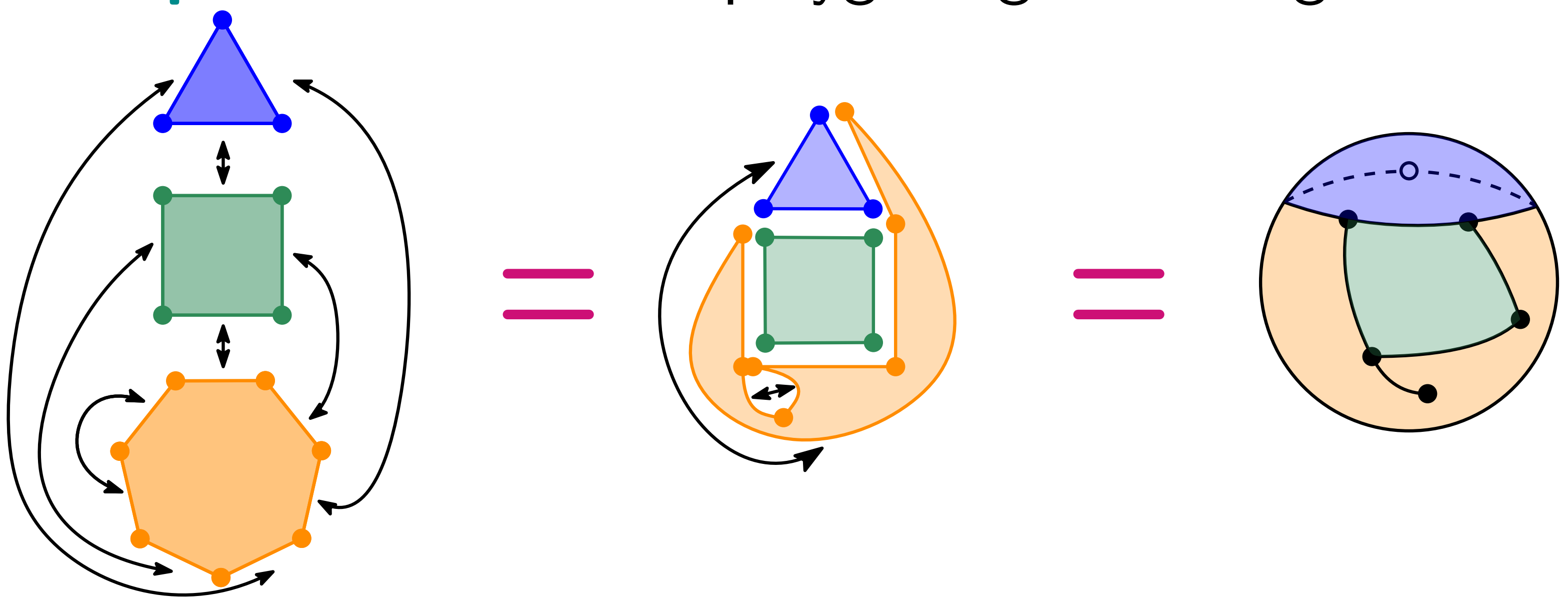
Combinatorial proof of the rationality scheme for maps in higher genus

Marie Albenque (CNRS, LIX, École Polytechnique)
joint work with **Mathias Lepoutre**

JCB, February 2021

Maps – Definition(s)

A **map** is a collection of polygons glued along their sides (with some technical conditions).



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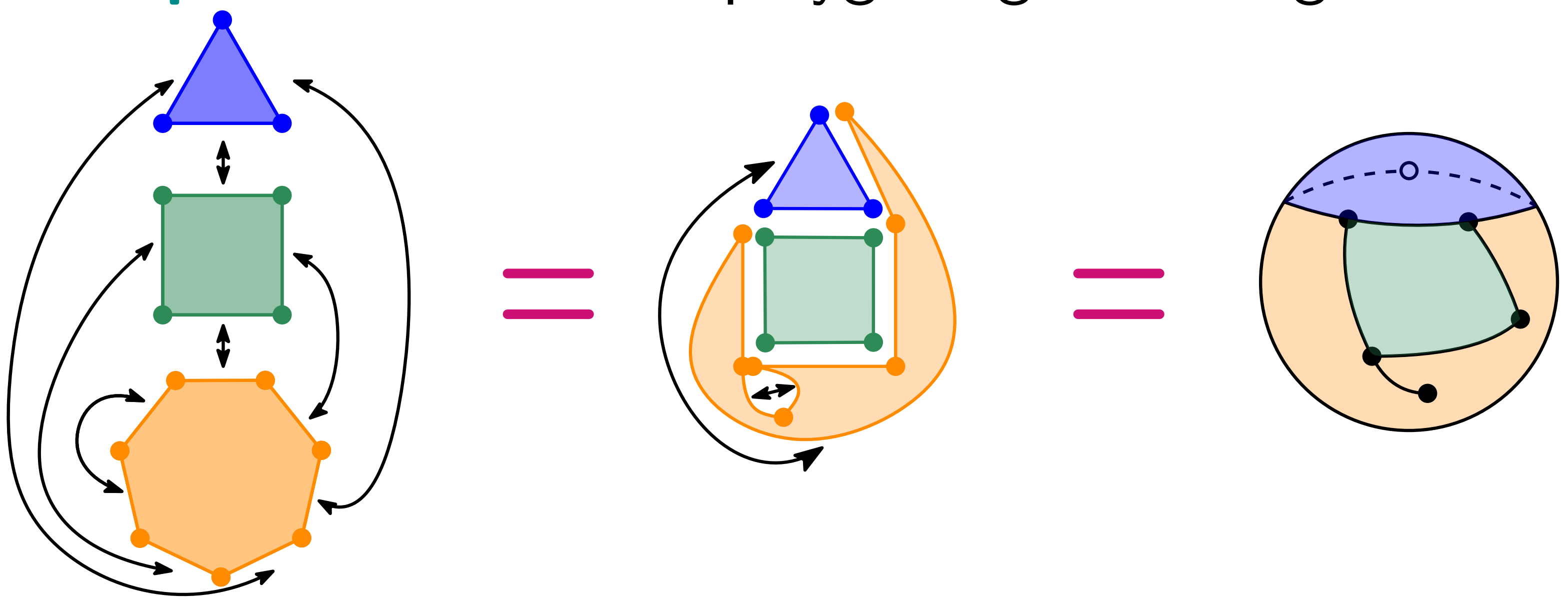

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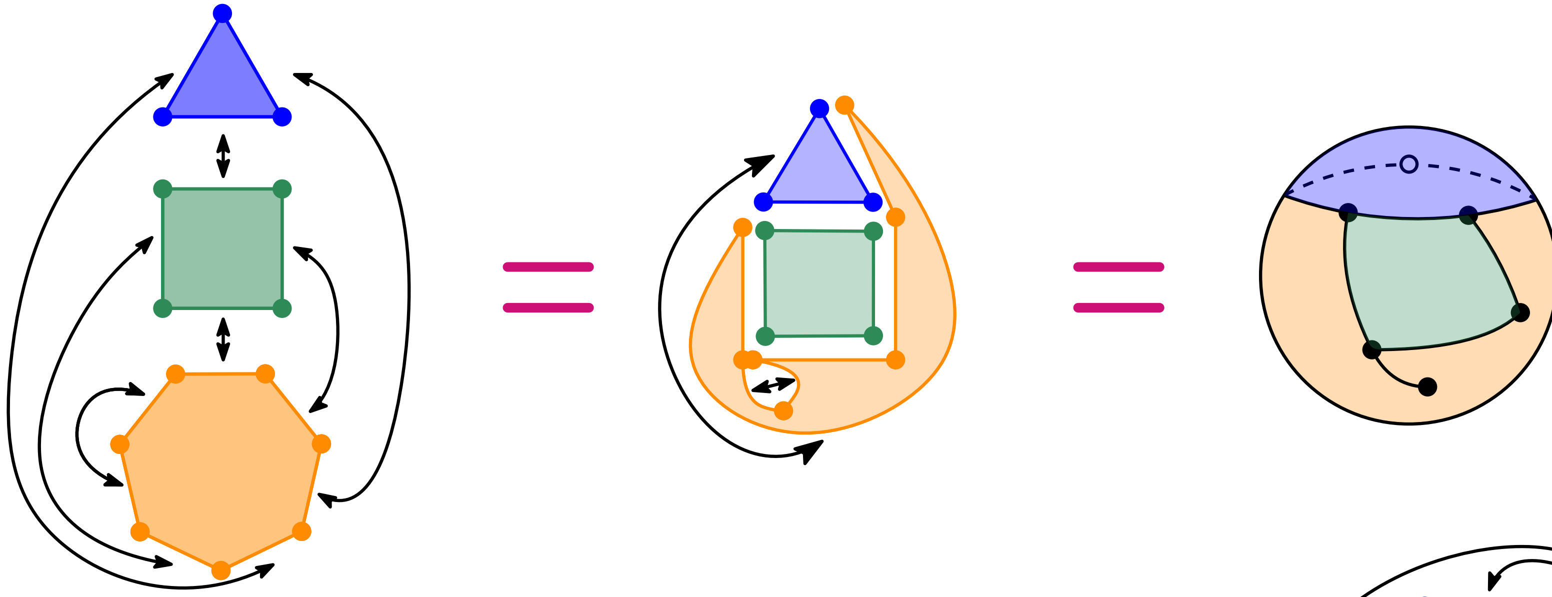

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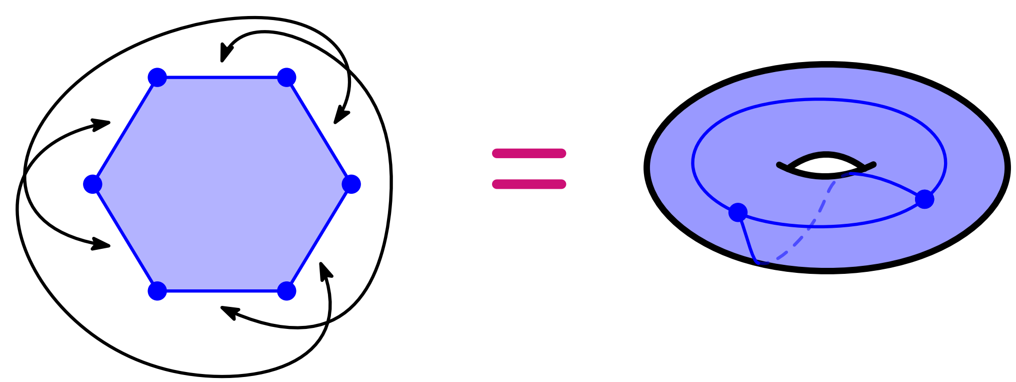

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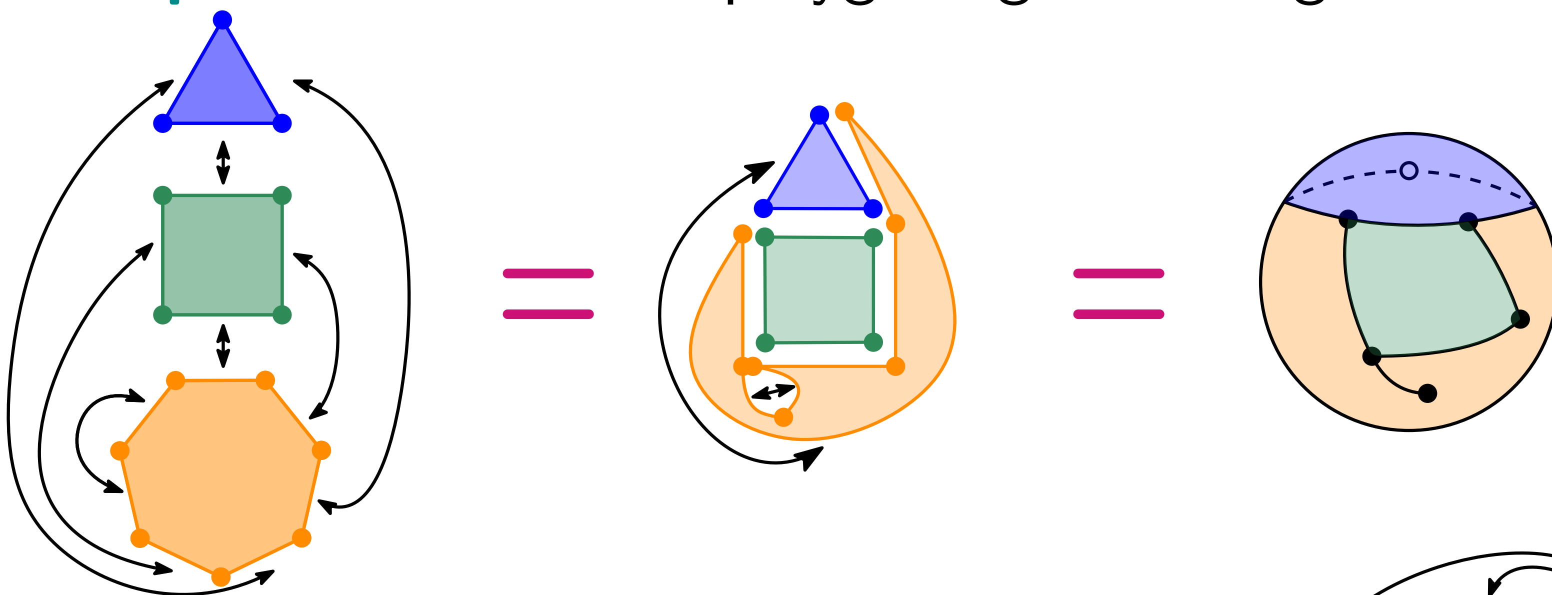

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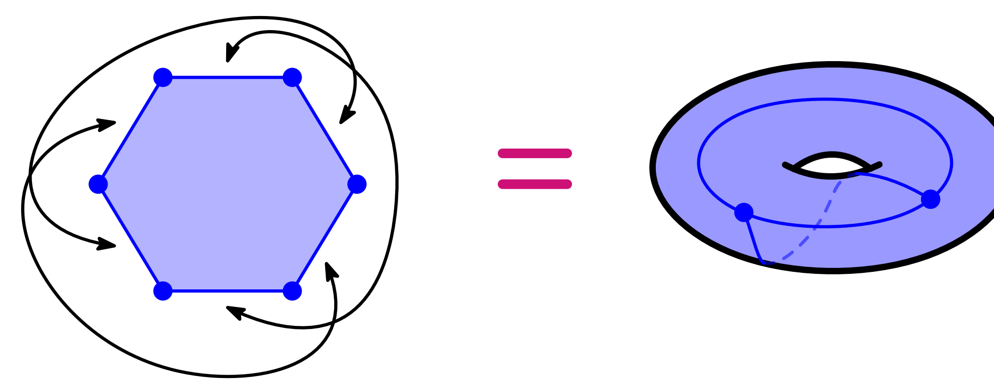


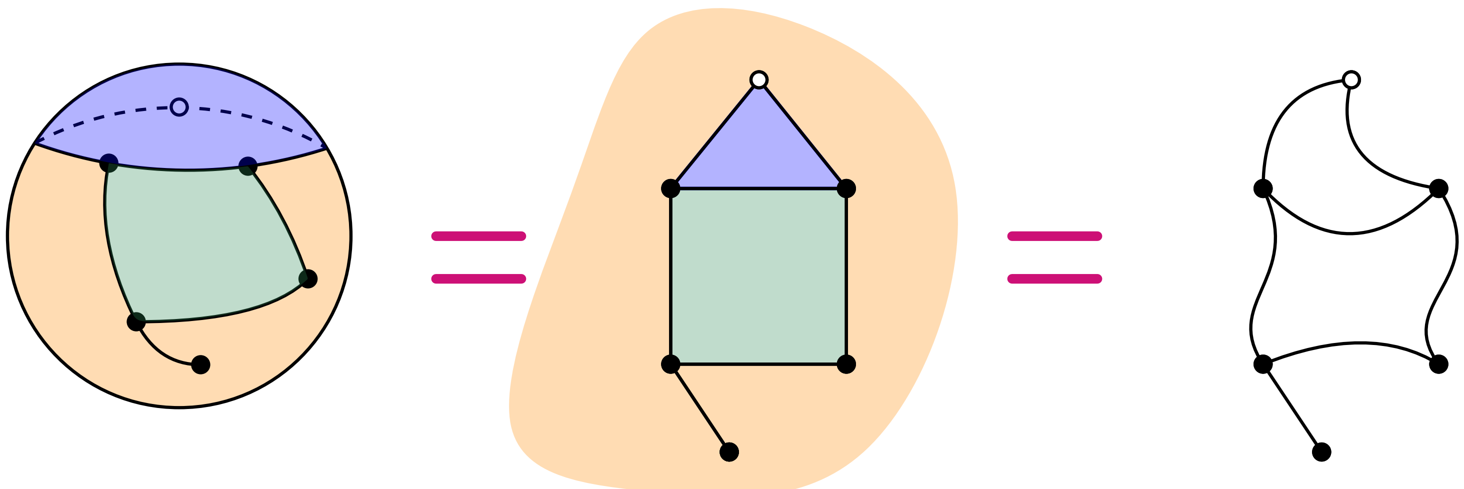
Euler's formula: for every map m (on a closed surface without boundary),

$$|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)$$

vertices, faces, edges, genus

Maps – Definition(s)

A **map of genus** g is a proper embedding of a connected graph in the torus with g holes (such that all its faces are homeomorphic to disks and considered up to orientation-preserving homeomorphisms).



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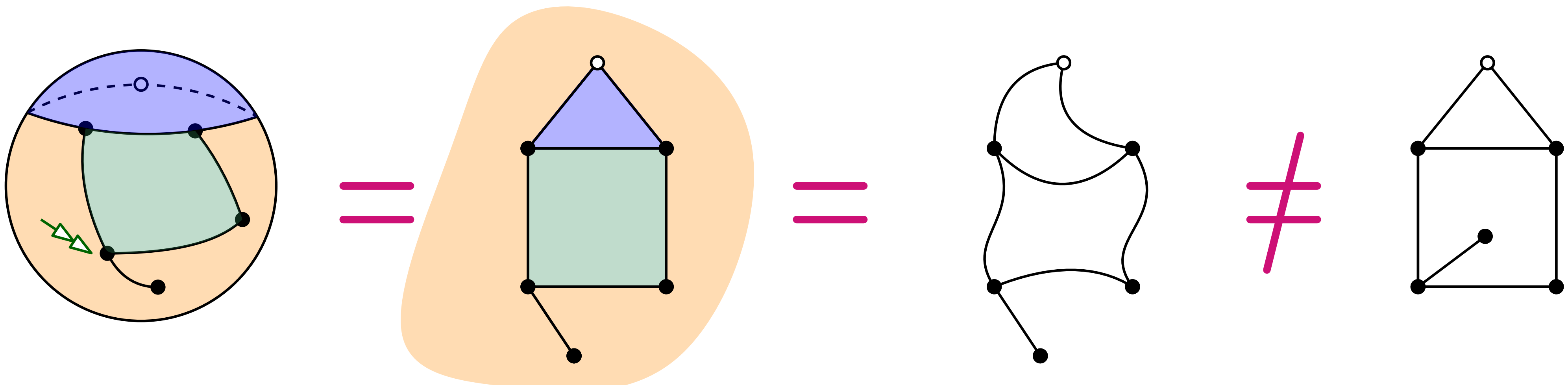

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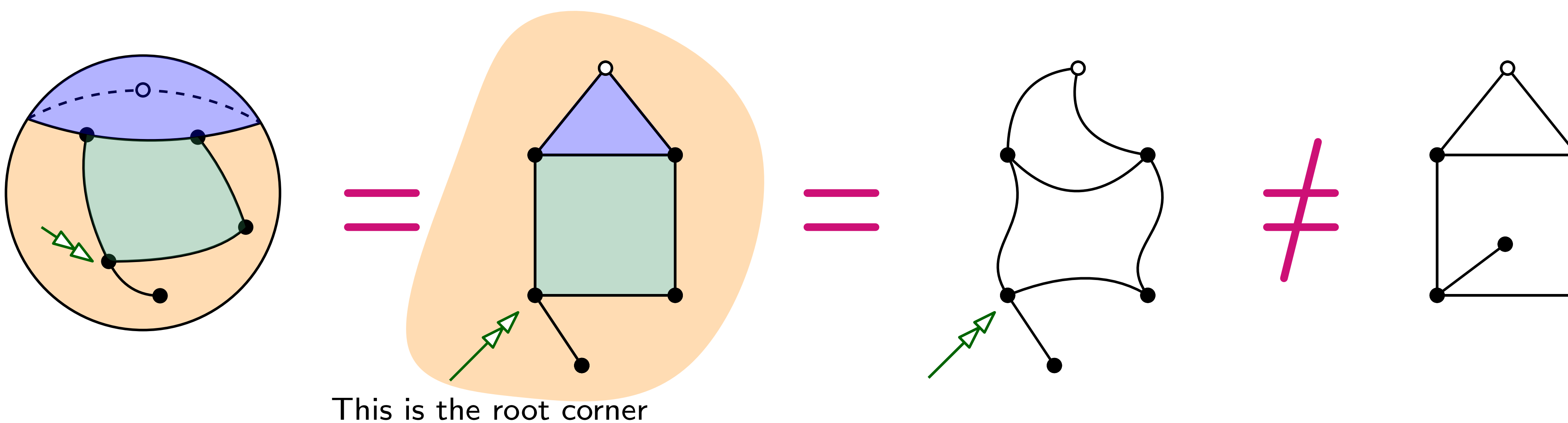

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To avoid dealing with symmetries: maps are **rooted** (a corner is marked).

Enumeration of planar maps

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e.g. $\#\Big\{\text{rooted planar maps with } n \text{ edges}\Big\} = \dfrac{2 \cdot 3^n}{n+2}\text{Catalan}(n)$ [Tutte 63]

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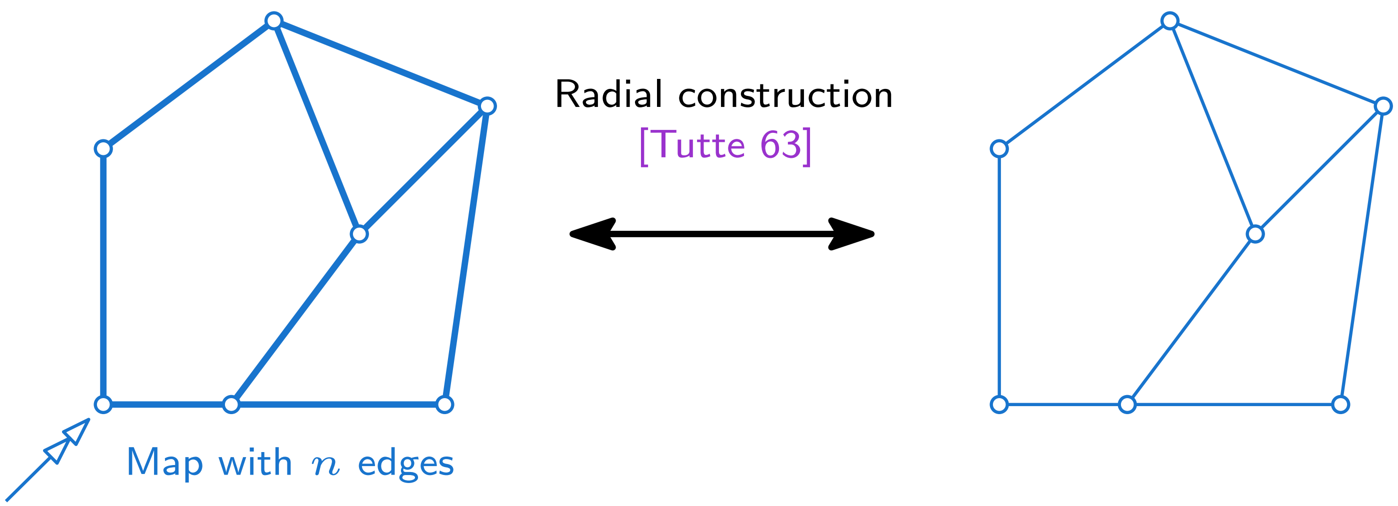

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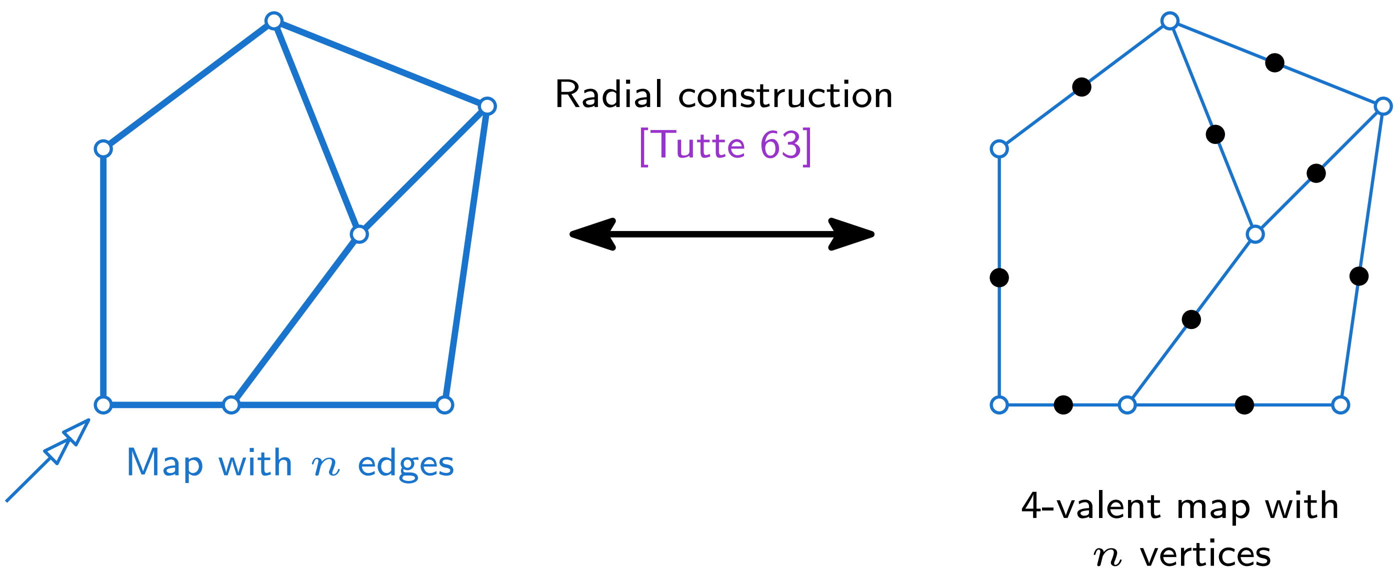

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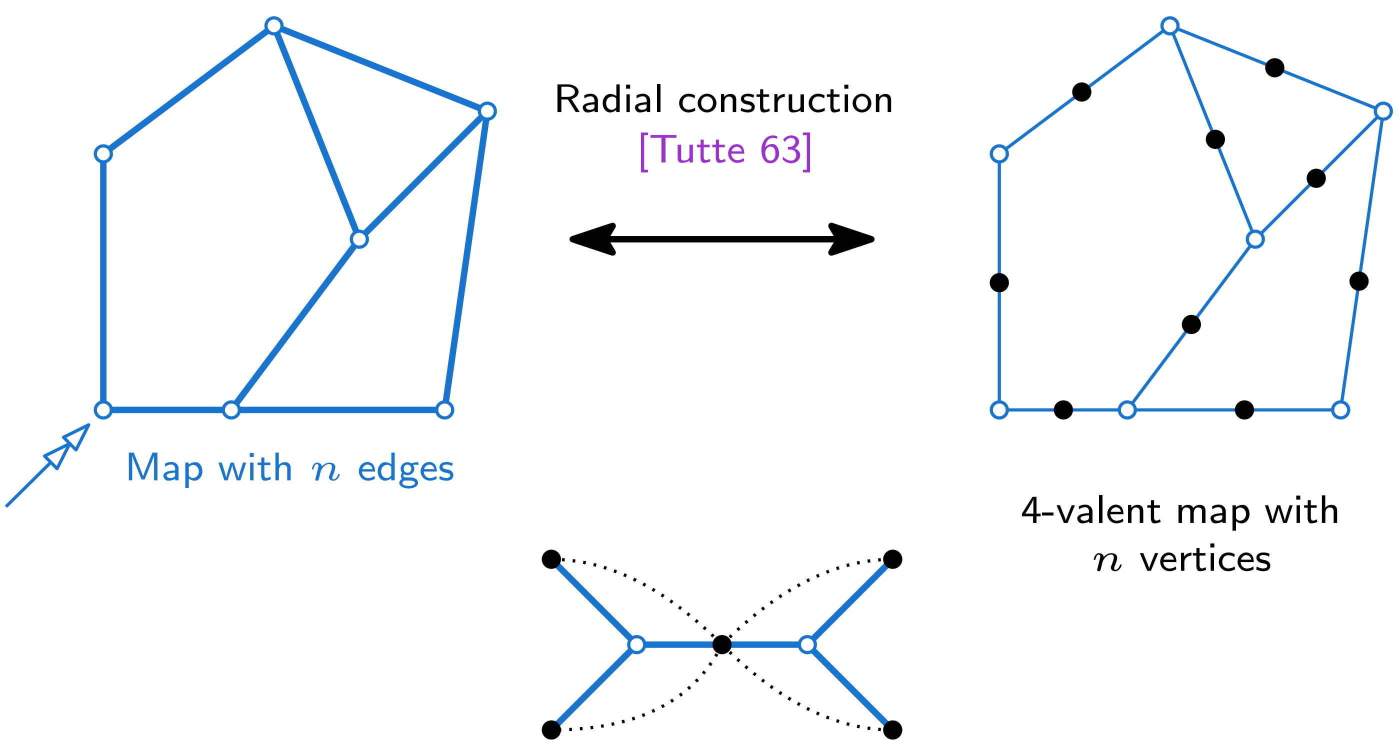

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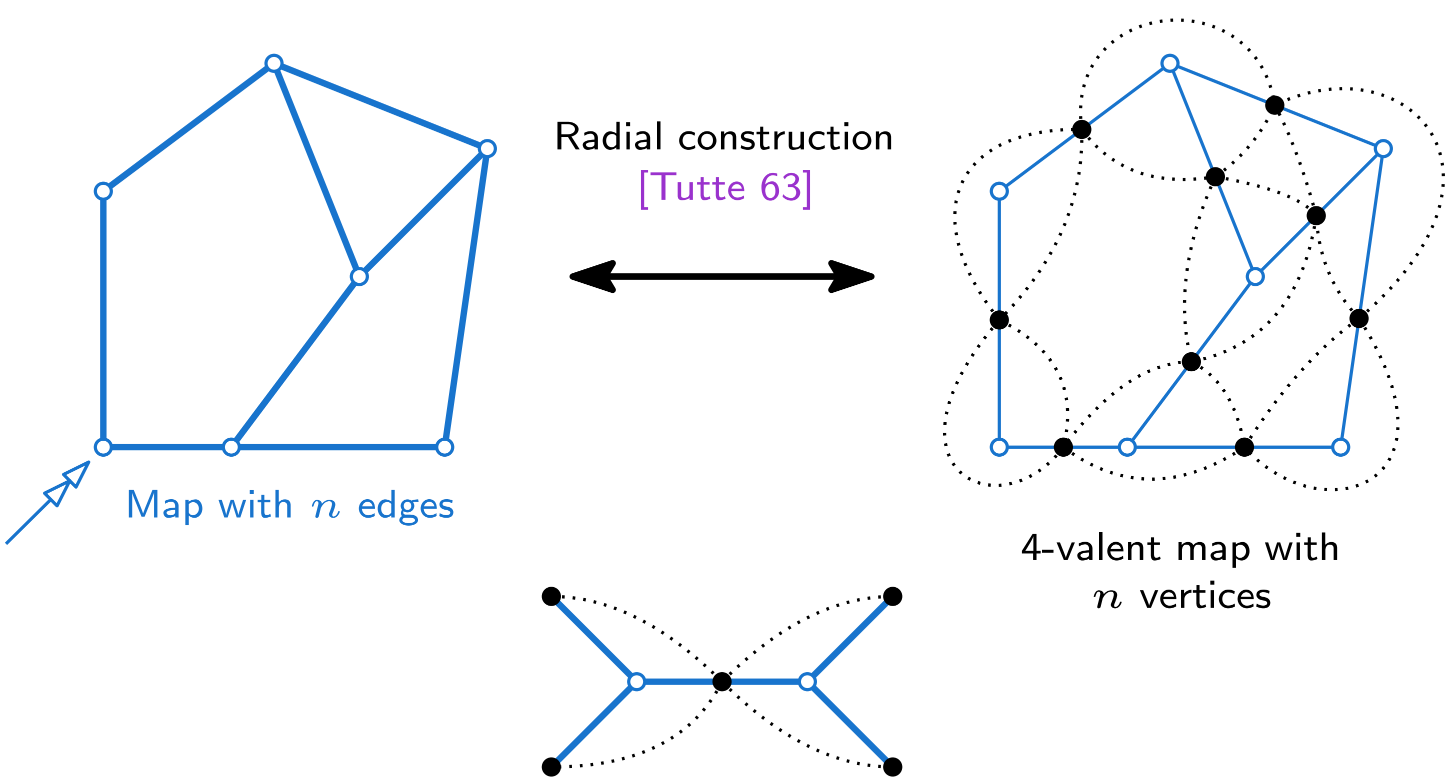

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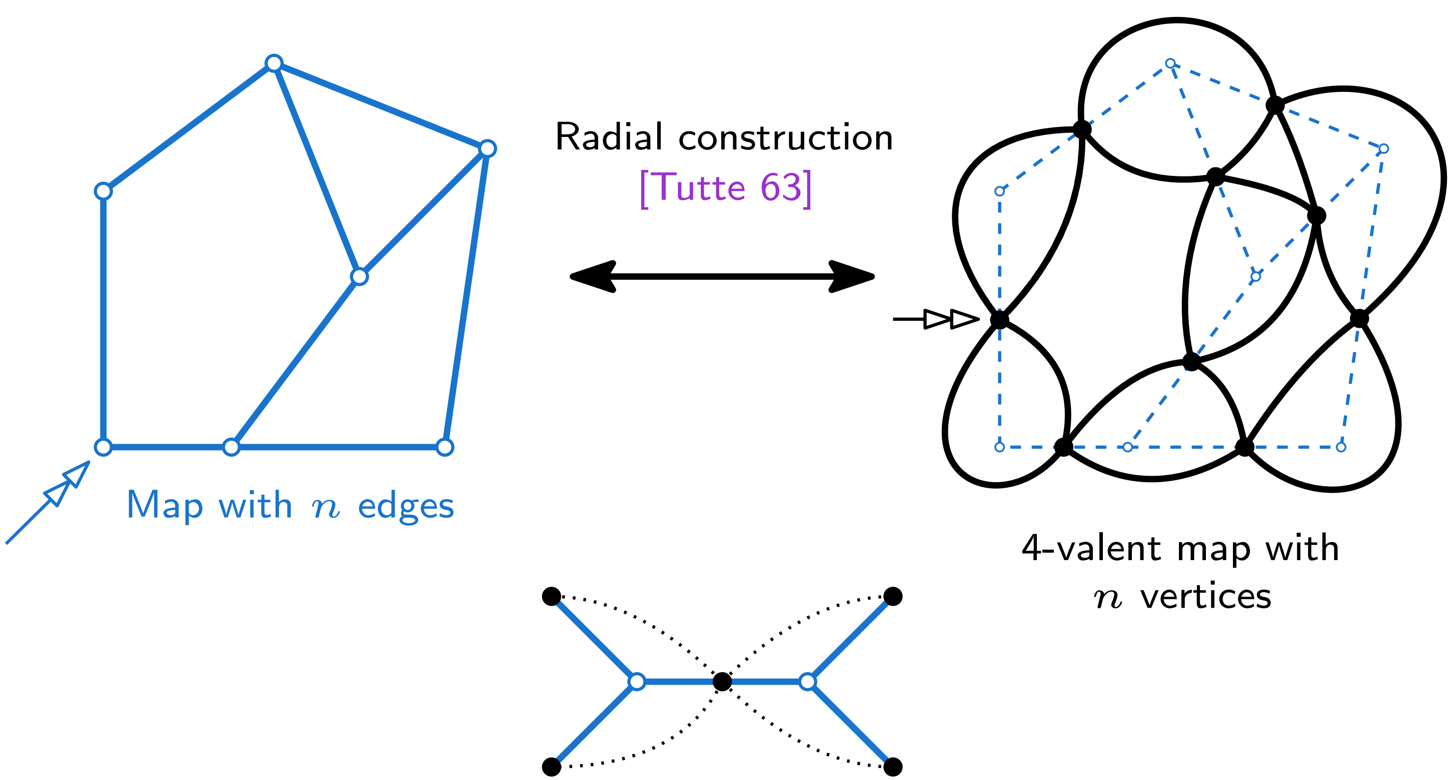

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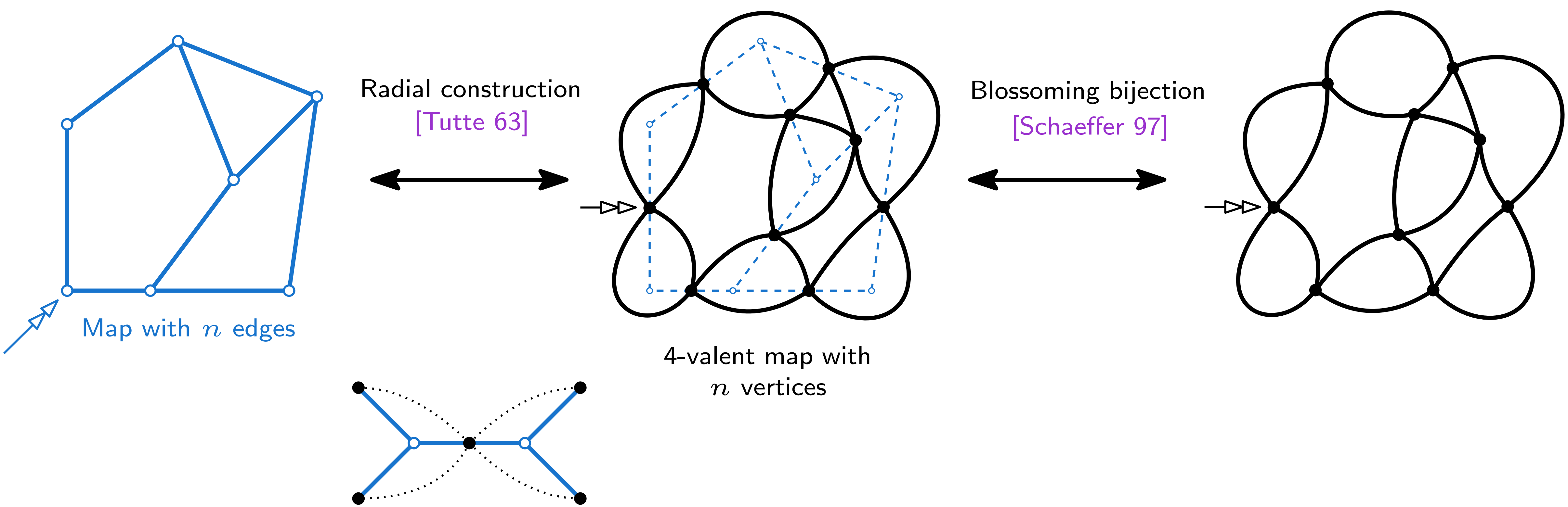

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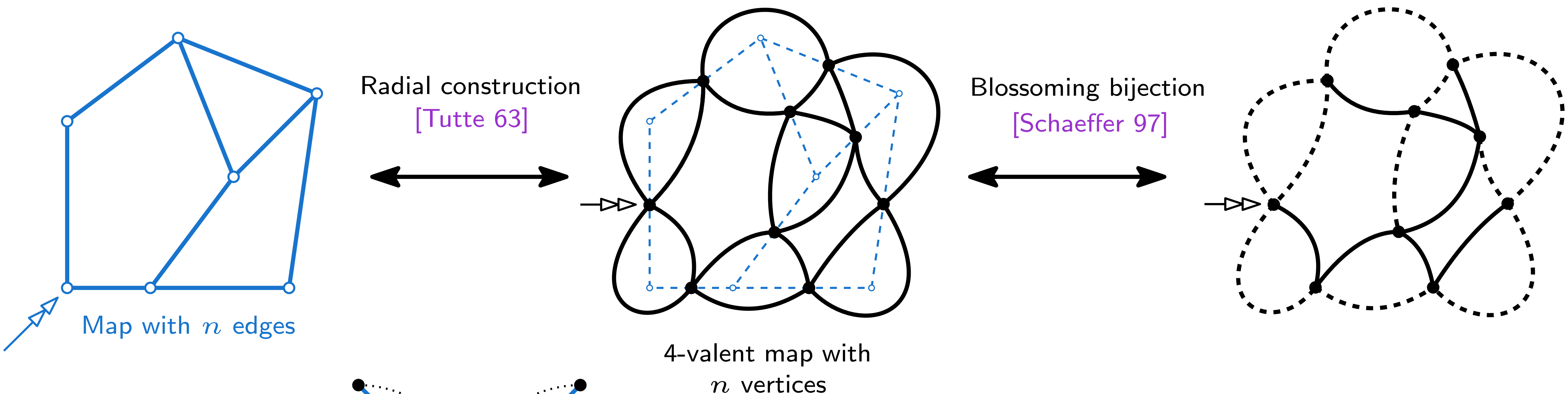

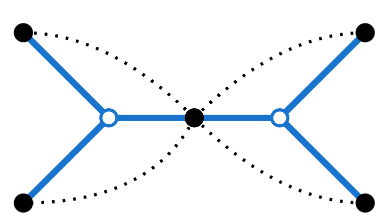

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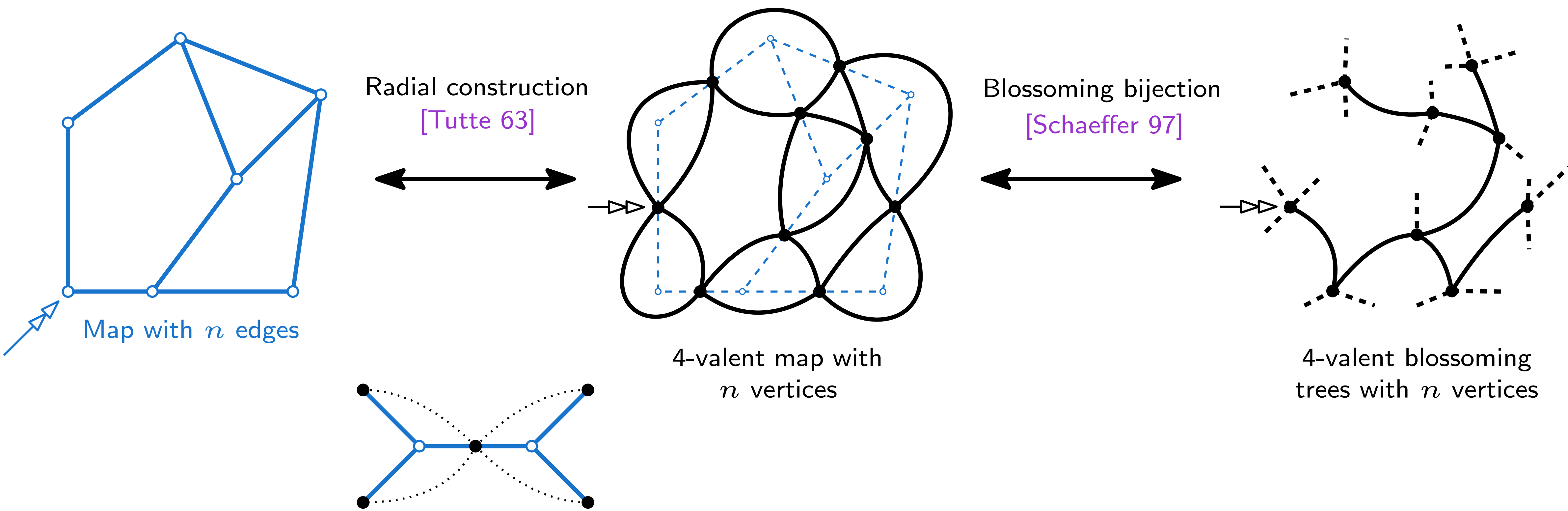

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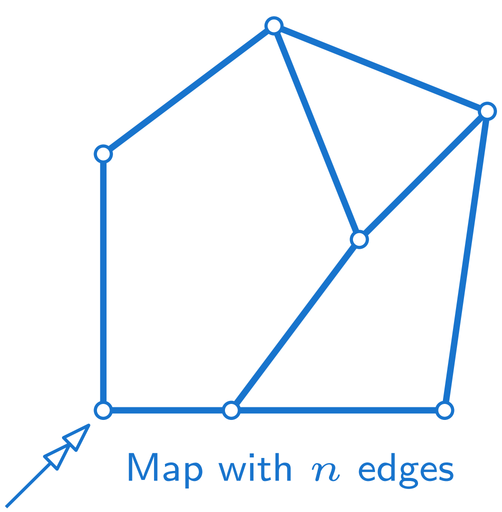
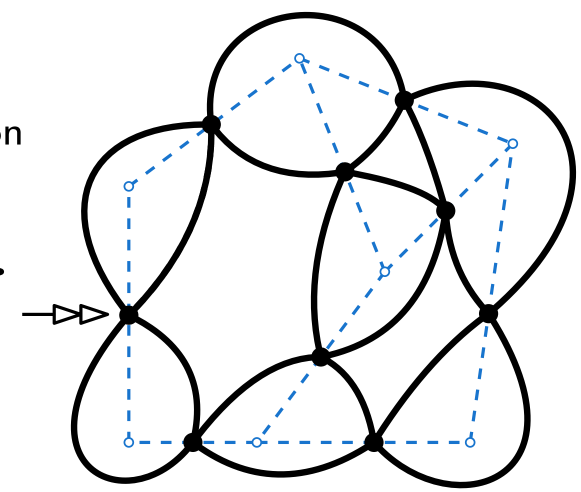


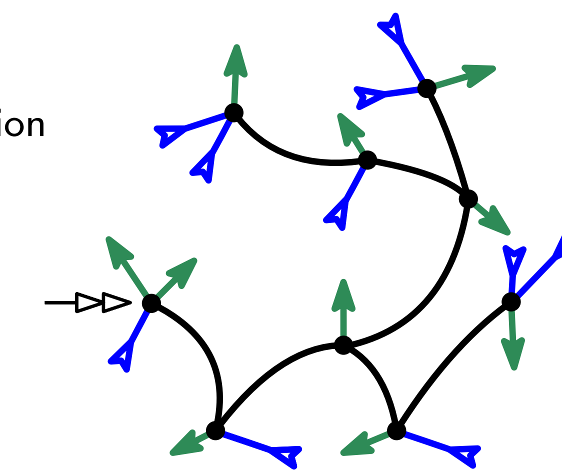
Map with n edges

Radial construction [Tutte 63]

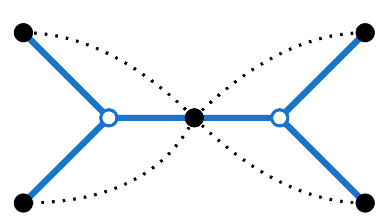


4-valent map with n vertices

Blossoming bijection [Schaeffer 97]



4-valent blossoming trees with n vertices



As a corollary:
combinatorial proof of Tutte's formula.

Bijections with blossoming trees

A **blossoming tree** is a plane tree where vertices can carry **opening stems** or **closing stems**:

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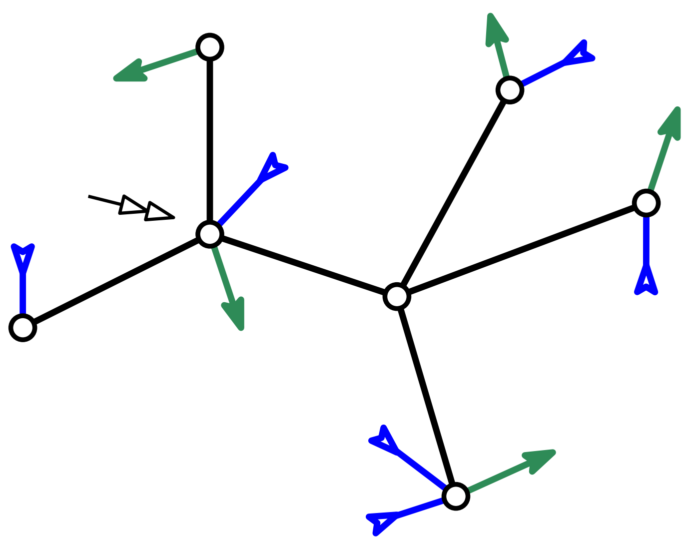

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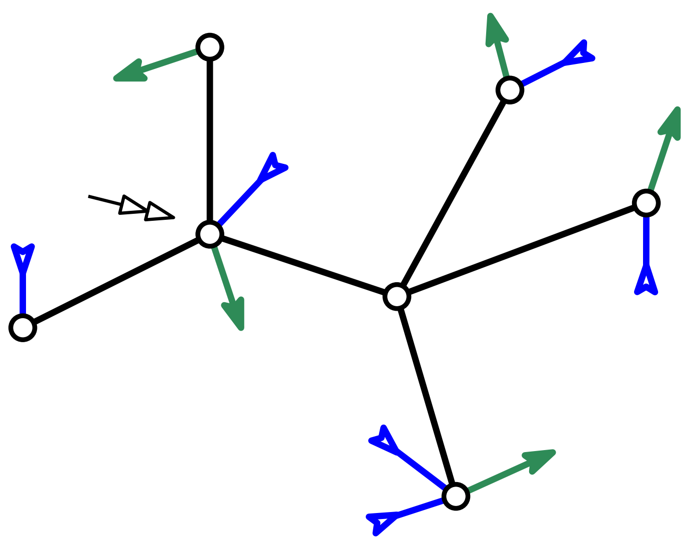

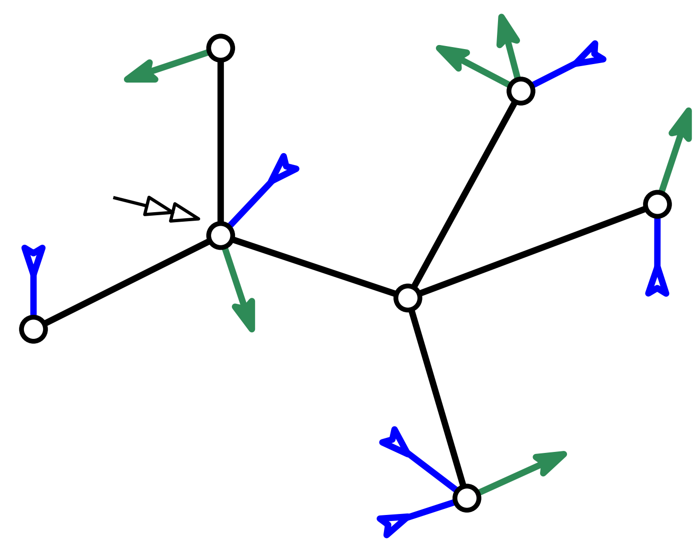

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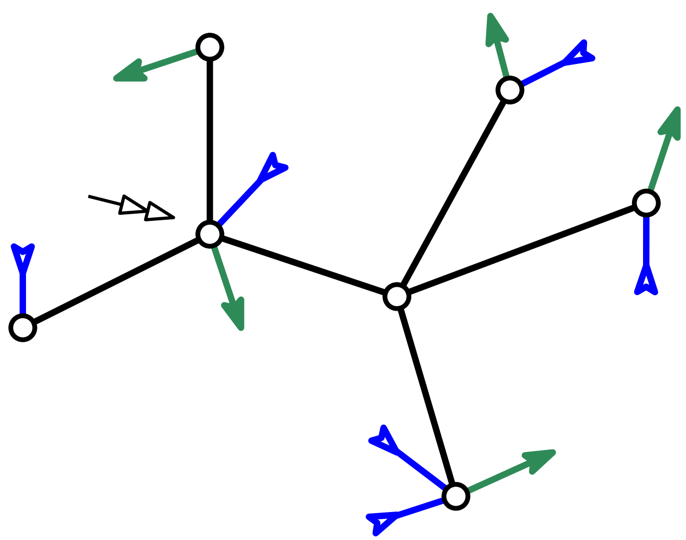

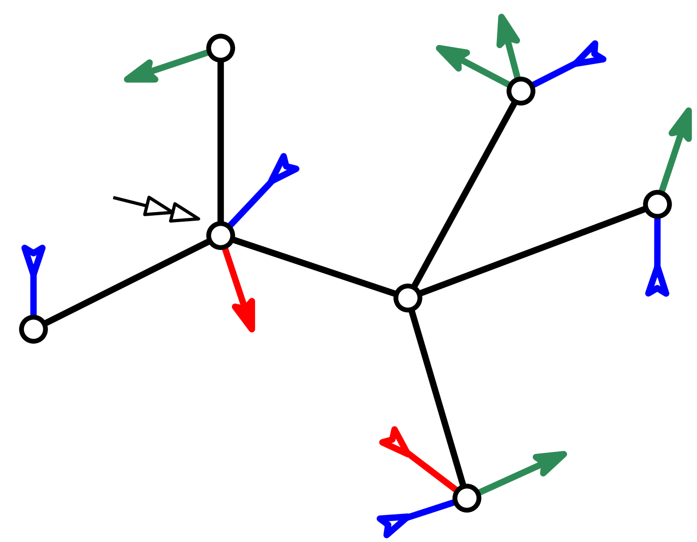

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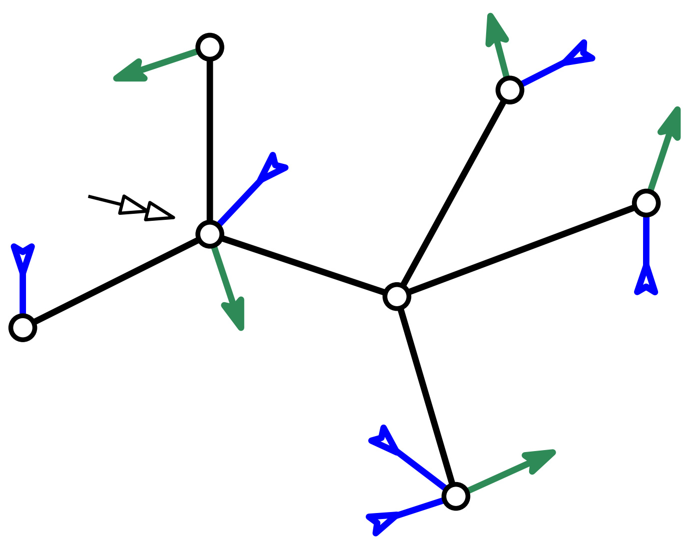

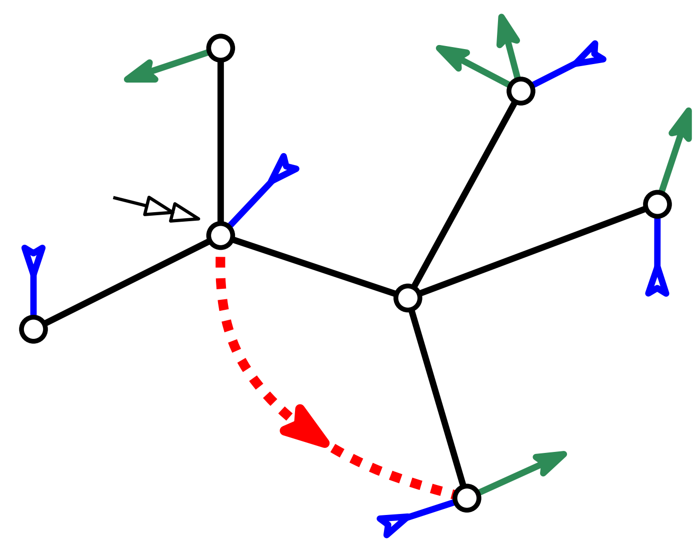

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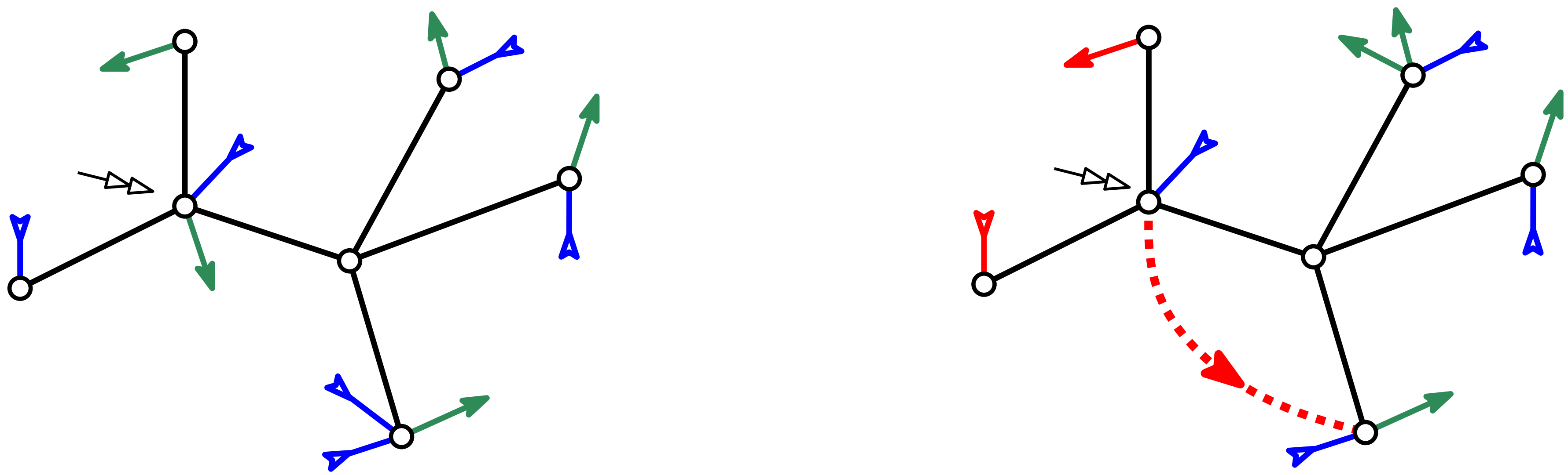

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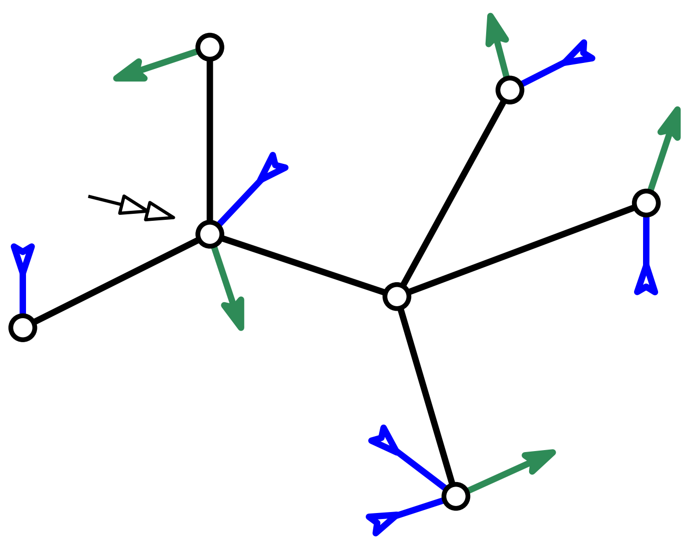

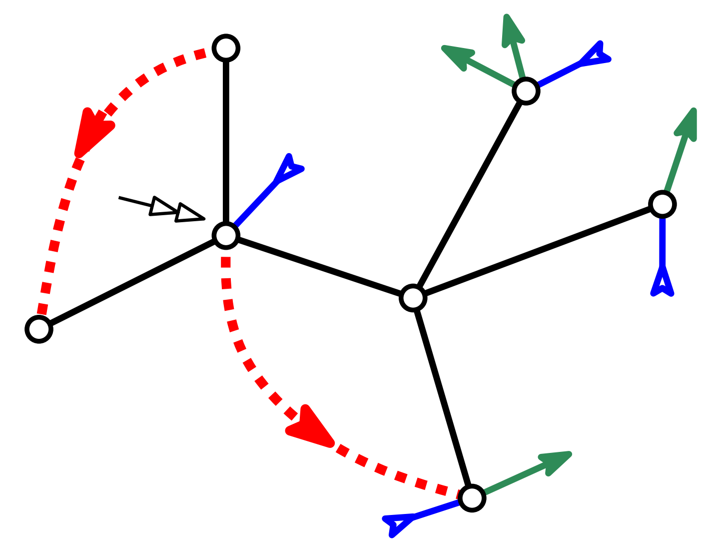

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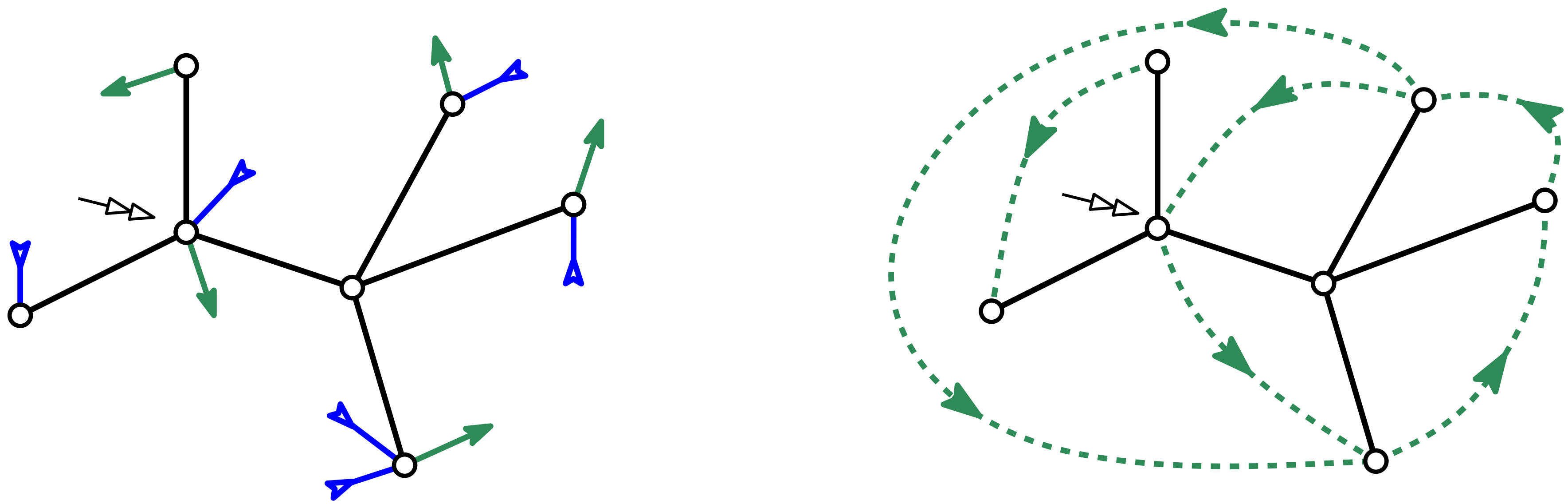

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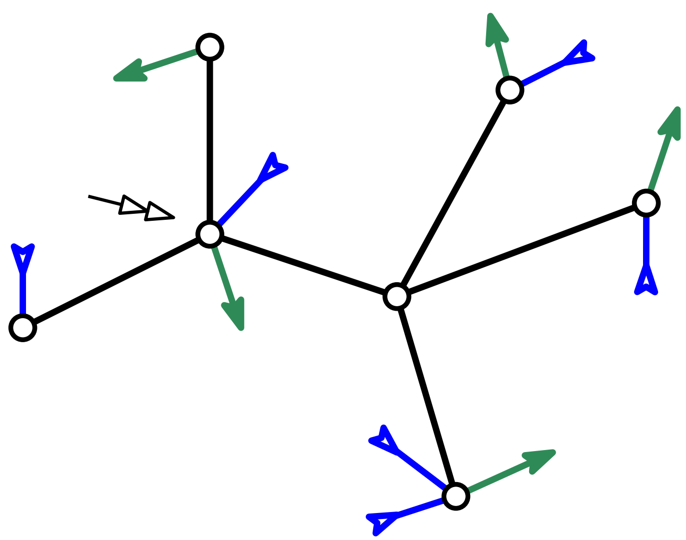

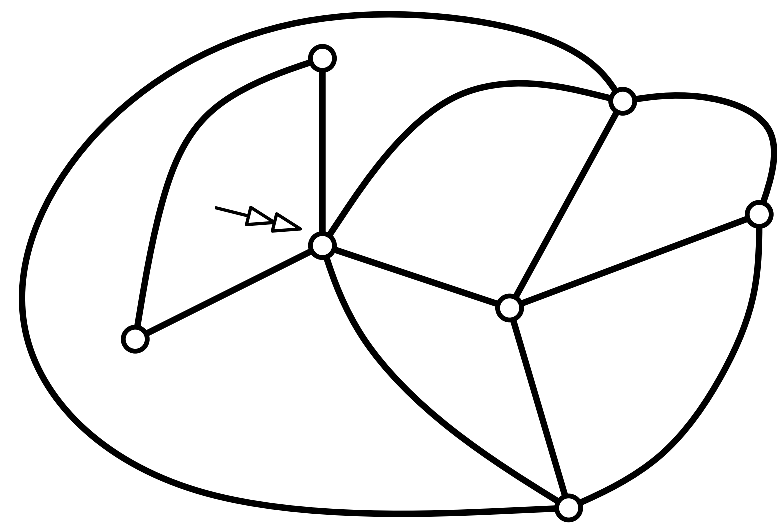

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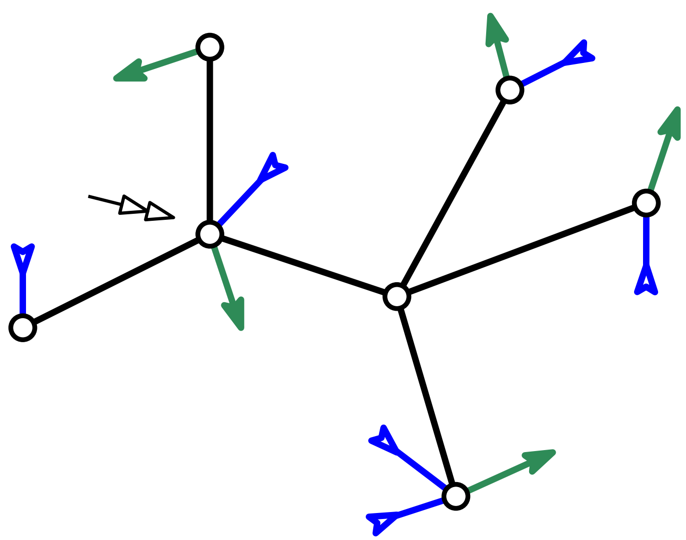

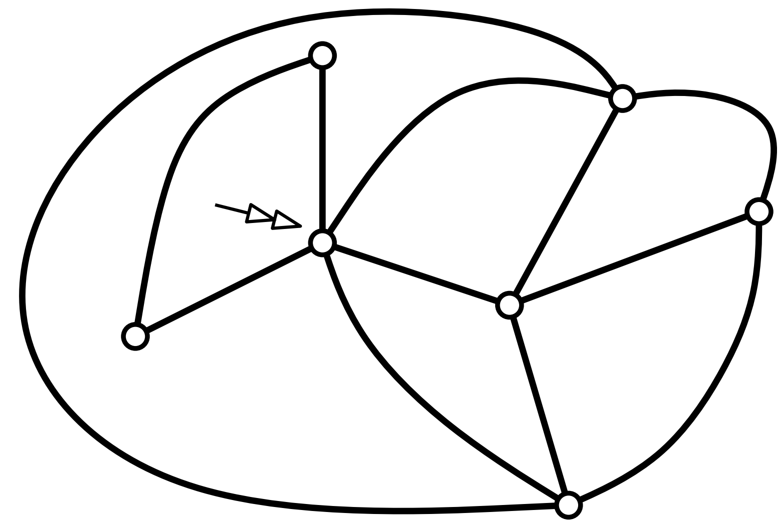

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Can we reverse the construction ?

i.e. can we determine a canonical spanning tree ?
and give a characterization of the possible trees ?

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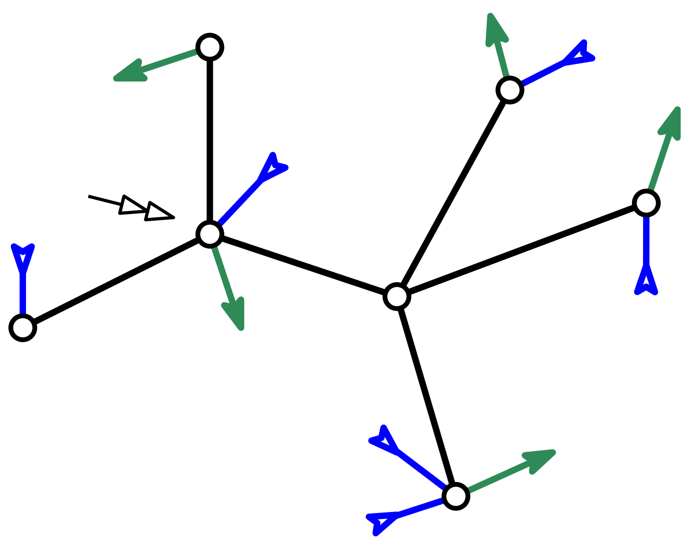

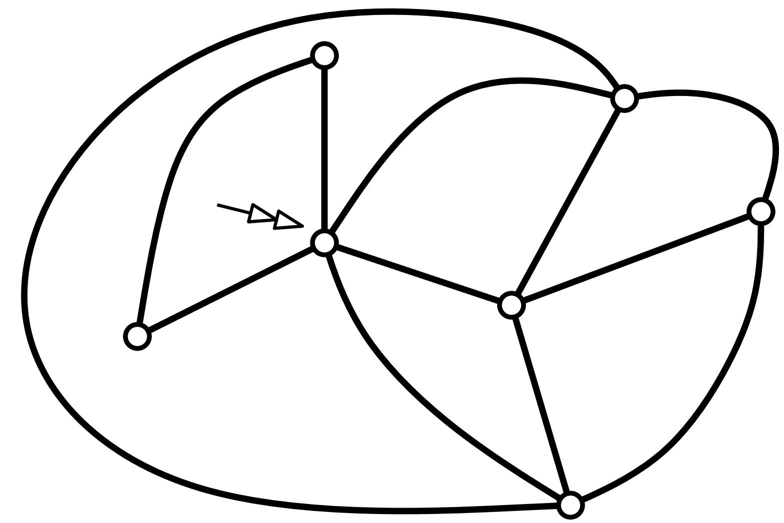

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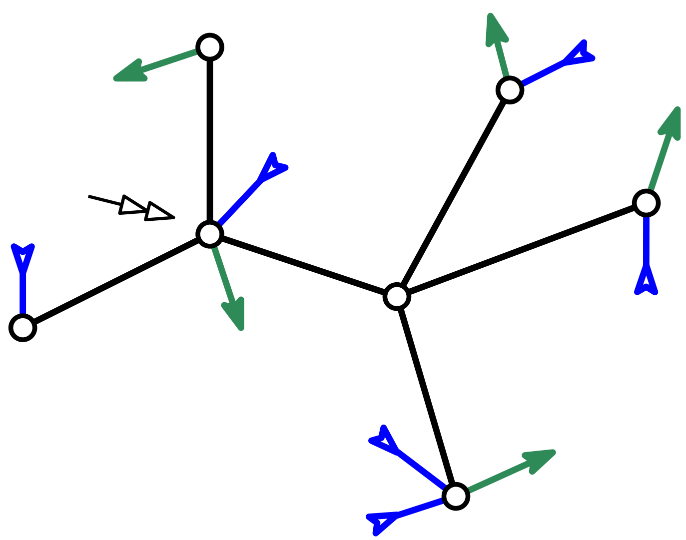

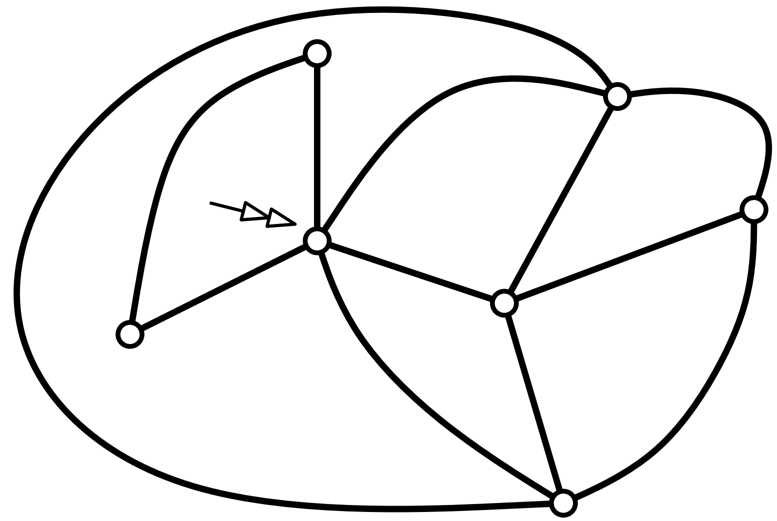

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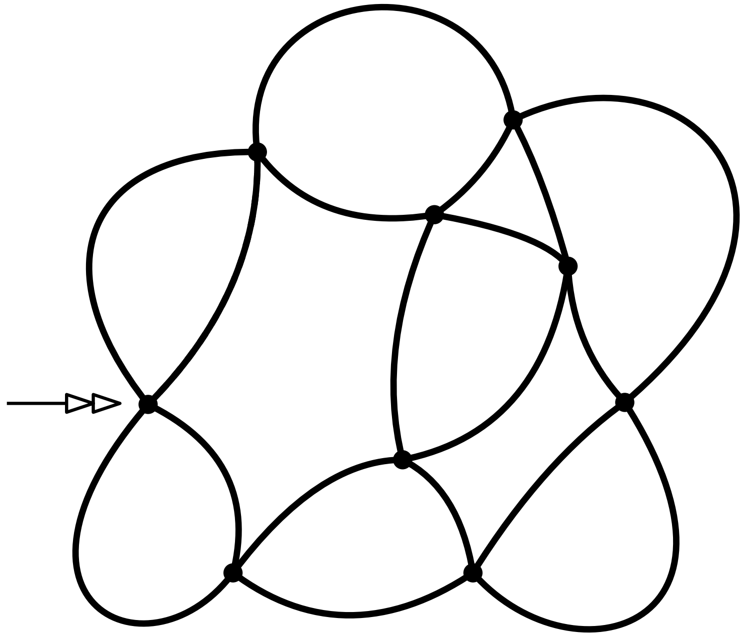
Many works in: [Schaeffer, Bousquet-Mélou, Bouttier, Di Francesco, Guitter, Poulalhon, Fusy, Bernardi, A.]

Schaeffer's blossoming bijection



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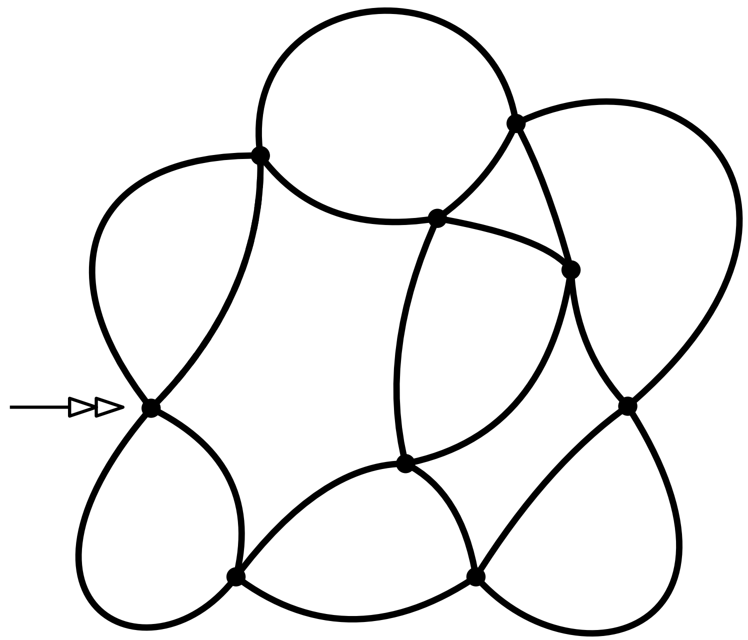

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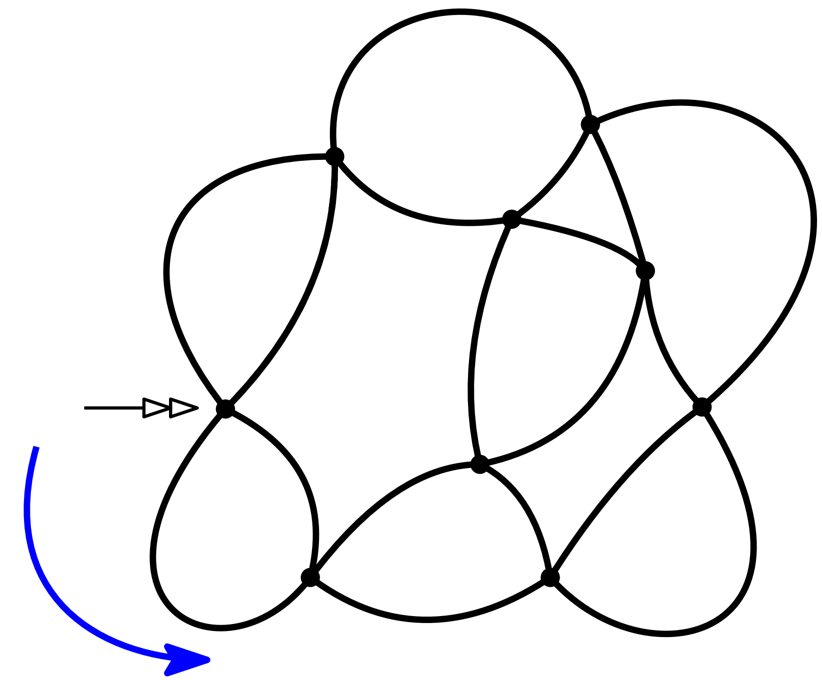

Turning ccw

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Otherwise, continue !

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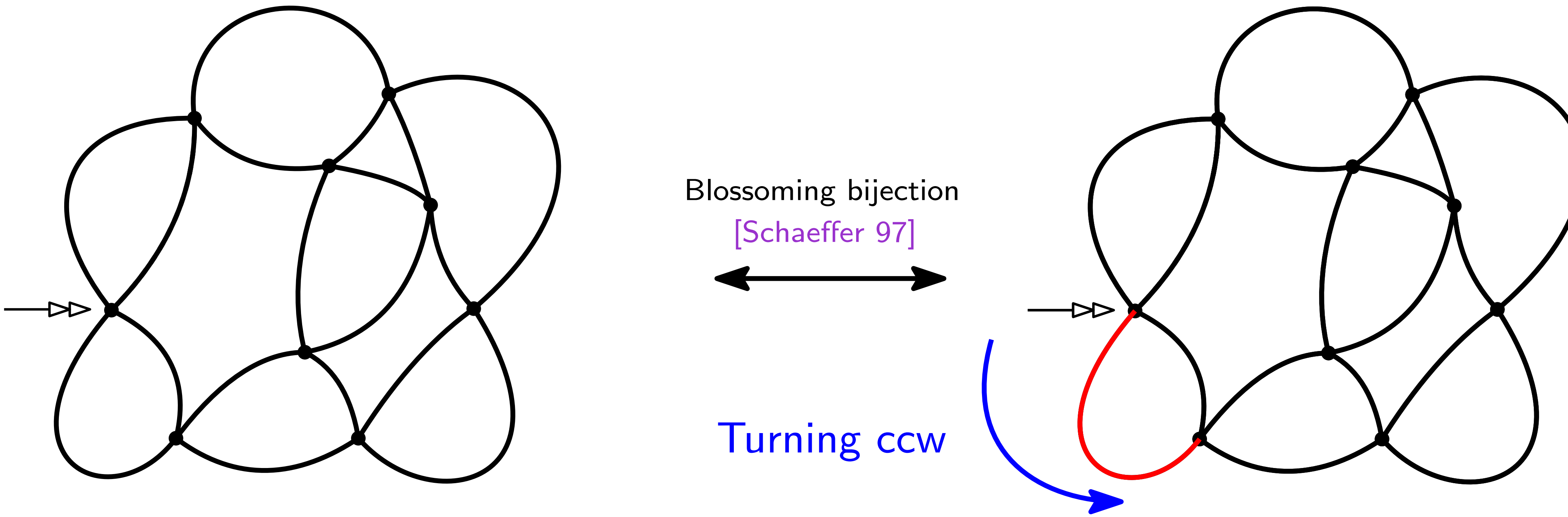

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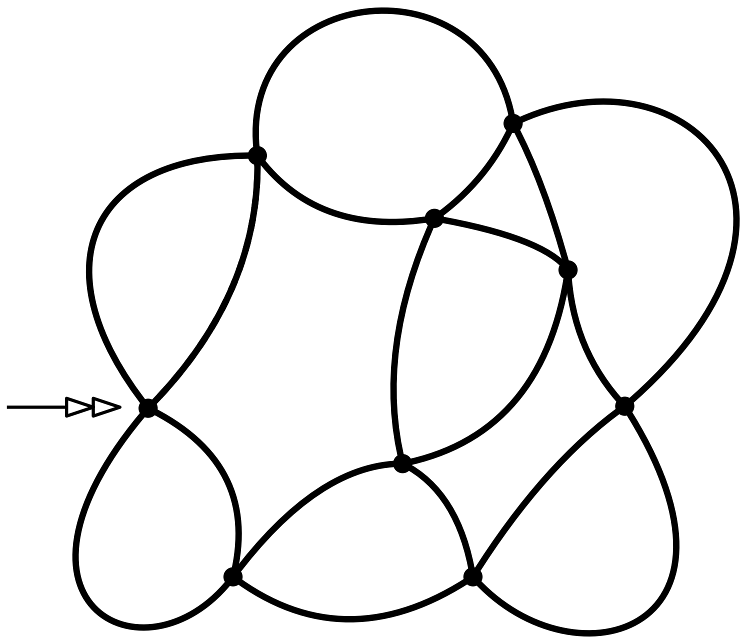

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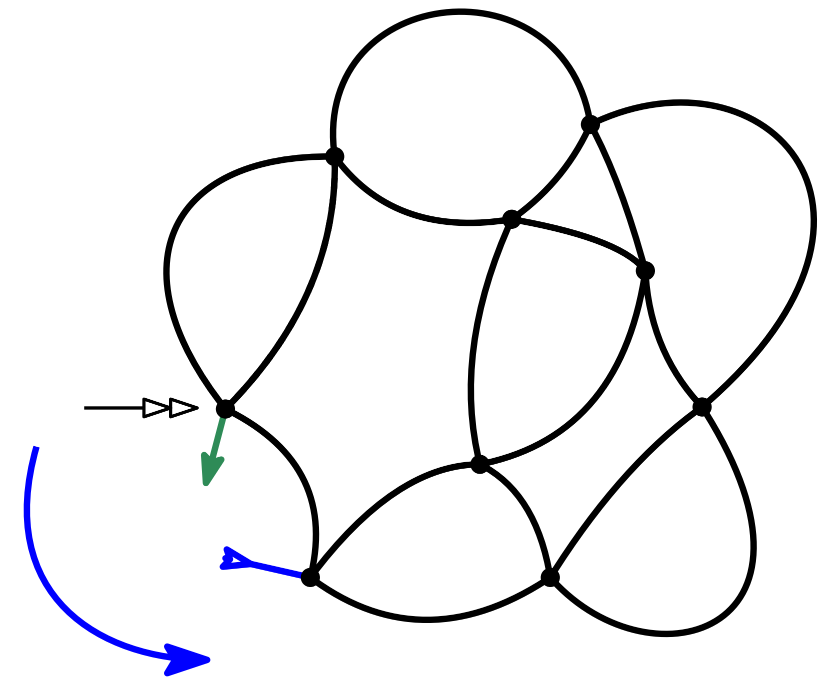

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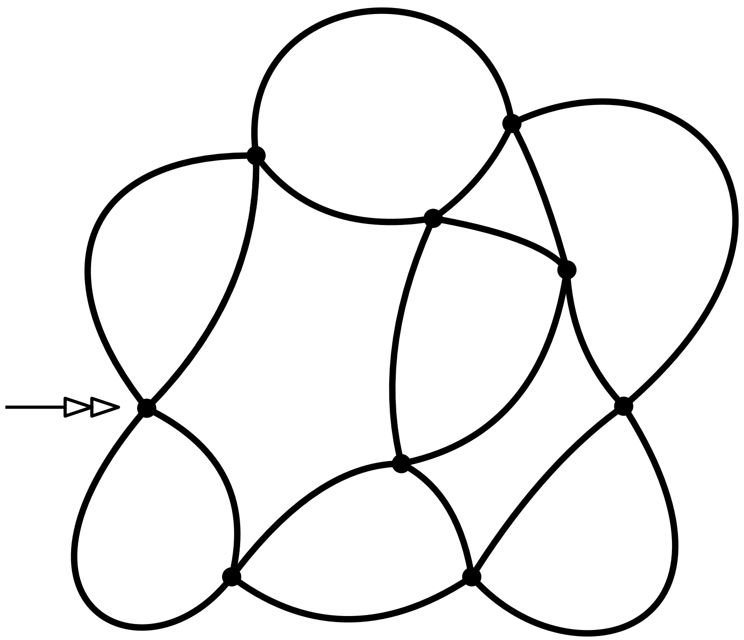

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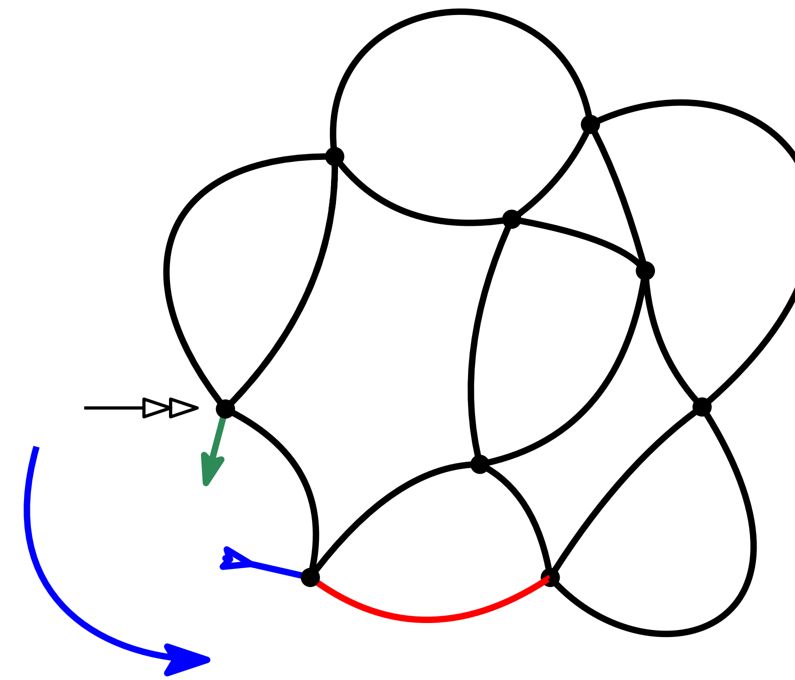

If the encountered edge is not a bridge, delete it !

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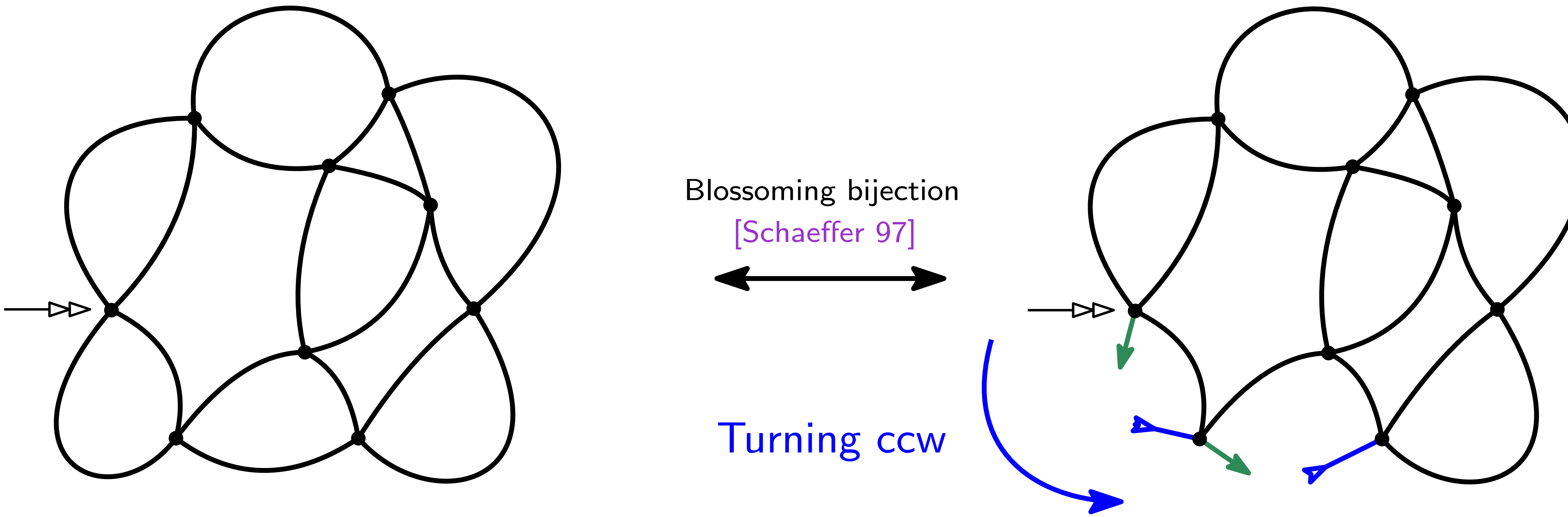
Schaeffer's blossoming bijection



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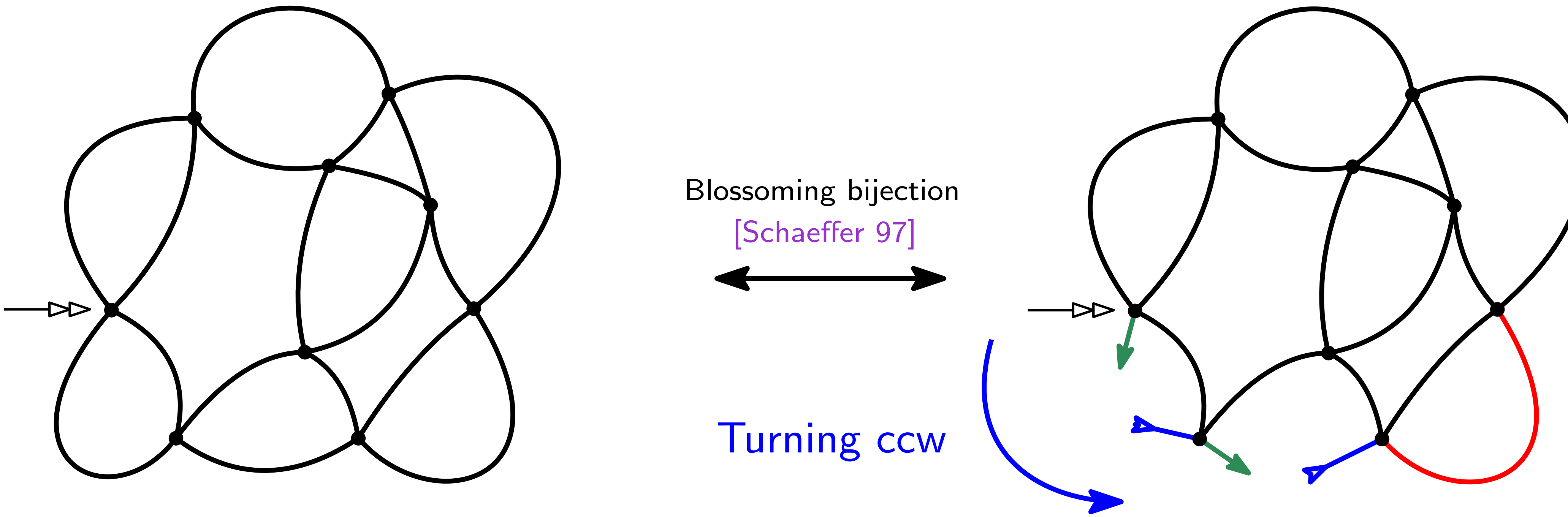

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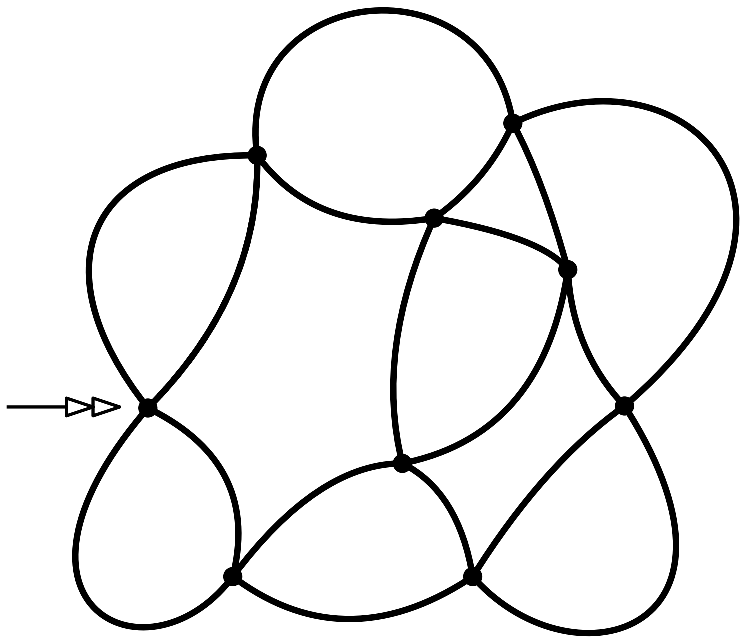

Blossoming bijection

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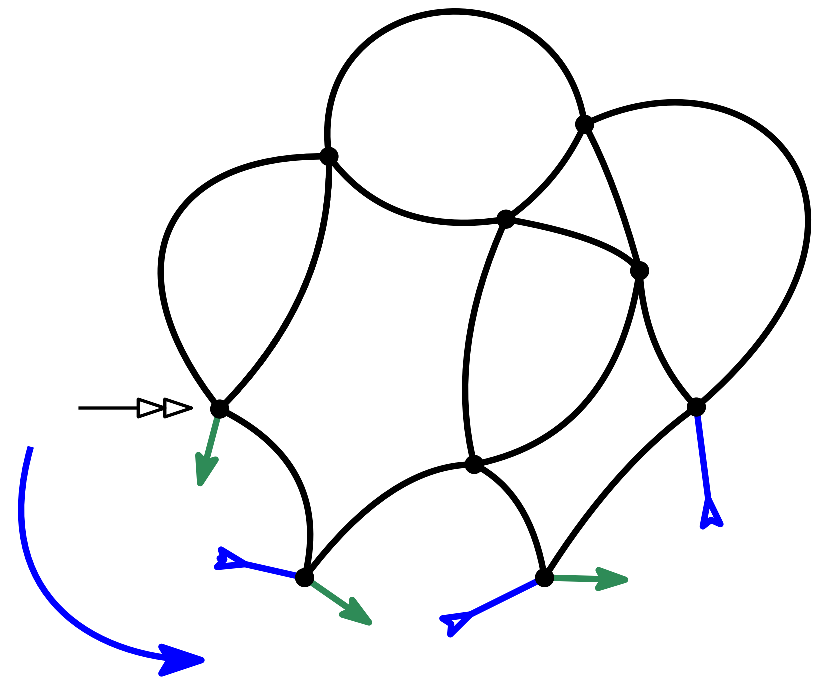

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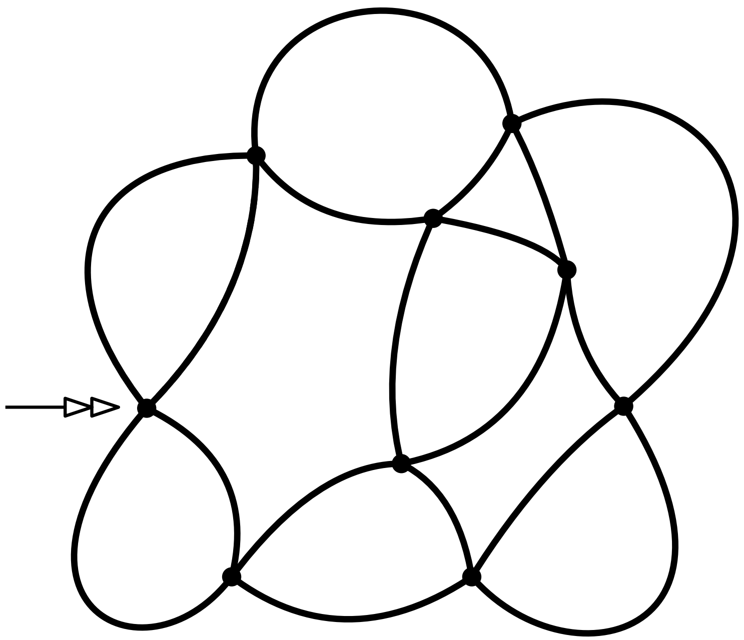

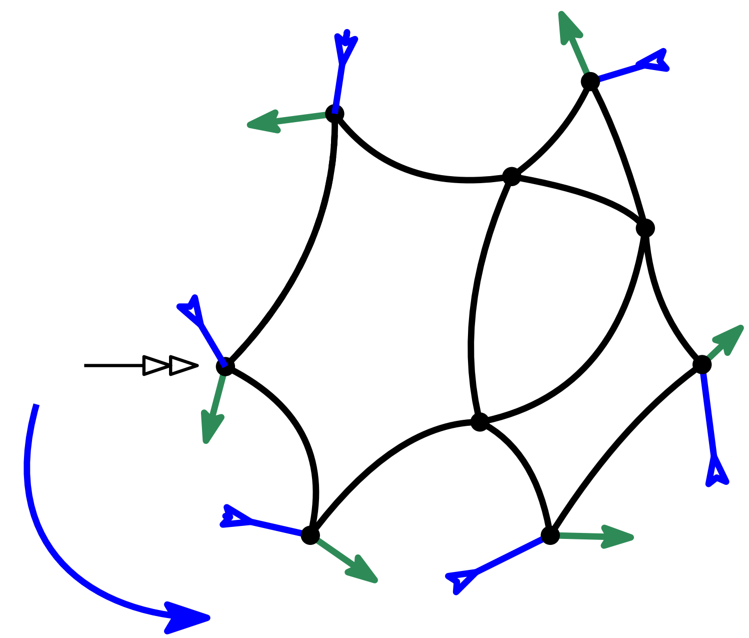

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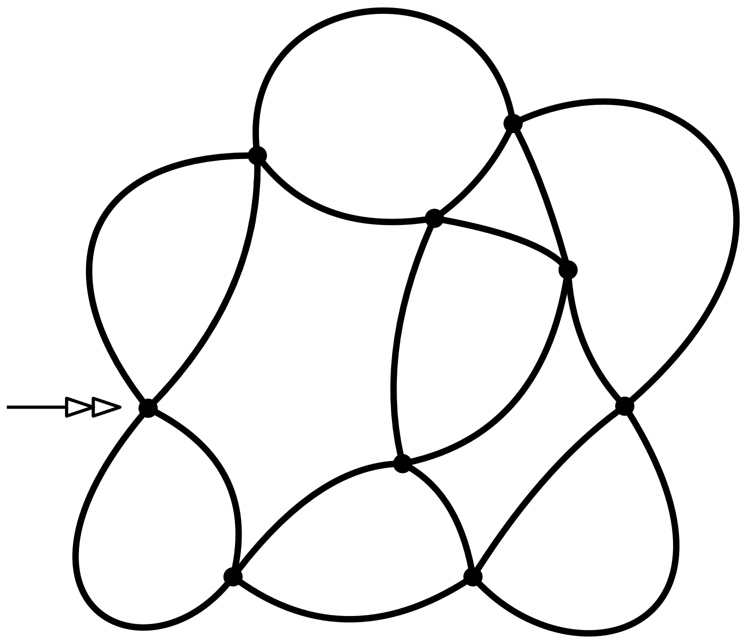

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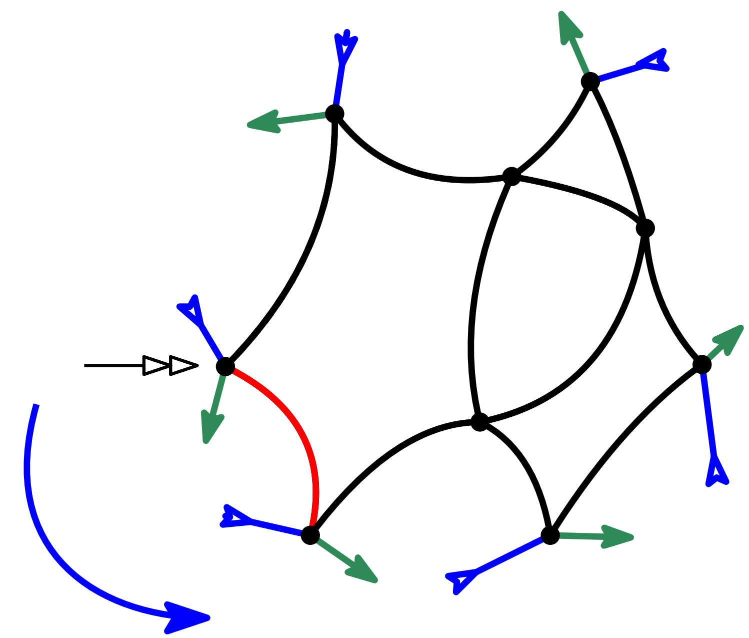

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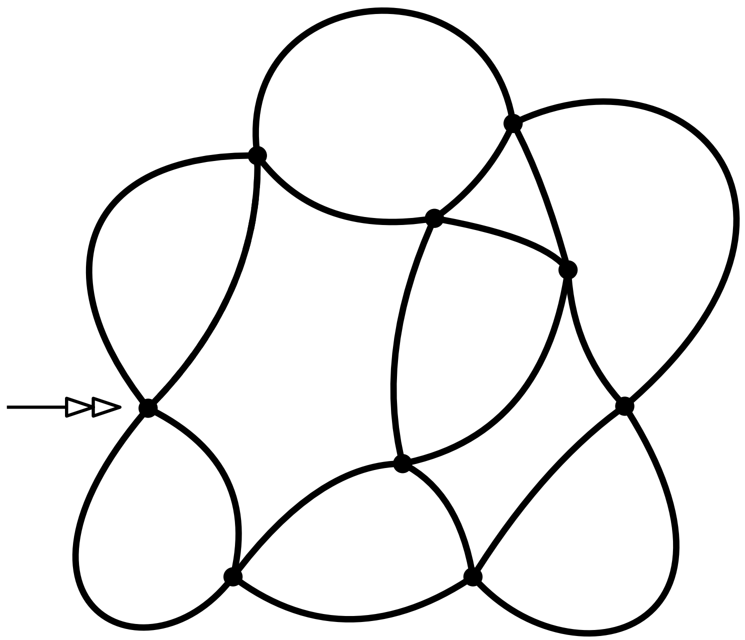

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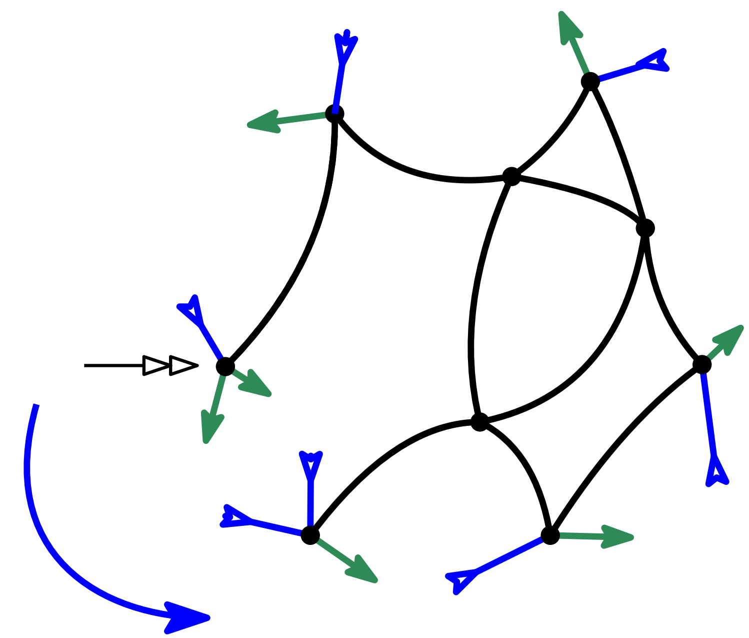

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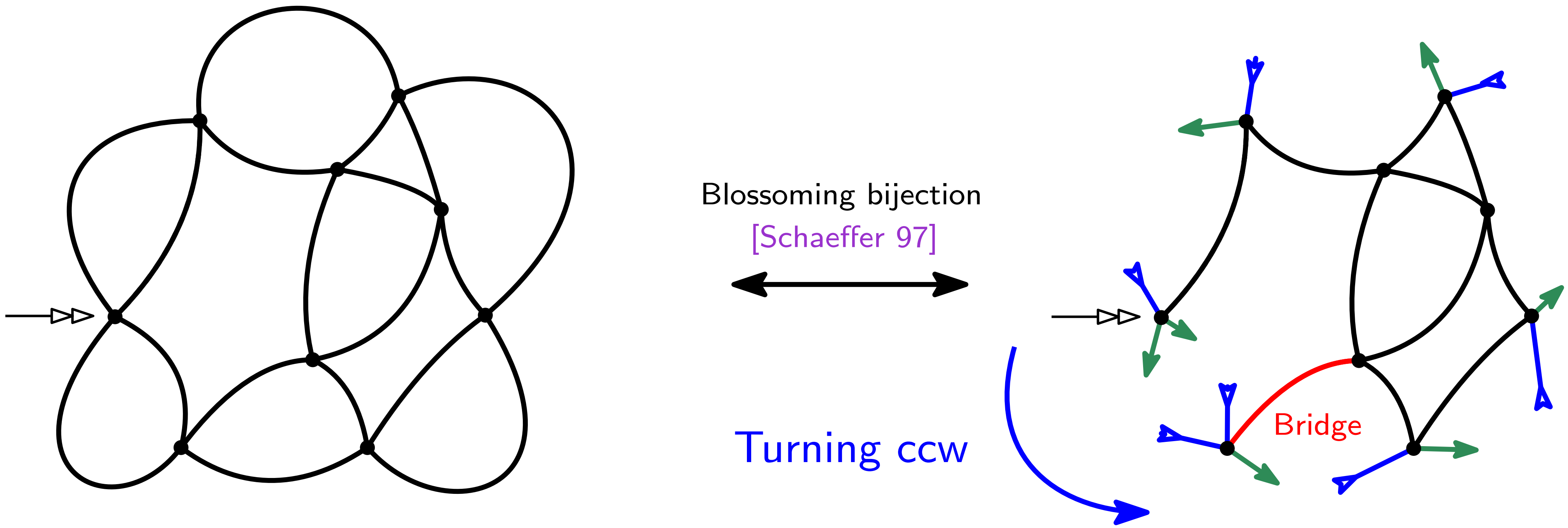

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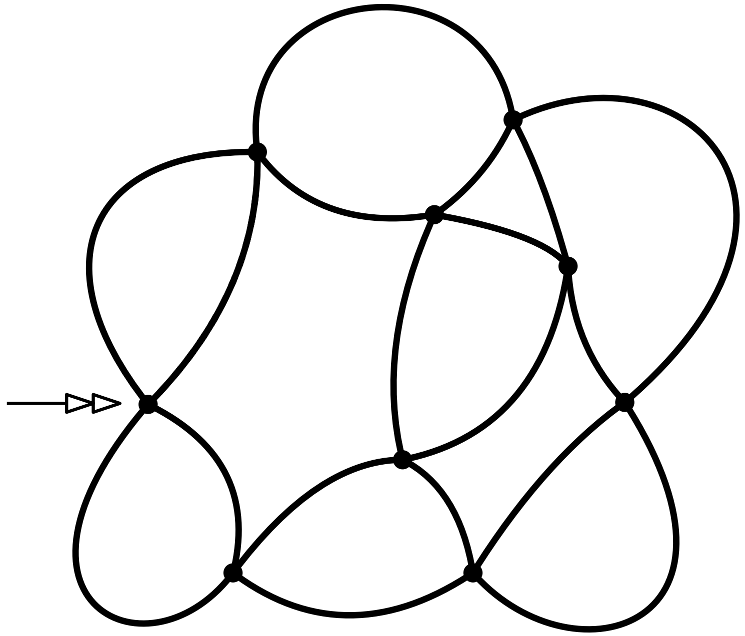

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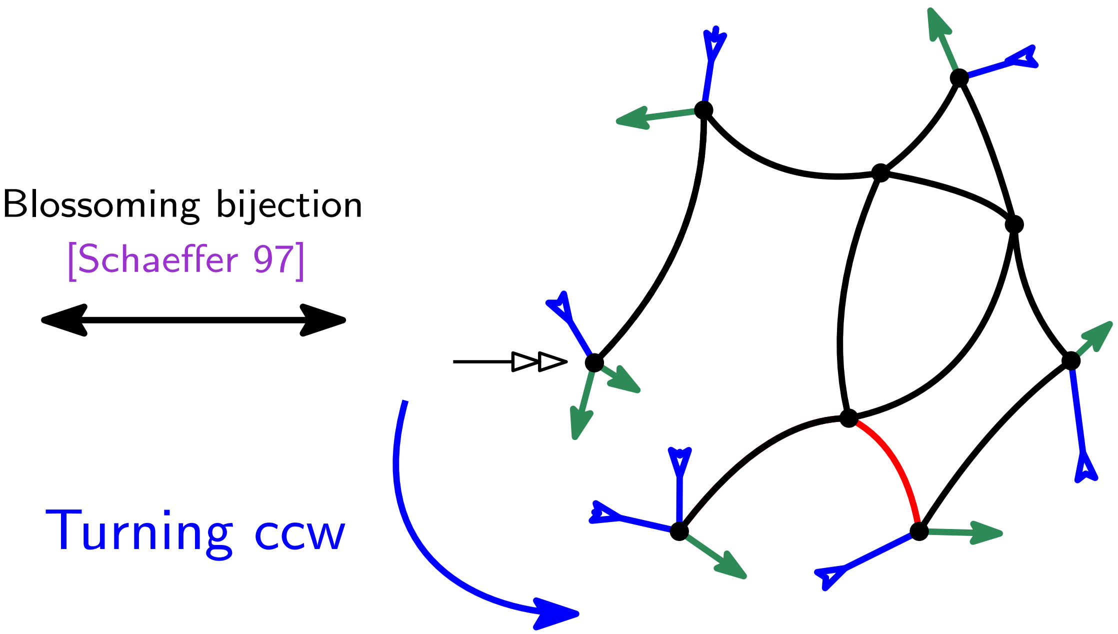

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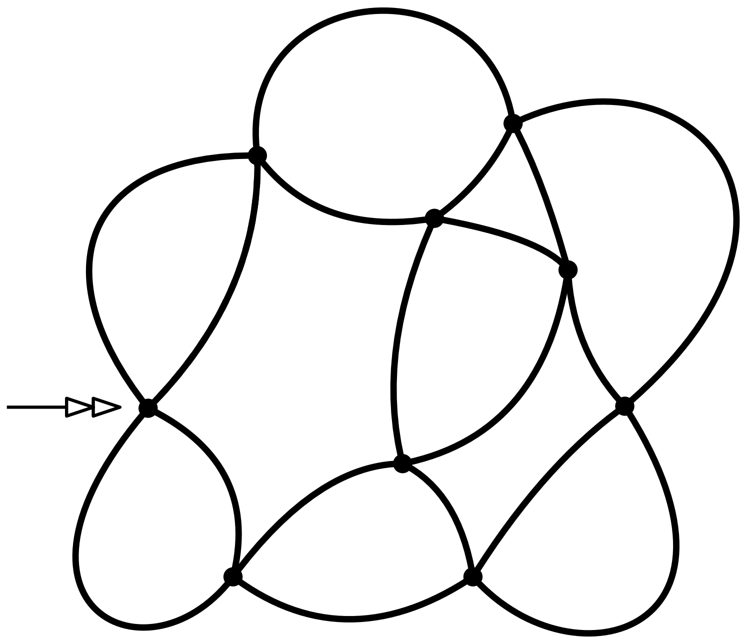

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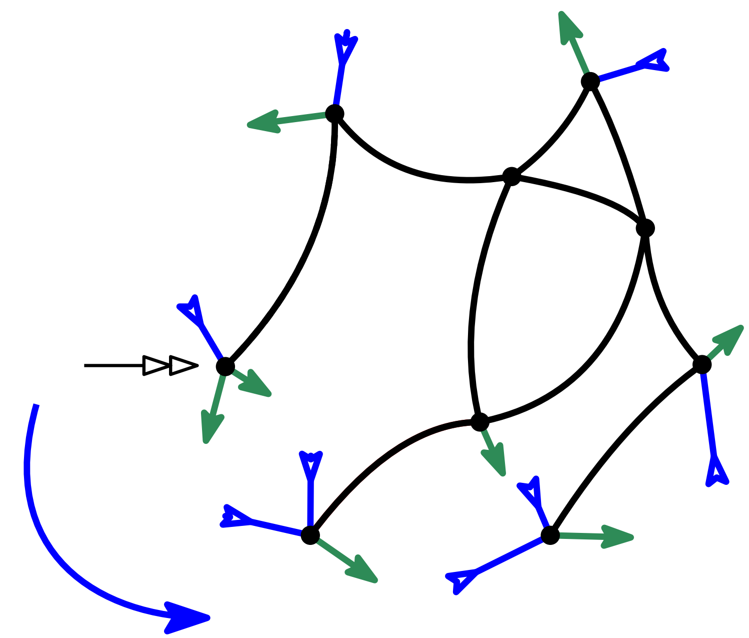

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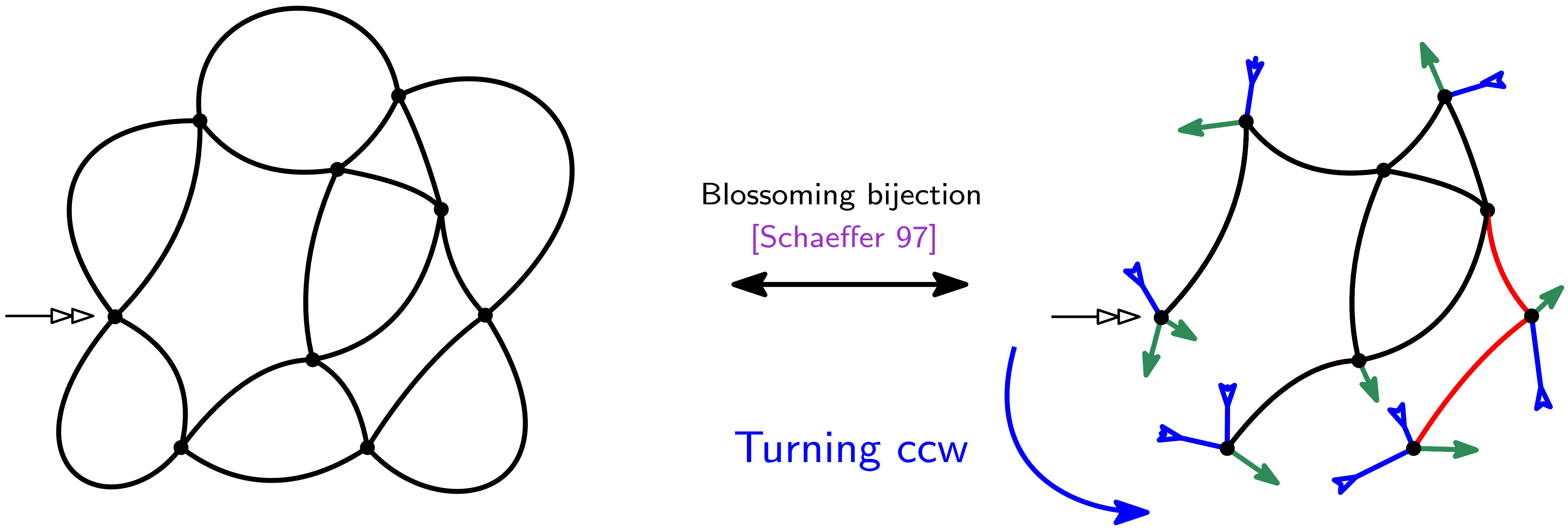

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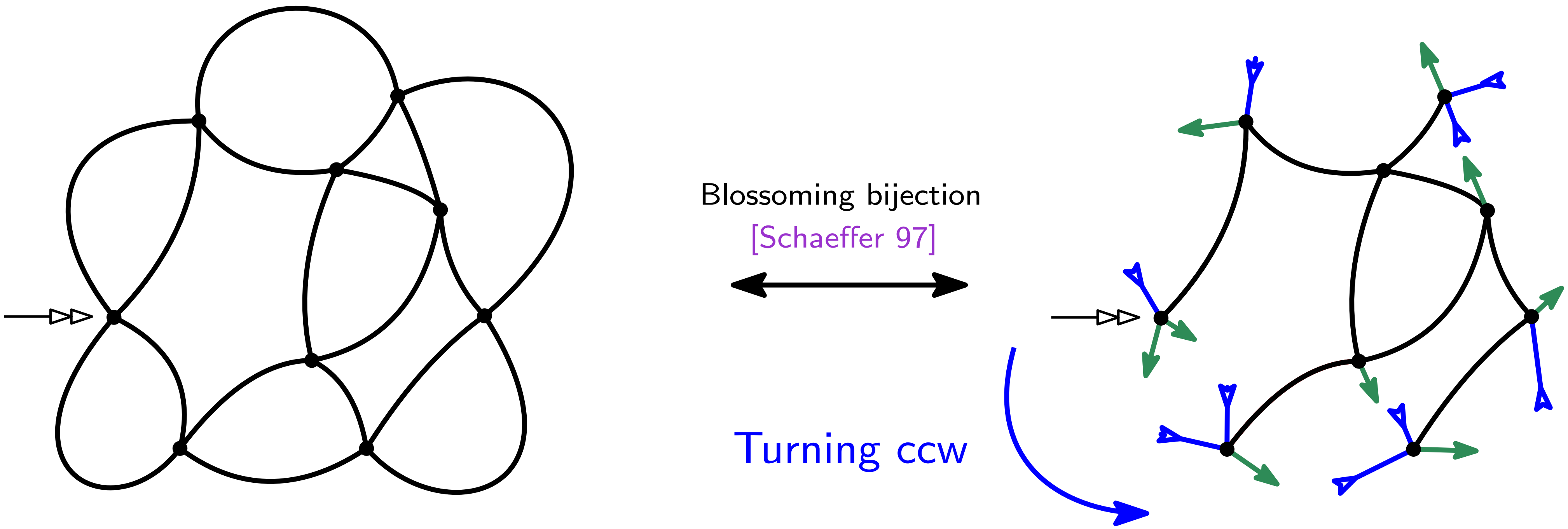

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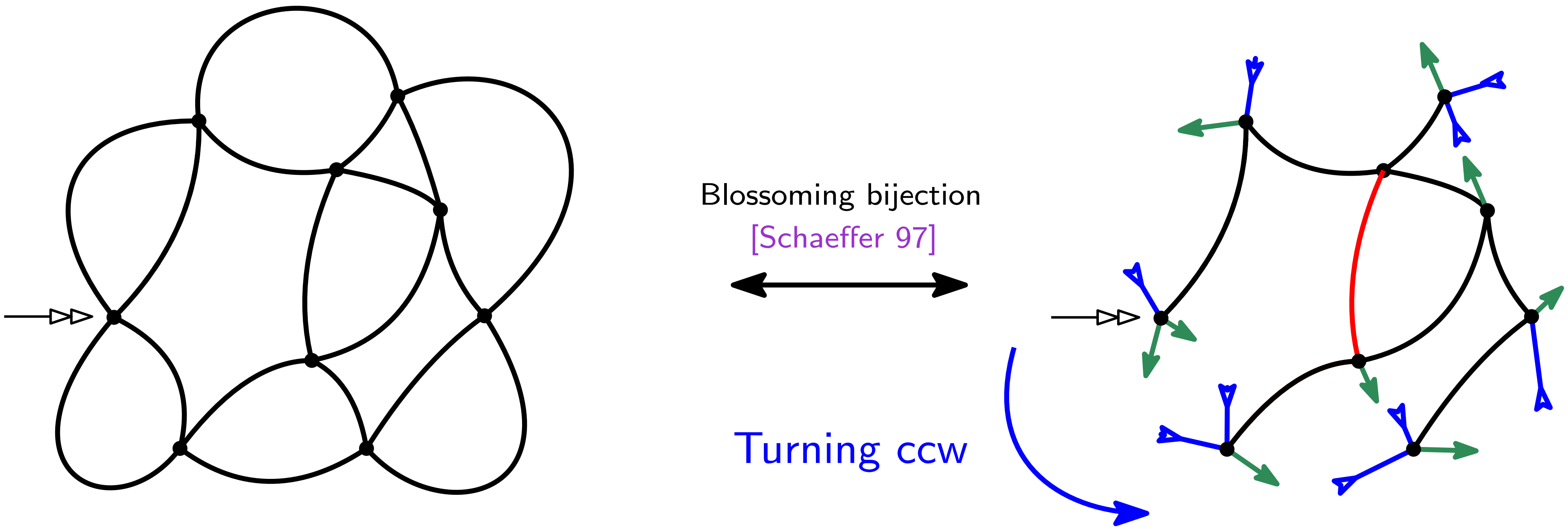

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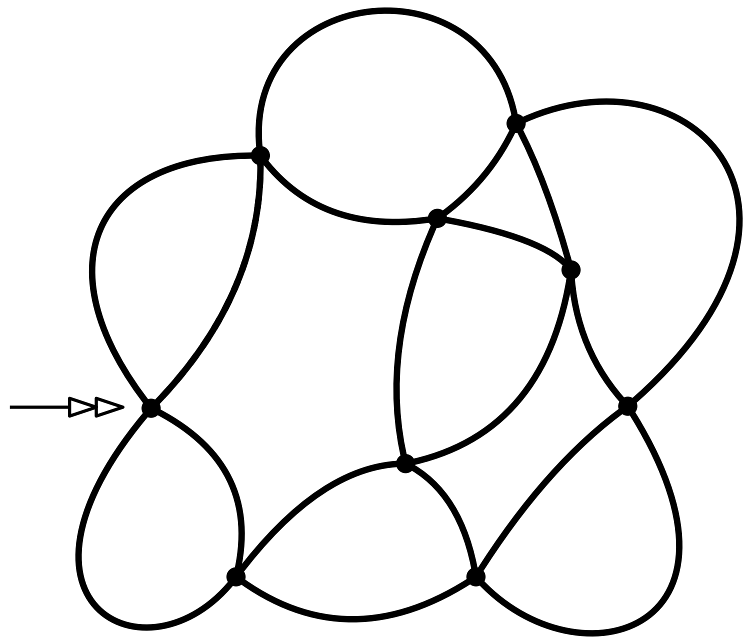

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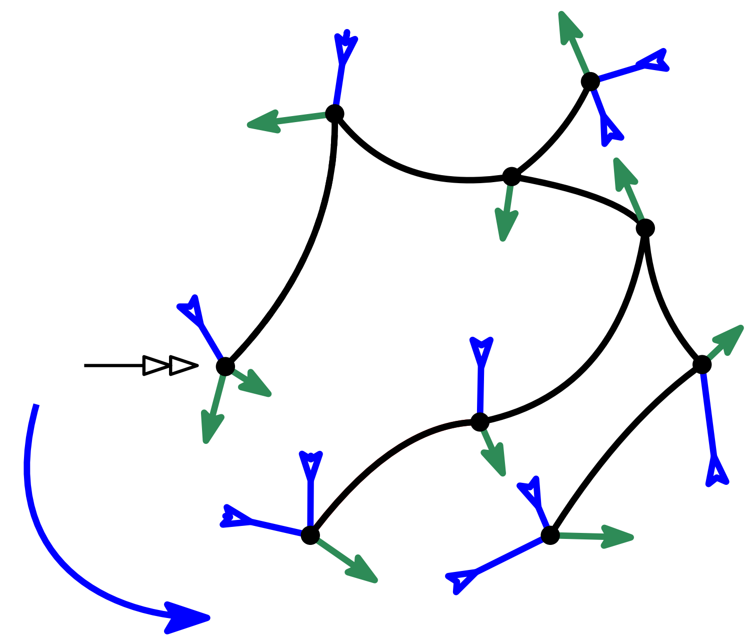

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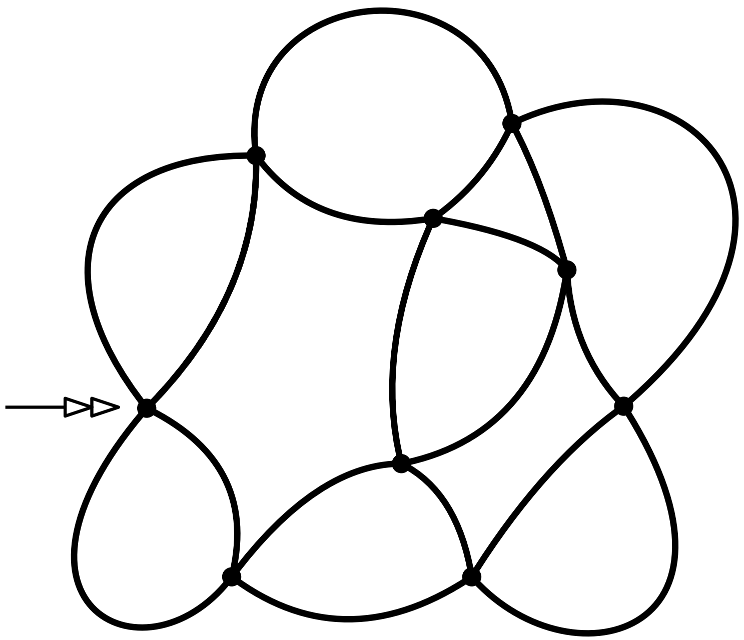

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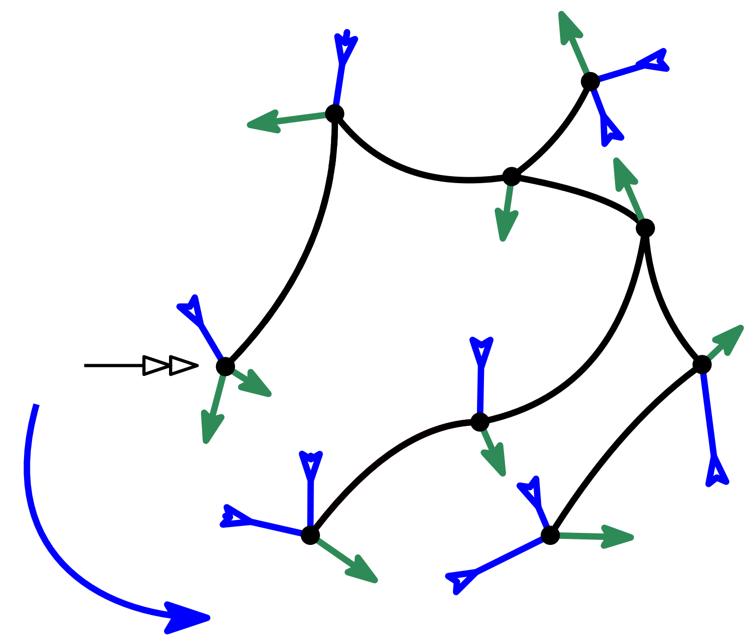

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Theorem: [Schaeffer 97]
This is a bijection between 4-valent maps with n vertices and a family of blossoming 4-valent plane trees with n vertices

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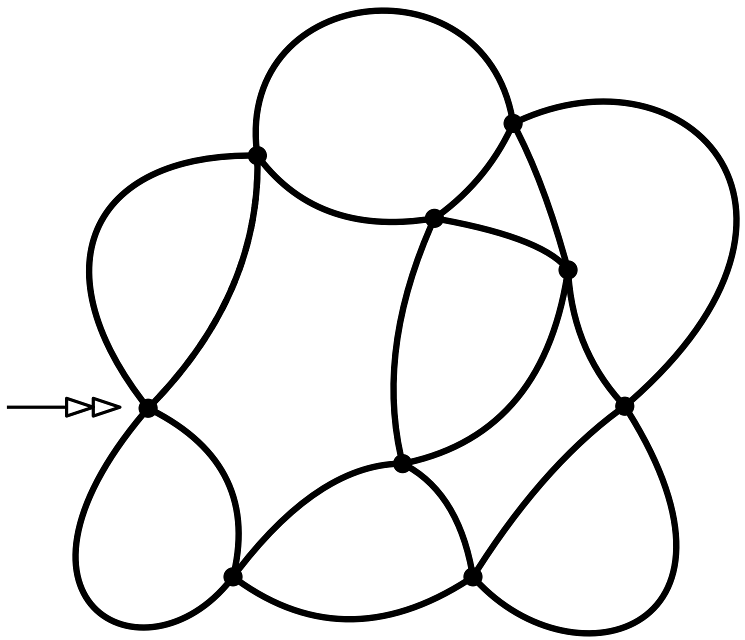

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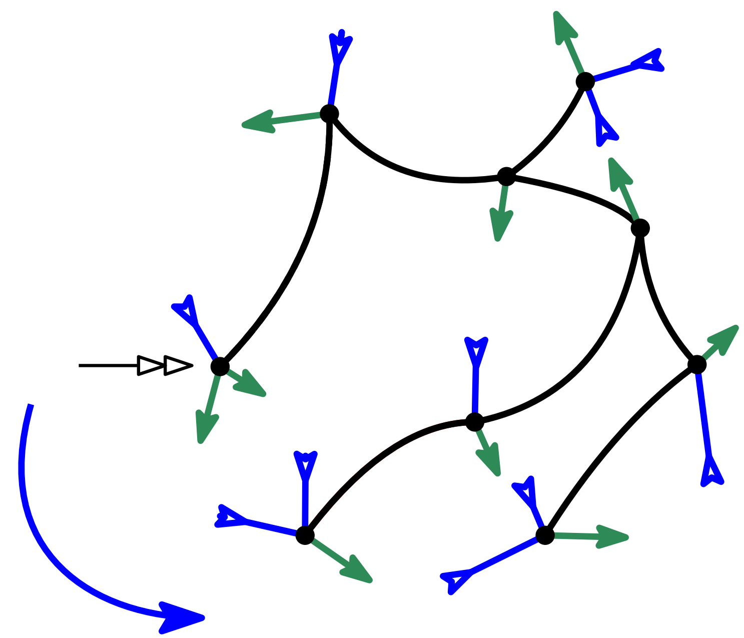

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Question:
Can we generalize it to 4-valent maps in higher genus ?

Rationality scheme in higher genus

Theorem: [Tutte 63], bijective proof in [Schaeffer 97]

$$M(z) = \sum_m z^{|E(m)|}, \text{ where } m \in \left\{\text{planar maps}\right\}.$$

Then:
$$M = \frac{1-4T}{(1-3T)^2} \quad \text{where} \quad T = \text{ unique formal power series defined by } T = z + 3T^2$$

Rationality scheme in higher genus

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Then: $M = \dfrac{1-4T}{(1-3T)^2}$ where $T =$ unique formal power series defined by $T = z + 3T^2$

Theorem: [Bender, Canfield 91], **first bijective proof** in [Lepoutre 19]

For any $g \geq 1$, let $M_g(z) = \sum_m z^{|E(m)|}$, where $m \in \Big\{\text{maps of genus } g\Big\}$.

Then M_g is a rational function of T.

Rationality scheme in higher genus

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Remark:
Result not available with the "mobile-type" bijection of [Chapuy – Marcus – Schaeffer]

Blossoming bijections in higher genus

Theorem: [Tutte 63], bijective proof in [Schaeffer 97]

$$M(z_\bullet, z_\circ) = \sum_m z_\bullet^{|V(m)|} z_\circ^{|F(m)|}, \text{ where } m \in \{\text{planar maps}\}.$$

Then $M = T_\circ T_\bullet (1 - 2T_\circ - 2T_\bullet)$ where $\begin{cases} T_\bullet &= z_\bullet + T_\bullet^2 + 2T_\circ T_\bullet \\ T_\circ &= z_\circ + T_\circ^2 + 2T_\bullet T_\circ \end{cases}$

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Euler's formula: $$|V(m)| + |F(m)| = 2 + |E(m)| - 2g(m)$$

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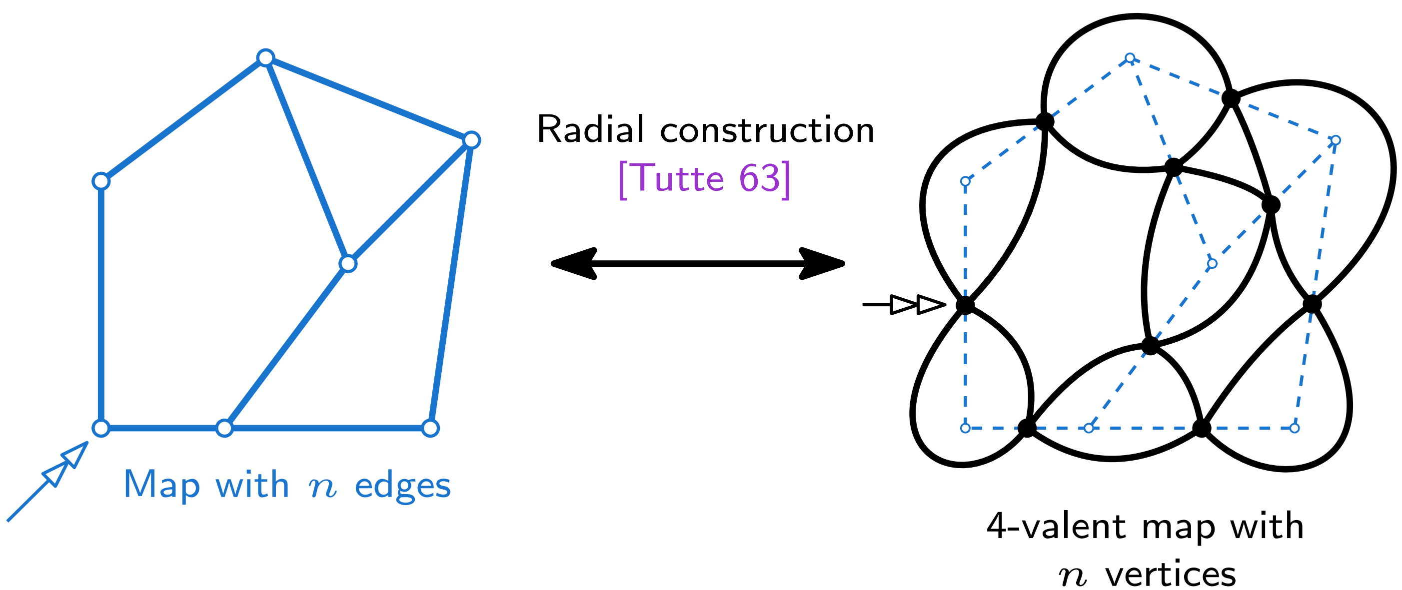

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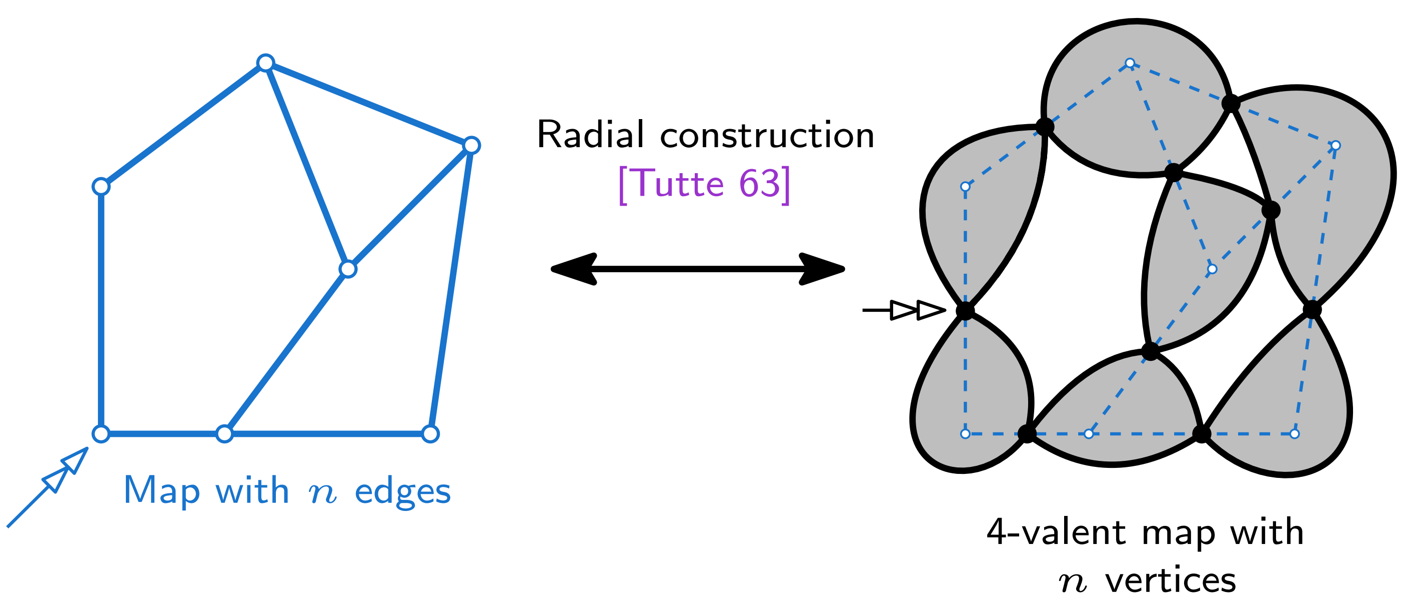

Already for planar maps, this result is not accessible with mobile-type bijections.

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For any $g \geq 1$, let

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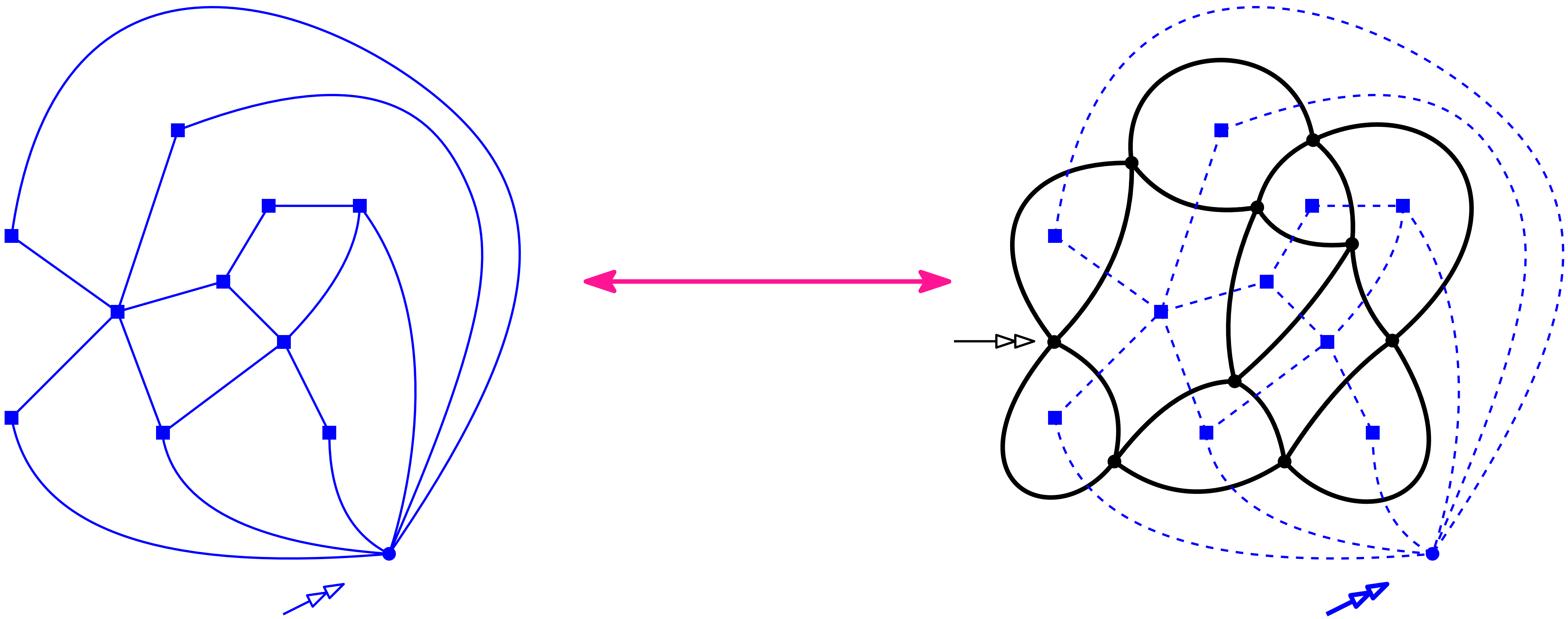
Reformulation of Schaeffer's blossoming bijection

Aparte: dual of a **tree-decorated map** (= map endowed with a spanning tree).



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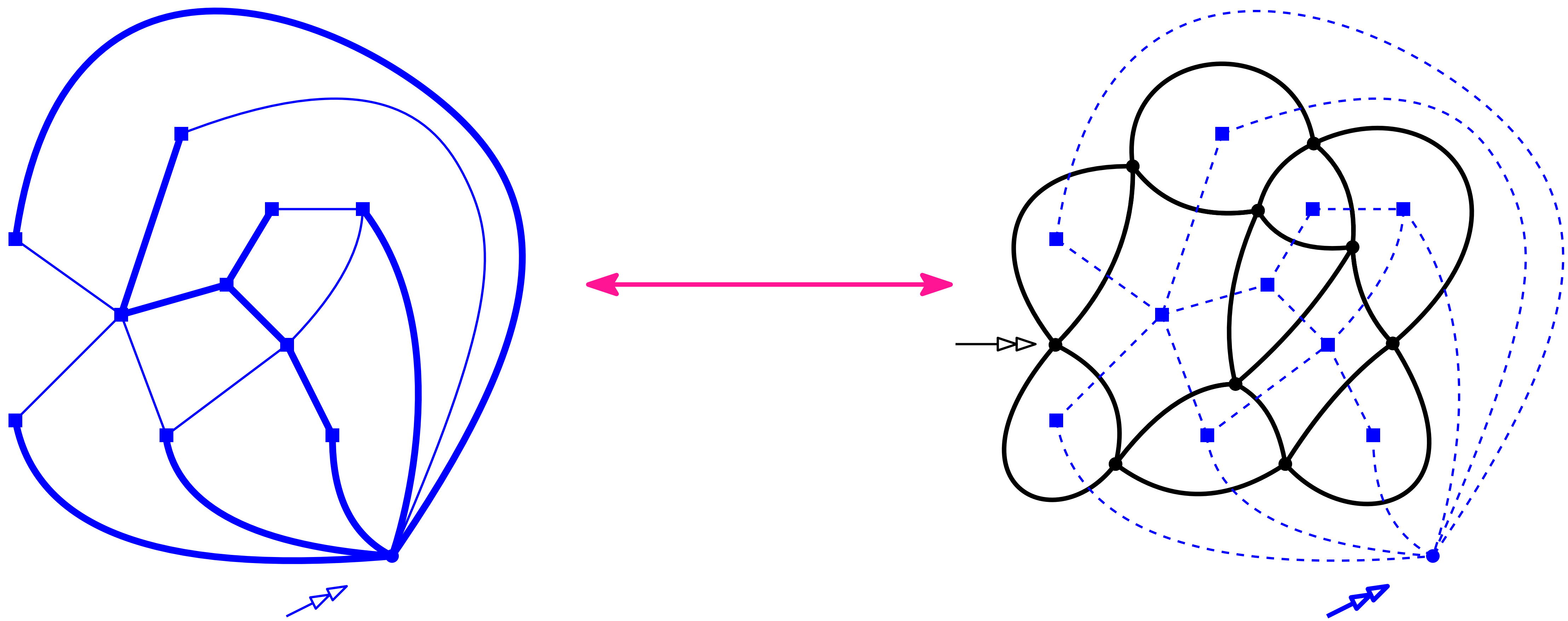

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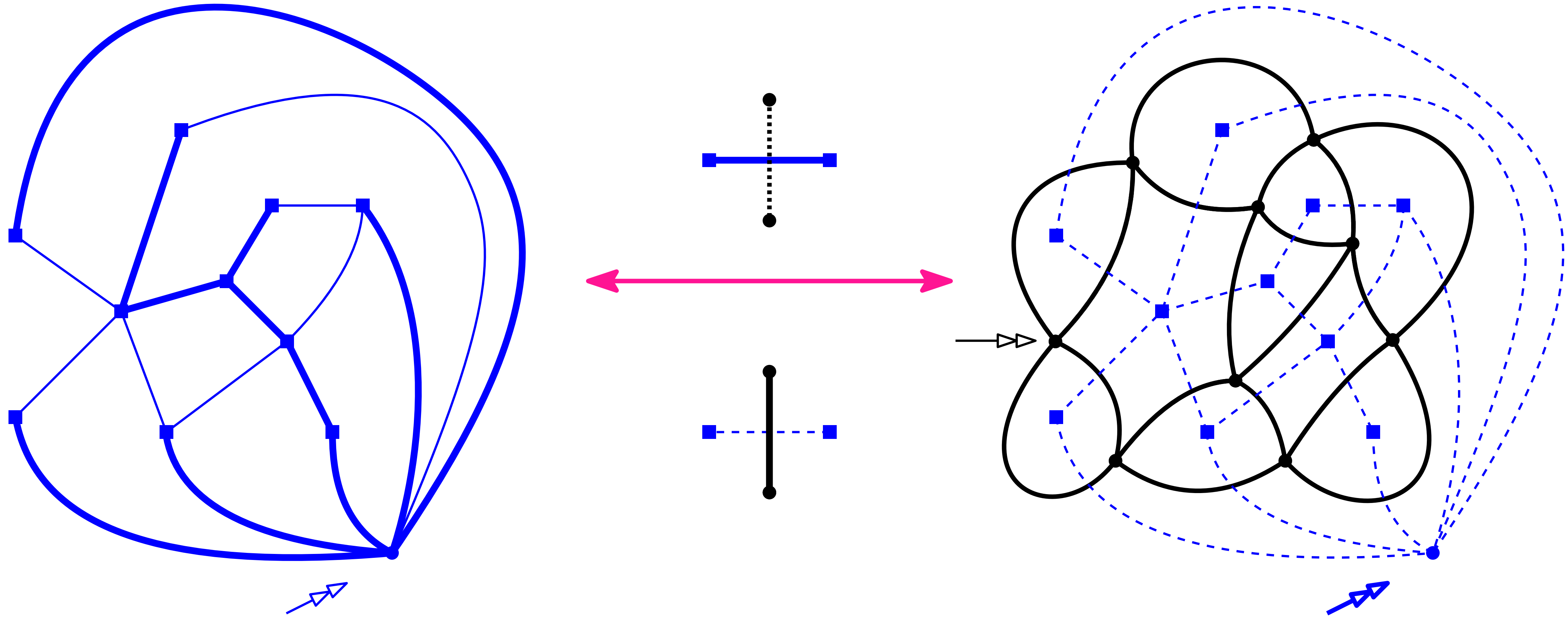

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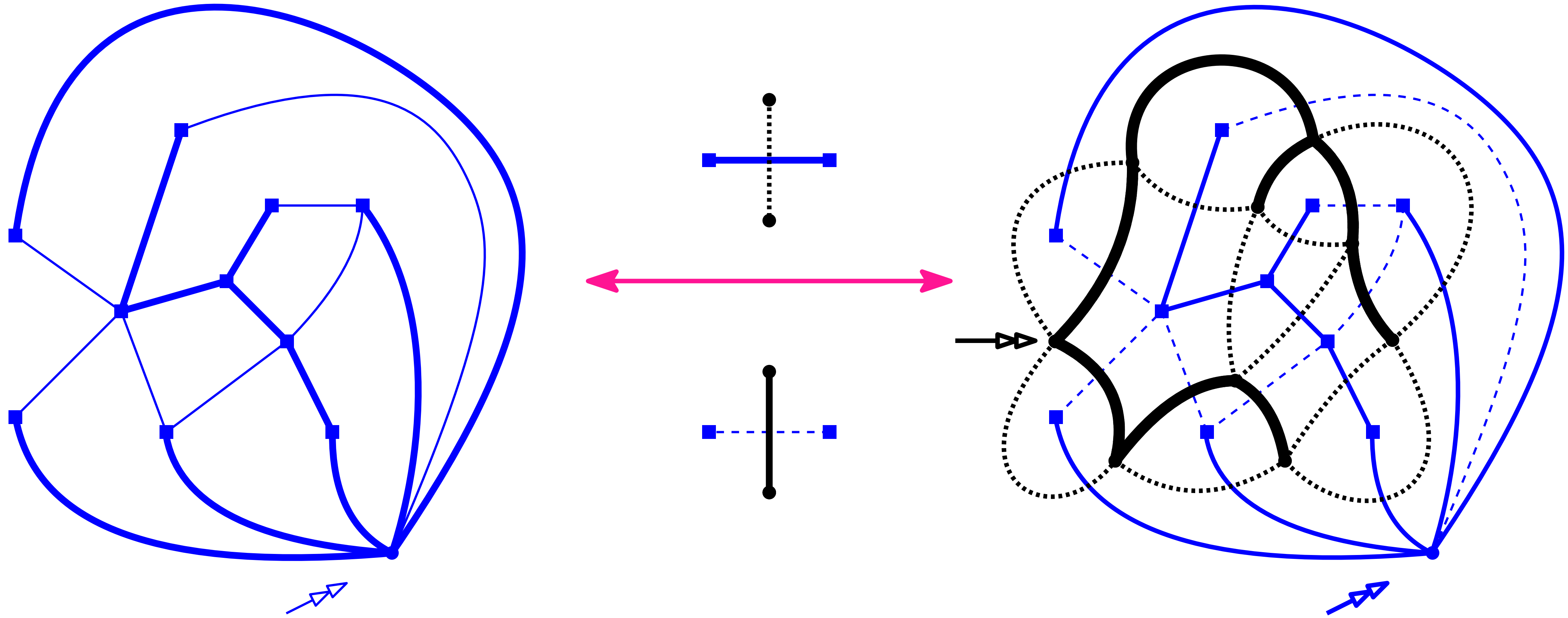

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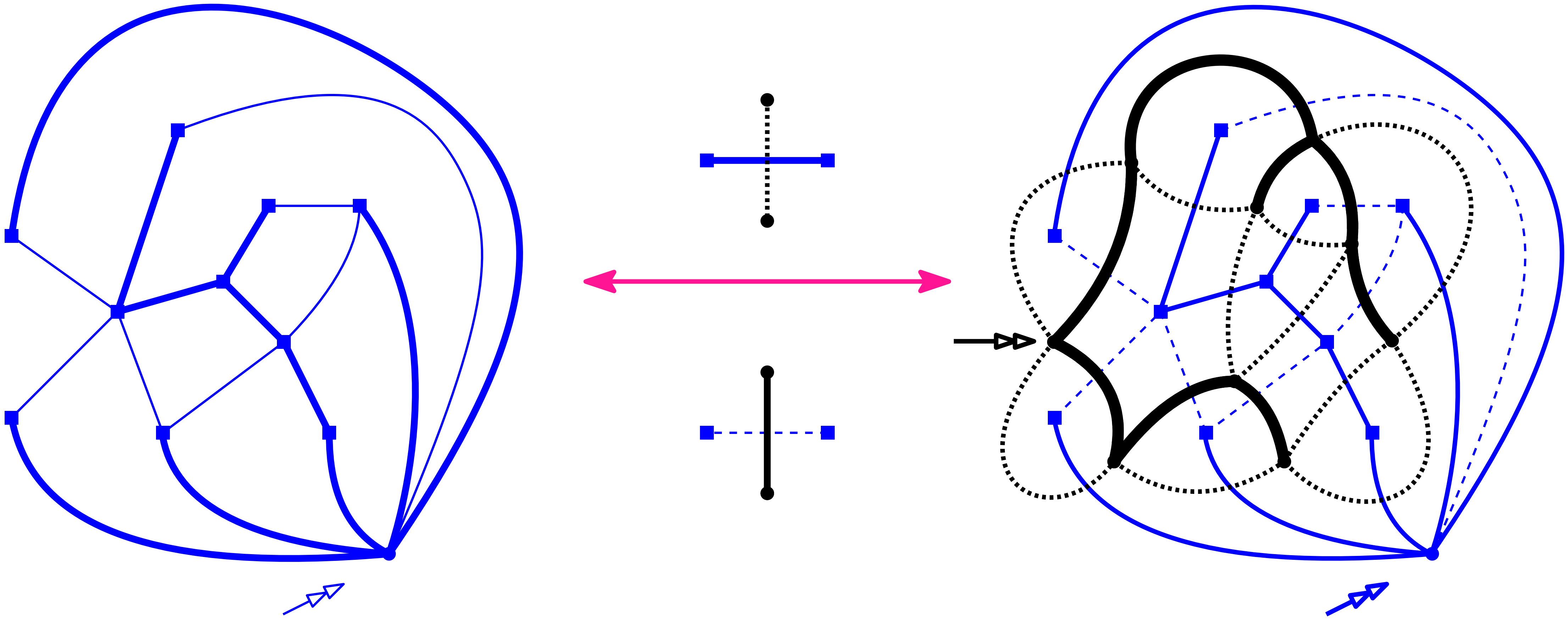
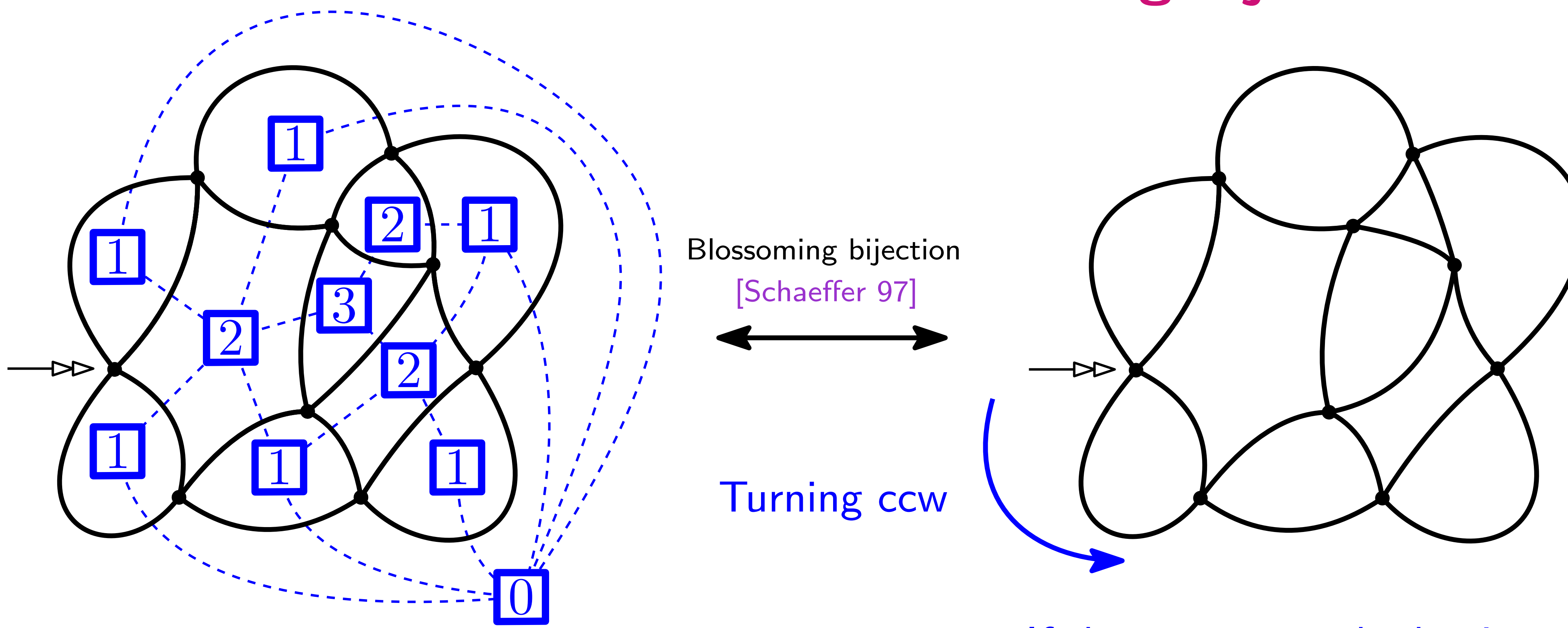


Prop (folklore): This is a **bijection** for the set of tree-decorated maps.

Abuse of language : "dual of a tree" = corresponding spanning tree of the dual map

Reformulation of Schaeffer's blossoming bijection



Label the faces by their distance to the root face in the dual graph

If the encountered edge is not a bridge, delete it !

Otherwise, continue !

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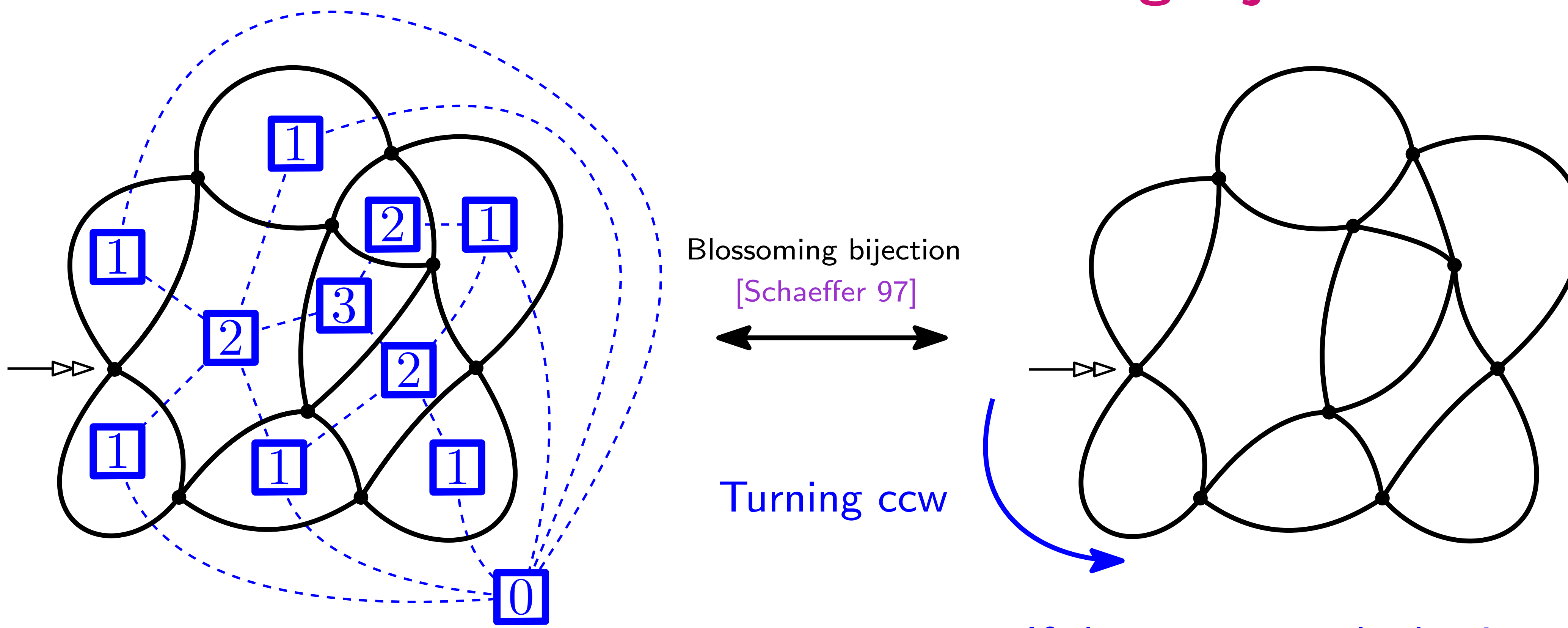

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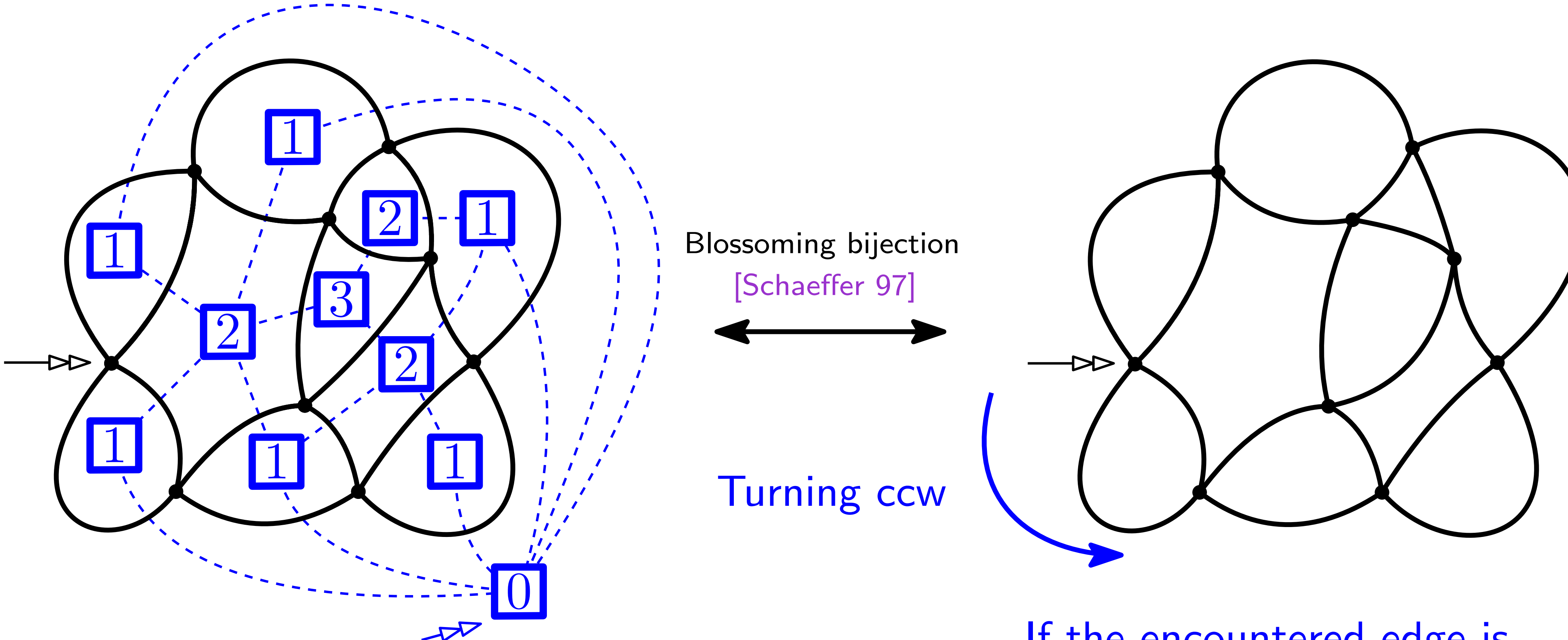

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Consider the "leftmost" breadth-first tree

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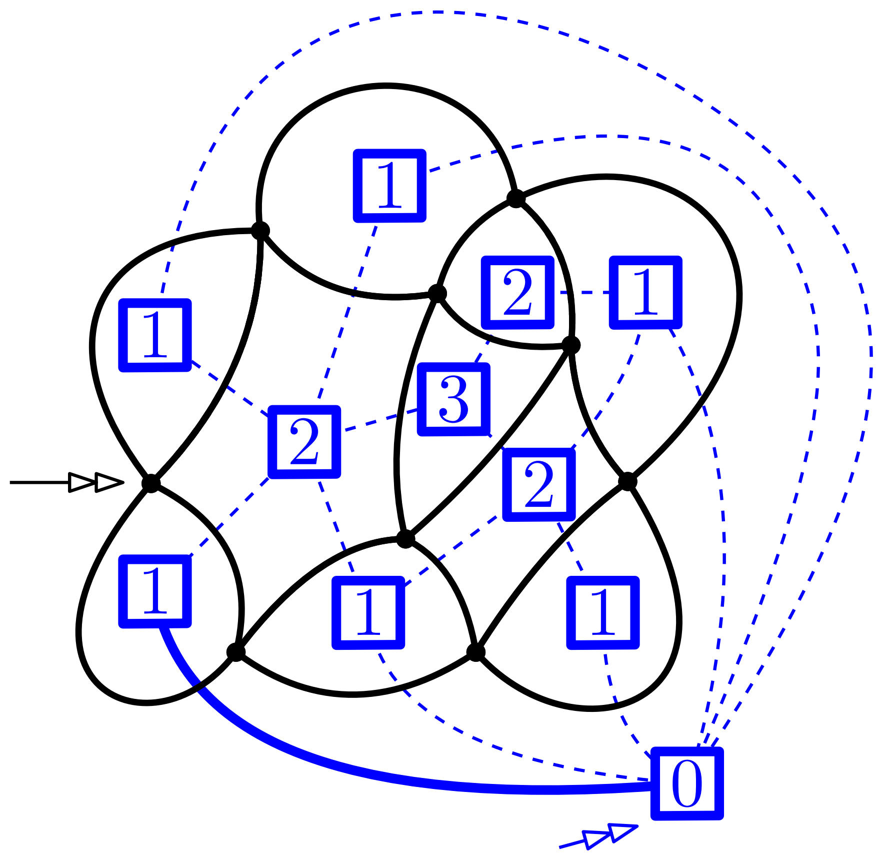

Blossoming bijection
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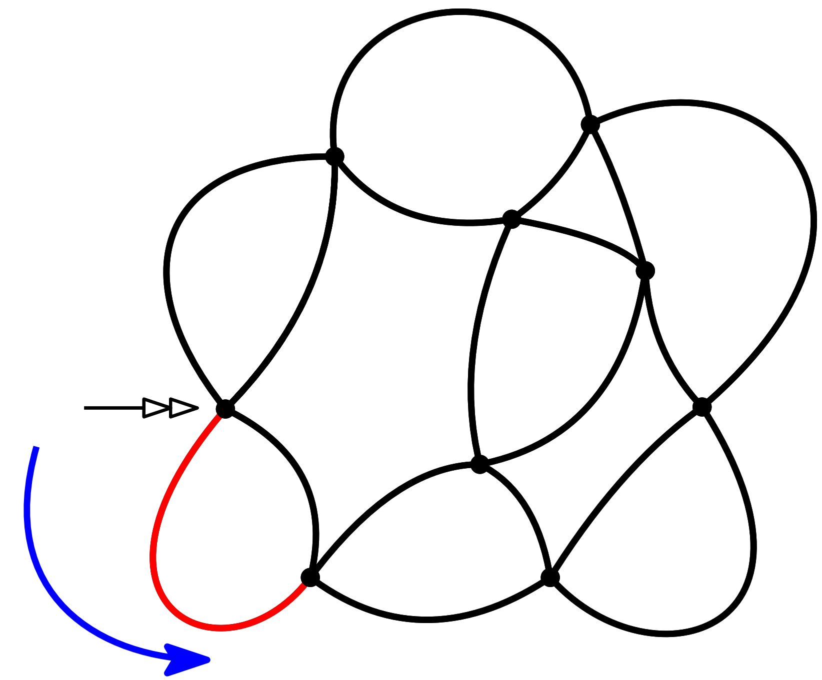

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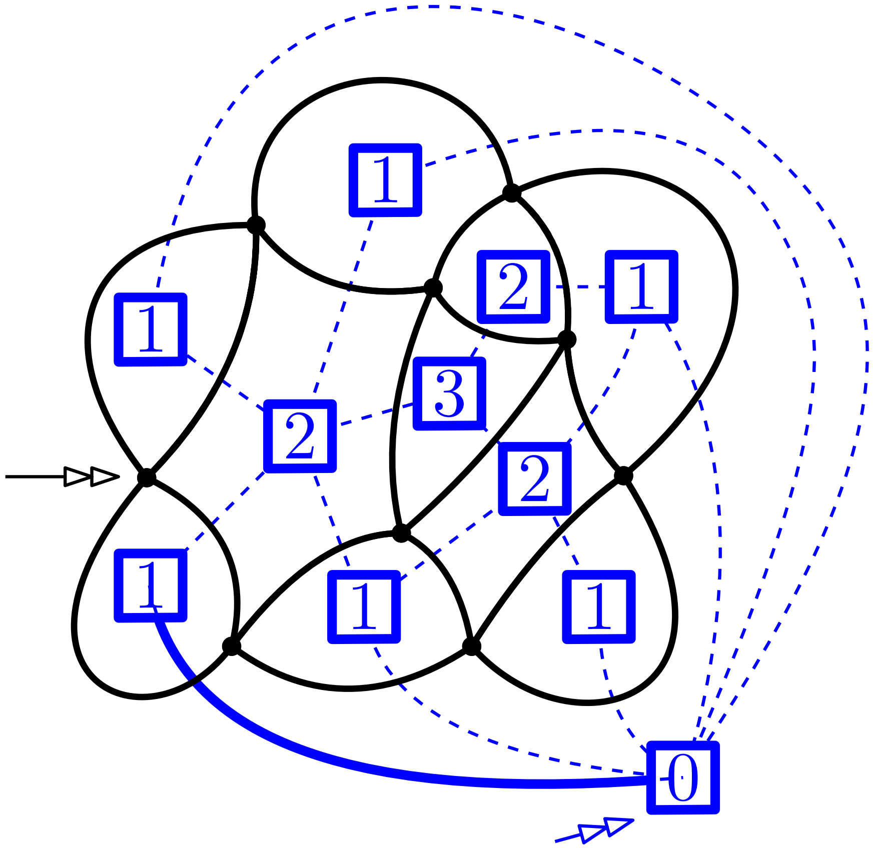

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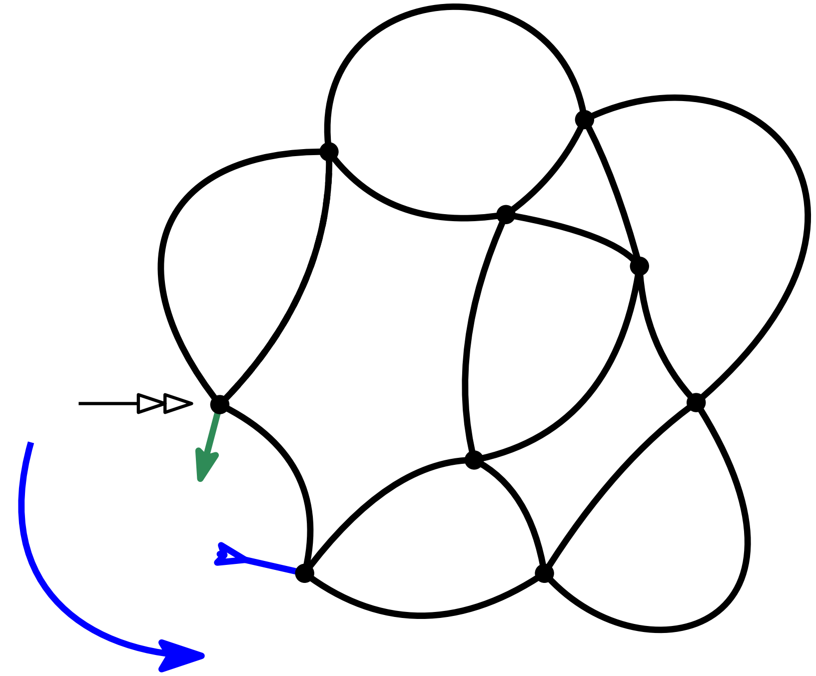

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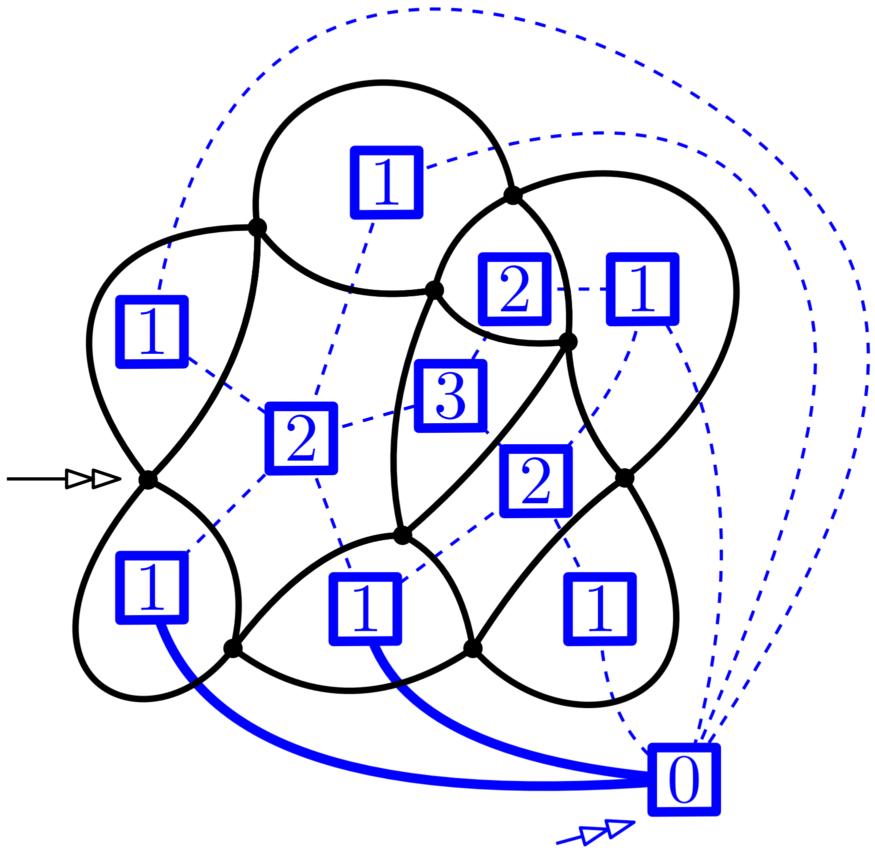

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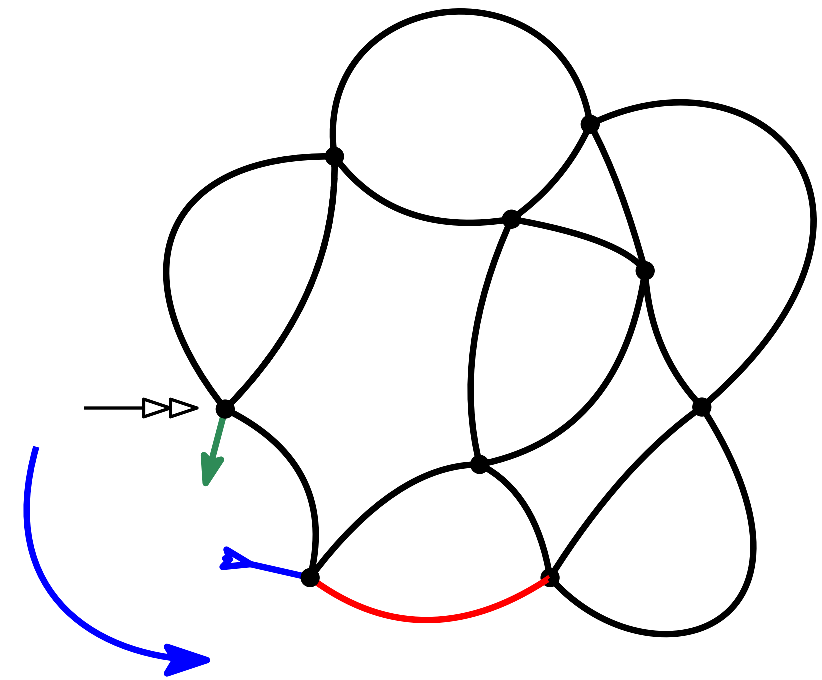

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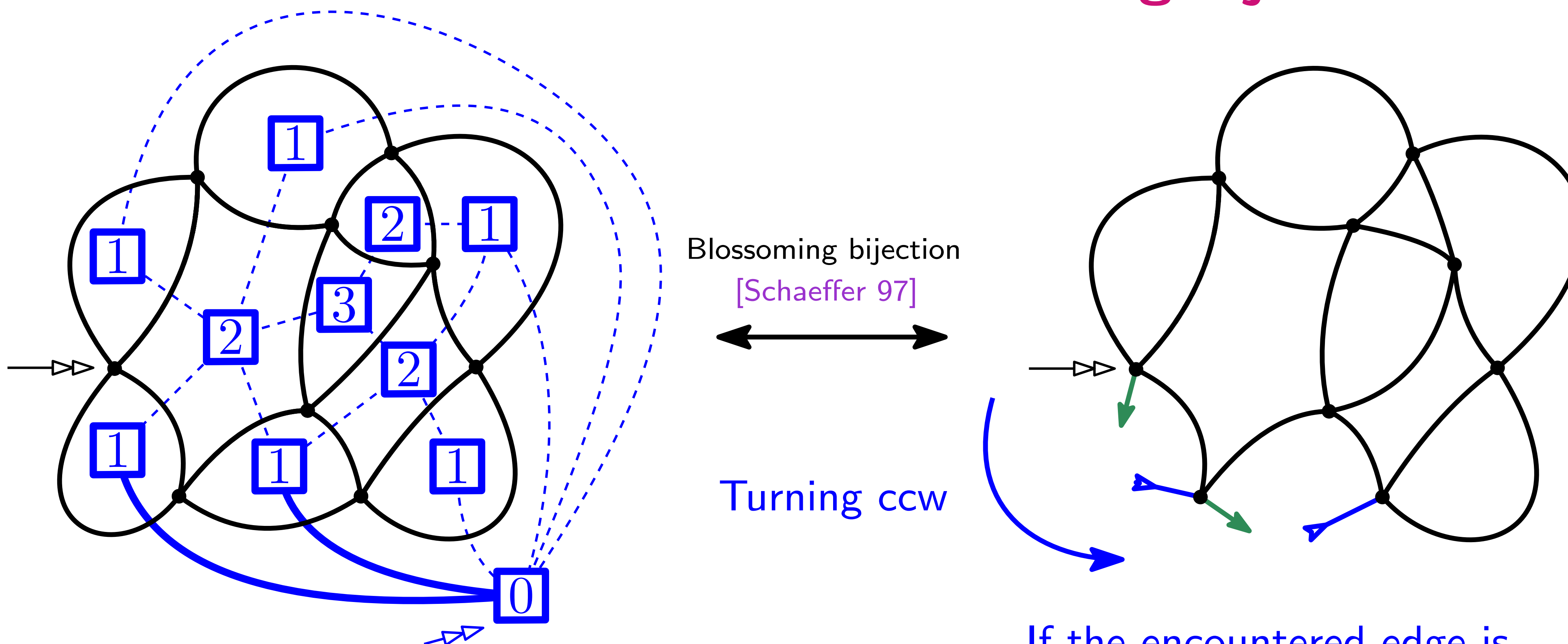

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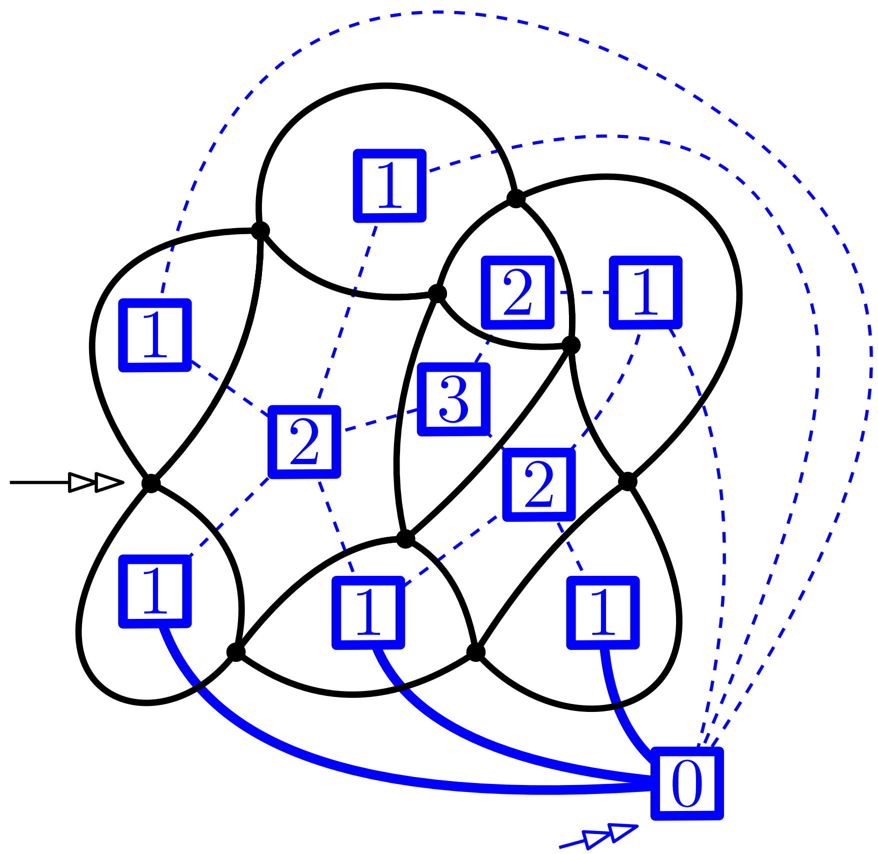

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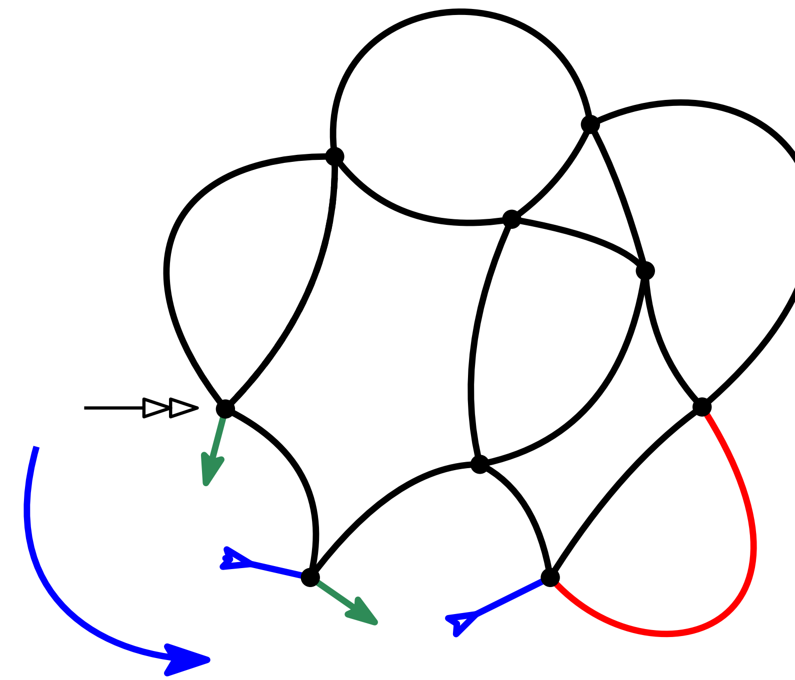

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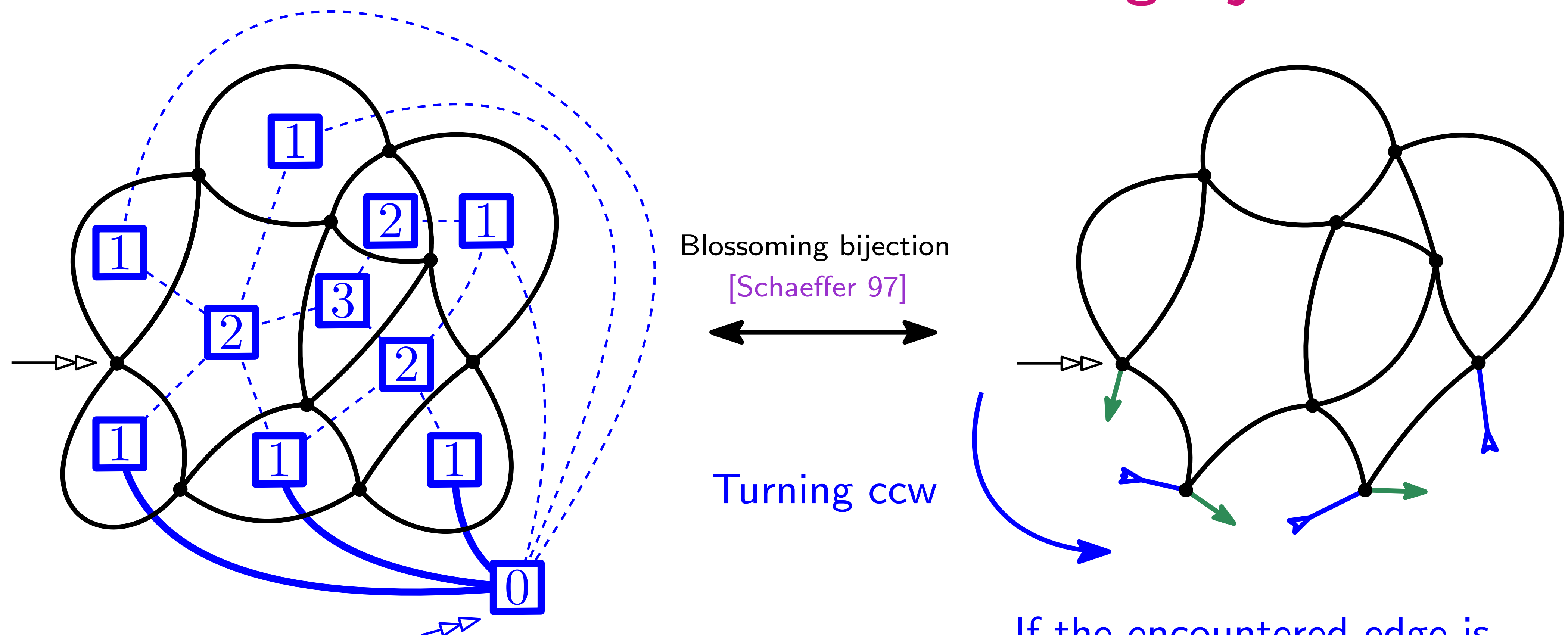

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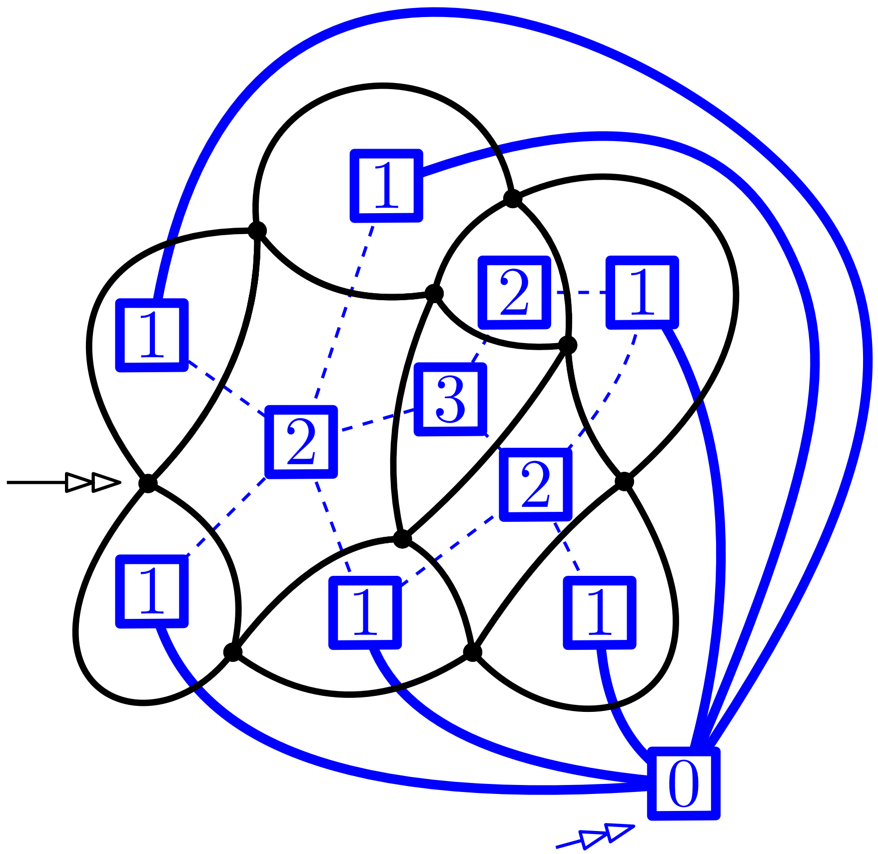

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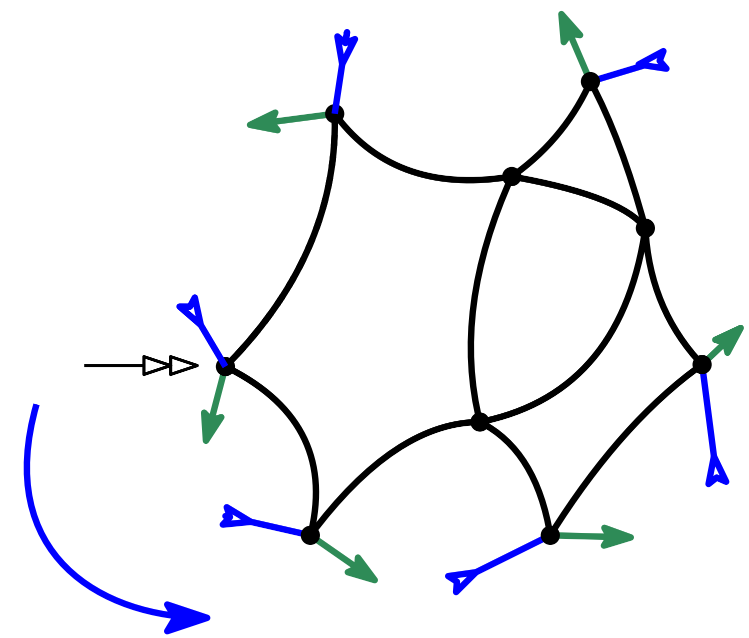

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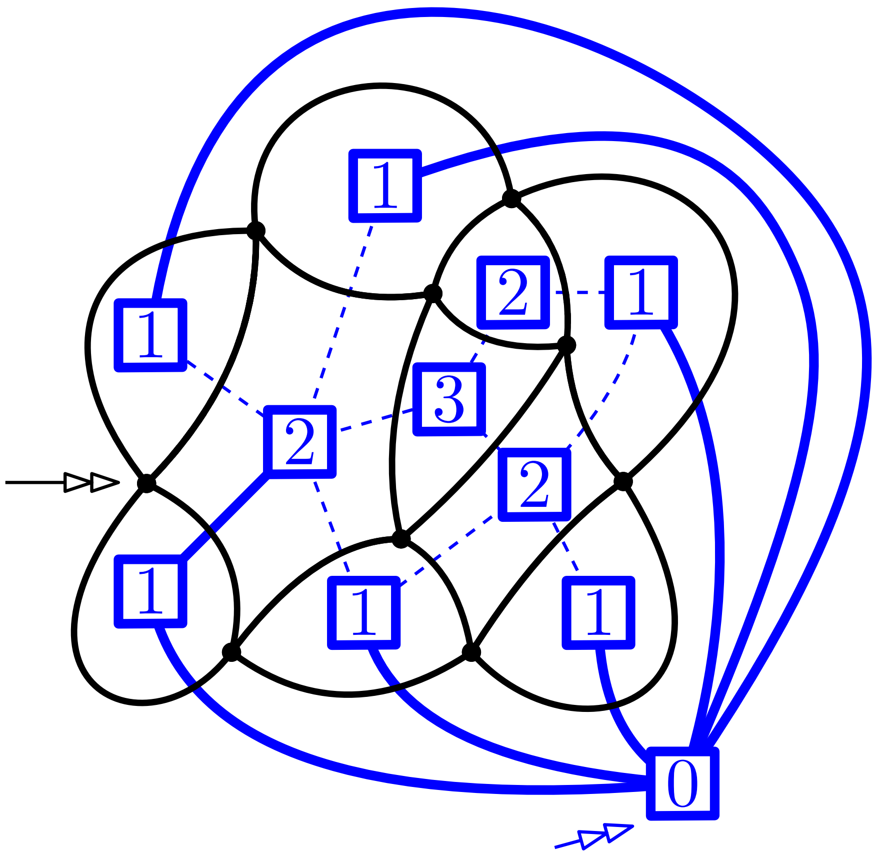
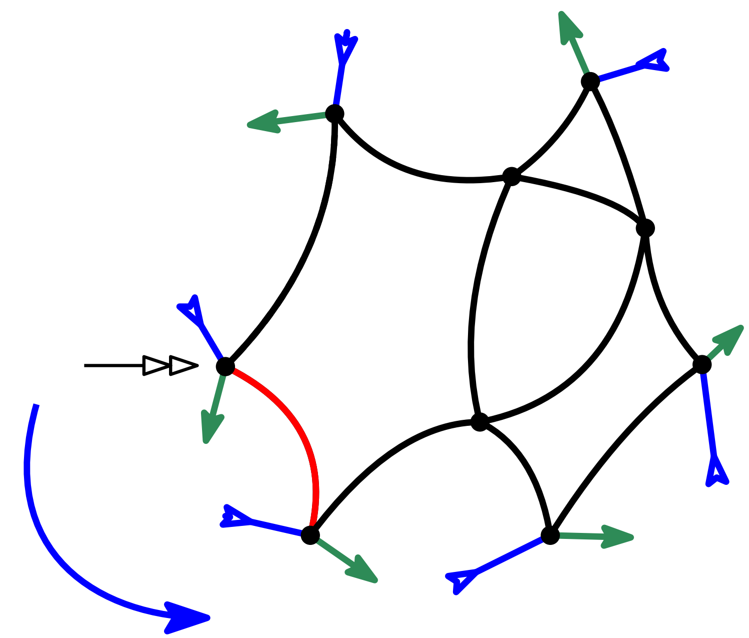
Reformulation of Schaeffer's blossoming bijection



Blossoming bijection

[Schaeffer 97]

Turning ccw



Label the faces by their distance to the root face in the dual graph

Consider the "leftmost" breadth-first tree

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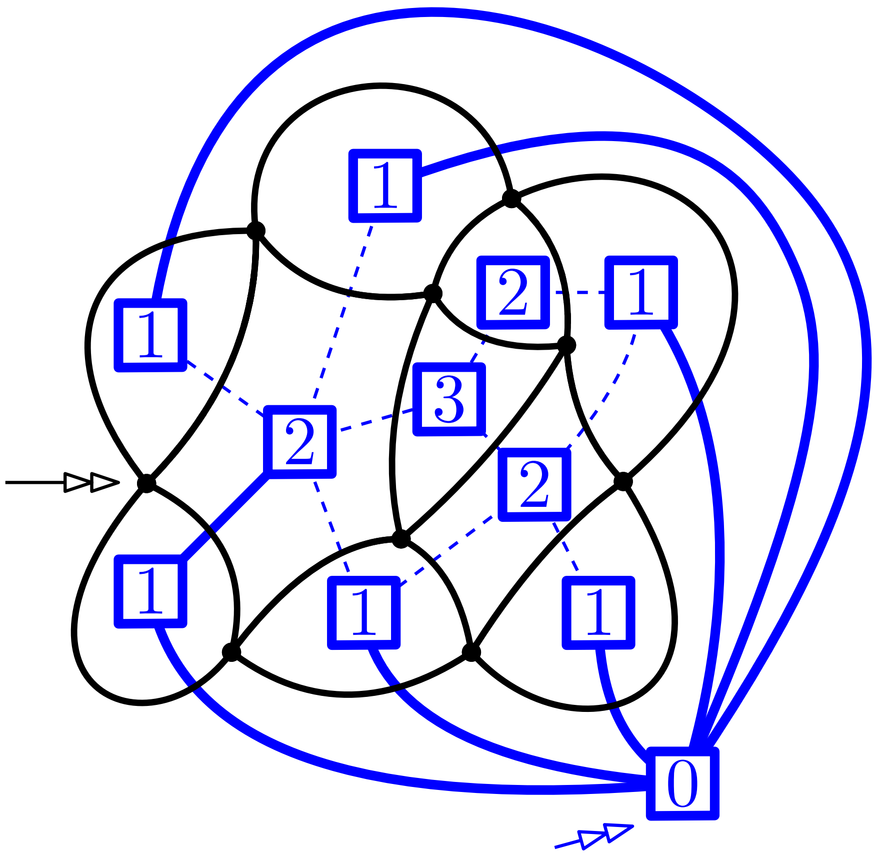

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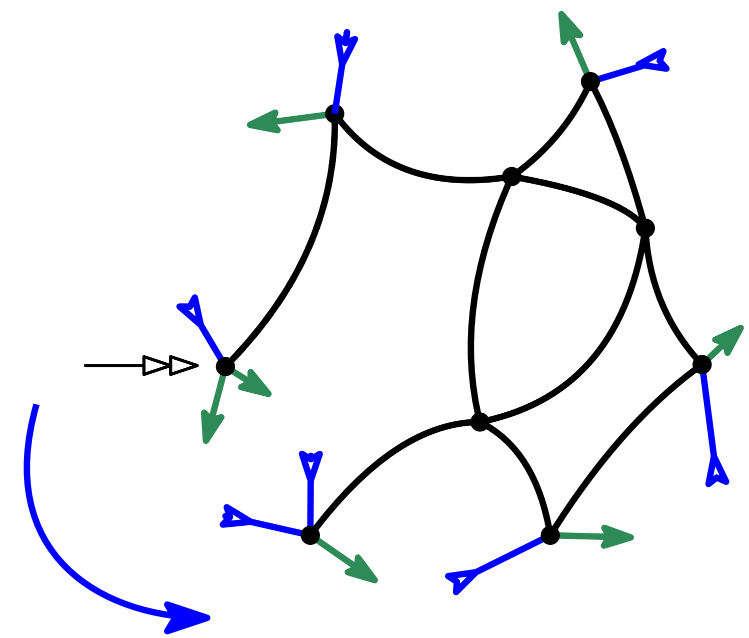

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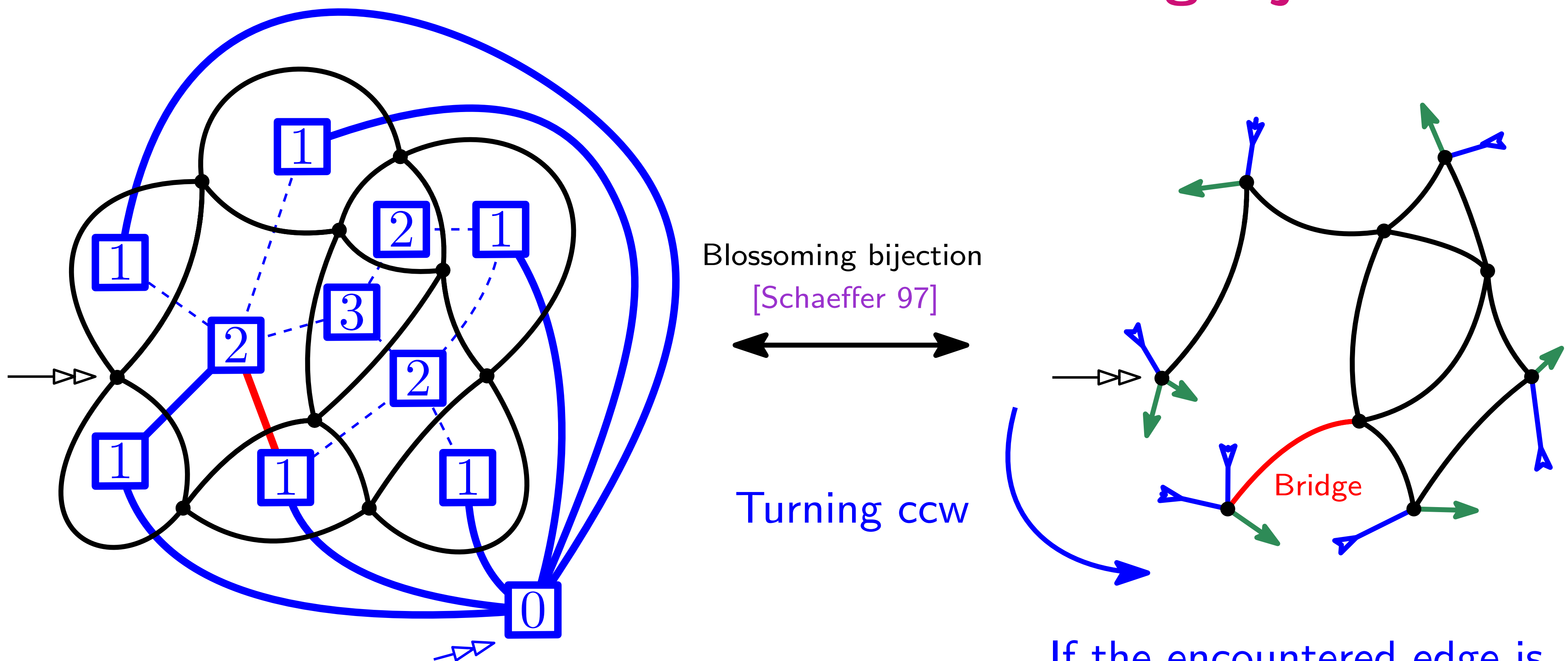

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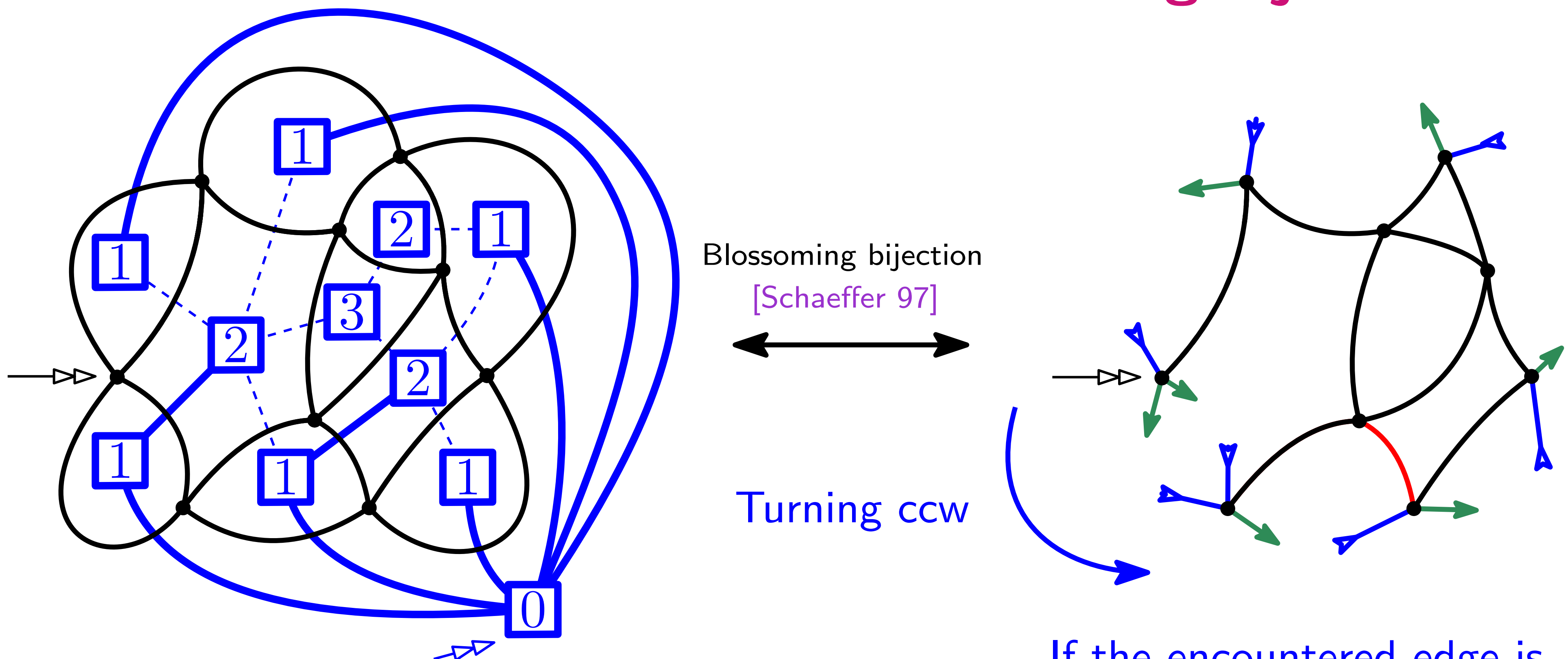

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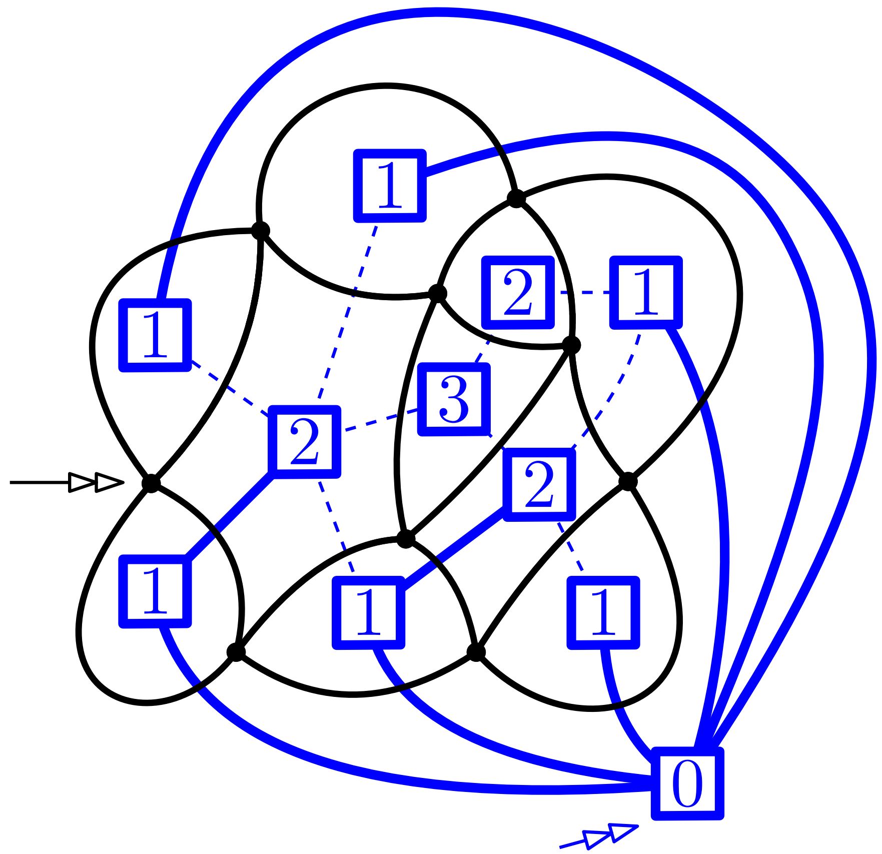

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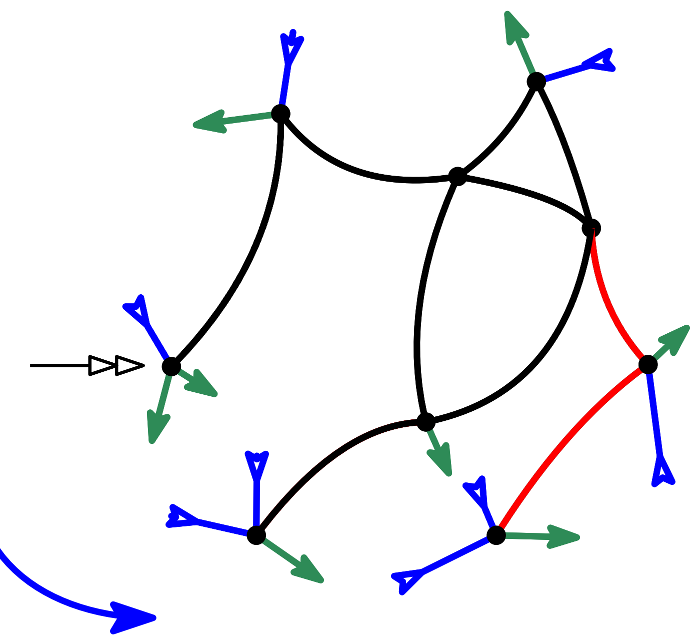

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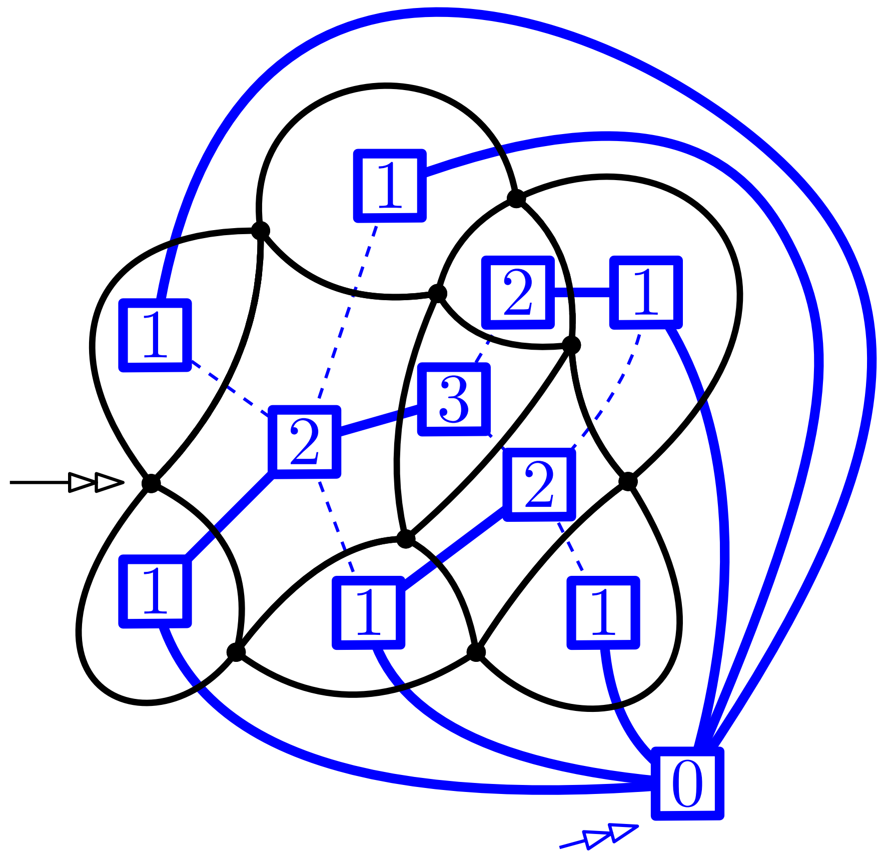

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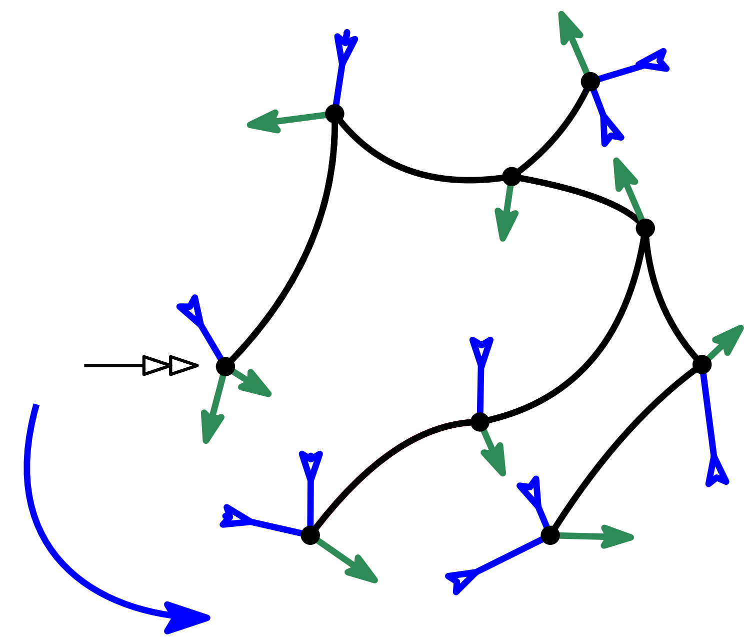

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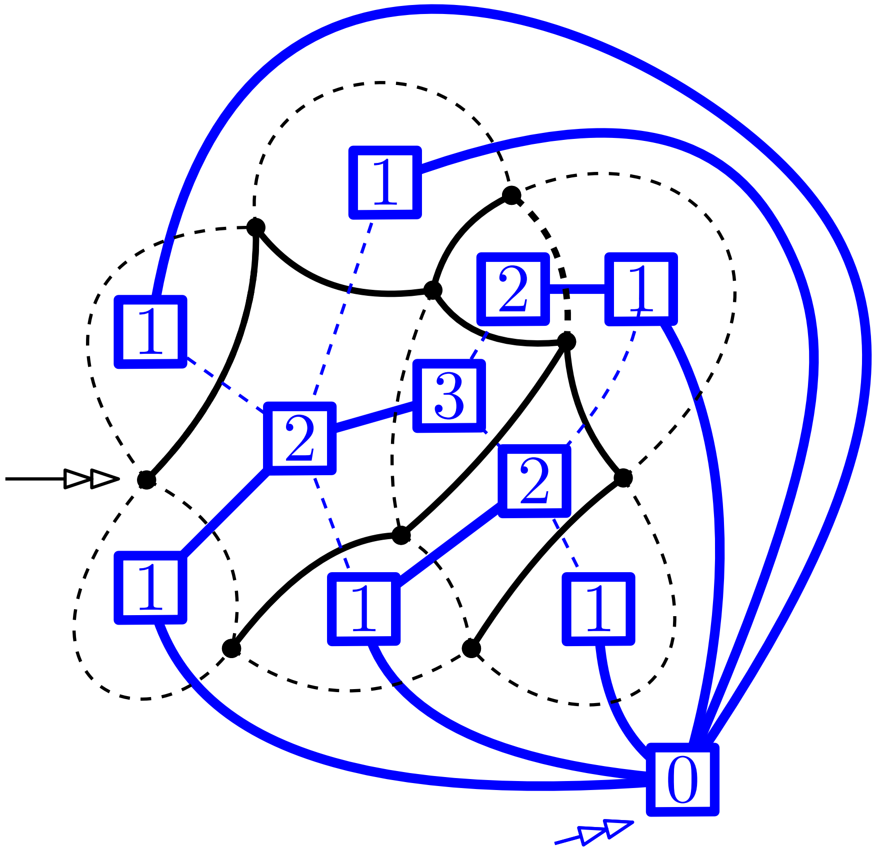

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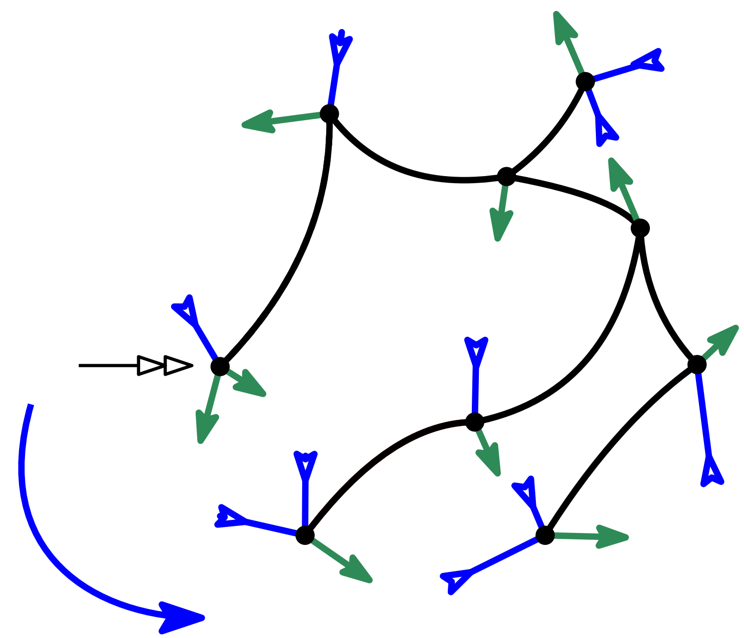


Turning ccw

Label the faces by their distance to the root face in the dual graph

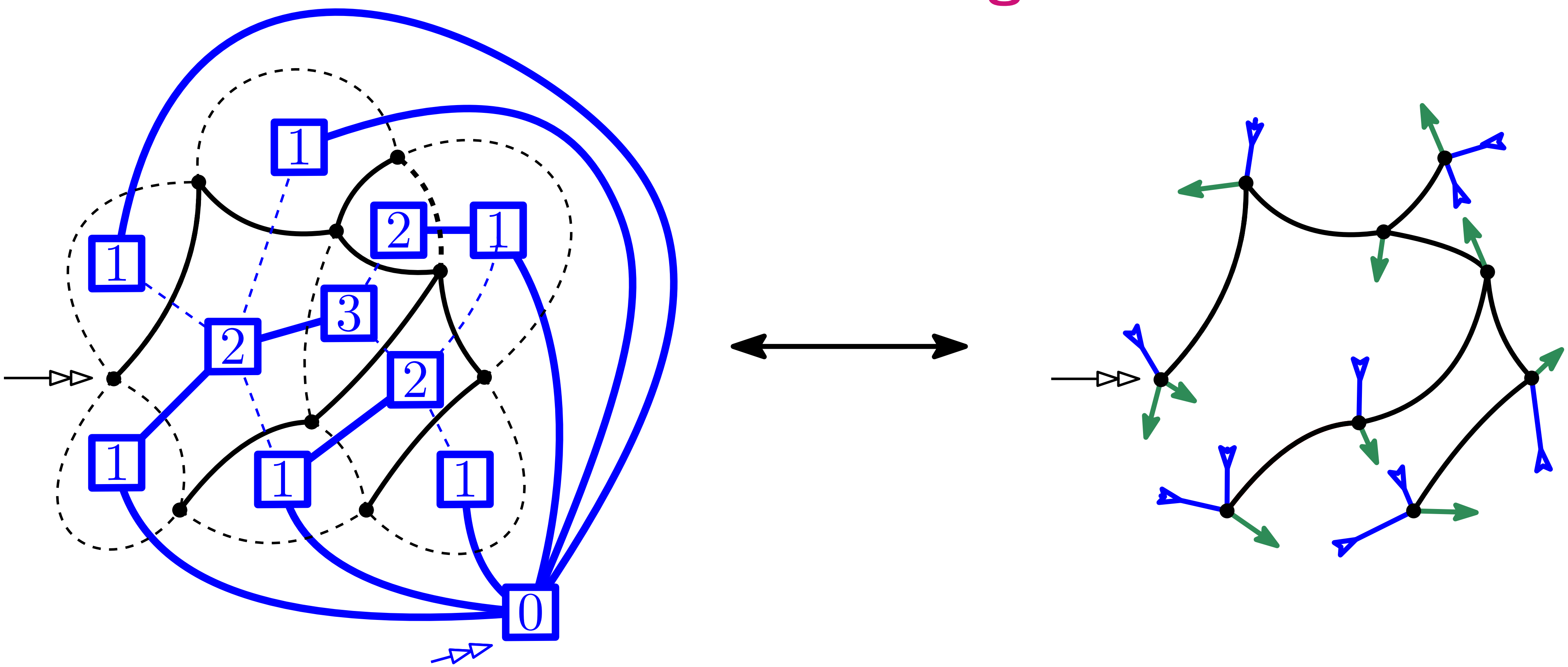
Consider the "leftmost" breadth-first tree

If the encountered edge is not a bridge, delete it !

Otherwise, continue !

Claim: The dual of the leftmost breadth-first tree is the blossoming tree given by the first description of the bijection.

Caracterization of the blossoming trees



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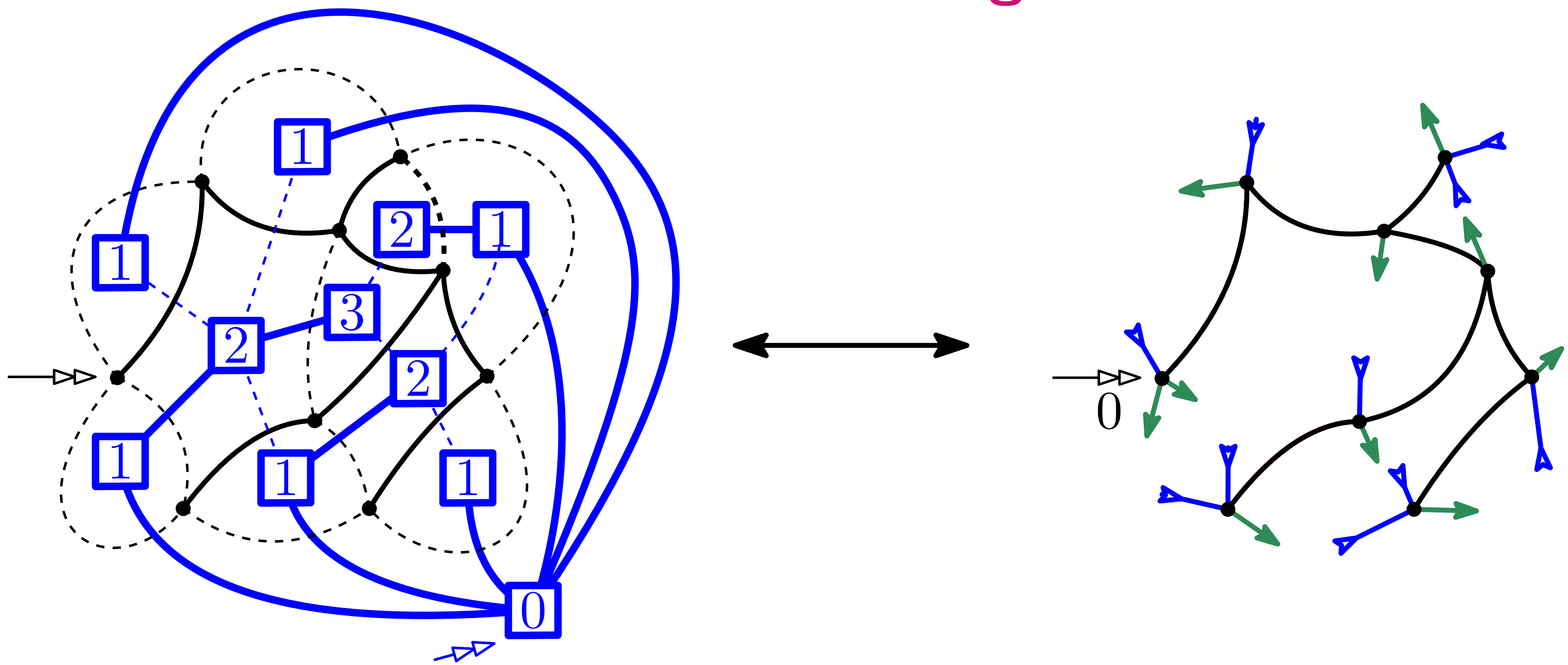

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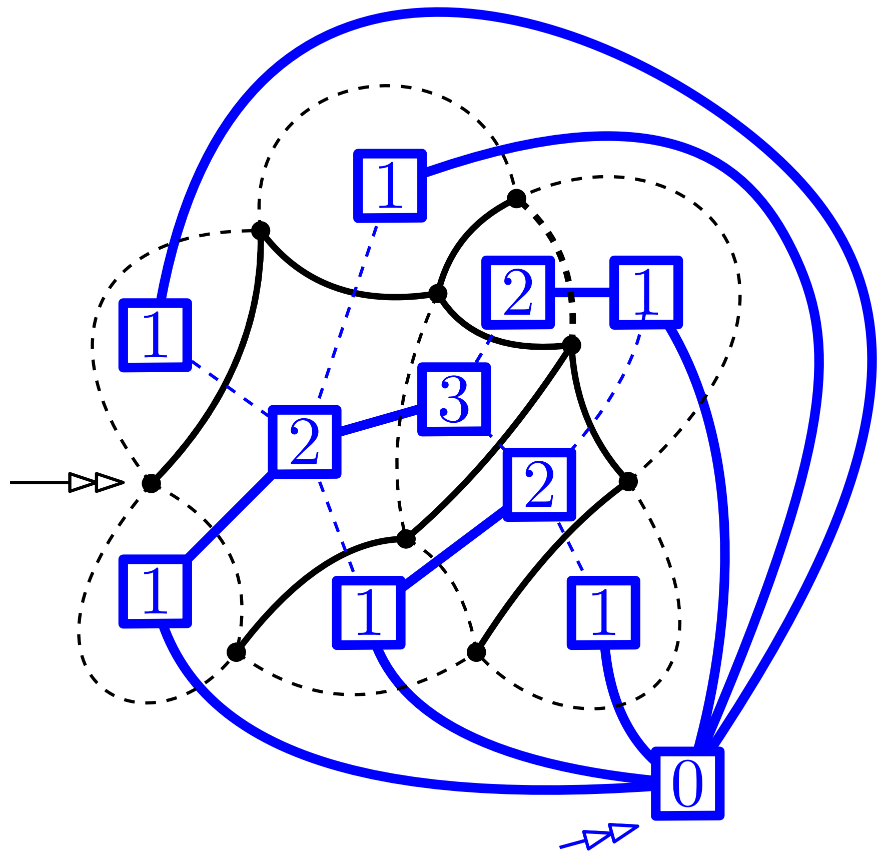

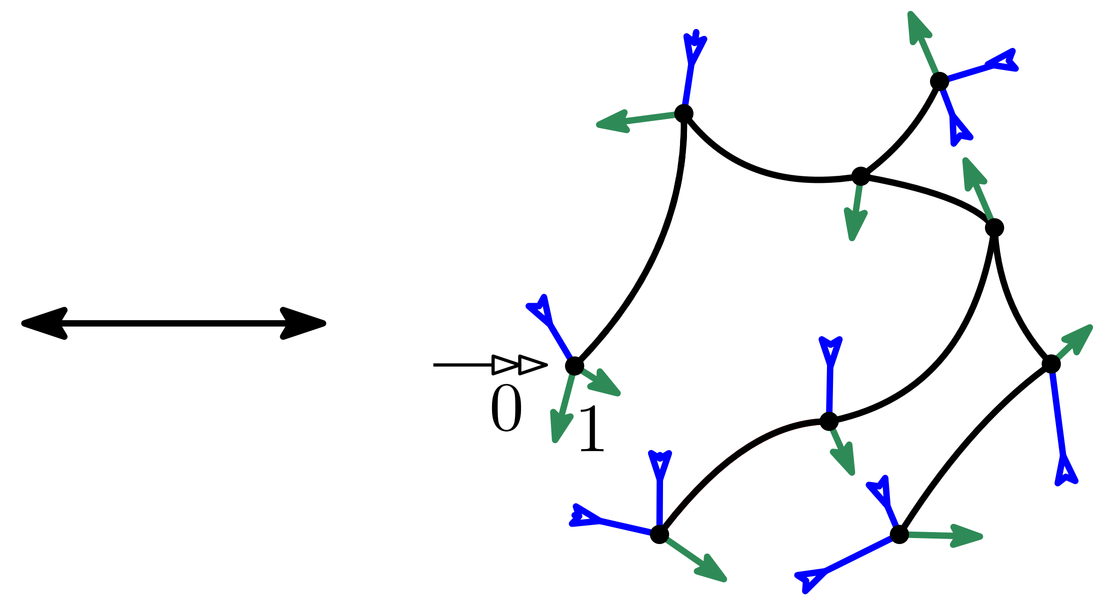

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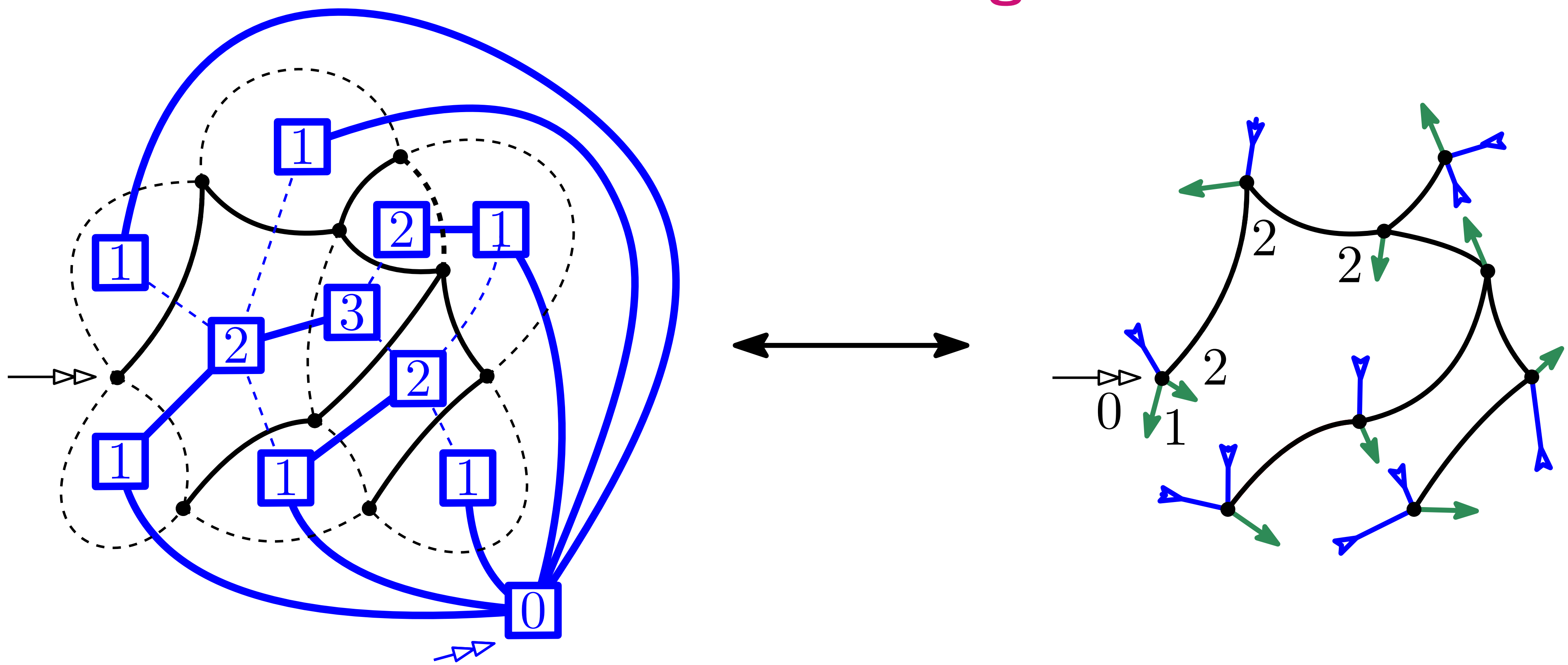

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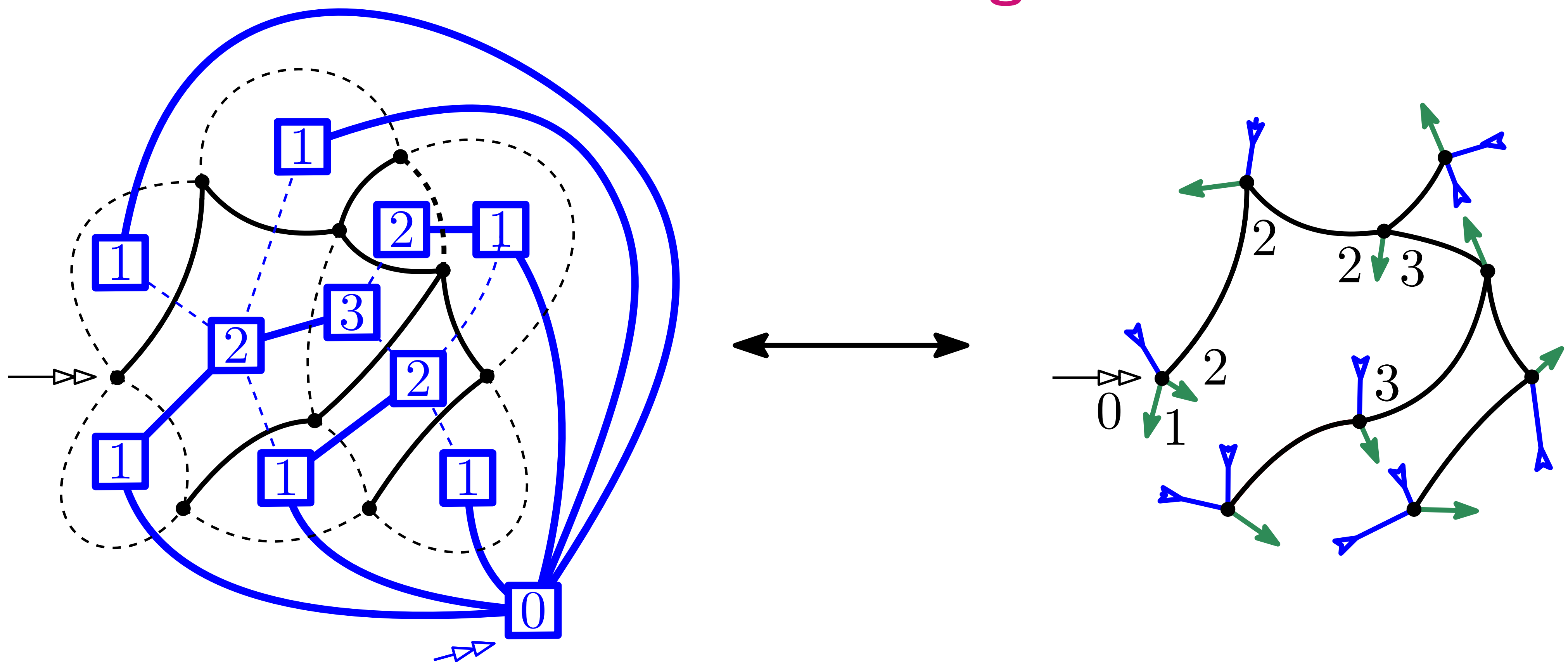

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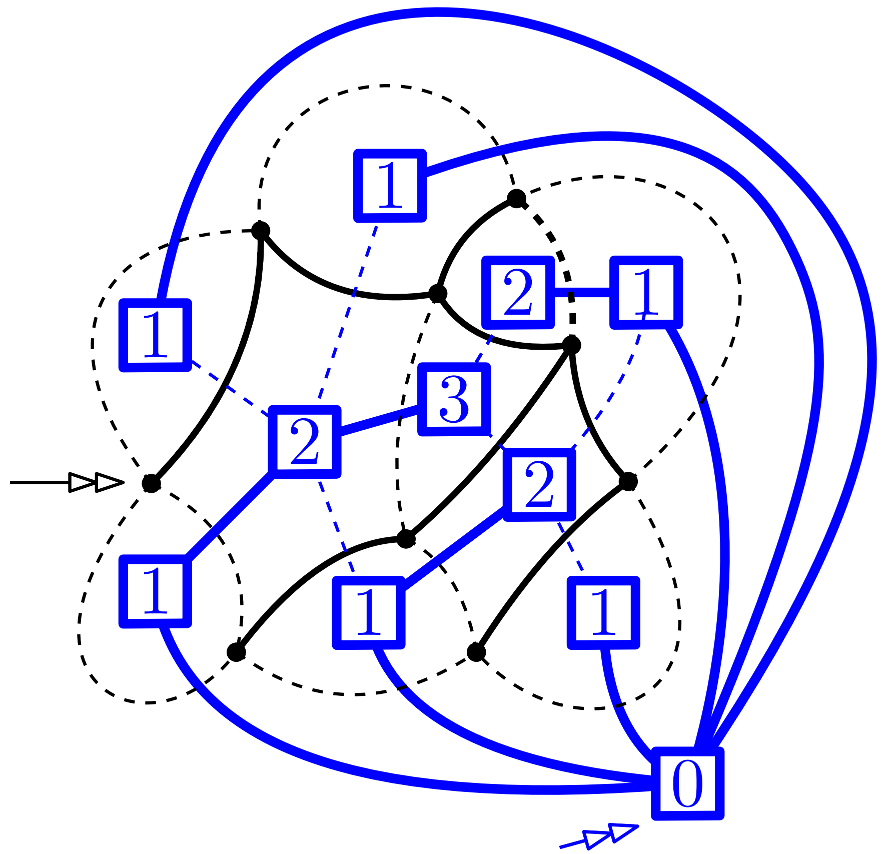

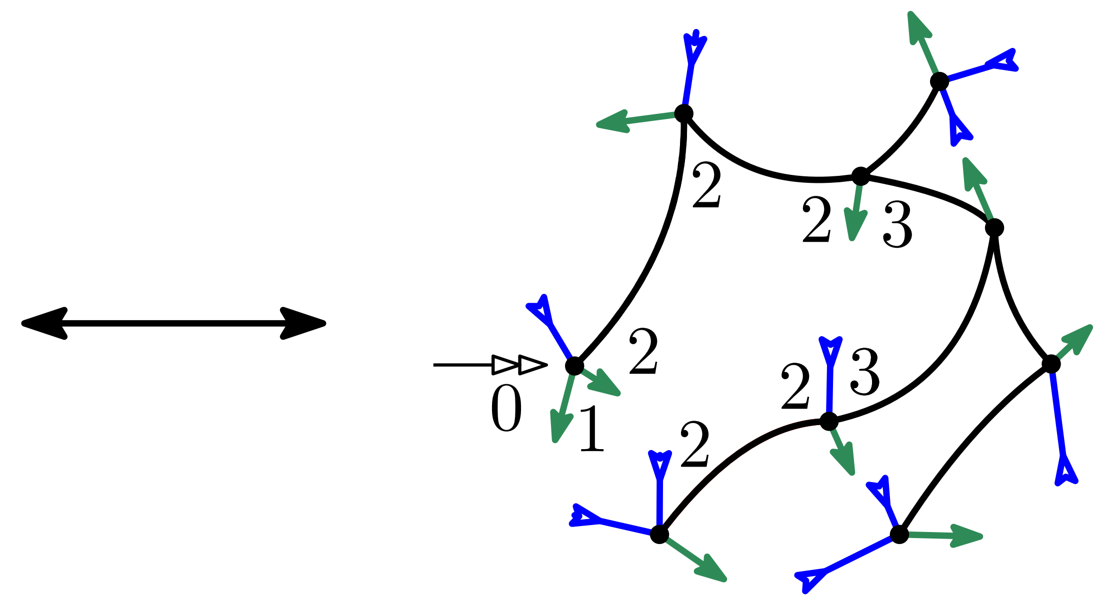

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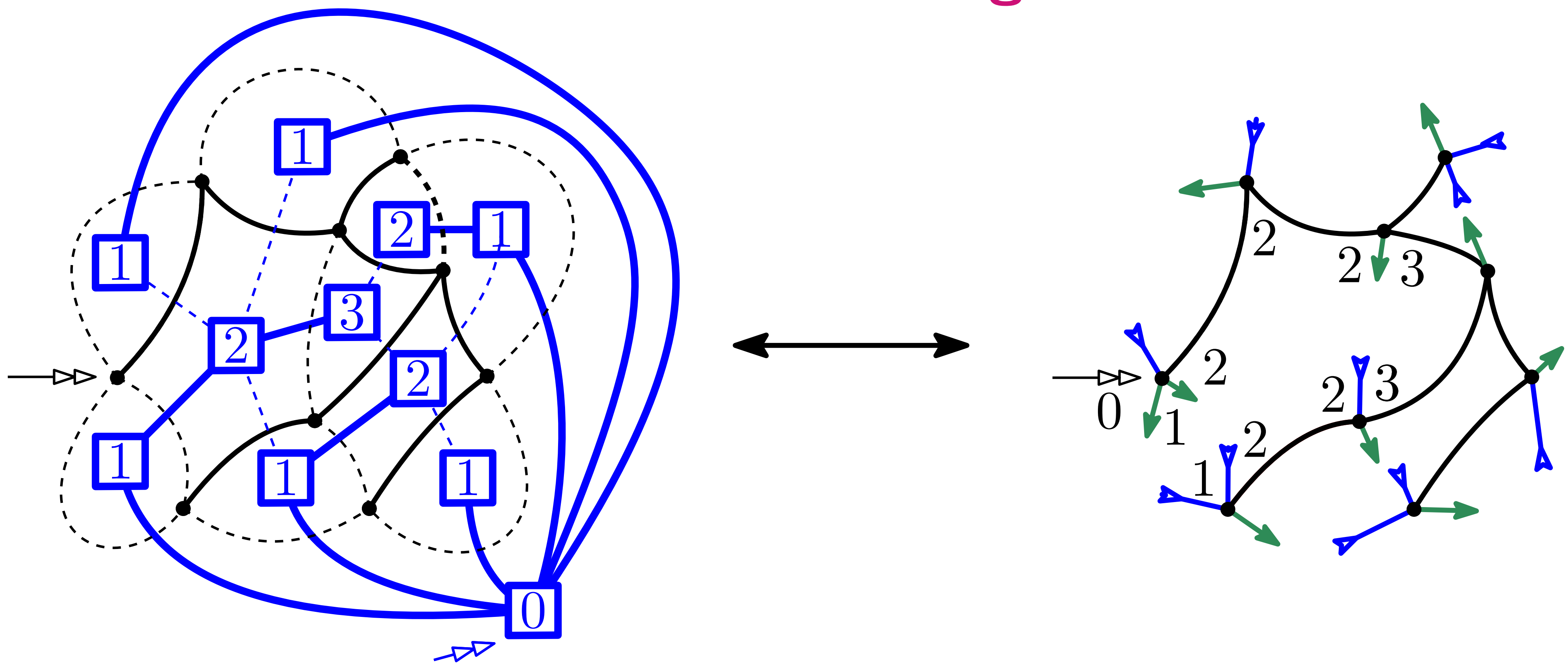

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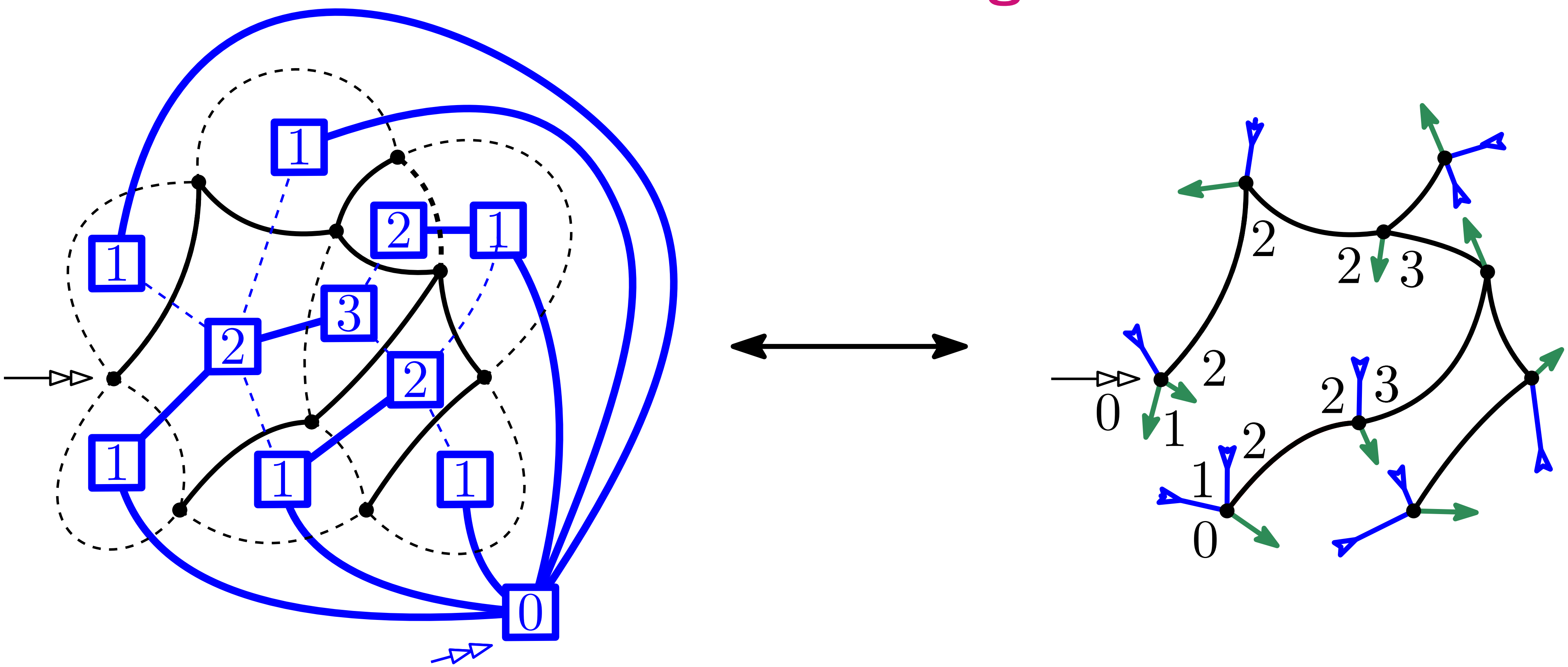

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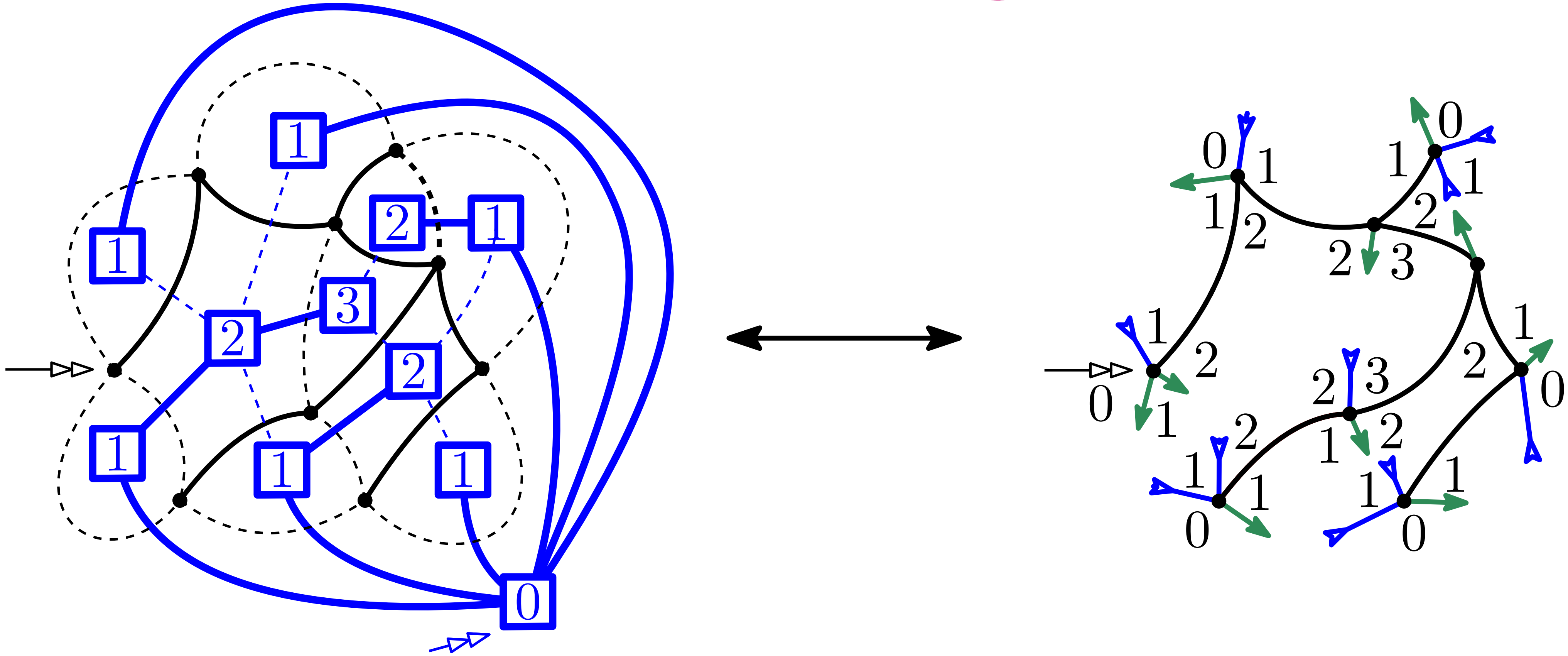

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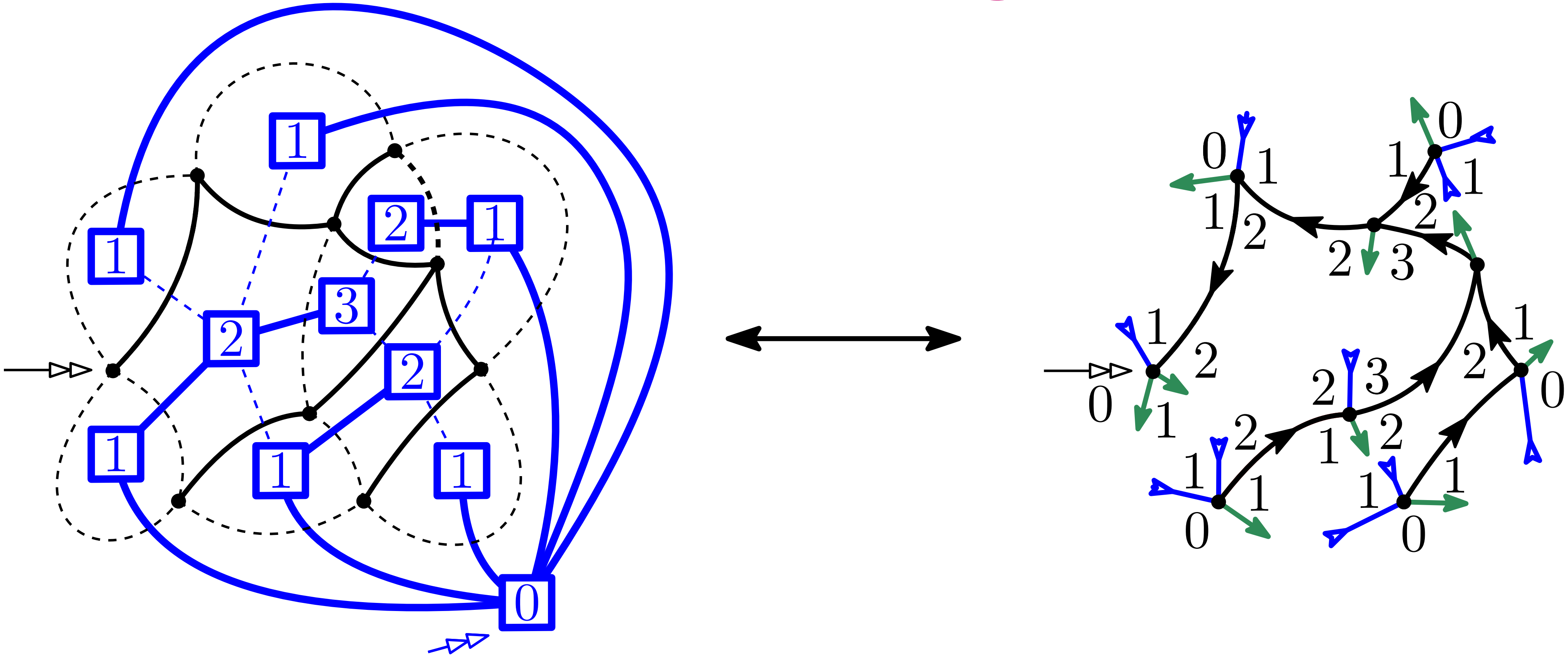
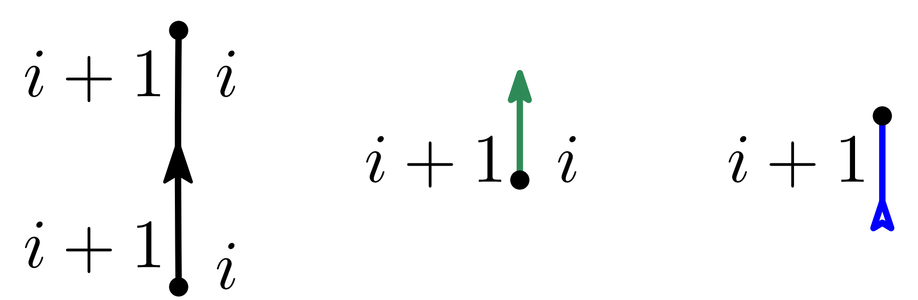


Good labeling of the corners:



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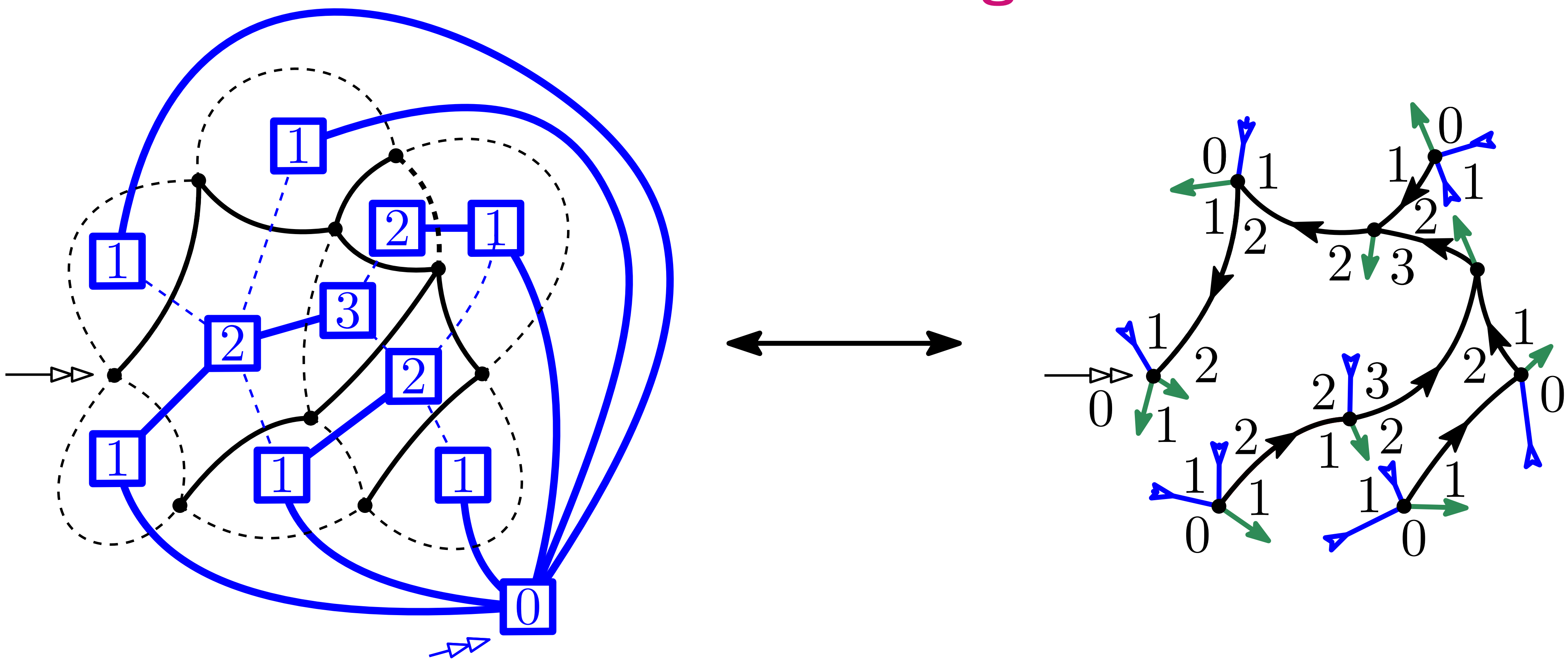

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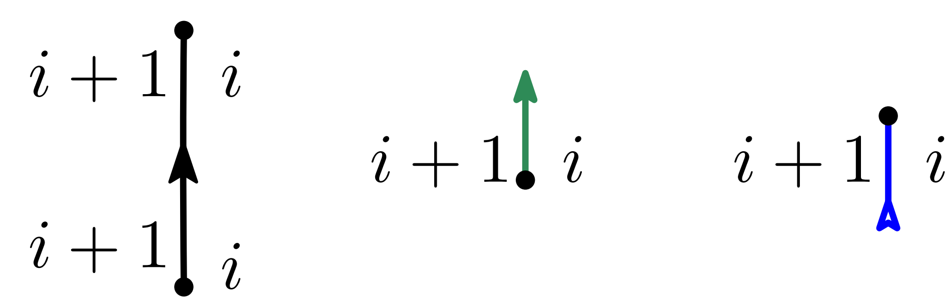
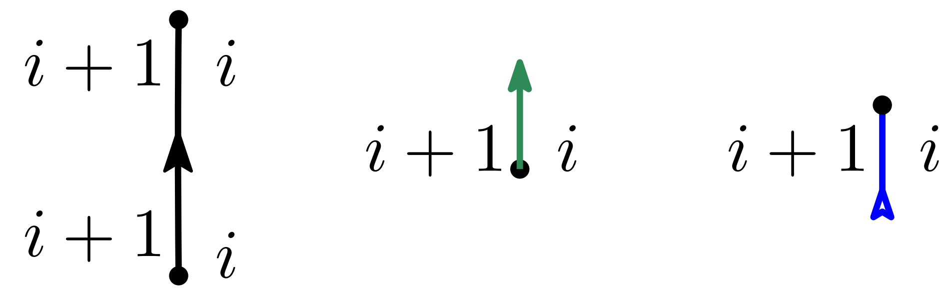


Theorem: The blossoming trees are 4-valent trees, that can be endowed with a **non-negative good labeling** of their corners.

Caracterization and enumeration of the blossoming trees

Good labeling of the corners:



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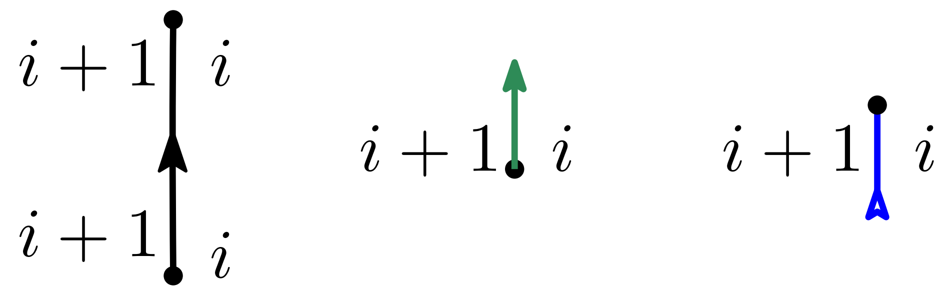
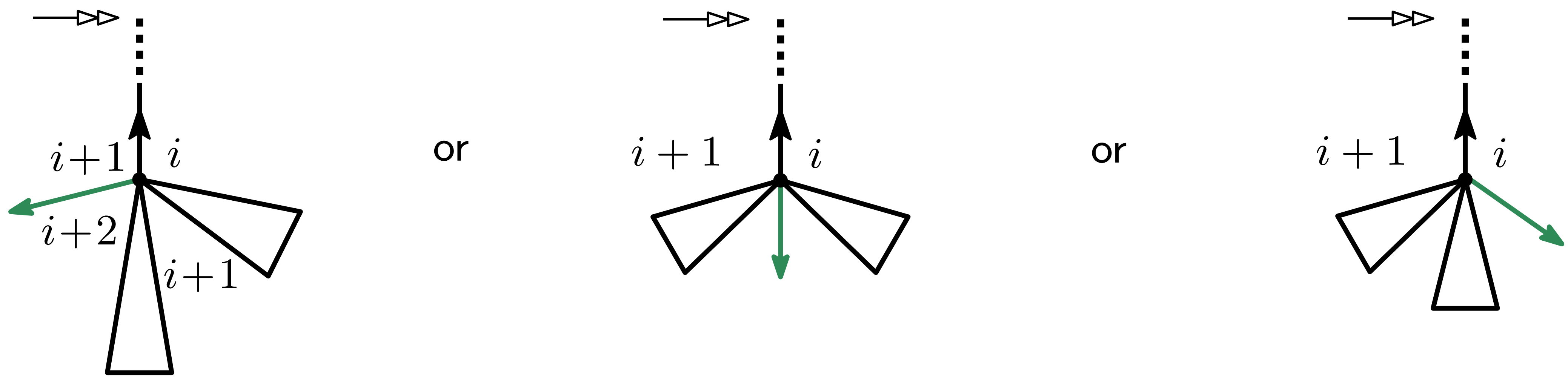


Locally around a vertex of a 4-valent tree with a ~~non negative~~ good labeling:

2 incoming edges and 2 outgoing edges :



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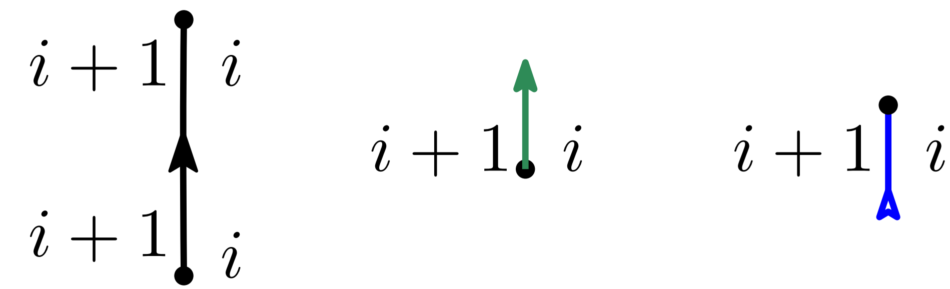

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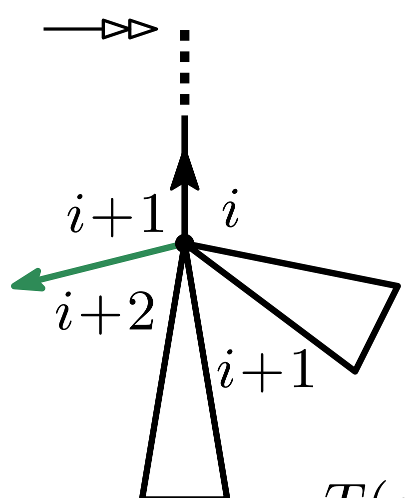

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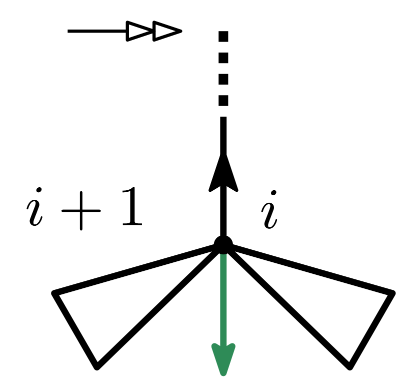

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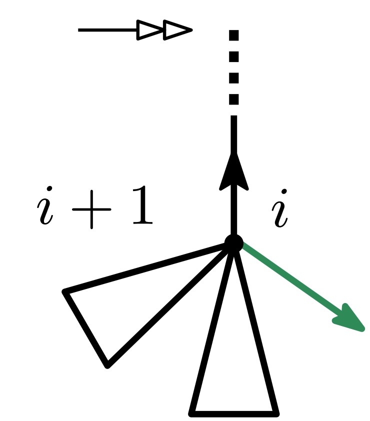

$T(z)$ = generating series of trees enumerated by number of closing stems:

$$T(z) = z + 3T(z)^2$$

we retrieve the enumerative result of [Schaeffer]

In higher genus

Theorem [Lepoutre '19]:

4-valent bicolorable maps of genus g

$\updownarrow$ bijection

4-valent blossoming unicellular maps of genus g,
that can be endowed with a good non-negative labeling

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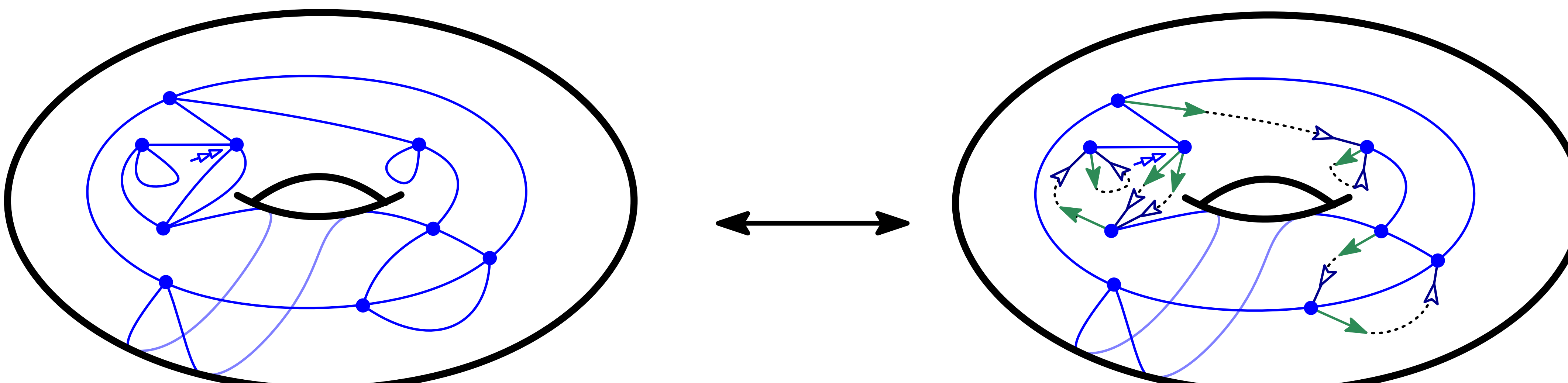

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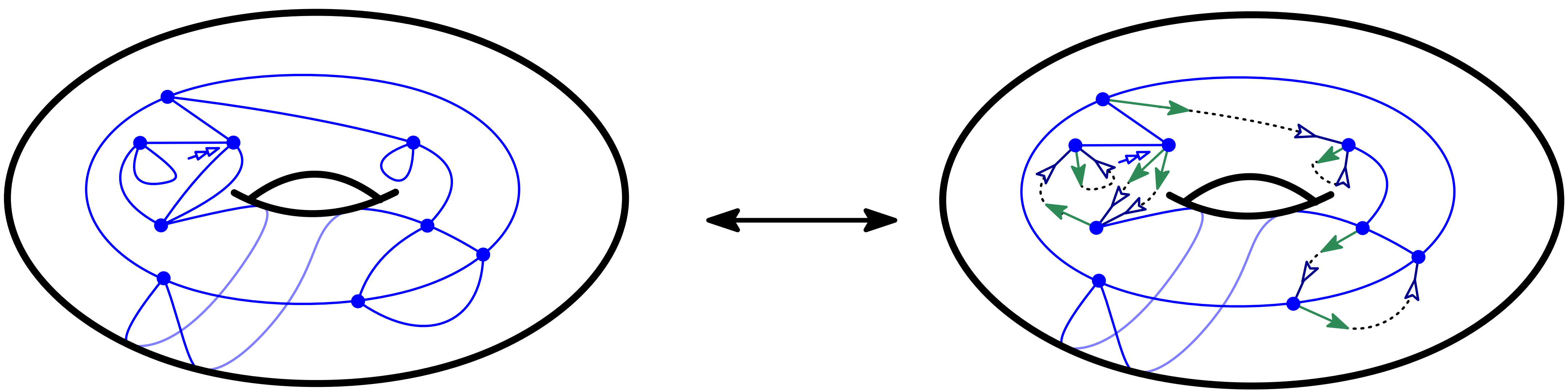

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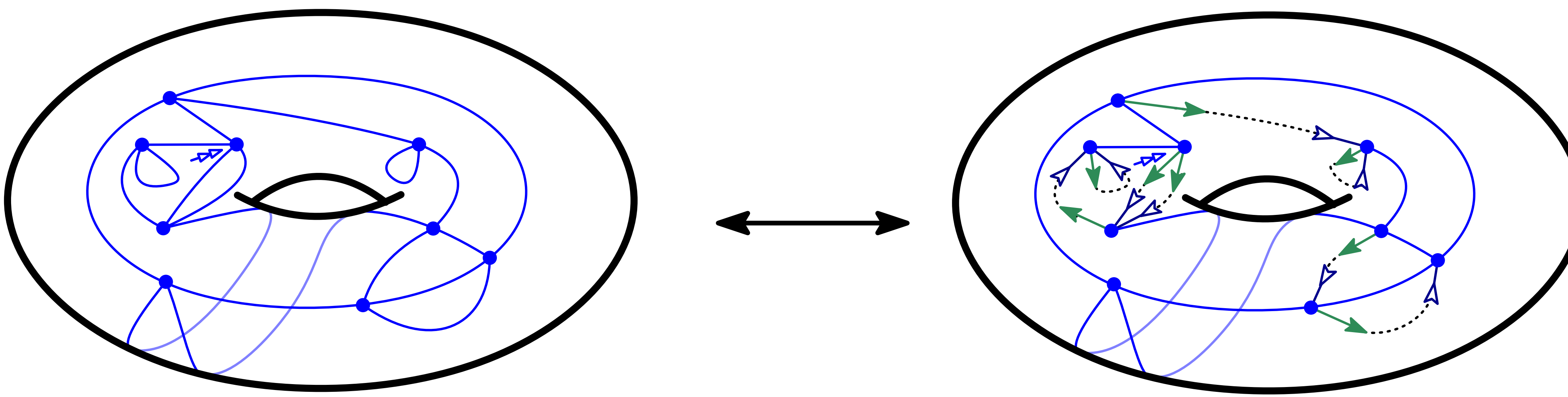

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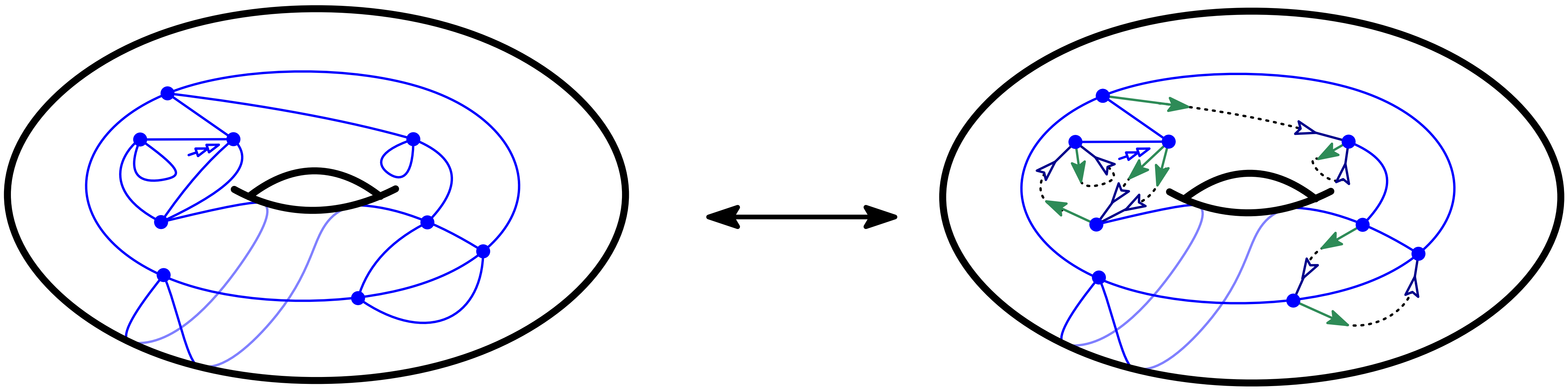

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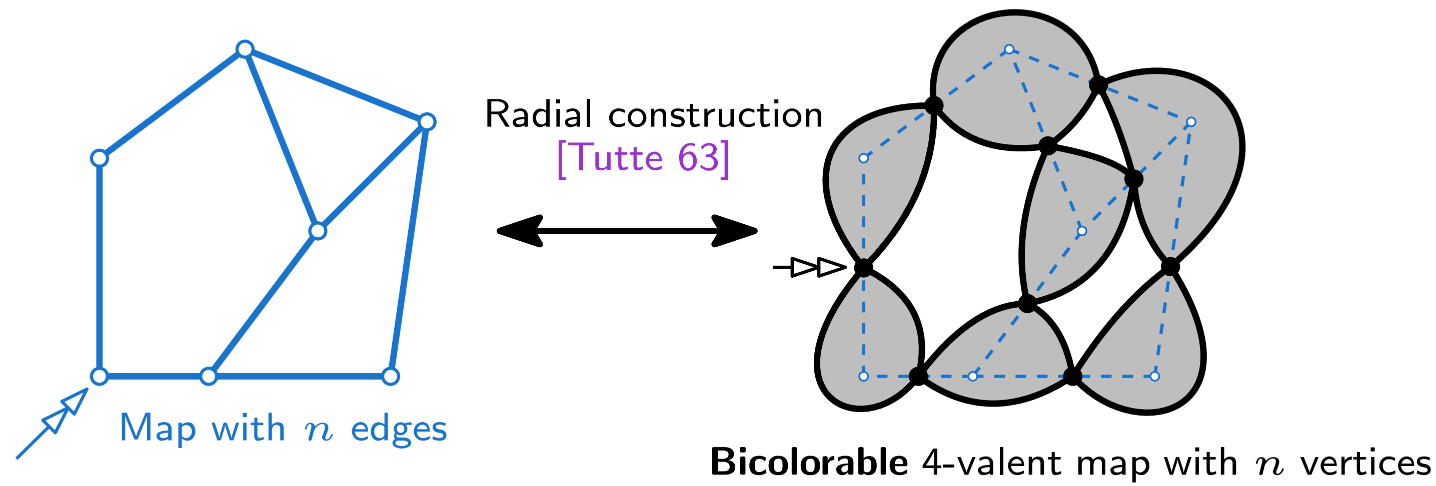

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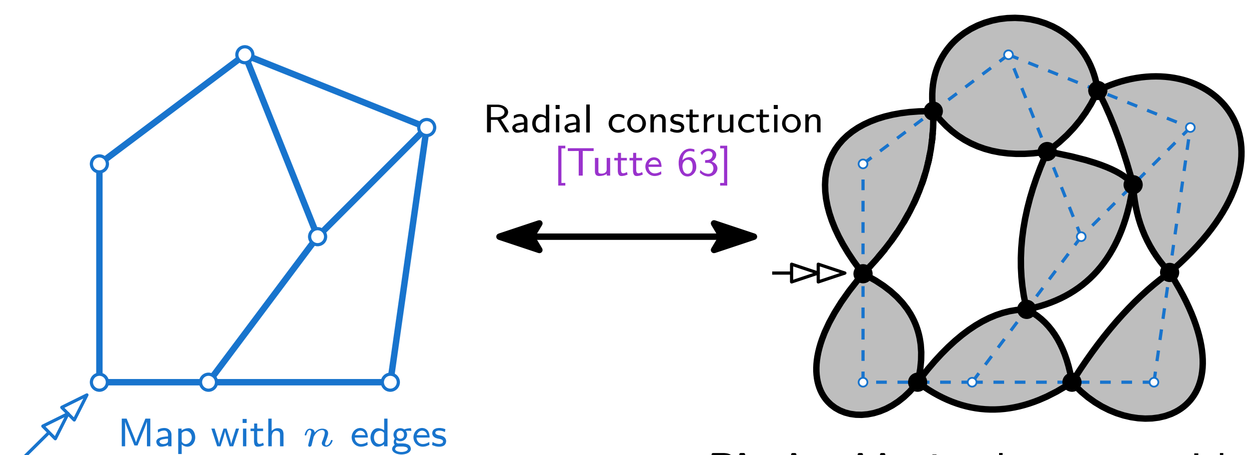
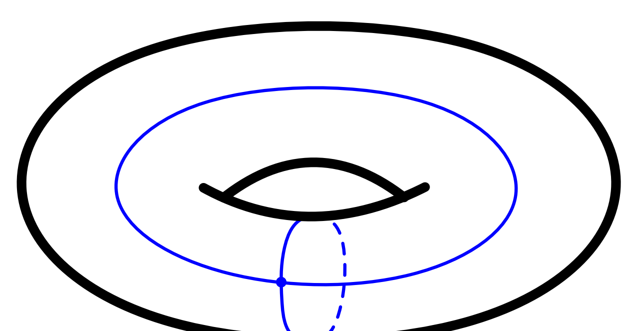


Bicolorable 4-valent map with n vertices

- **Planar** 4-valent maps are bicolorable, not true in general in higher genus.



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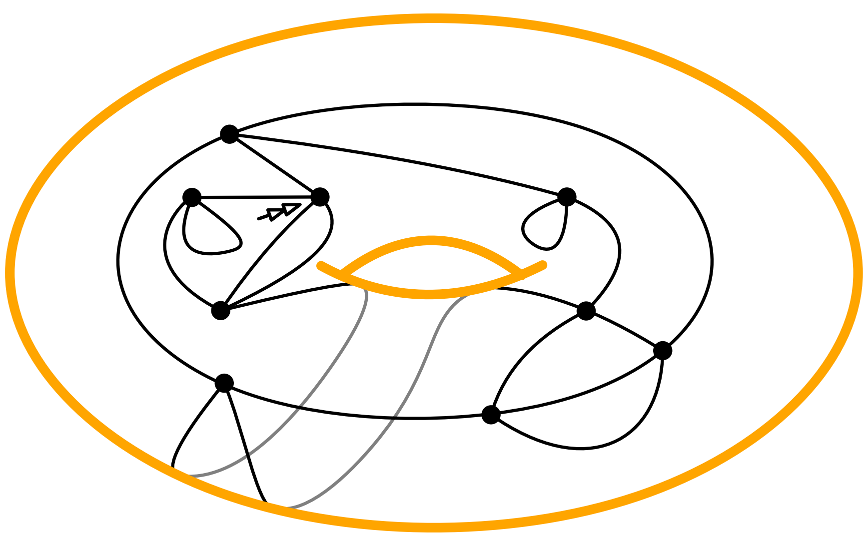
Dual of a tree-decorated map in **higher genus**.



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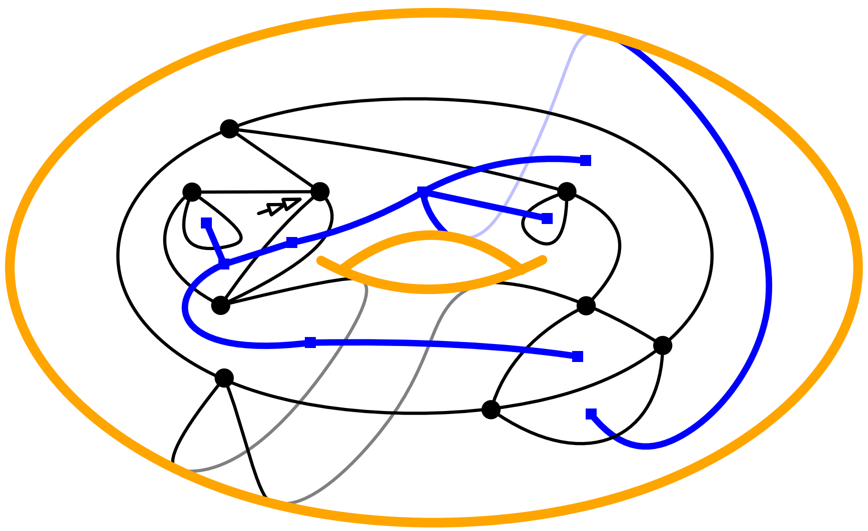

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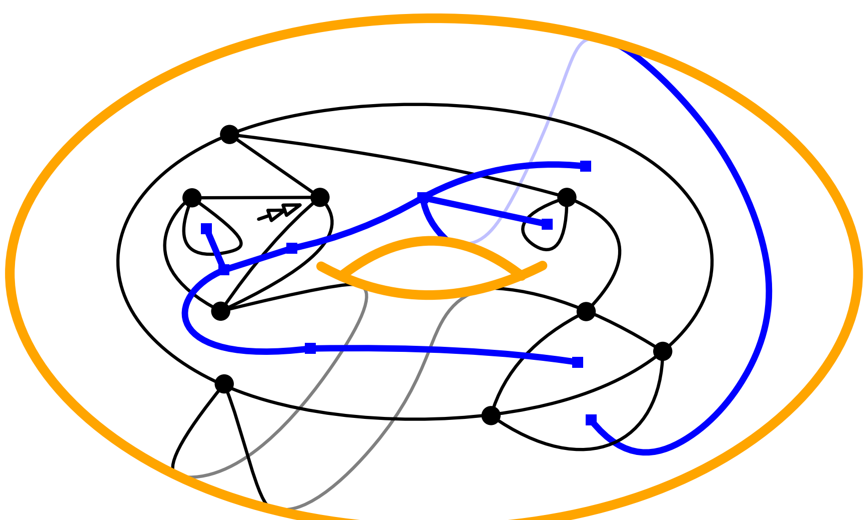

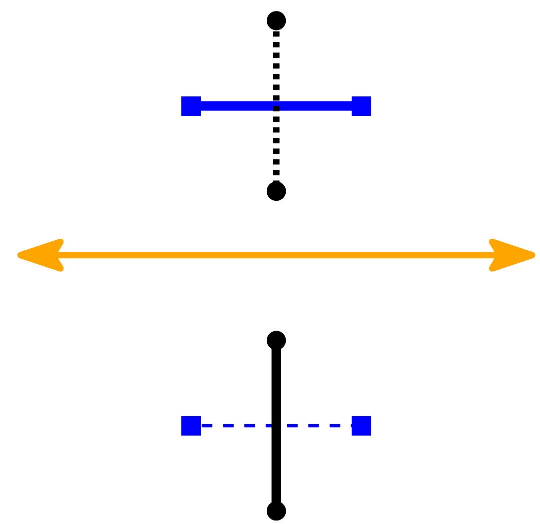

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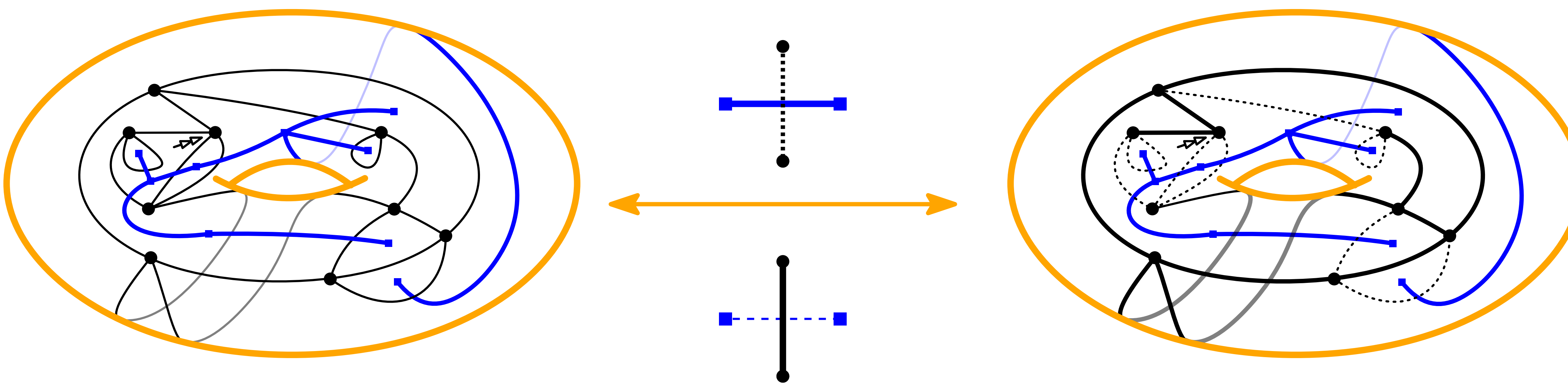

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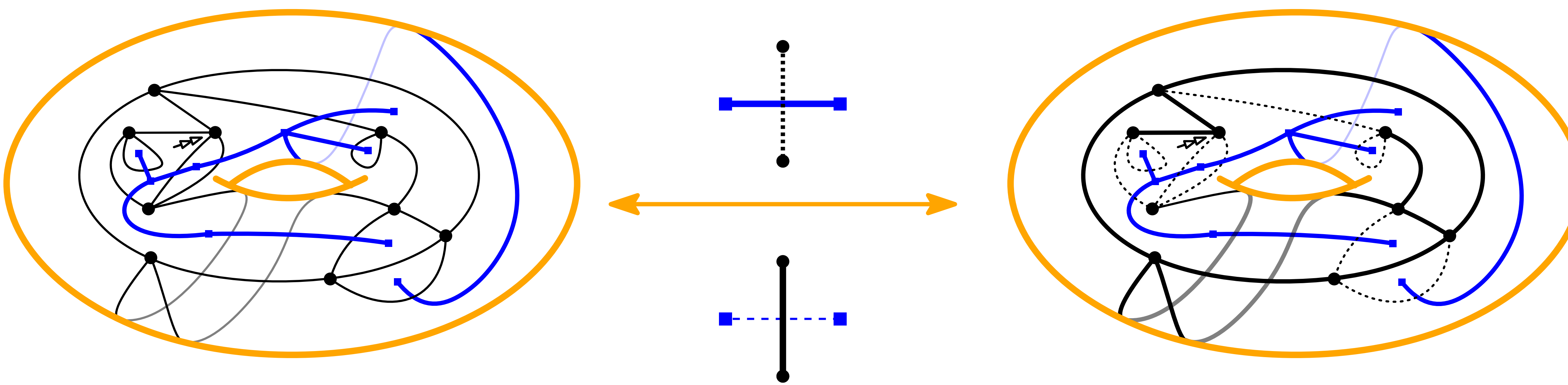

Prop (folklore): The dual of a **tree-decorated map** of genus g is a map with a **spanning unicellular map** of genus g.

Theorem [Lepoutre '19]:

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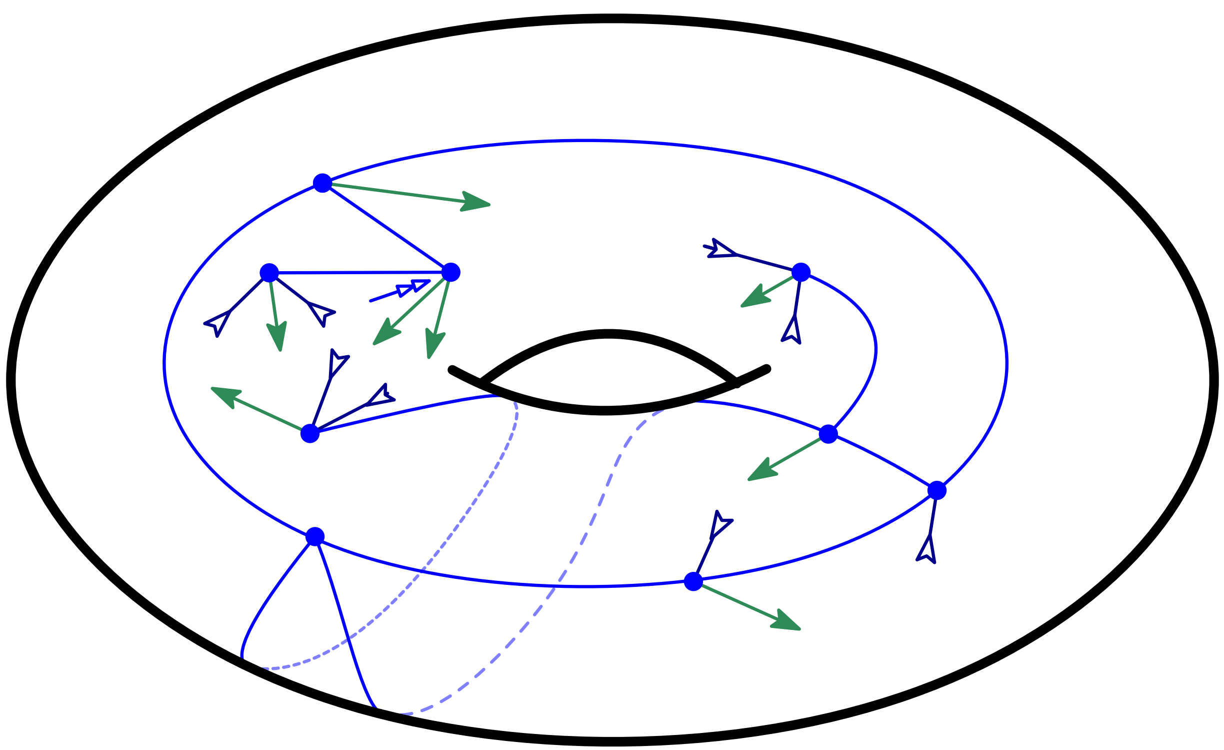

As in the planar case, the labeling is uniquely determined by the opening/closing stems.

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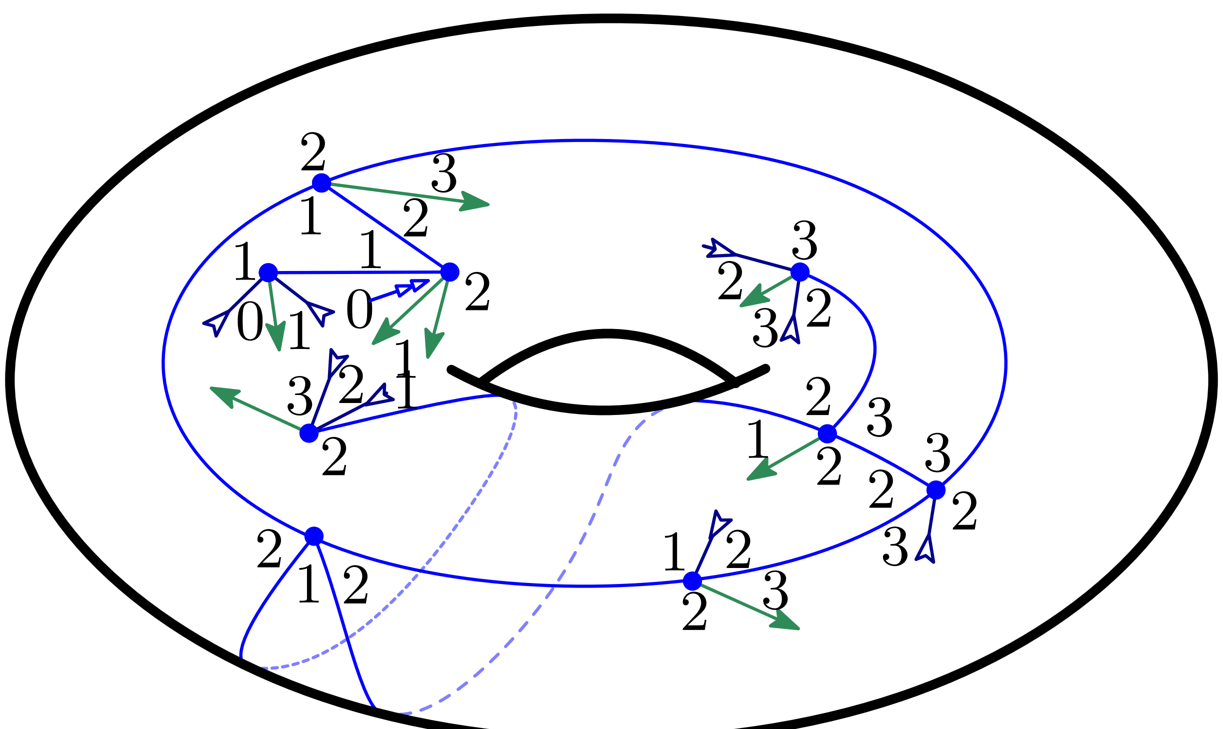

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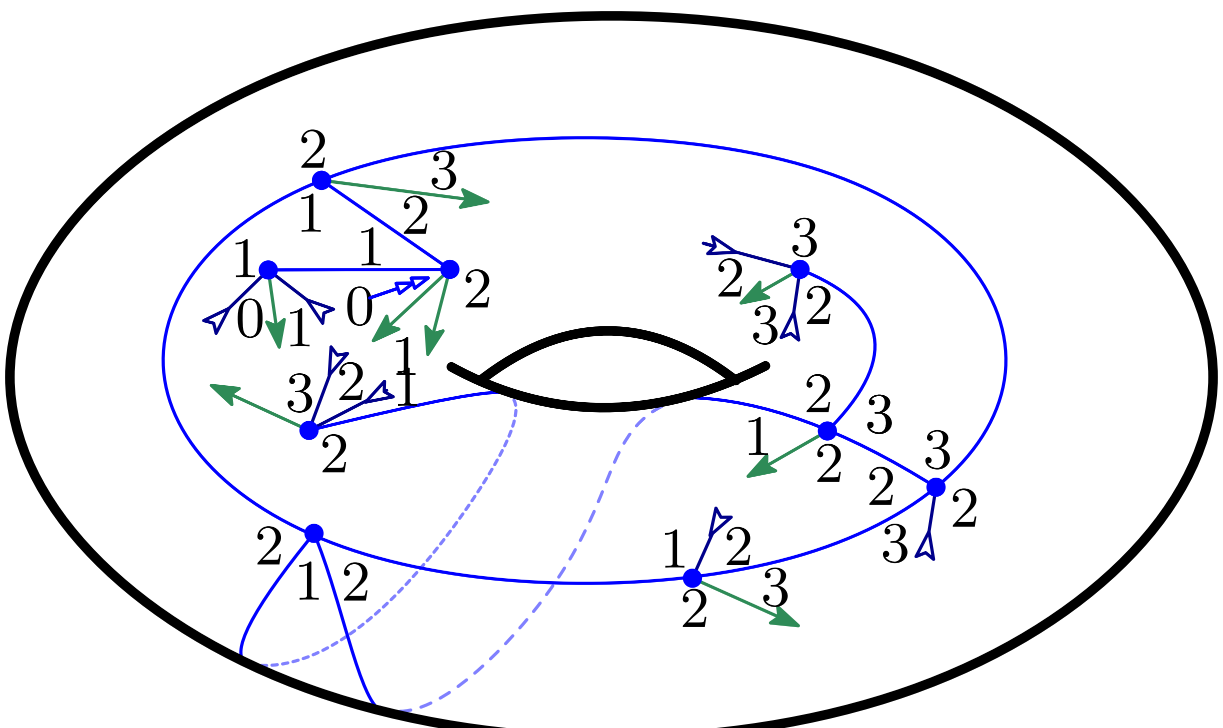

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good labeling with respect to the orientation obtained by orienting **backwards** the edges in the contour of the unique face.

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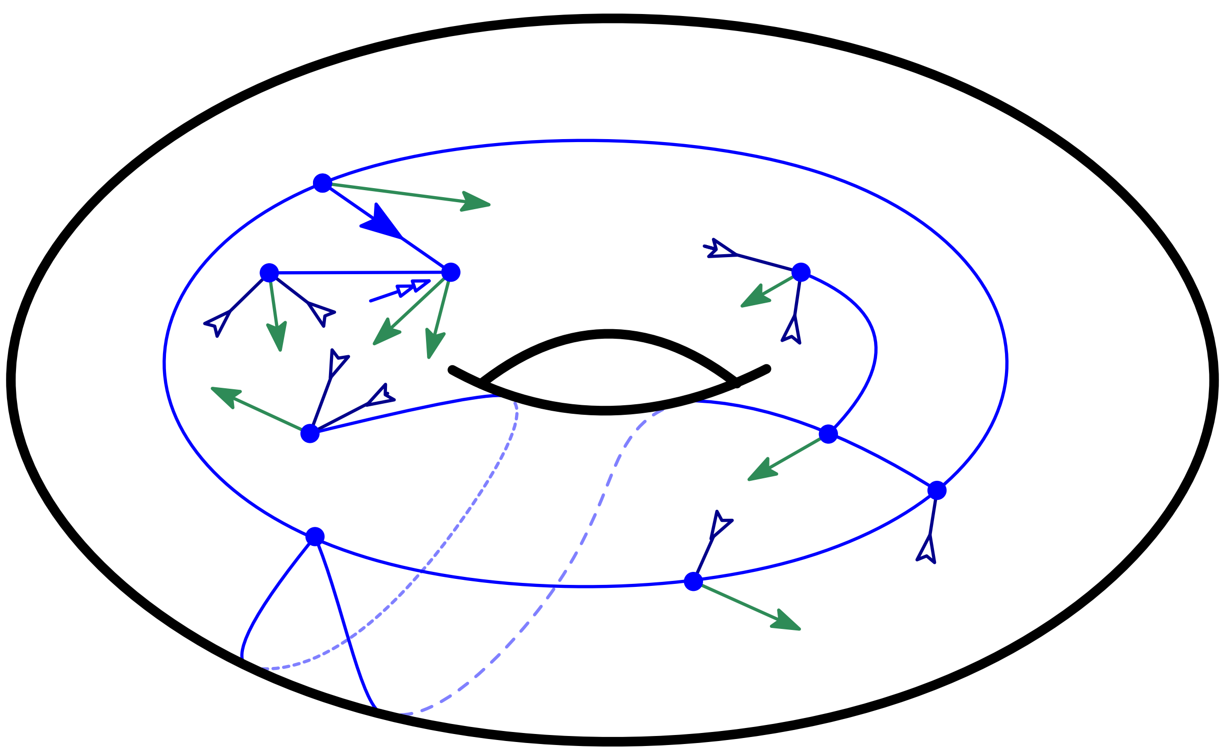

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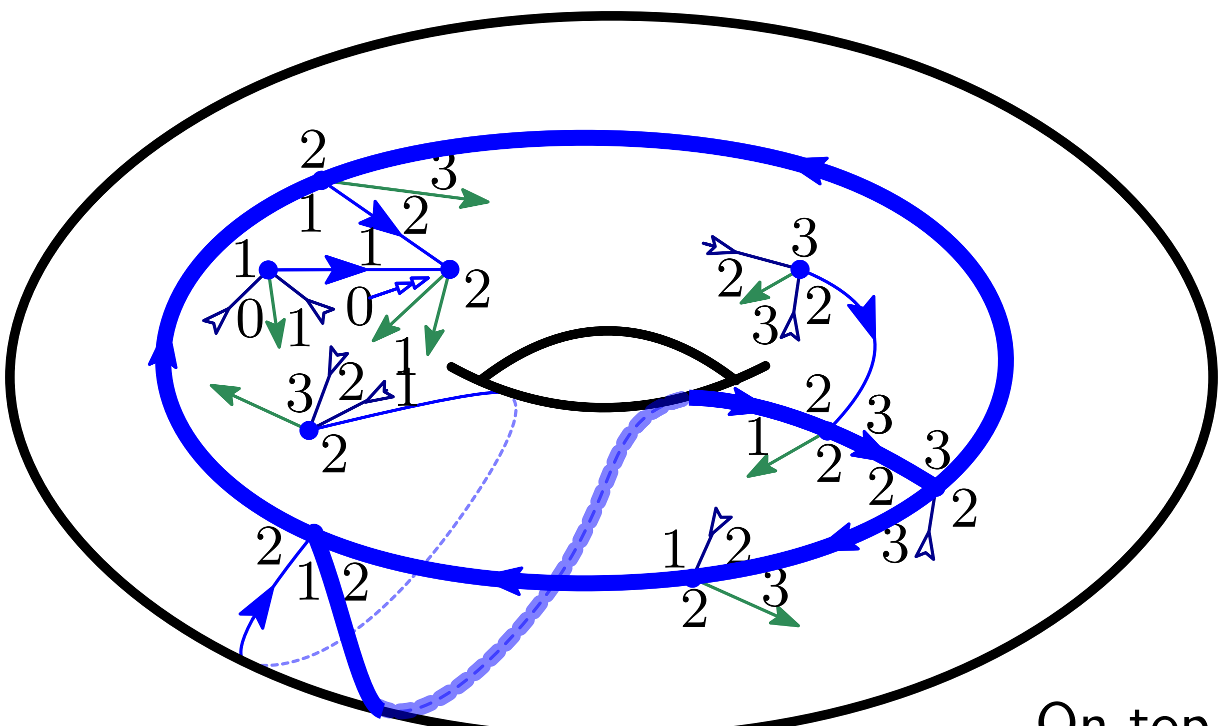

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On top of the local constraints around each vertex, the fact that the labeling is **good** gives some compatibility constraints for the edges of the **non-contractible cycles**.

In higher genus

Theorem [Lepoutre '19]:

4-valent bicolorable maps of genus g

$\updownarrow$ bijection

4-valent blossoming unicellular maps of genus g,
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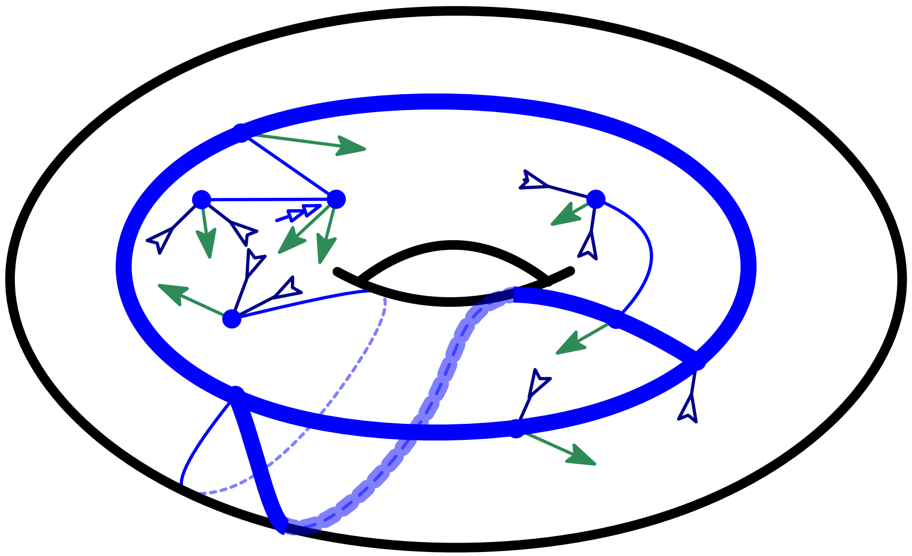
How to enumerate these objects ?

How to prove the rationality schemes with this bijection?

In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the **core** + tree-like parts



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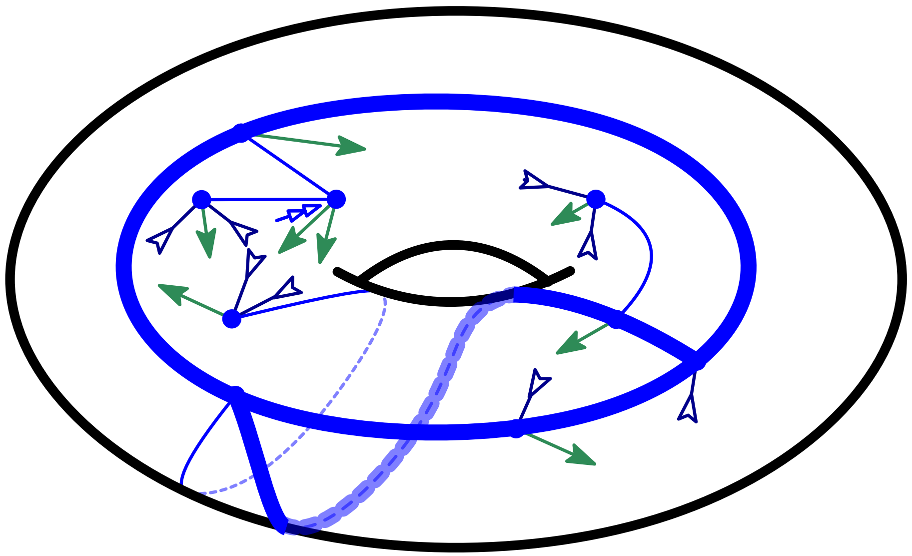
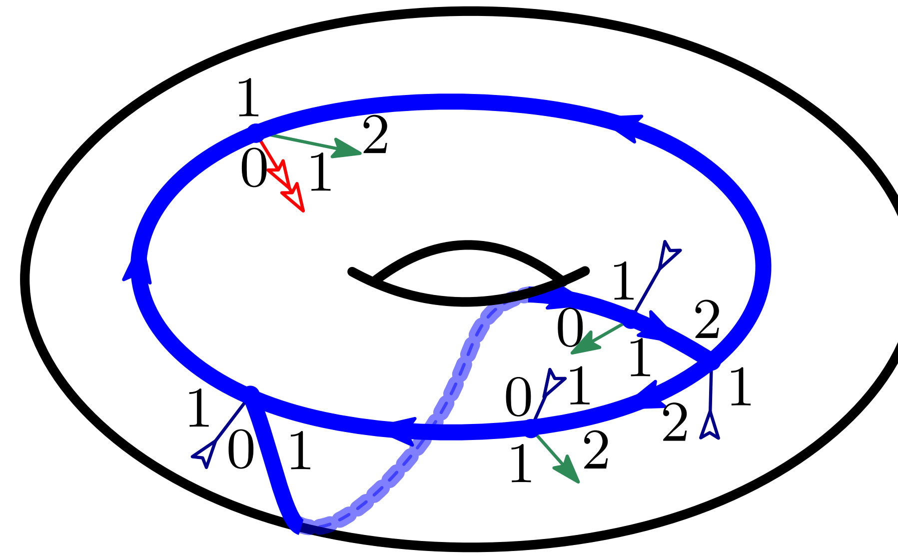


1st step:

erase the trees

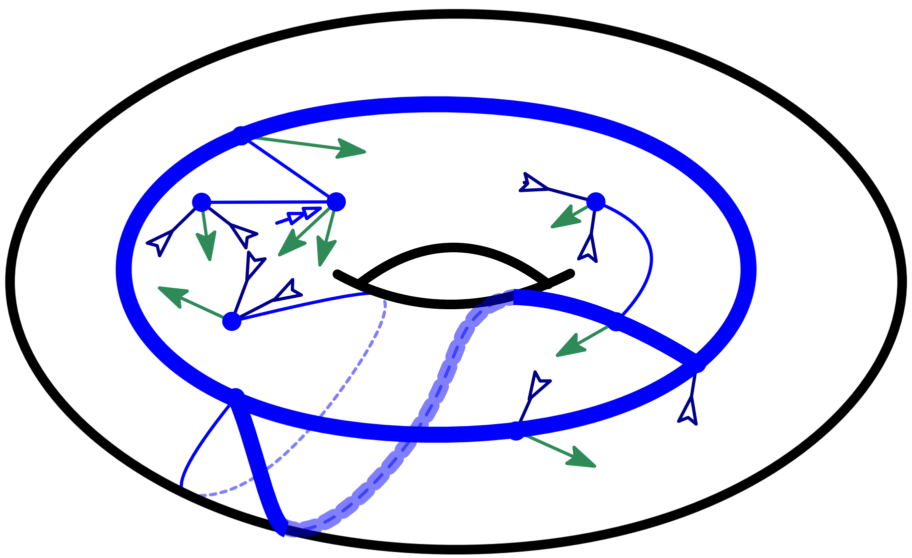
i.e.

- replace trees by
- tree containing the root by



In higher genus: scheme reduction

Unicellular map = non-contractible cycles, the **core** + tree-like parts



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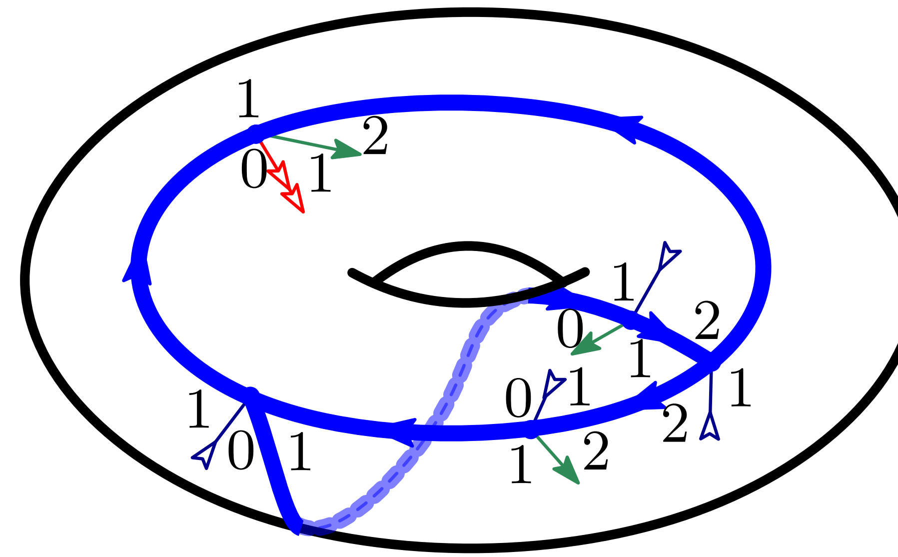

still a non-negative good labeling !

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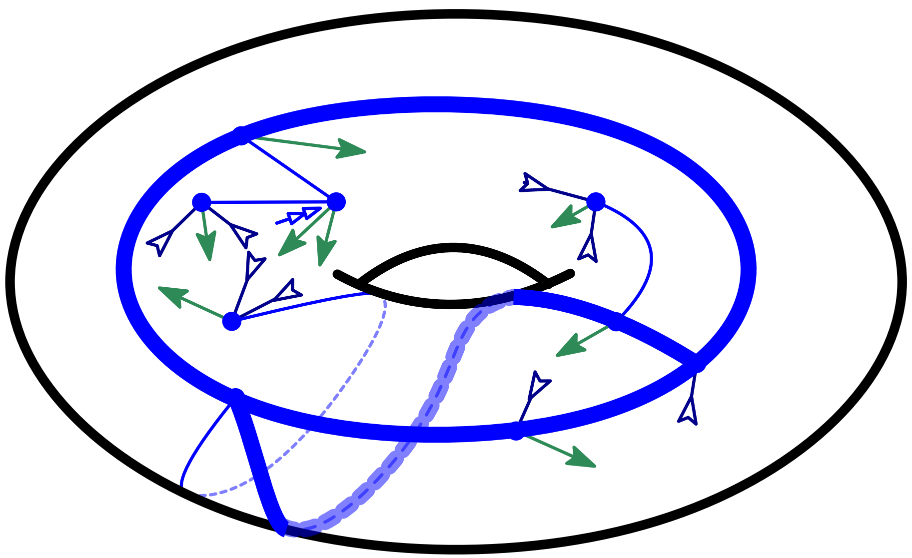

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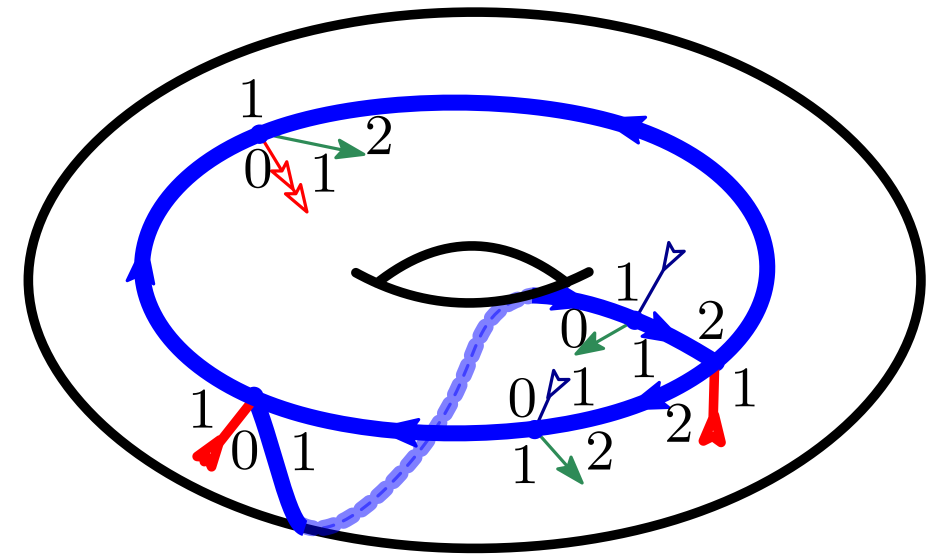

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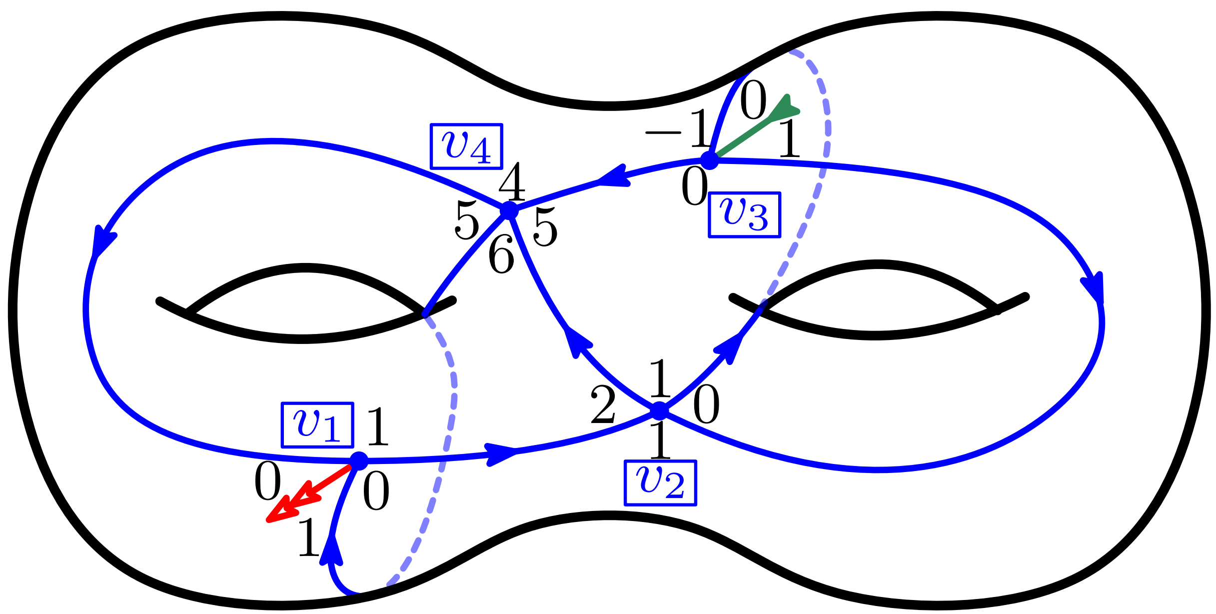
2nd step:

reroot at a "scheme stem"

numbers of such stems depends on the **shape of the scheme**.



v_4
v_3
v_1
v_2
0
-1
1
0
4
5
5
6
1
2
0
1
1
0
0
1

In higher genus: scheme reduction

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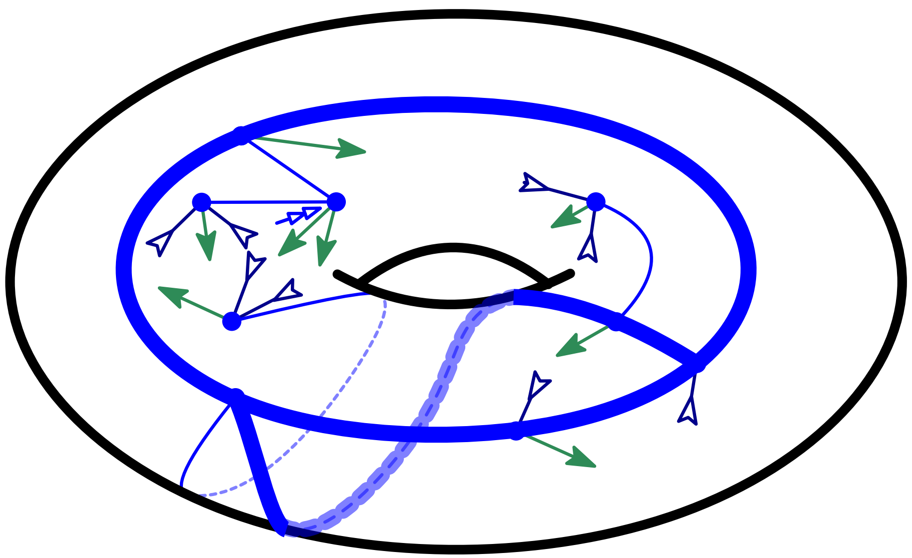

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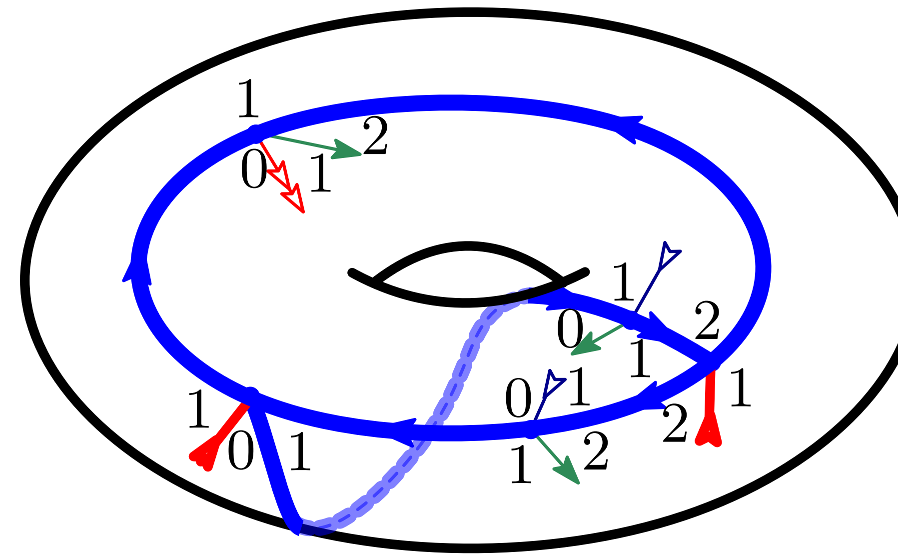

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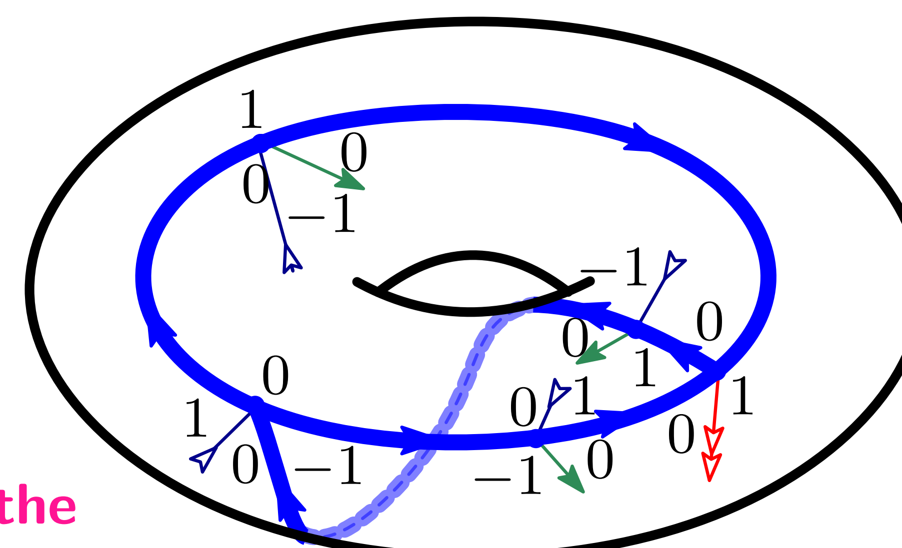

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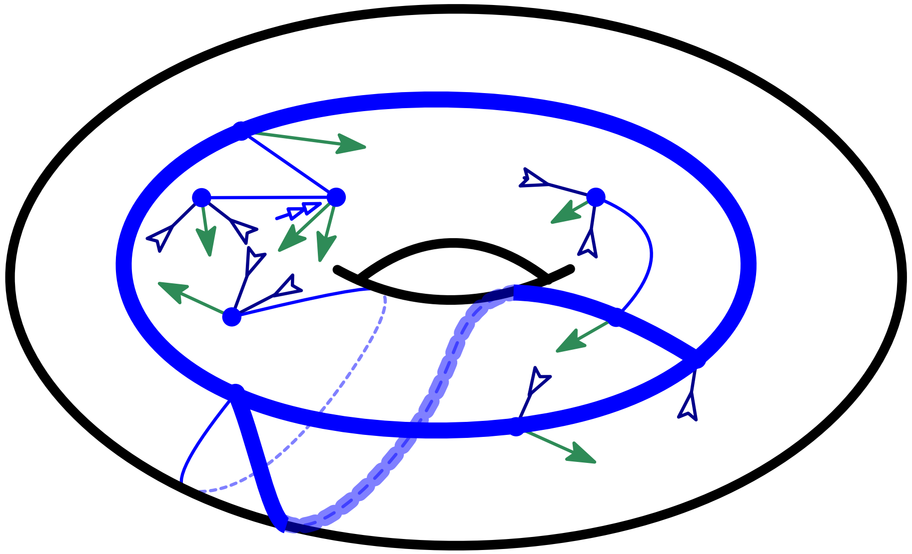

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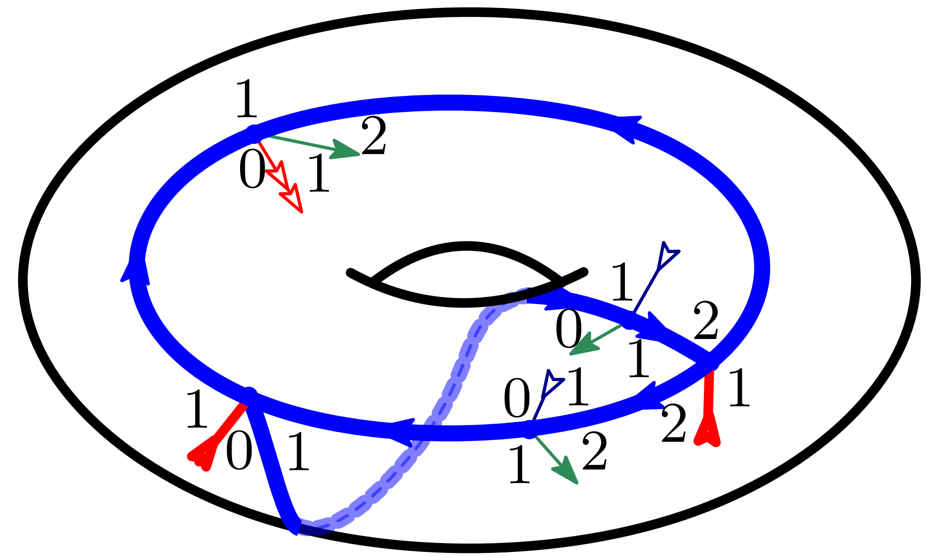

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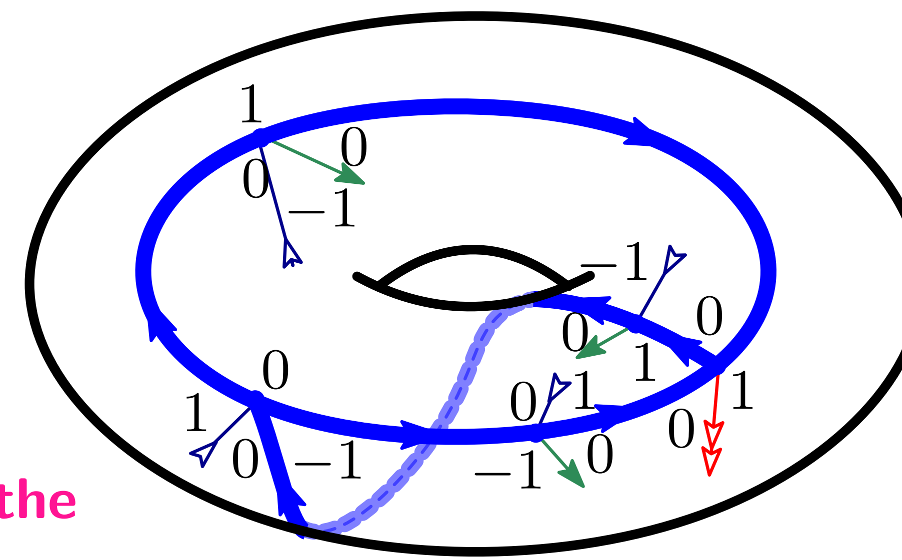

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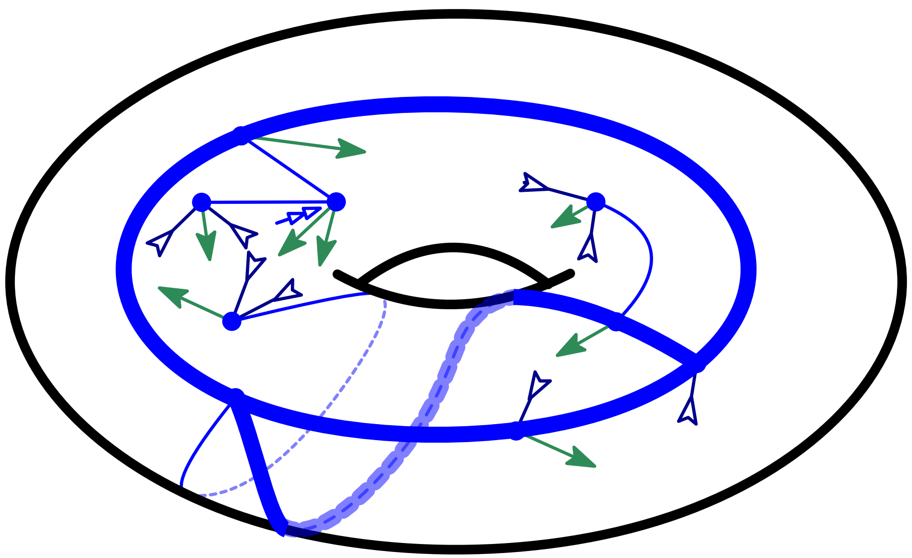

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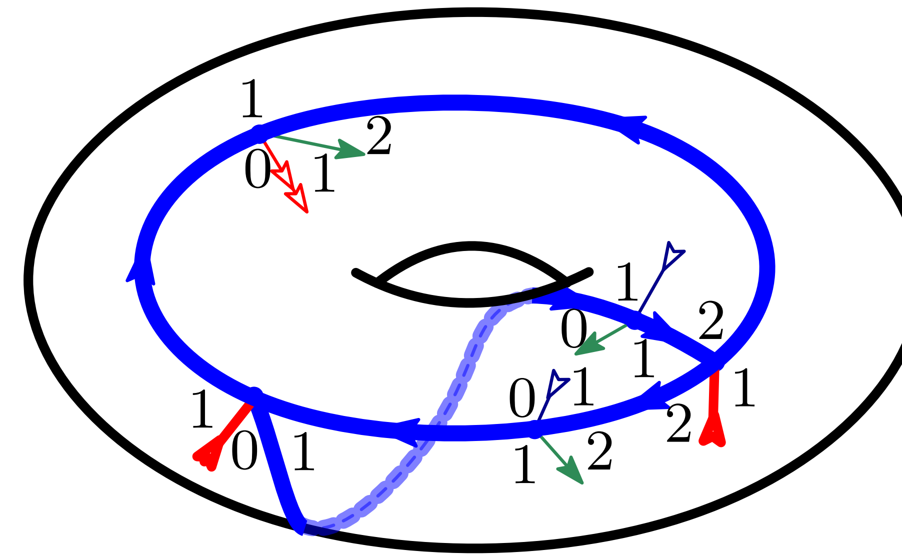

still a non-negative good labeling !

$M_s(z)$ = gen. series of maps that admit s as scheme.
after applying the radial construction + Lepoutre's bijection + erasing the trees !

then:

$$M_s(z) = \kappa_s \cdot R_s(T(z))$$

R_s = blossoming cores that admit s as scheme

κ_s = cst which depends on s

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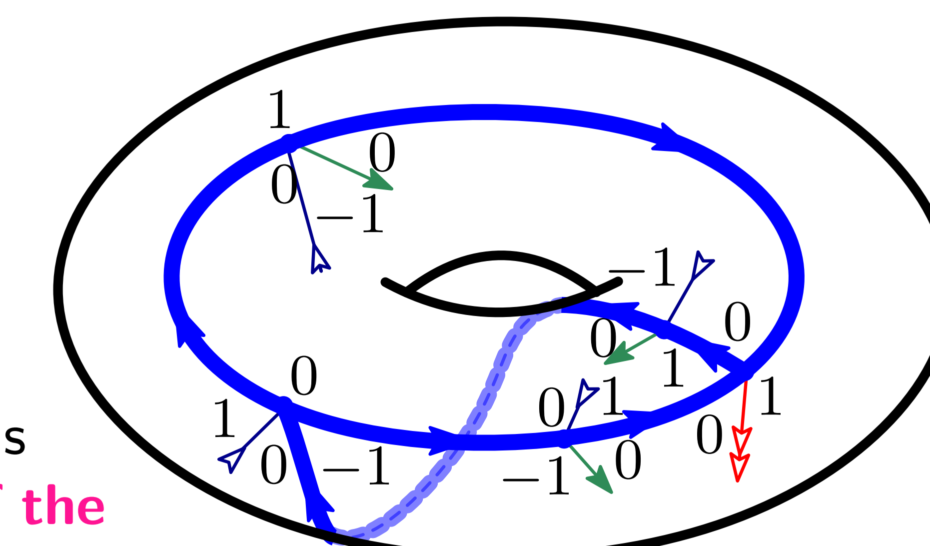

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Back to the theorems

Theorem: [Bender, Canfield 91], **first bijective proof** in [Lepoutre 19]

For any $g \geq 1$, let $M_g(z) = \sum_m z^{|E(m)|}$, where $m \in \Big\{\text{maps of genus } g\Big\}$.

Then M_g is a rational function of T, where:

$$T = \text{ unique formal power series defined by } T = z + 3T^2$$

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Remark: an analogous statement does not hold for the bijection of [Chapuy – Marcus – Schaeffer]

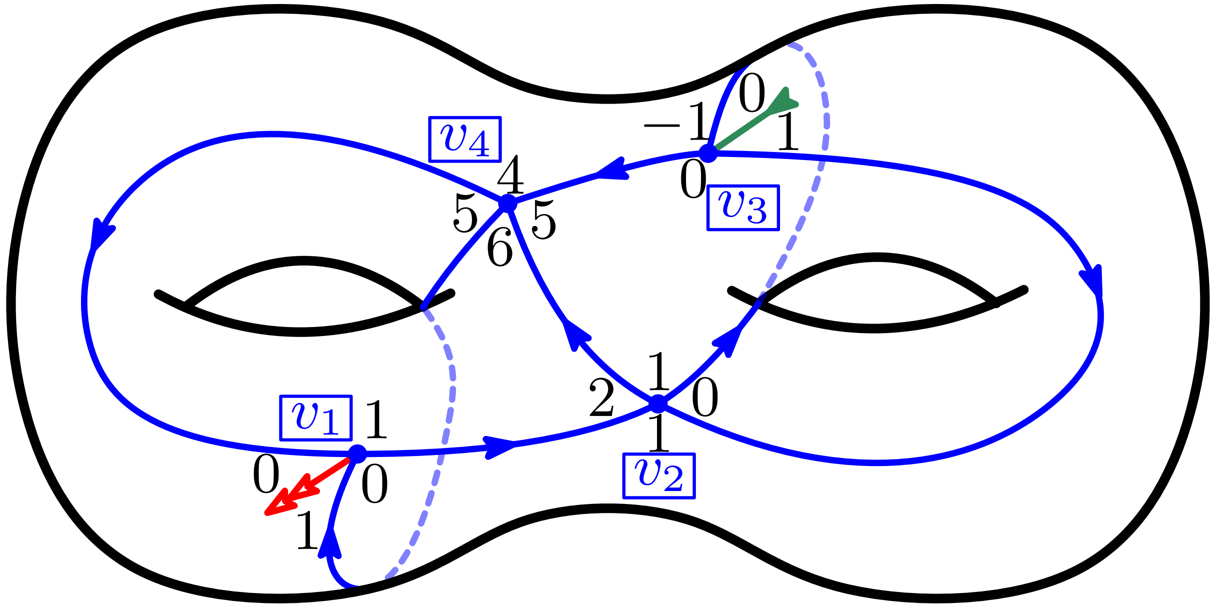
Kind of a miracle that it does work for this bijection.

But, this seems robust: extension to **bivariate enumeration** and to **Eulerian k-angulations**

(w.i.p with Castellvi and Fusy)

Thank you for your attention !

Schemes in higher genus



In higher genus: labeled scheme