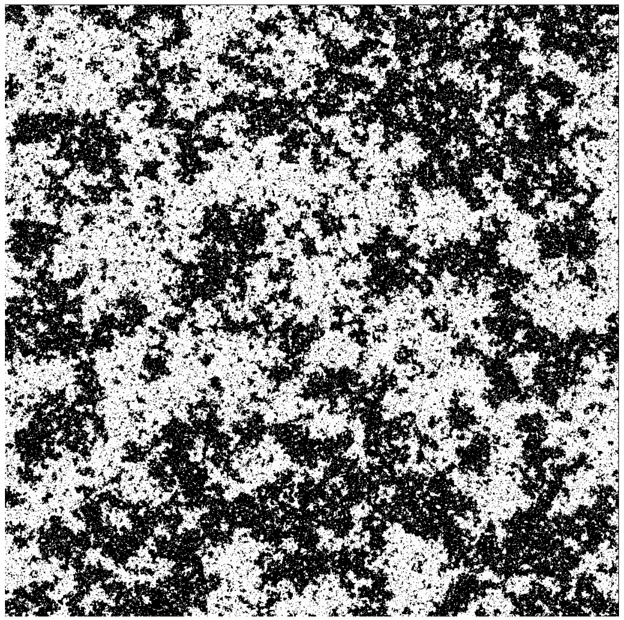
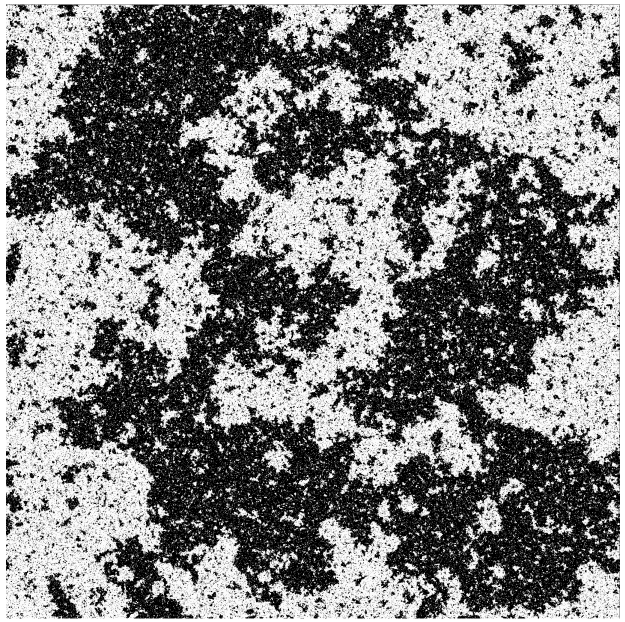


Same universality class as critical percolation.
CLE_6 ??

[Smirnov 2001]
for critical percolation

$\nu = \nu_c$



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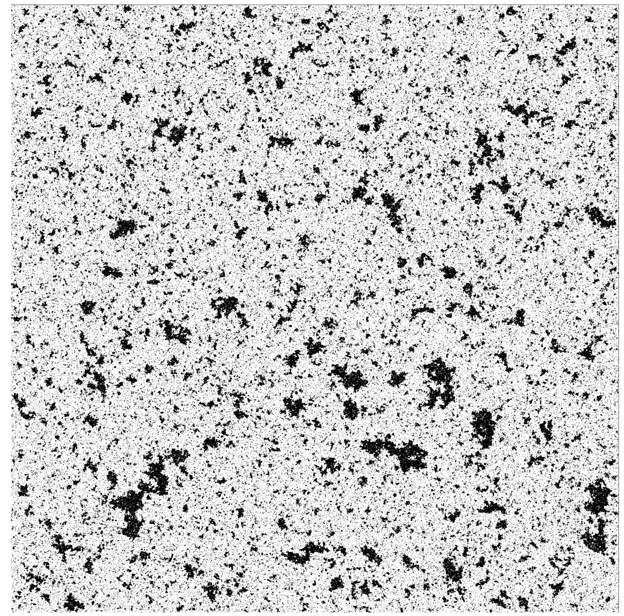

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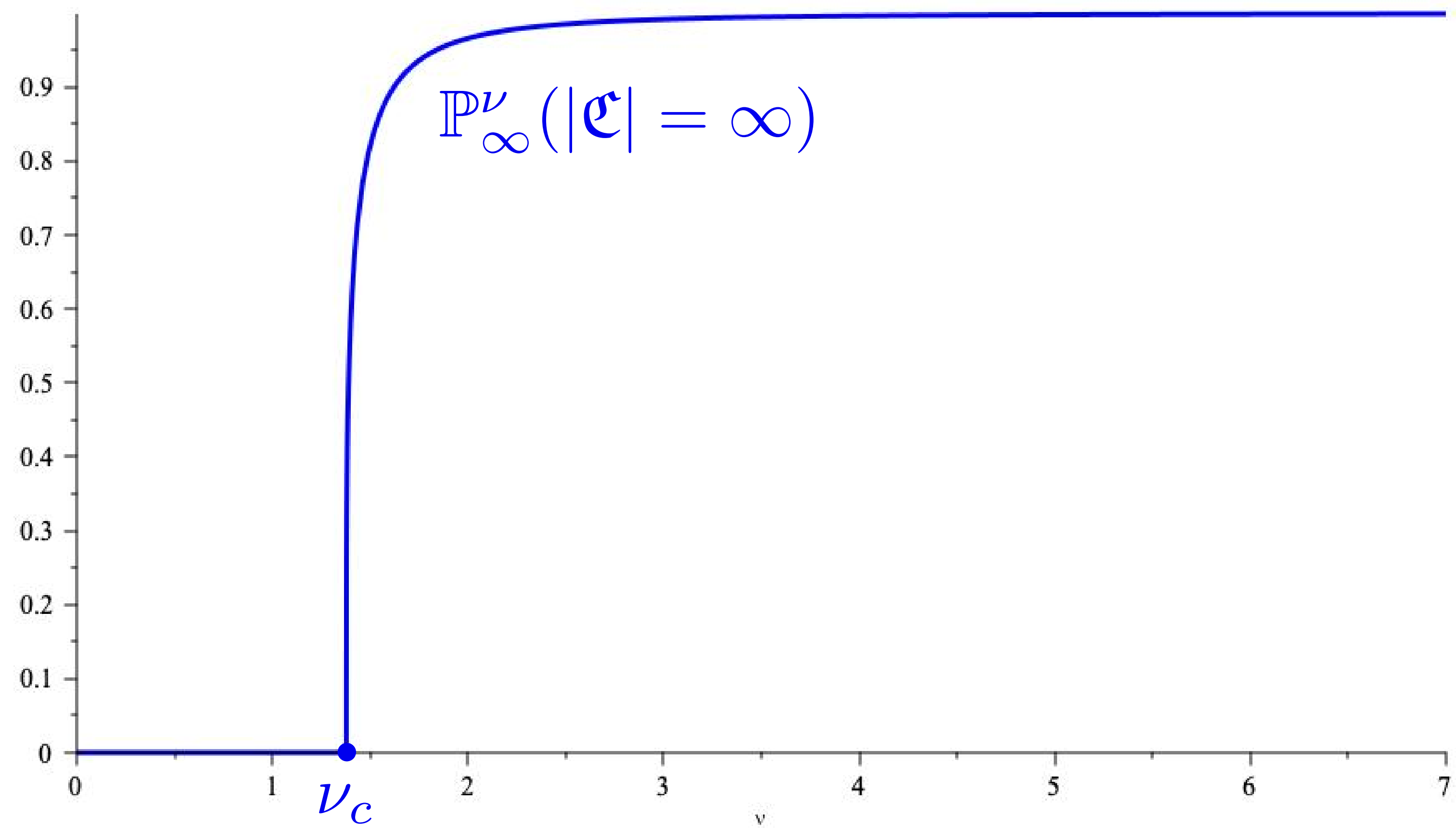
Clusters in the ν-IIPT: phase transition

Theorem [A. – Ménard, 22+]

Under $\mathbb{P}_\infty^\nu$, the cluster of the root vertex is:

- finite almost surely for $\nu \leq \nu_c$
- infinite with ("explicit"!) positive probability for $\nu > \nu_c$.



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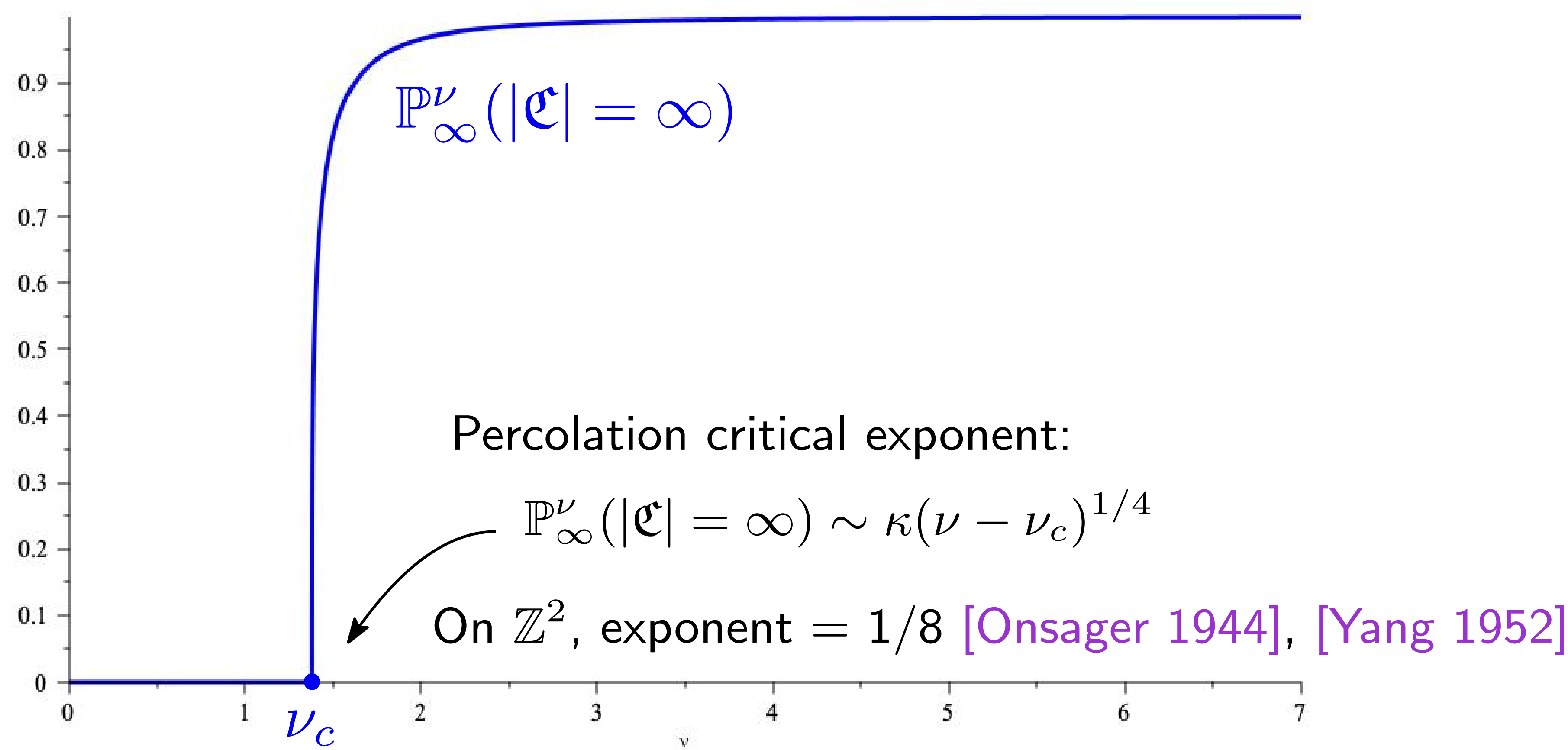
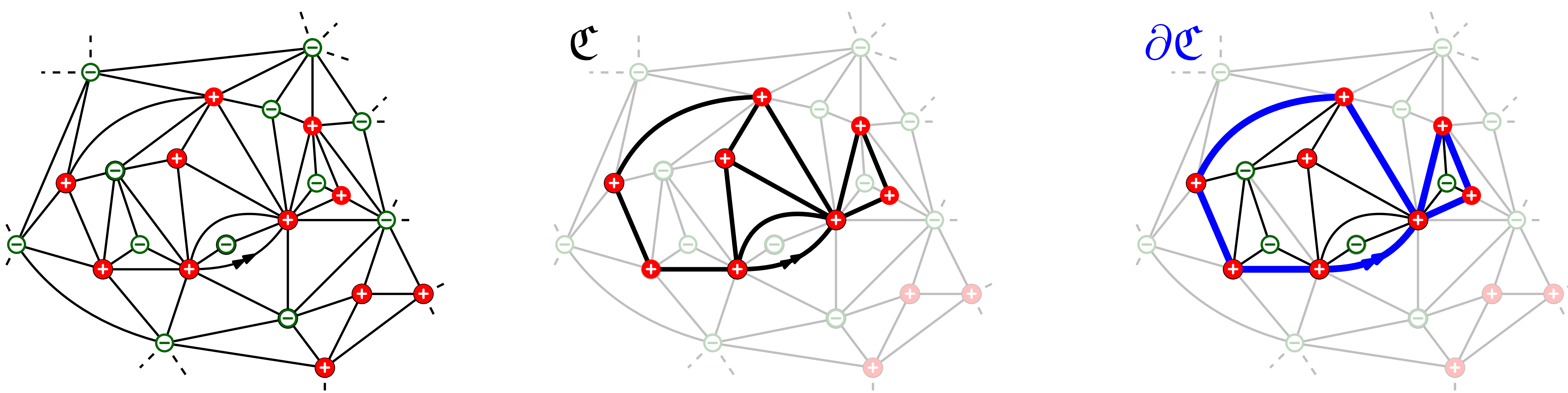


Clusters in the ν-IIPT: cluster size exponents

Theorem [A. – Ménard, 22+]

Denote by $\mathfrak{C}$ the spin cluster of the root vertex.

	for $\nu < \nu_c$	for $\nu = \nu_c$	for $\nu > \nu_c$
$\mathbb{P}^{\nu}_{\infty}(\lvert\mathfrak{C}\rvert \geq n)$	$\propto n^{-1/7}$	$\propto n^{-1/11}$	not relevant
$\mathbb{P}^{\nu}_{\infty}(\lvert\partial\mathfrak{C}\rvert = p)$	$\propto p^{-2}$	$\propto p^{-4/3}$	$\propto \exp(-\alpha p)$



The special case $\nu = 1$: UIPT with critical percolation

Recall that for a triangulation T with spin configuration σ, $\mathbb{P}_n^\nu\Big(\{(T,\sigma)\}\Big) = \frac{\nu^{m(T,\sigma)}\delta_{|e(T)|=3n}}{\mathcal{Z}_n}$.

For $\nu = 1$, all configurations (= trig. + spins) have the same probability

$\Leftrightarrow$ uniform triangulation of size n where spins are independent and +/- with probability 1/2.

$\Leftrightarrow$ uniform triangulation of size n with a percolation of parameter 1/2 on its vertices.

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Percolation on the UIPT much studied:

$p_c = 1/2$ + no infinite cluster at p_c [Angel 04]

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Percolation of the UIPT via its clusters in [Bernardi, Curien, Miermont, 17].

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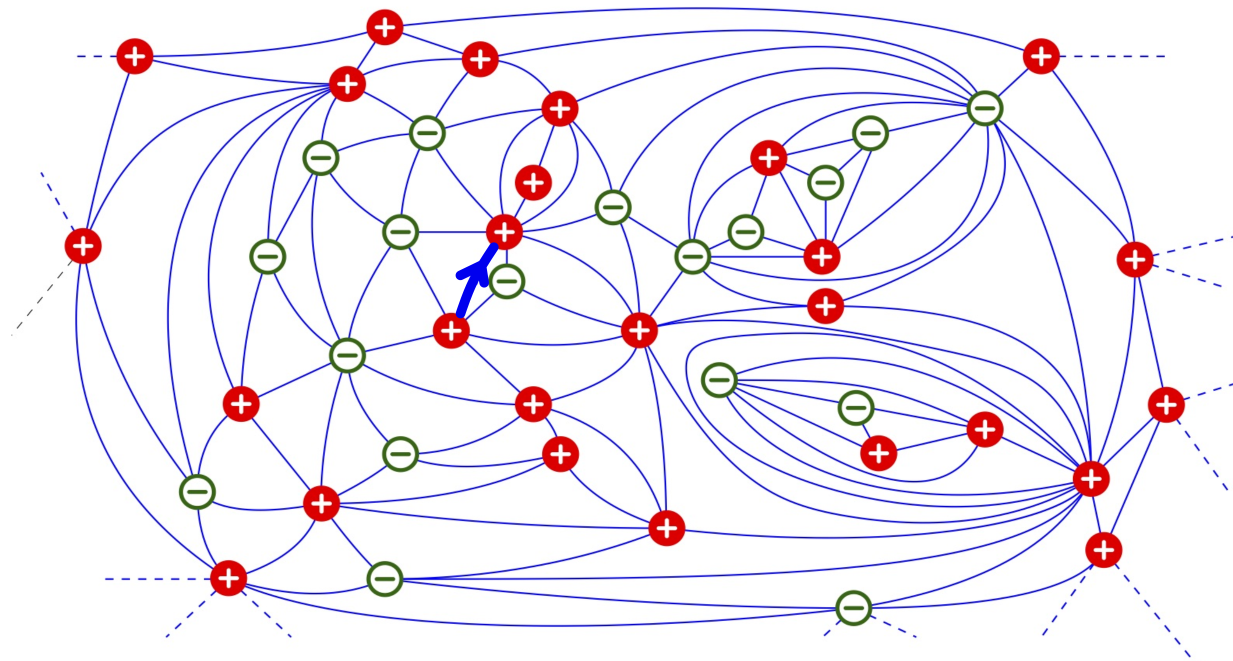
Our results reinforce the idea that:

Ising model in high-temperature (i.e. $\nu < \nu_c$) $\sim$ Critical percolation

Idea of the proof I: Gasket decomposition

[Borot – Bouttier – Guitter, 2012], [Bernardi – Curien – Miermont, 2017]



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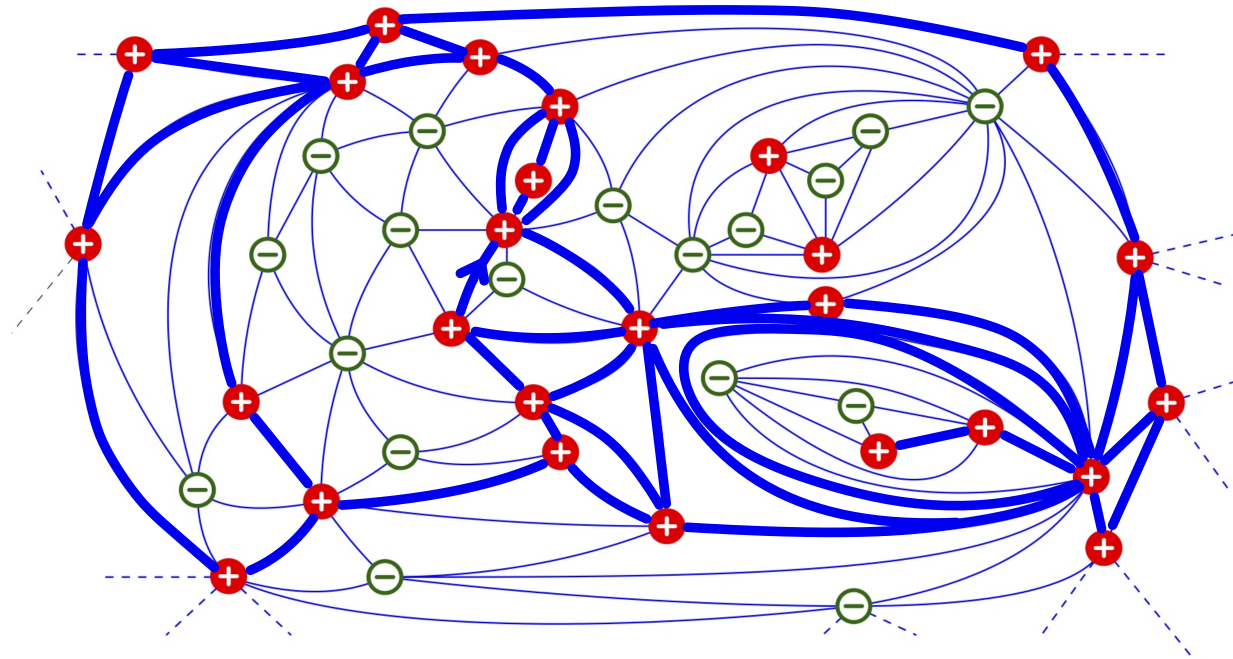

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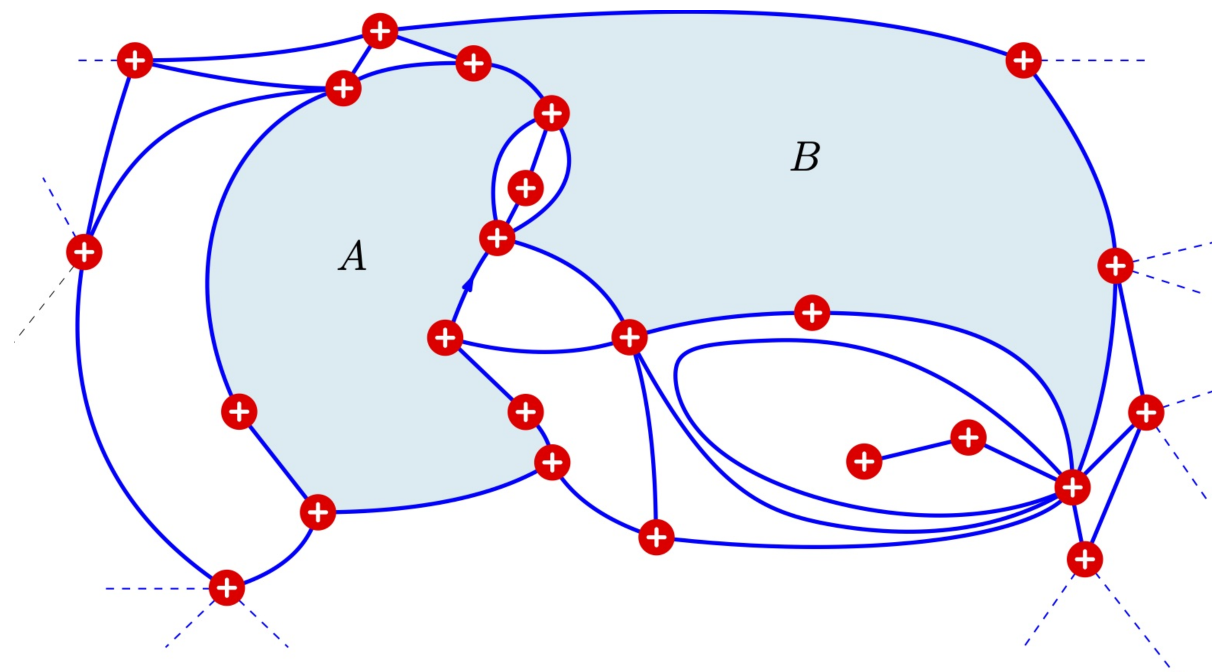

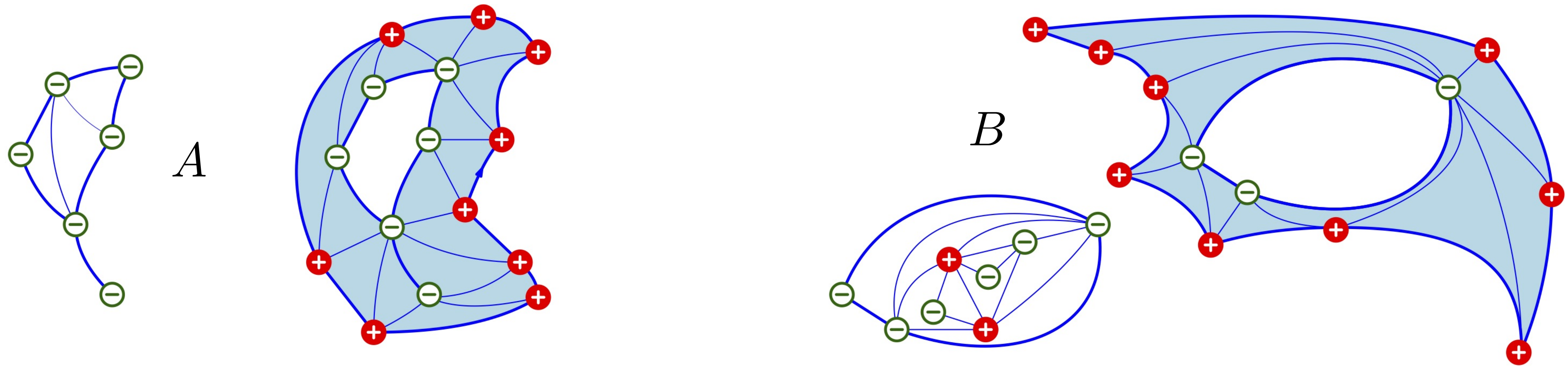

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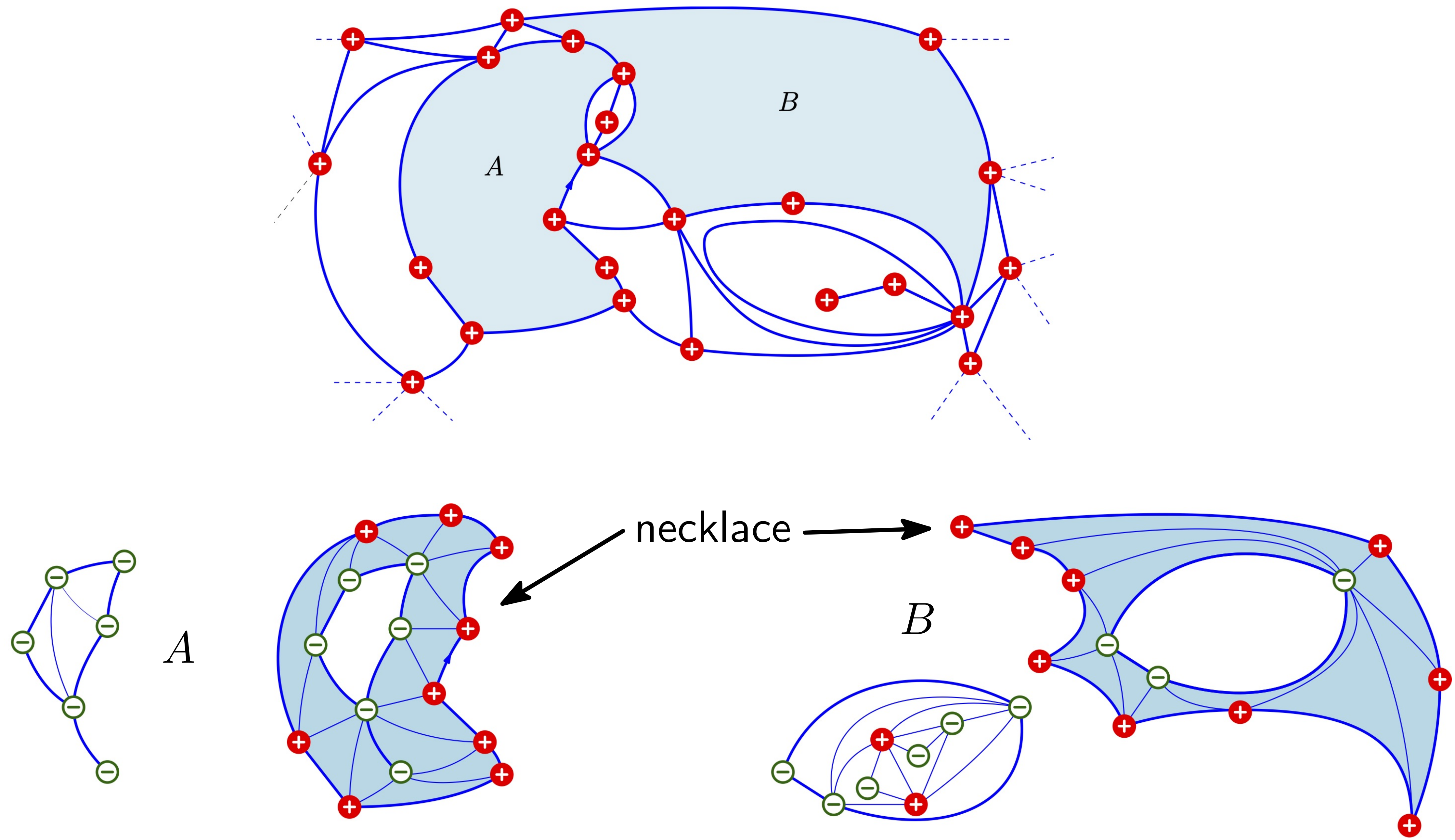

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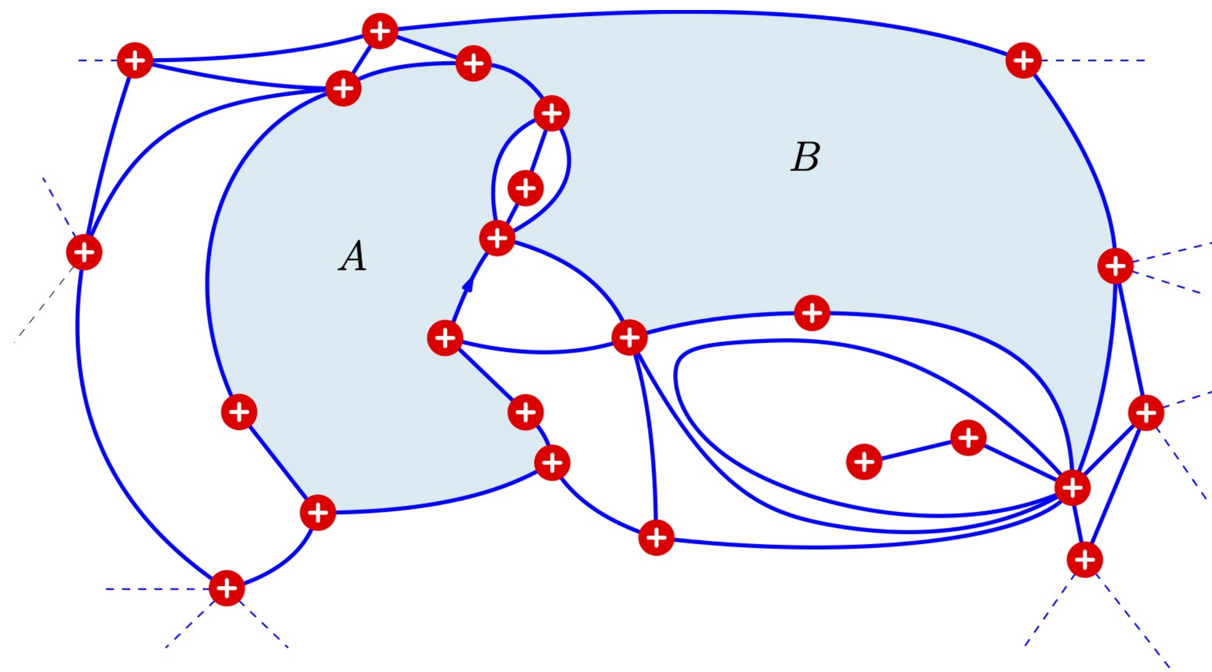

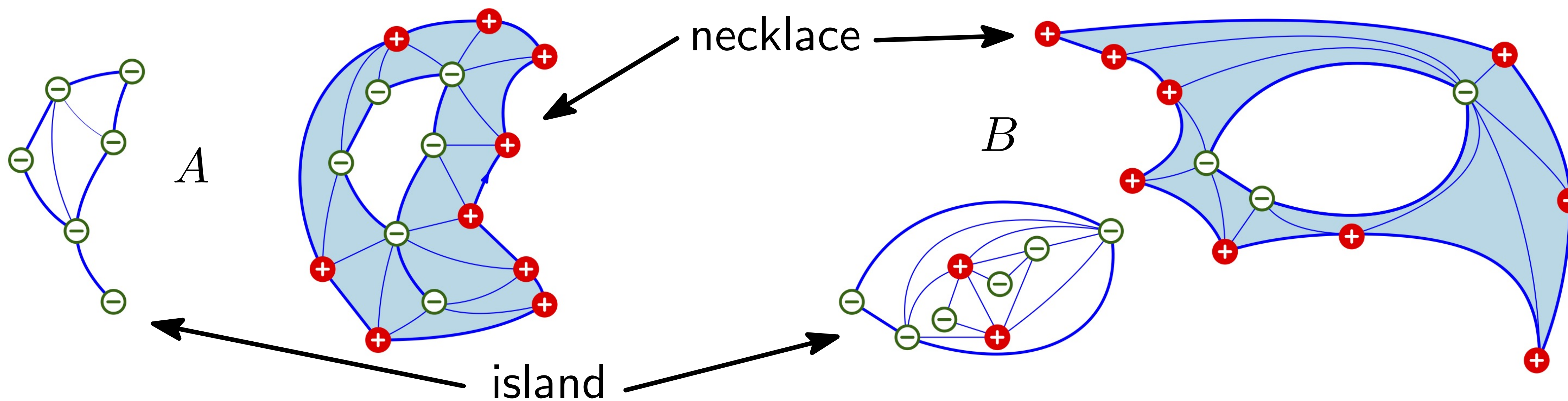

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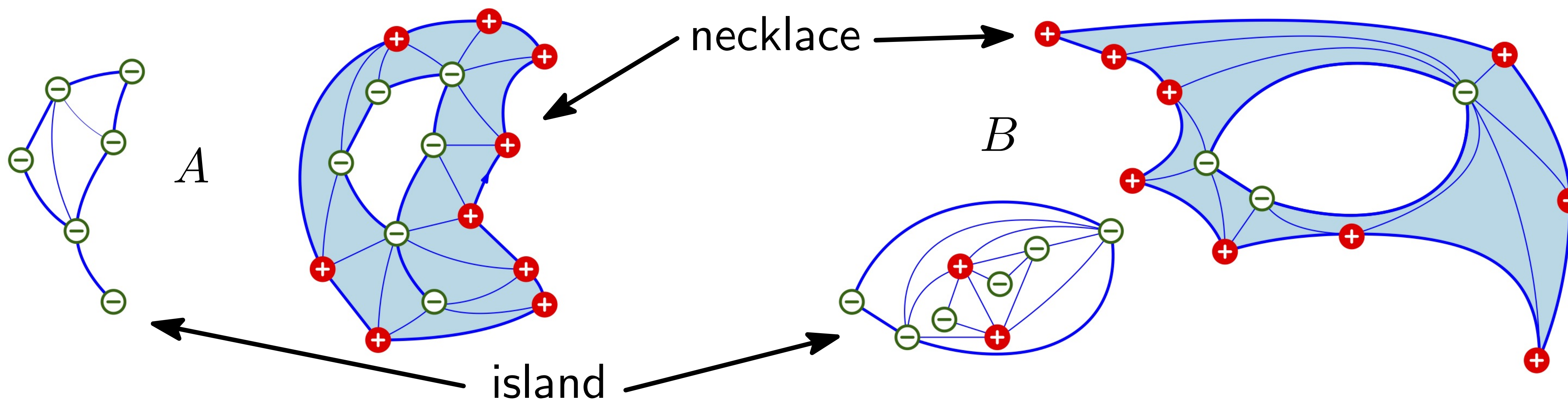

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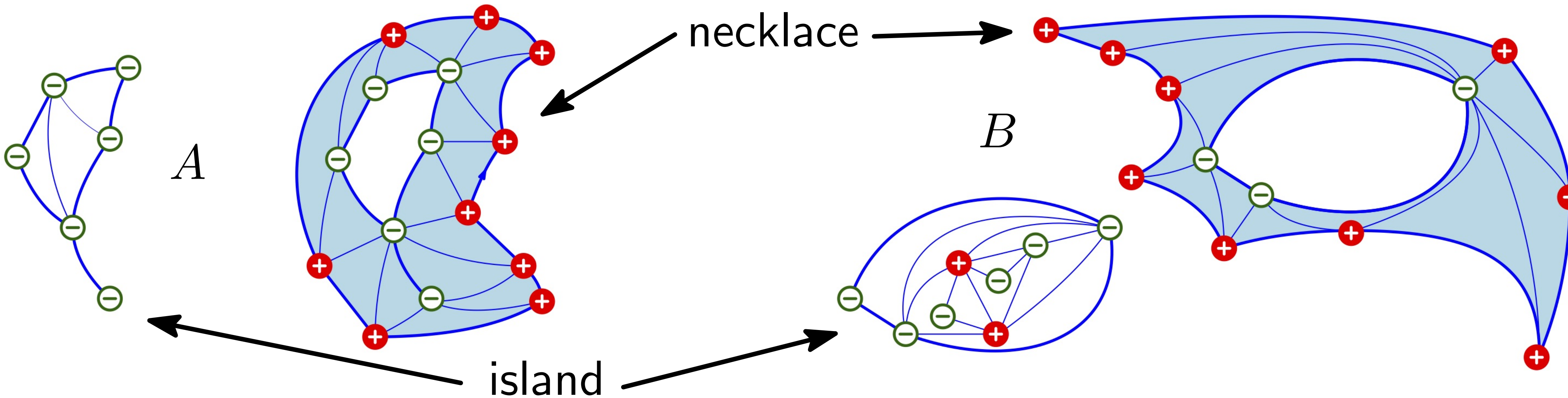

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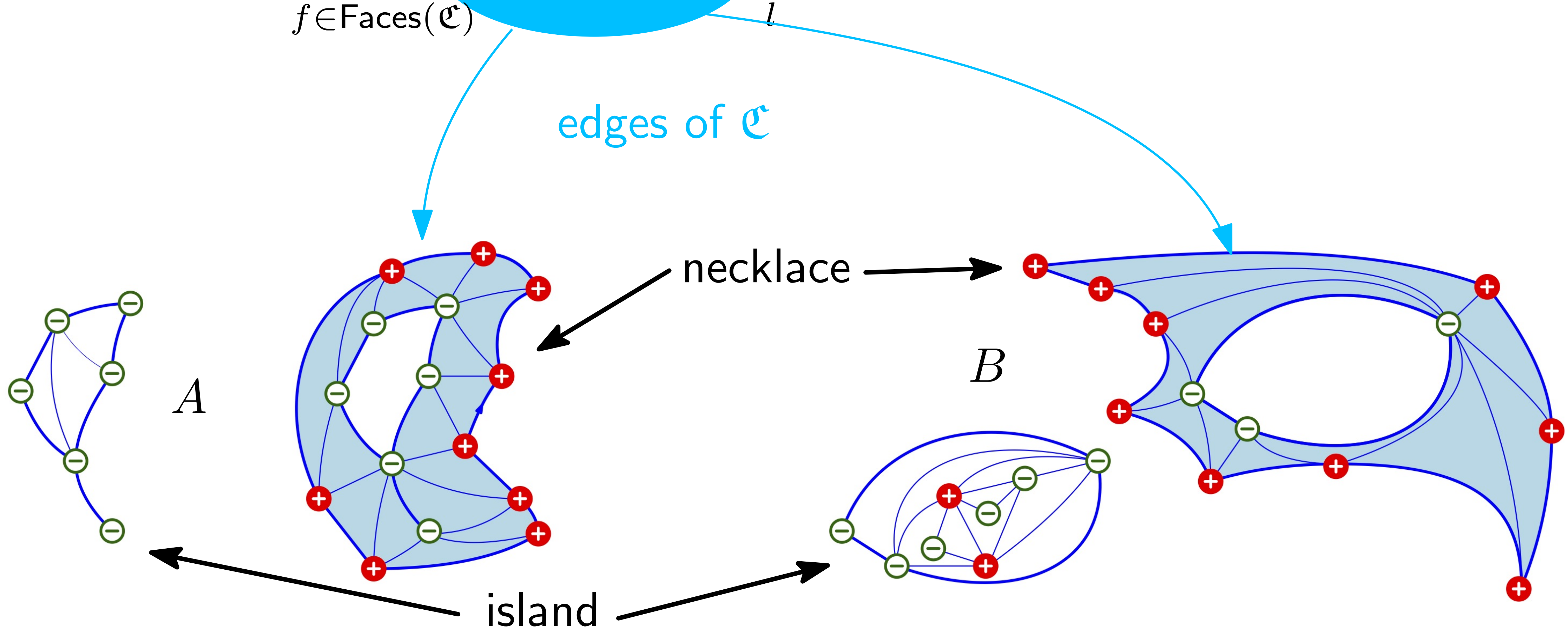

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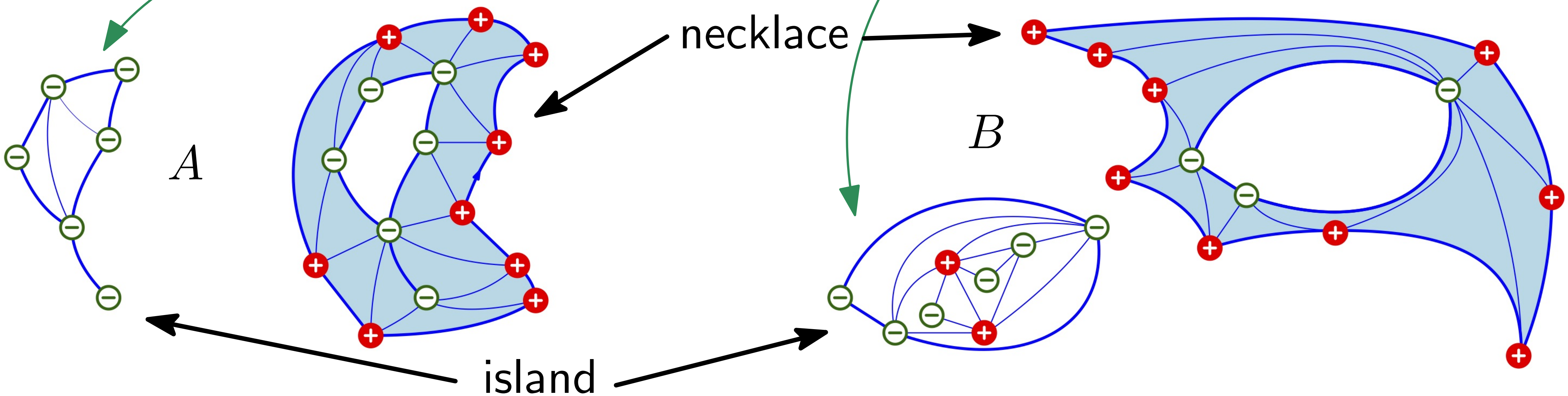
Triangulations with boundary length l and **monochromatic** boundary conditions



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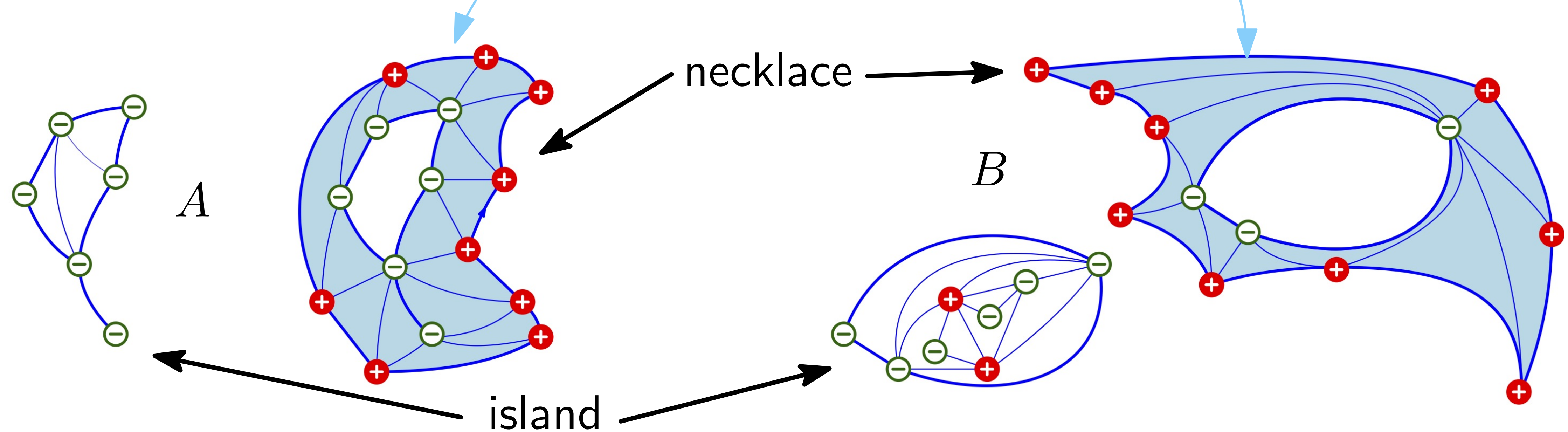

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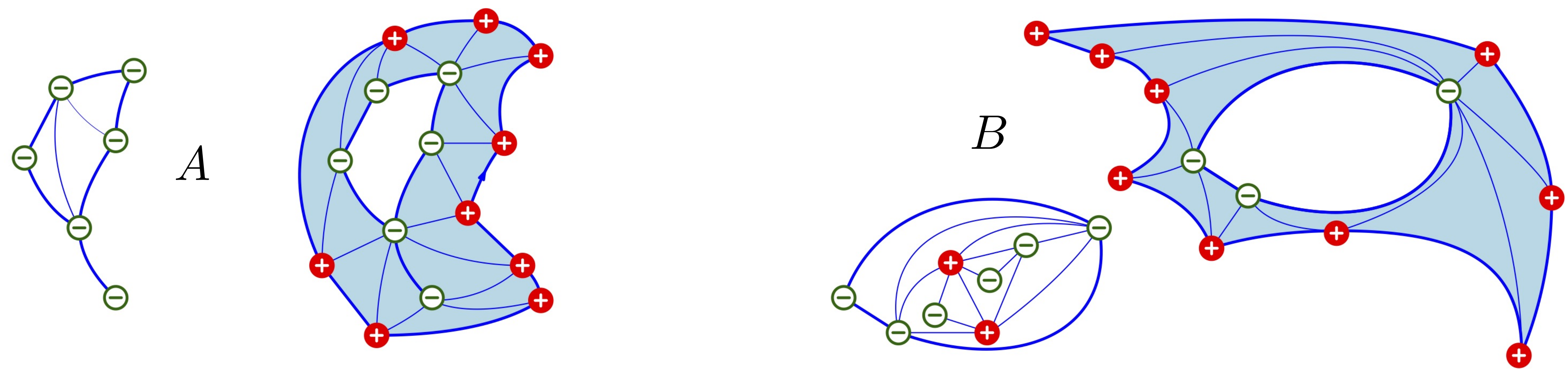

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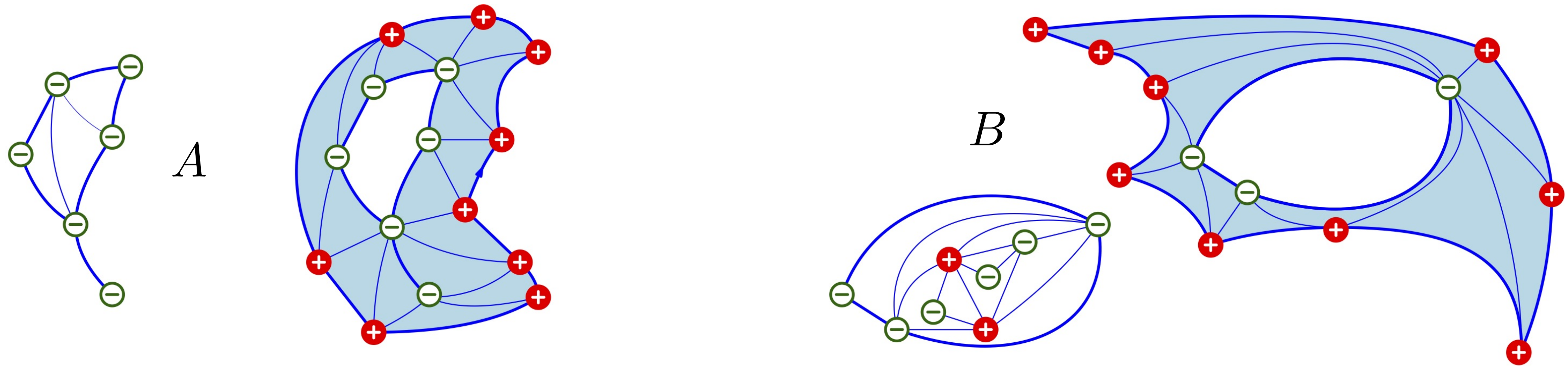
, where $q_k(\nu, t) = (\nu\, t)^{k/2}\, \mathbf{1}_{\{k=3\}} + \left(\nu\, t^3\right)^{k/2} \cdot \sum_{l \geq 0} \binom{k+l-1}{k-1} t^l\, Q_l^+(\nu, t)$



Idea of the proof II: Boltzmann maps

Boltzmann map associated to $(q_k) =$

Probability distribution on the set of rooted planar maps such that:

$$\mathbb{P}^{\text{bol}}(\mathfrak{m}) \propto \prod_{f \in F(\mathfrak{m})} q_{\text{deg}(f)} \qquad \text{for any rooted planar map } \mathfrak{m}$$

$\mathbb{P}^{\mathrm{bol}}$ is **admissible** if $\sum_{\mathfrak{m} \in \mathcal{M}} \prod_{f \in F(\mathfrak{m})} q_{\mathrm{deg}(f)} < \infty$

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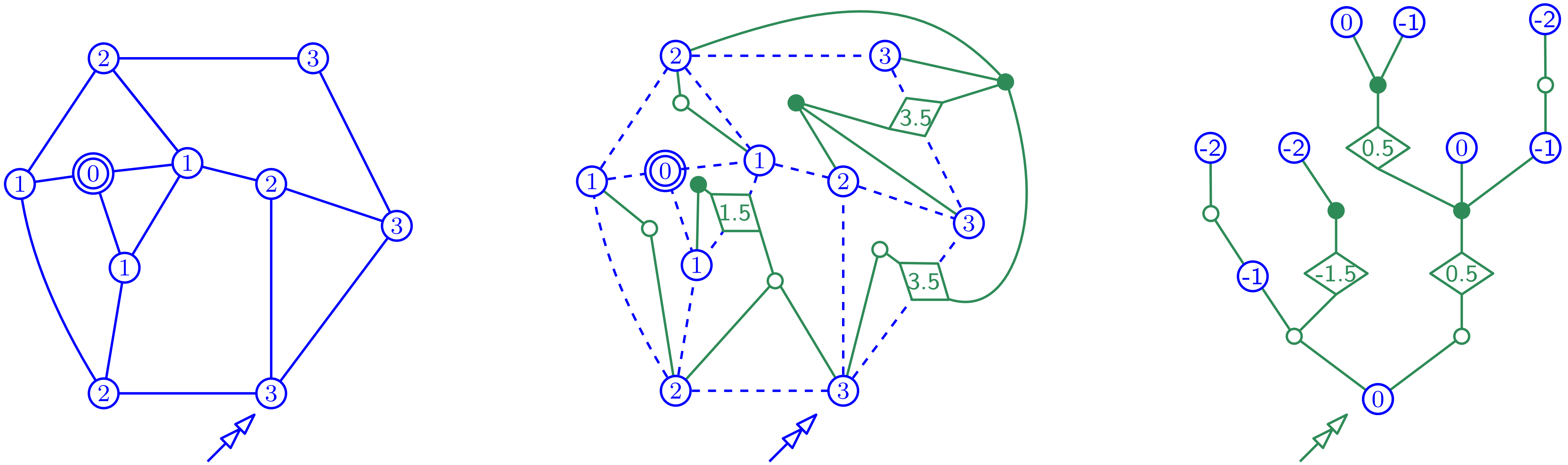

The Bouttier – Di Francesco – Guitter bijection (a.k.a the BDG bijection).

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- $\mathbb{P}^{\text{bol}}$ is **non-regular critical** if $\mathbb{P}^{\text{bol}}(\text{degree of a typical face} > k)$ decreases polynomially.

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$\mathbb{P}^{\text{bol}}$ is **critical** if $\mathbb{E}^{\text{bol}}(|\mathrm{m}|) < \infty$ and $\mathbb{E}^{\text{bol}}(|\mathrm{m}|^2) = \infty$.

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Idea of the proof II: Boltzmann maps

Boltzmann map associated to (q_k) $=$

Probability distribution on the set of rooted planar maps such that:

$$\mathbb{P}^{\text{bol}}(\mathfrak{m}) \propto \prod_{f \in F(\mathfrak{m})} q_{\text{deg}(f)} \qquad \text{for any rooted planar map } \mathfrak{m}$$

$\mathbb{P}^{\mathrm{bol}}$ is **admissible** if $\sum_{\mathfrak{m} \in \mathcal{M}} \prod_{f \in F(\mathfrak{m})} q_{\deg(f)} < \infty$

Properties of the random map depends on the properties of (q_k).

[Marckert-Miermont, Miermont-Le Gall, Borot-Bouttier-Guitter, Budd, Bernardi-Curien-Miermont, Marzouk]

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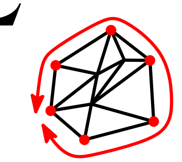
Idea of the proof III:
singularity analysis via rational parametrization

$$Q^+(\nu,t,y) := \sum_{l\geq 0} Q_l^+(\nu,t)y^l, \qquad \text{where } Q_l^+ := \sum_{(T,\sigma)=} t^{|T|}\nu^{m(T,\sigma)}$$



Theorem [A. – Ménard, 22+]

Study of the singular developments of Q^+ in t and in y.

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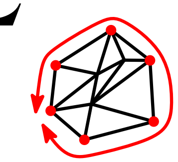
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Theorem [A. – Ménard, 22+]

Study of the singular developments of Q^+ in t and in y.

Sketch of the proof:

- Obtained in [AMS 21] an algebraic equation for Q^+, by Tutte's invariants method [Bernardi, Bousquet-Mélou].
- We use the rational parametrization (for t) given in [Bernardi, Bousquet-Mélou] for Q_1.
- With Maple, we compute a rational parametrization (for y) for different values of ν.
- We interpolate the coefficients given in the different parametrizations.
- With the rational parametrizations (and Maple), can compute the asymptotics.

Same strategy used in a slightly different context by [Chen, Turunen]

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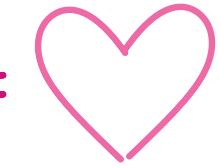
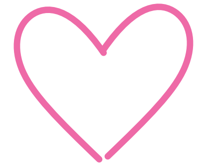
$$t^3 = U\,\frac{\Big((1+\nu)U-2\Big)\,P(\nu,U)}{32\nu^3(1-2U)^2}$$

$$y = \frac{8\nu(1-2U)}{U\Big((1+\nu)\cdot U-2\Big)}\cdot\frac{V(V+1)}{V^3+\frac{9(1+\nu)\cdot U^2-2(3+10\nu)U+8\nu}{U\Big((1+\nu)\cdot U-2\Big)}V^2-\frac{9(1+\nu)\cdot U-2(2\nu+3)}{U\Big((1+\nu)\cdot U-2\Big)}V\ -}$$

$$\begin{aligned}\hat{Q}^+(\nu,U,V) = U\cdot\frac{\Big((1+\nu)\cdot U-2\Big)(1-\nu)}{(V+1)^3\cdot P(\nu,U)} \\ \times\left(V^3+\frac{9(1+\nu)\cdot U^2-2(3+10\nu)\cdot U+8\nu}{U\cdot\Big((1+\nu)\cdot U-2\Big)}\cdot V^2-\frac{9(1+\nu)\cdot U-2(2\nu+3)}{U\cdot\Big((1+\nu)\cdot U-2\Big)}\cdot V-1\right) \\ \times\left(V^2+\frac{5(1+\nu)\cdot U^2-2(3\nu+2)\cdot U+2\nu}{U\cdot\Big((1+\nu)\cdot U-2\Big)}\cdot 2V-\frac{P(\nu,U)}{U\cdot\Big((1+\nu)\cdot U-2\Big)(1-\nu))}\right)\end{aligned}$$

Idea of the proof IV: Computations + Maple = 

Idea of the proof IV: Computations + Maple = 

We compute explicitely:

$$\mathbb{P}^{\nu}_{\infty}\left(|\text{cluster}| < \infty\right) = \sum_{\mathrm{c}\in\mathcal{M}} \mathbb{P}^{\nu}_{\infty}\left(\text{cluster} = \mathrm{c}\right)$$

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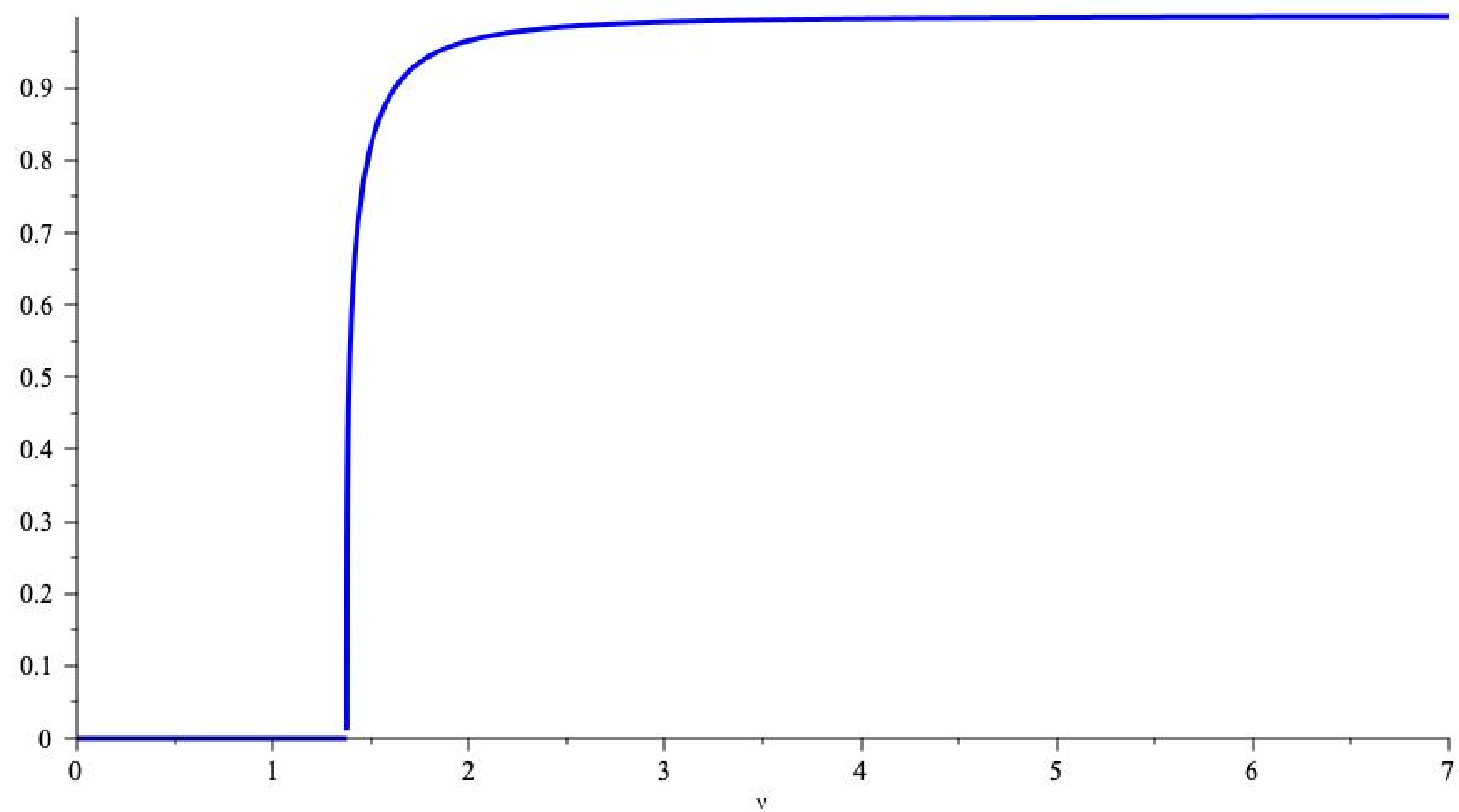
For $\nu > \nu_c$, we obtain an expression with an integral.



Additional results:

We obtain similar tail estimates for the size of the clusters for related models:

- Ising-weighted Boltzmann triangulations
  We recover in particular the results obtained in [Bernardi, Curien, Miermont]
  Connections with some results obtained in [Borot, Bouttier, Guitter] and [Borot, Bouttier, Duplantier].

- Expected size of the cluster for Ising-weighted triangulations of size n.

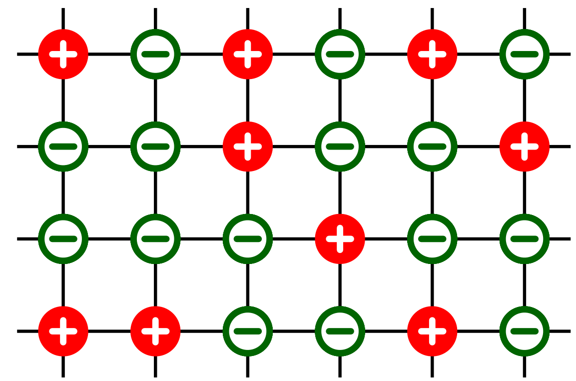
	for $\nu < \nu_c$	for $\nu = \nu_c$	for $\nu > \nu_c$
$\mathbb{E}_n^{\nu}(\lvert\text{cluster}\rvert) \sim$	$c(\nu)\, n^{3/4}$	$c(\nu_c)\, n^{5/6}$	$c(\nu)\, n$

- Geometry of cluster interfaces, via looptrees [Curien, Kortchemski 15].

IV - Link with Liouville Quantum Gravity and KPZ relation

Motivations from statistical physics

Originally the Ising model was studied on **regular lattices** such as $\mathbb{Z}^2$ [Ising, Onsager].



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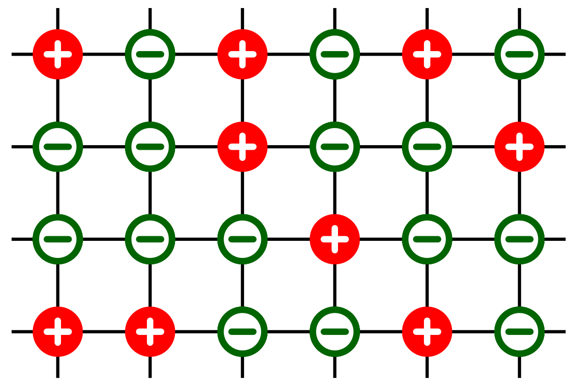

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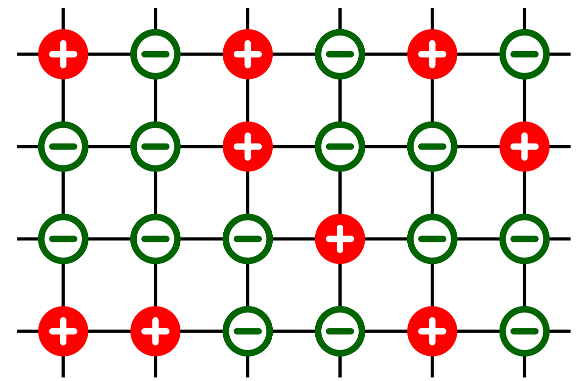

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In **general relativity**, the underlying space is not Euclidian anymore but is a Riemannian space, whose curvature describes the gravity.

One of the main challenge of modern physics is to make two theories consistent:

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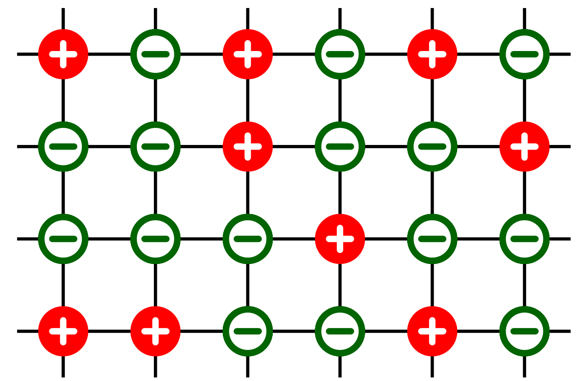

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One of the main challenge of modern physics is to make two theories consistent:

- quantum mechanics (which governs microscopic scales)
- general relativity (which governs macroscopic scales)

One attempt to reconcile these two theories, is the **Liouville Quantum gravity** which replaces the **deterministic** Riemannian space by a **random** metric space.

Liouville Quantum Gravity

For $\gamma \in (0, 2)$, γ-**Liouville Quantum Gravity** (or γ-LQG)
$=$ measure on a surface defined as the "exponential of the Gaussian Free Field"
[Polyakov, 1981], [Duplantier, Sheffield 2011].

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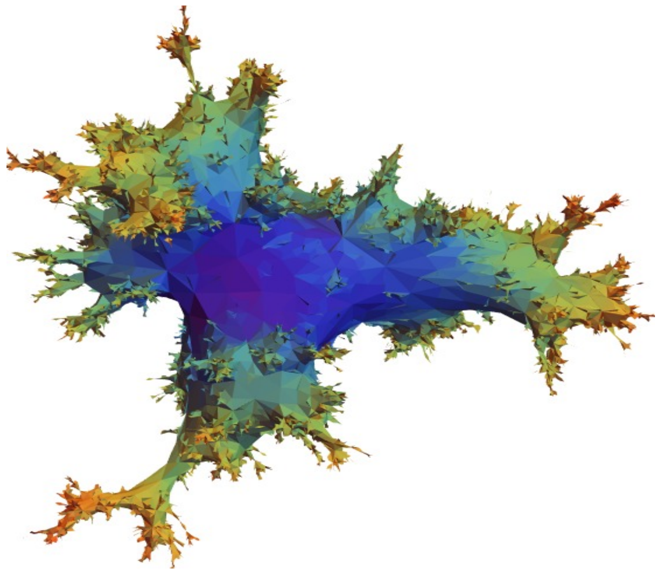
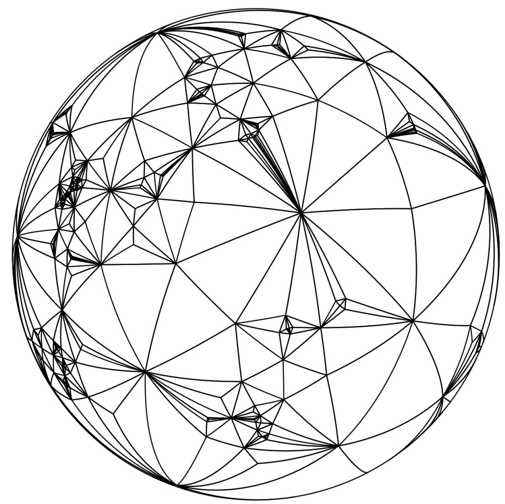
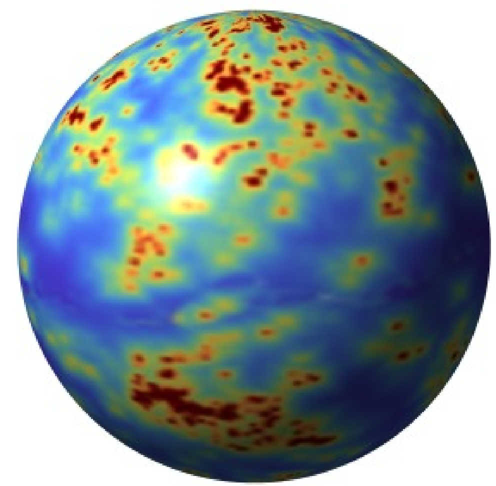


Simulation of the Brownian map
by T.Budd



Simulation of a large simple triangulation
embedded in the sphere by circle packing.
Software CirclePack by K.Stephenson.



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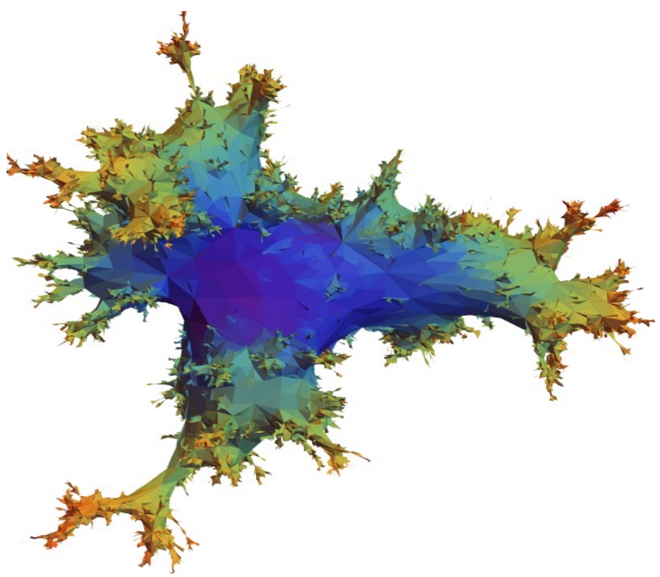

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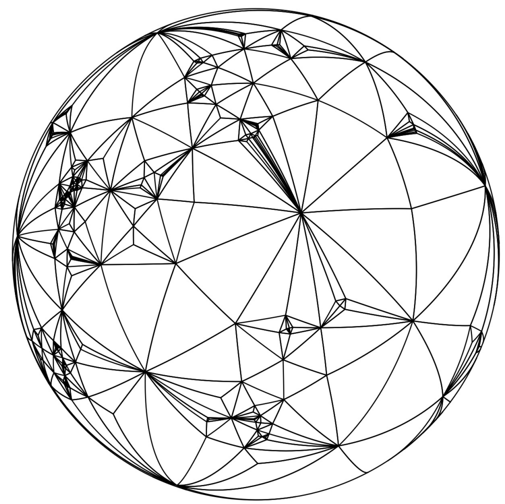

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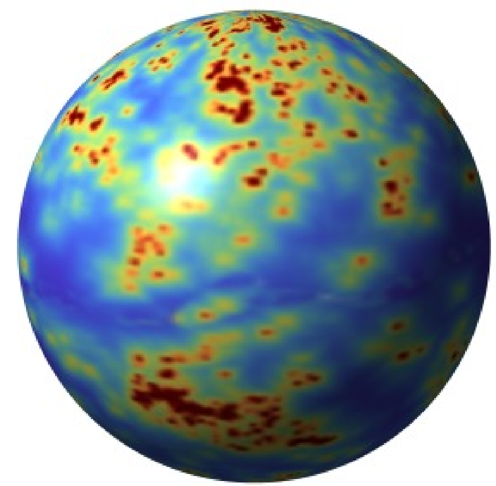

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Other statistical models on random maps are believed to converge towards γ-LQG:
For critical Ising model on maps, $\gamma = \sqrt{3}$ (for non-critical Ising, $\gamma = \sqrt{8/3}$).

Decorated γ-LQG

Statistical models on random maps are believed to converge towards γ-LQG:

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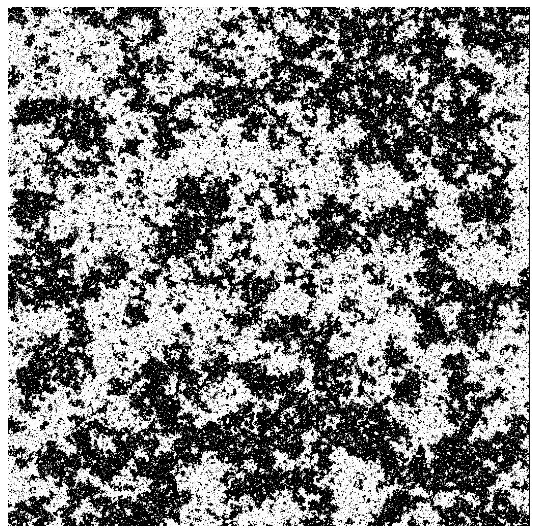
What about the clusters ? And their boundary ?

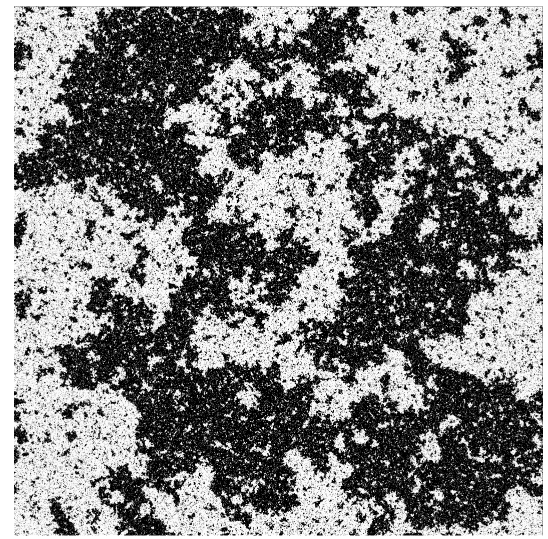
Recall the behaviour in the Euclidean case:

$\nu < \nu_c$



CLE_6 ?

Critical Ising model



CLE_3

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What about the clusters ? And their boundary ?

Recall the behaviour in the Euclidean case:

We expect the same behaviour but on the corresponding γ-LQG.

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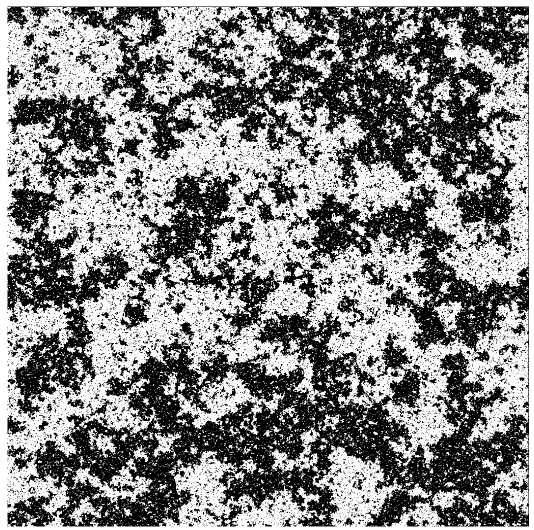
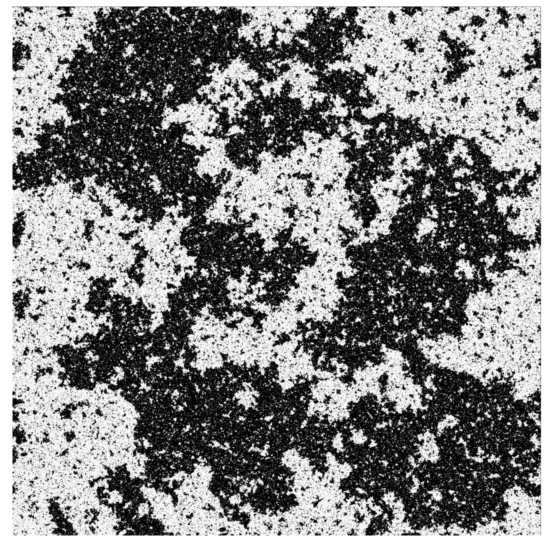


CLE$_6$?

Critical Ising model



CLE$_3$

For **critical percolation on uniform triangulations**, proved by [Holden-Sun 20], building on earlier works [Bernardi-Holden-Sun 18] and [Gwynne-Holden-Sun 21].

Decorated γ-LQG and KPZ

The KPZ relation [Knizhnik, Polyakov,Zamolodchikov, 1988], [Duplantier, Sheffield 2011]:

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$

links the Eucliden conformal weight x of a fractal to its quantum counterpart Δ.

i.e. We could “transfer” volume and perimeter exponents from **deterministic** to **random** geometry and vice versa.

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For $\nu < \nu_c$ ⟷ KPZ, $\gamma = \sqrt{8/3}$ ⟷ Dimension of SLE_6 [Beffara 08]

For $\nu = \nu_c$ ⟷ KPZ, $\gamma = \sqrt{3}$ ⟷ Dimension of SLE_3

All exponents match !

Exponents for the volume $|\mathfrak{C}|$:

For $\nu < \nu_c$ ⟷ KPZ, $\gamma = \sqrt{8/3}$ ⟷ Dimension of the gasket of CLE_6 [Miller, N.Sun, Watson 14]

For $\nu = \nu_c$ ⟷ KPZ, $\gamma = \sqrt{3}$ ⟷ Dimension of the gasket of CLE_3

Perspectives

- Singularity with respect to the UIPT for $\nu \neq 1$.
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Thank you for your attention !