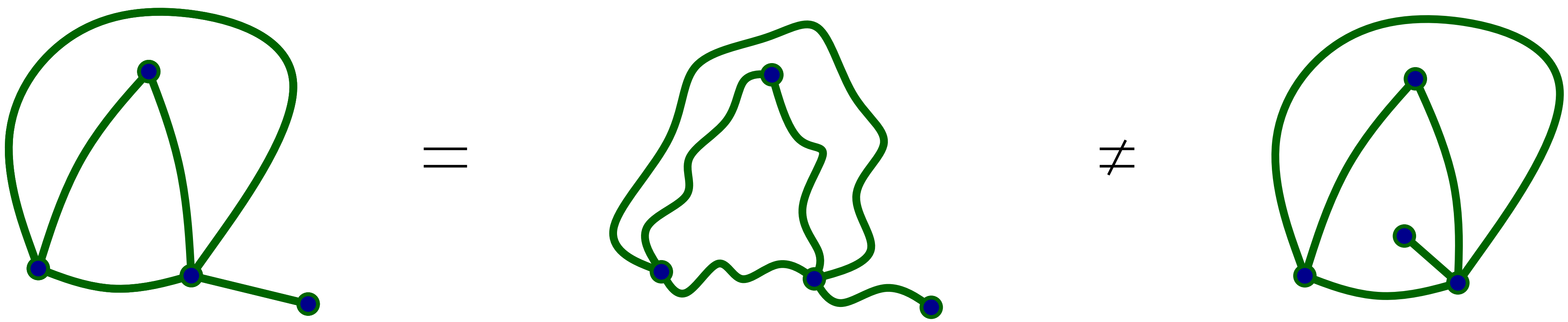
Bipolar orientations and blossoming trees

Marie Albenque (CNRS, Paris)
Joint work with Dominique Poulalhon

Kolkom, November 17th 2012

Plane Maps.

A **plane map** is the embedding of a connected graph in the plane up to continuous deformations.



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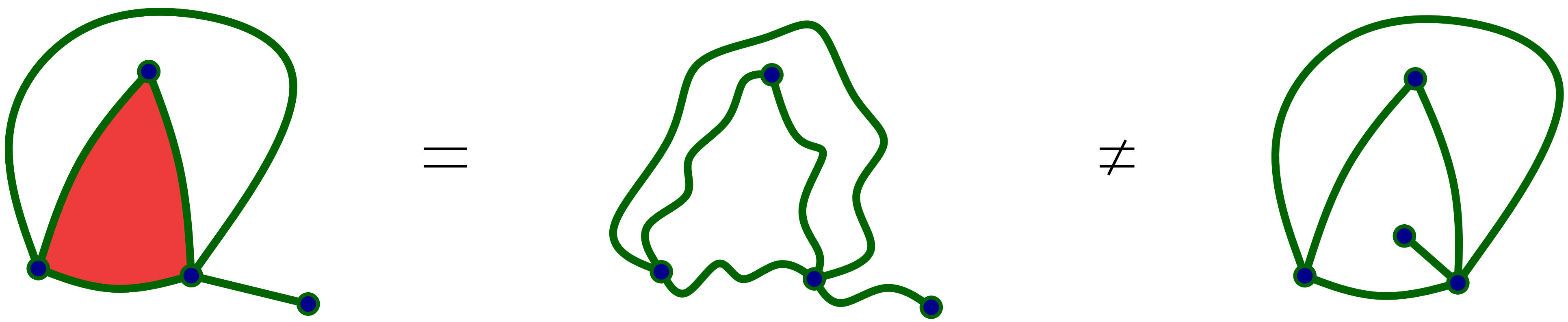

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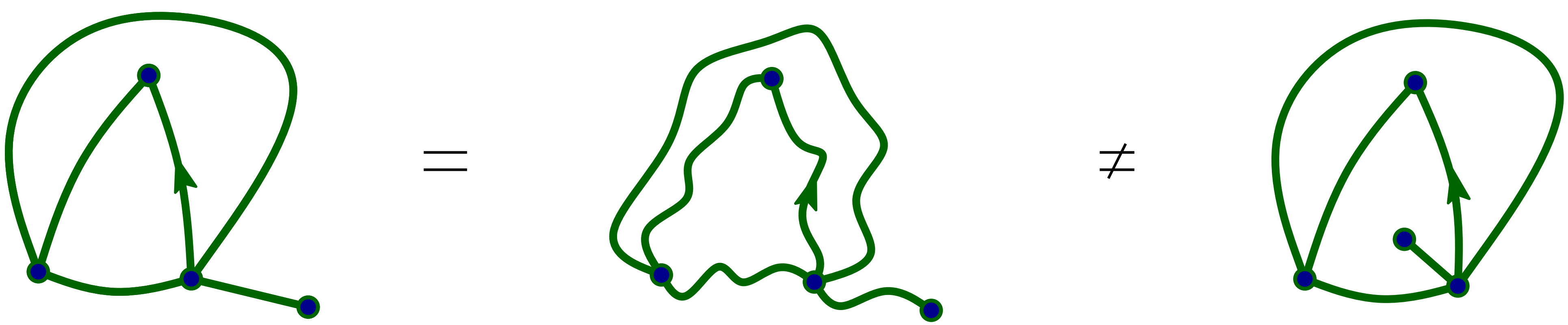

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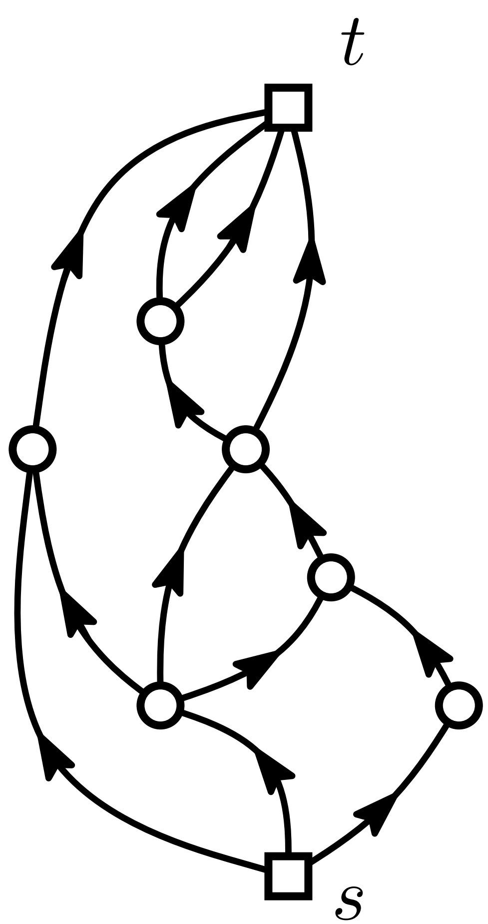
Plane maps are **rooted**.

There is one special face which is infinite: the **outer face**.

Plane Bipolar Orientations

A **plane bipolar orientation** is a plane map:

- endowed with an acyclic orientation,
- with a unique **source** vertex (without ingoing edges),
- with a unique **sink** vertex (without outgoing edges).



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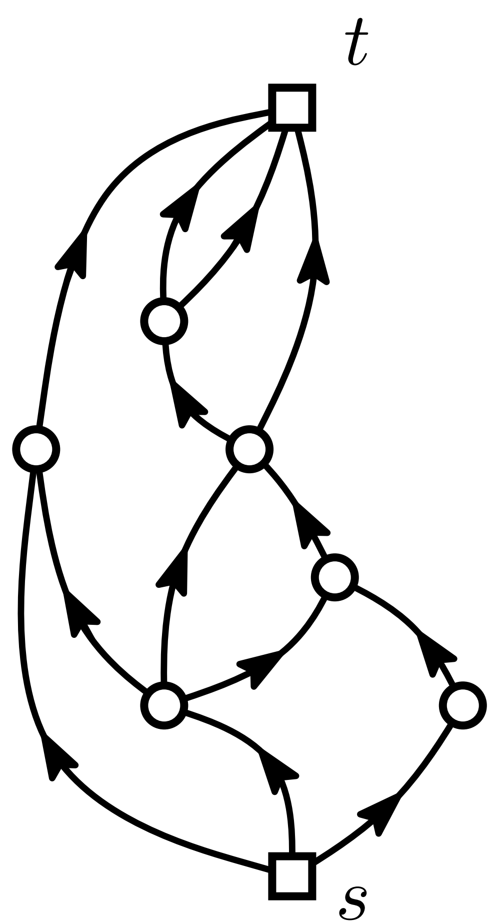

Theorem : The number Θ_{ij} of bipolar orientations with $i+2$ vertices and $j+1$ faces is equal to:

$$\Theta_{ij} = \frac{2(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}.$$

[Baxter '01]
[Fusy, Poulalhon, Schaeffer '09]
[Bonichon, Bousquet-Mélou, Fusy '10]
[Felsner, Fusy, Noy, Orden '11]

Enumeration

One of the main question when studying some families of maps:

How many maps belong to this family ?

- **Recursive decomposition**: Tutte '60s. Baxter '01
- **Matrix integrals**: t'Hooft '74,Brézin, Itzykson, Parisi and Zuber '78.
- **Representation of the symmetric group**: Goulden and Jackson '87.
- **Bijective approach with labeled trees**: Cori-Vauquelin '81, Schaeffer '98, Bouttier, Di Francesco and Guitter '04, Bernardi, Chapuy, Fusy, Miermont, ...
- **Bijective approach with blossoming trees**: Schaeffer '98, Schaeffer and Bousquet-Mélou '00, Poulalhon and Schaeffer '05, Fusy, Poulalhon and Schaeffer '06.

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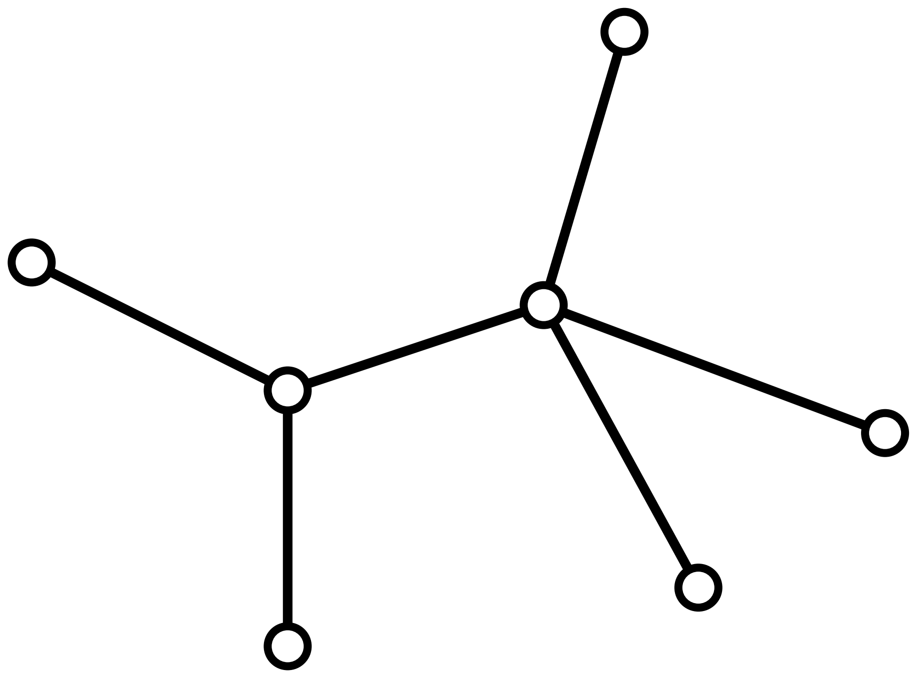
What is a blossoming tree ?

A **blossoming tree** is a plane tree where vertices can carry **opening stems** or **closing stems**, such that :

$$\# \text{ closing stems} = \# \text{ opening stems}$$



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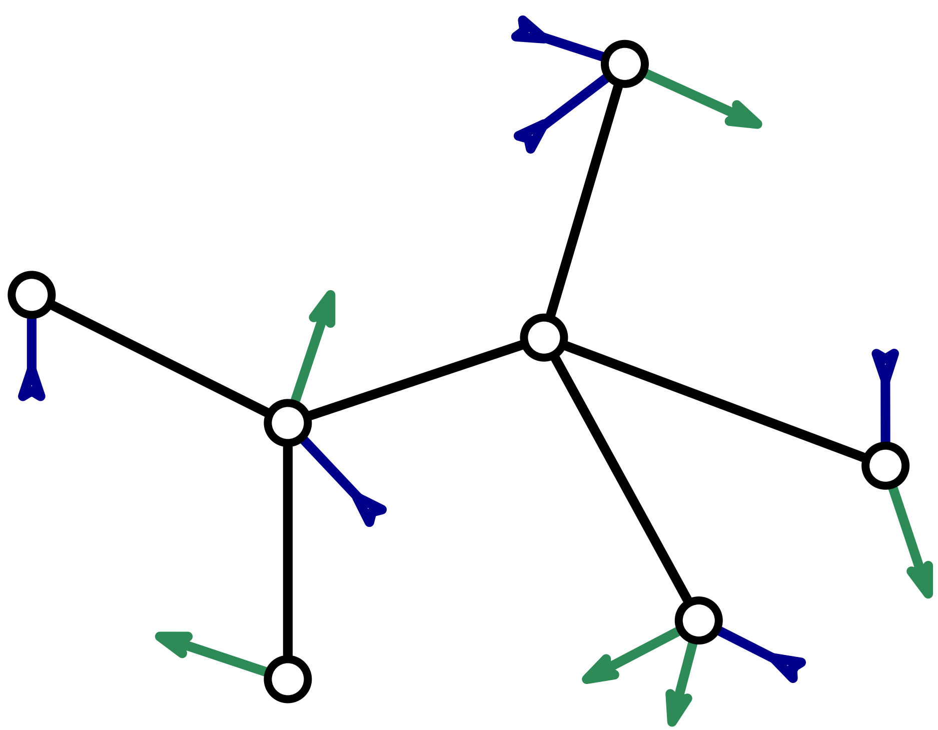

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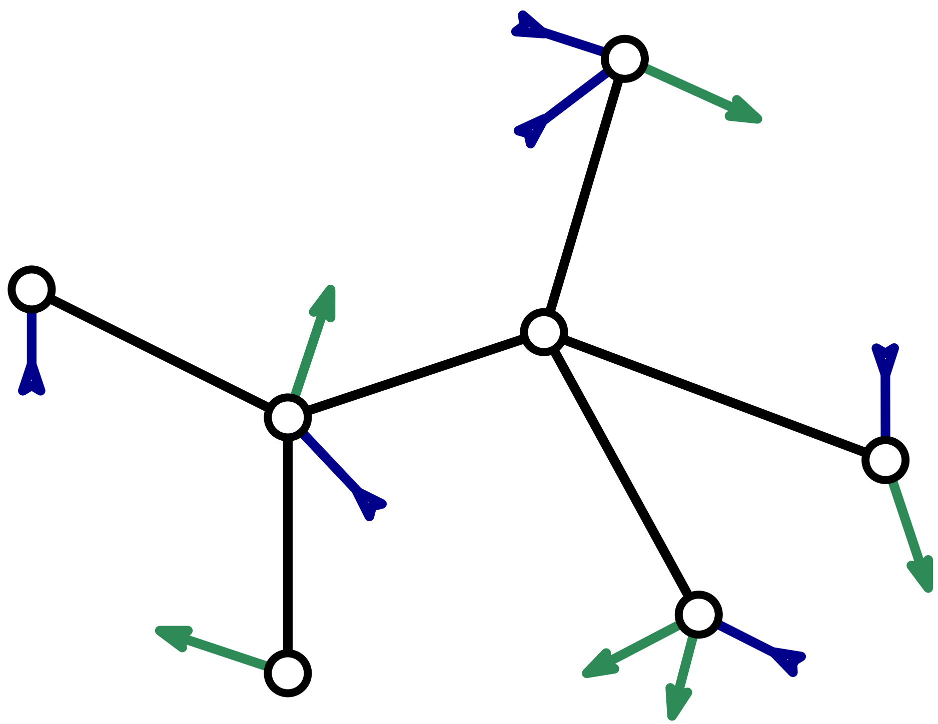

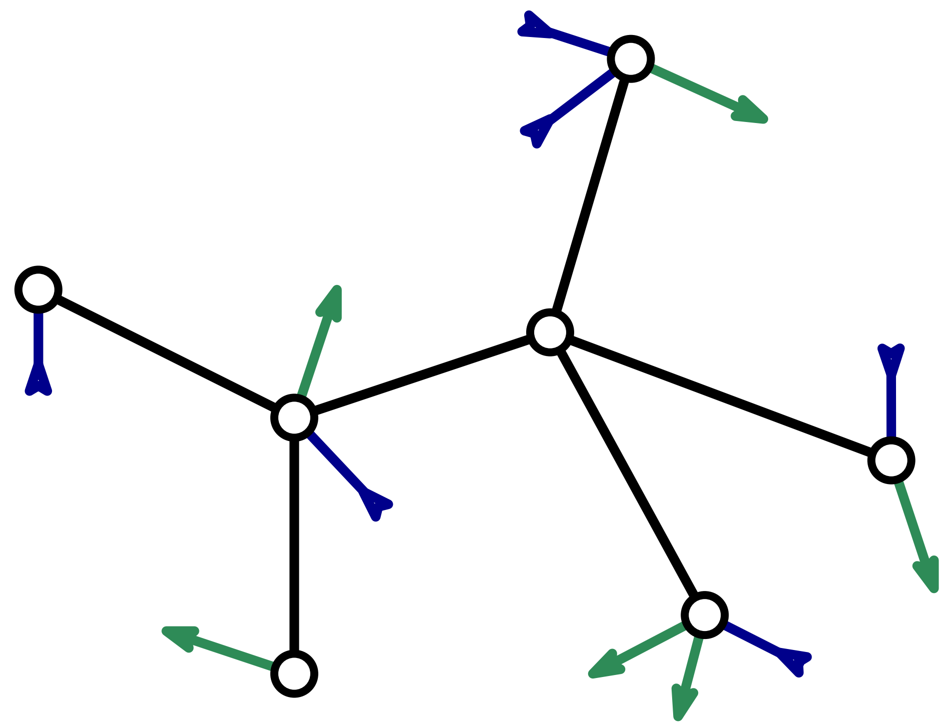

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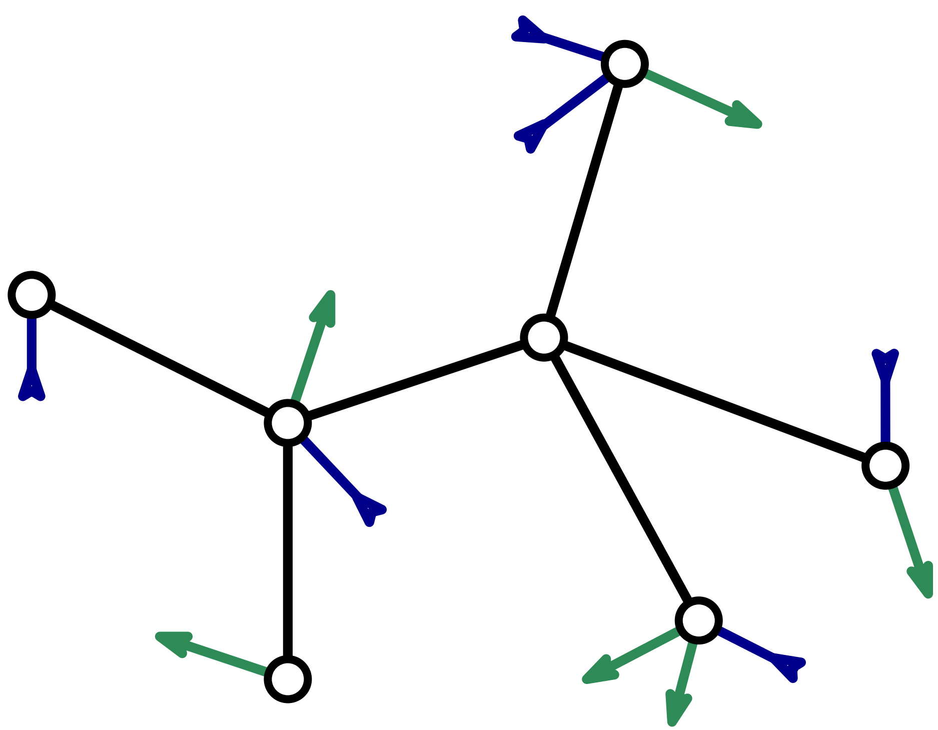

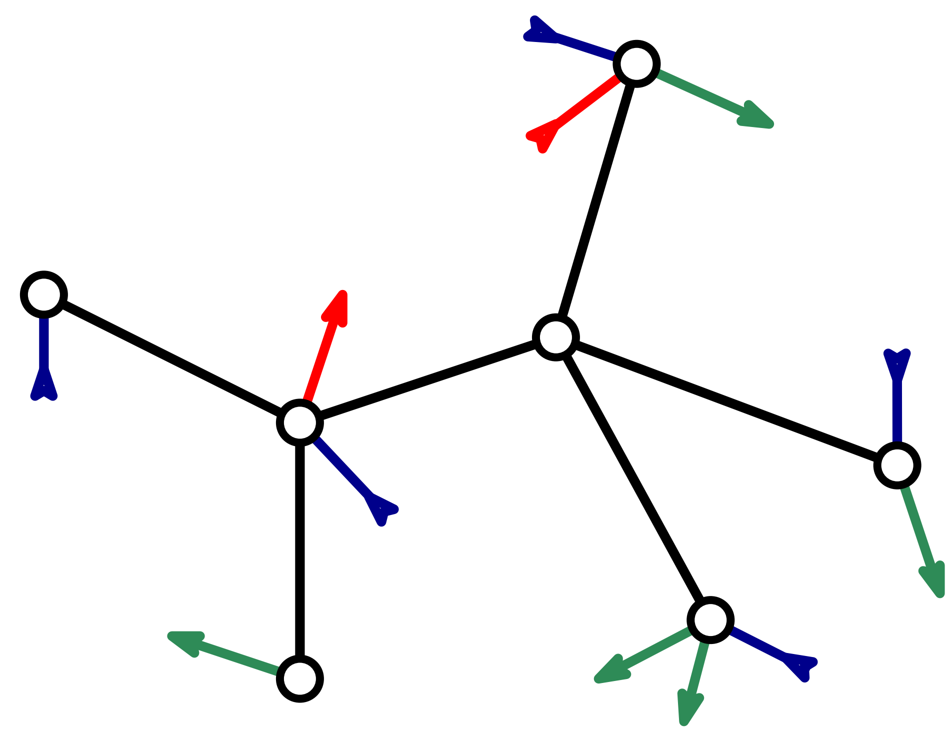

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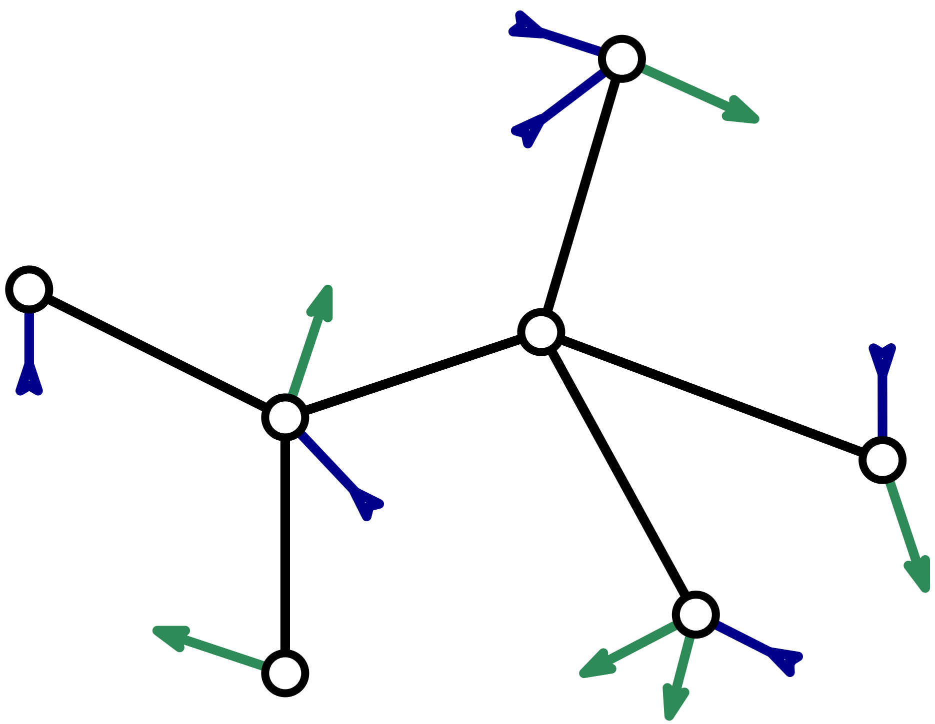

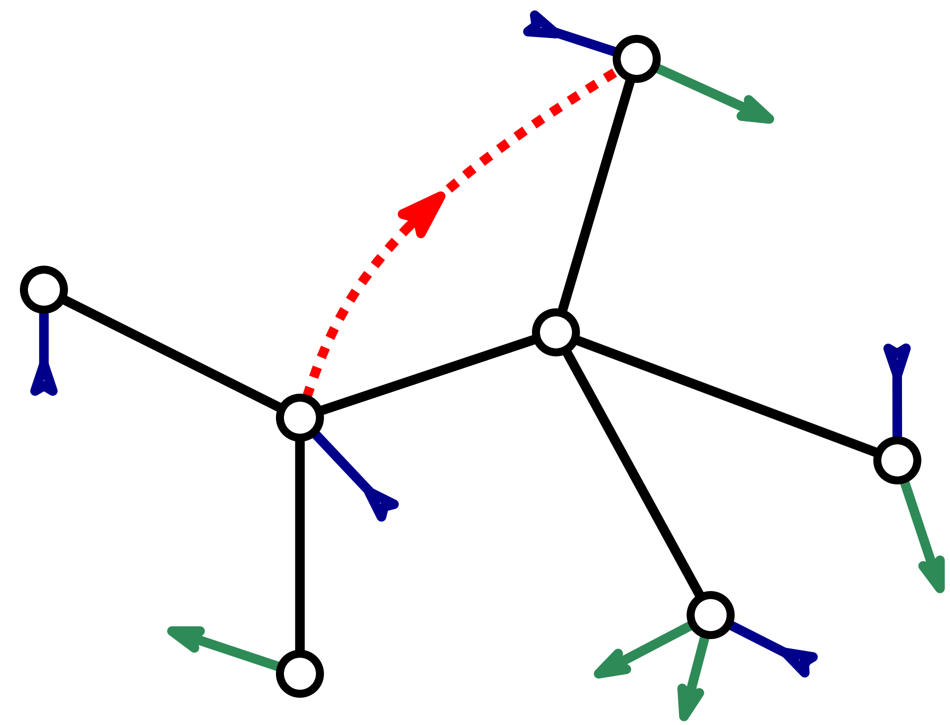

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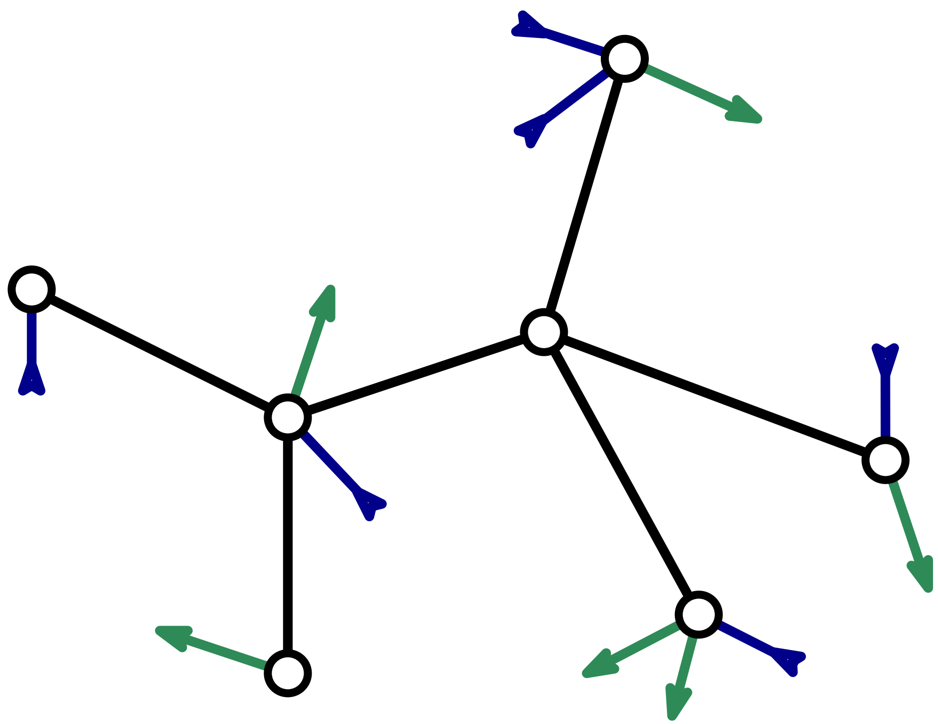

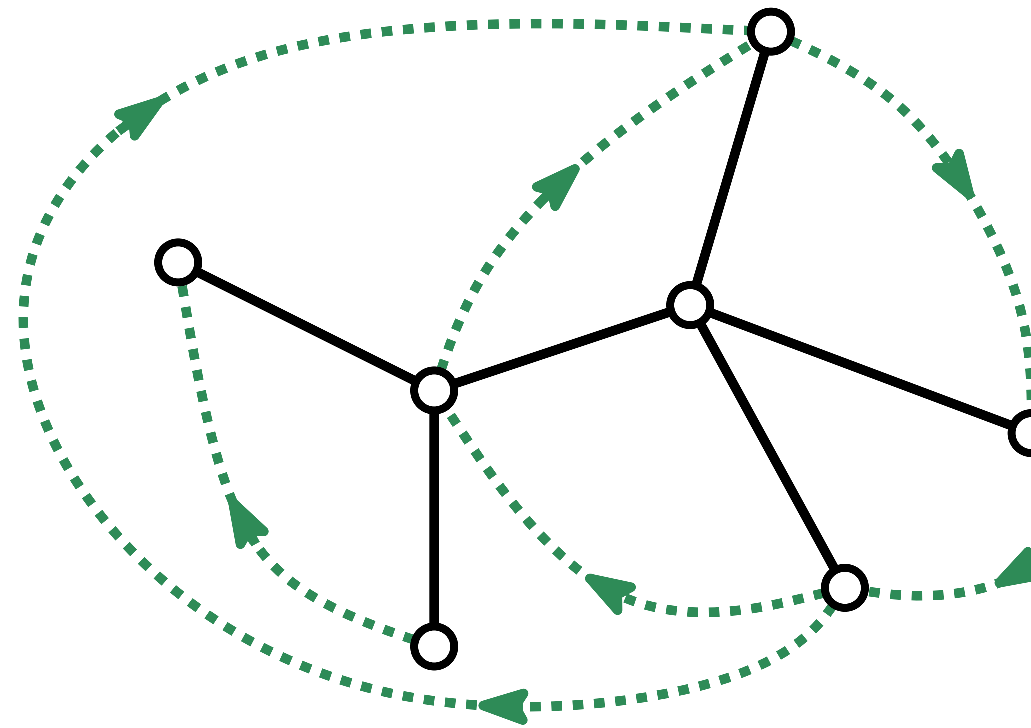

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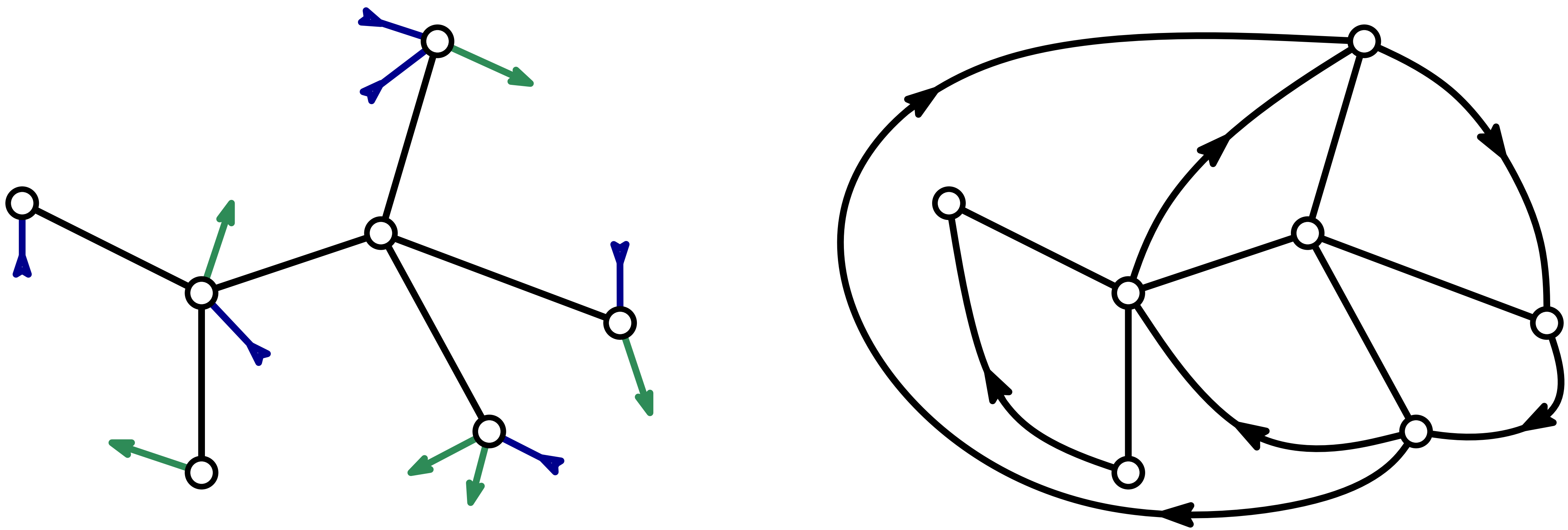

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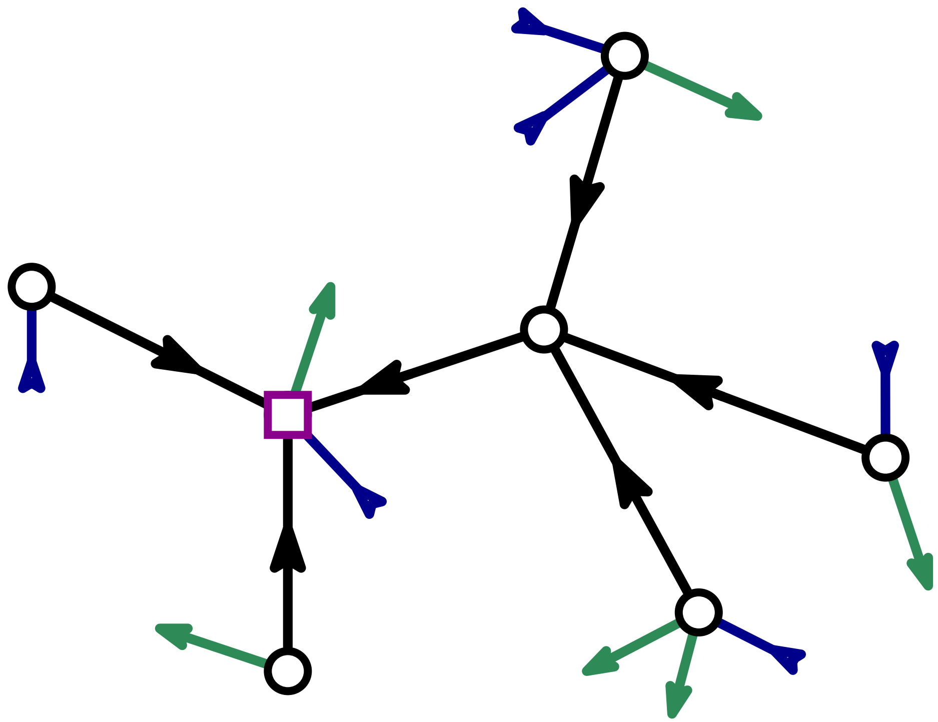

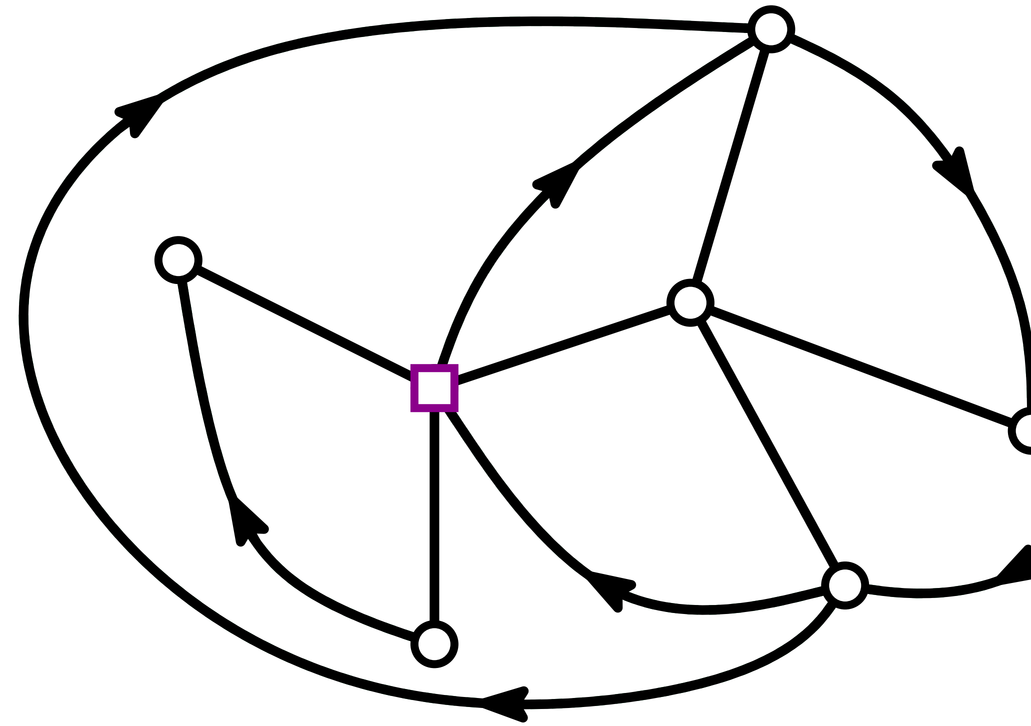

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If the tree is rooted and its edges oriented towards the root + closure edges oriented naturally

$\Rightarrow$ Accessible orientation of the map without ccw cycles.

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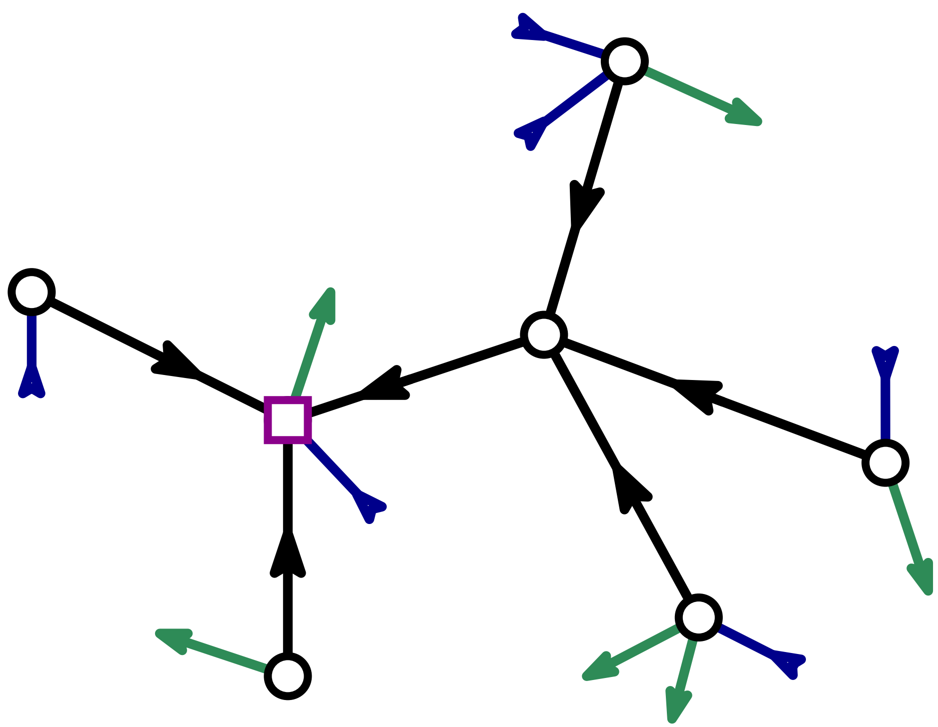

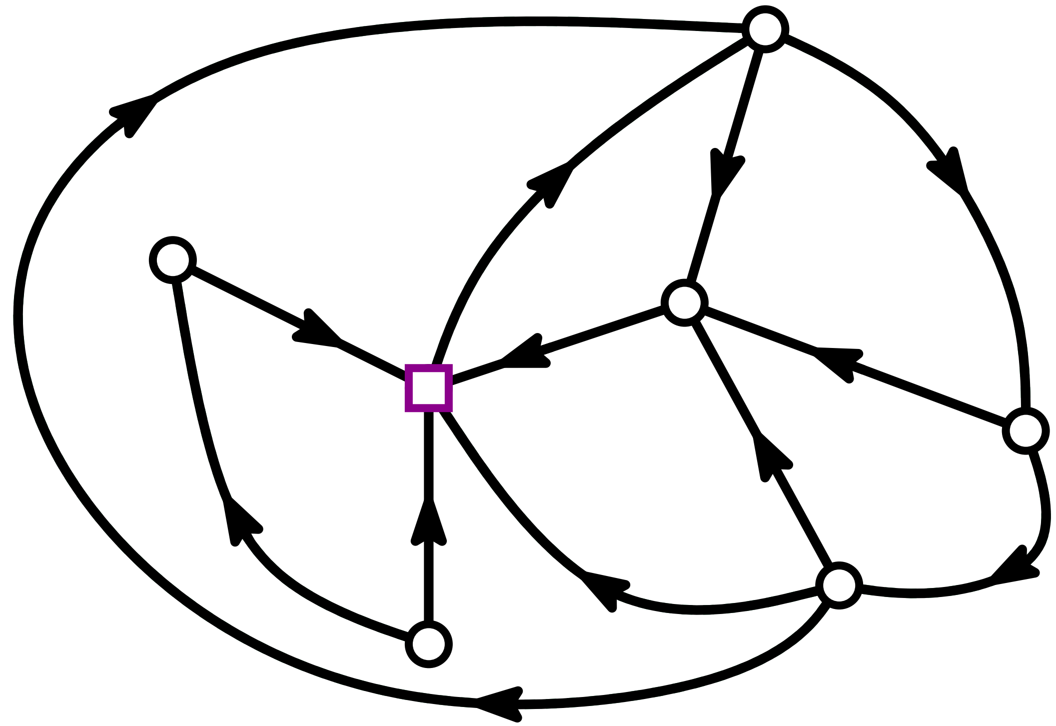

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Can we always find a blossoming tree from a plane map ?

Theorem : [A., Poulalhon]
If a plane map M has a marked vertex v is endowed with an orientation such that :

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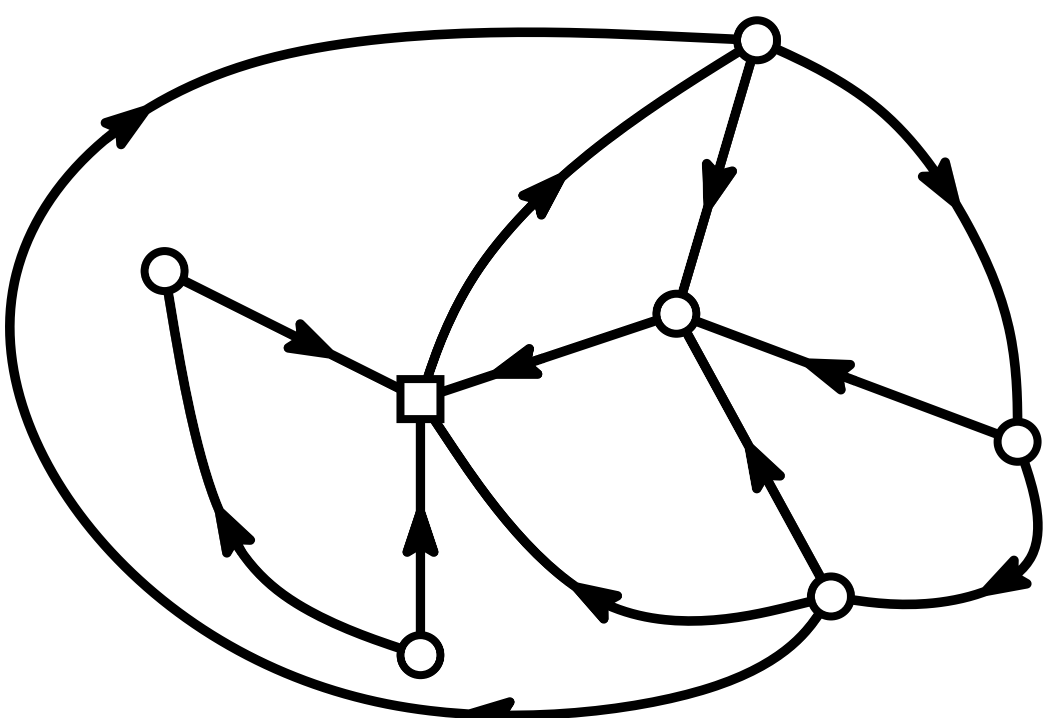

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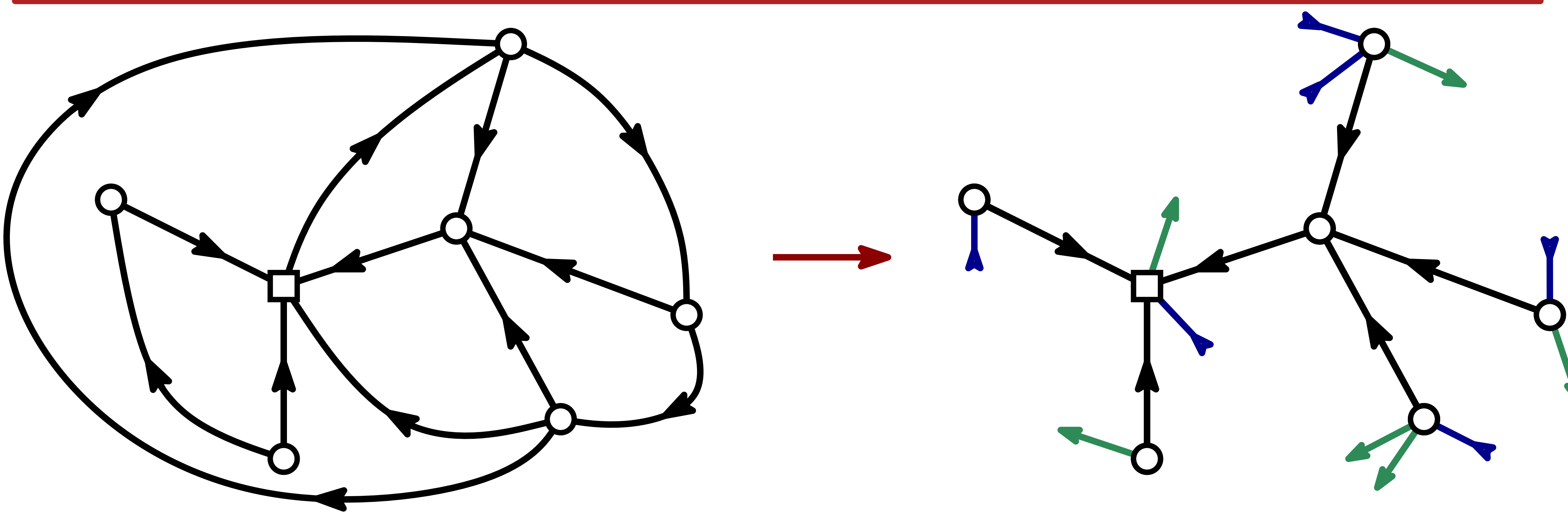

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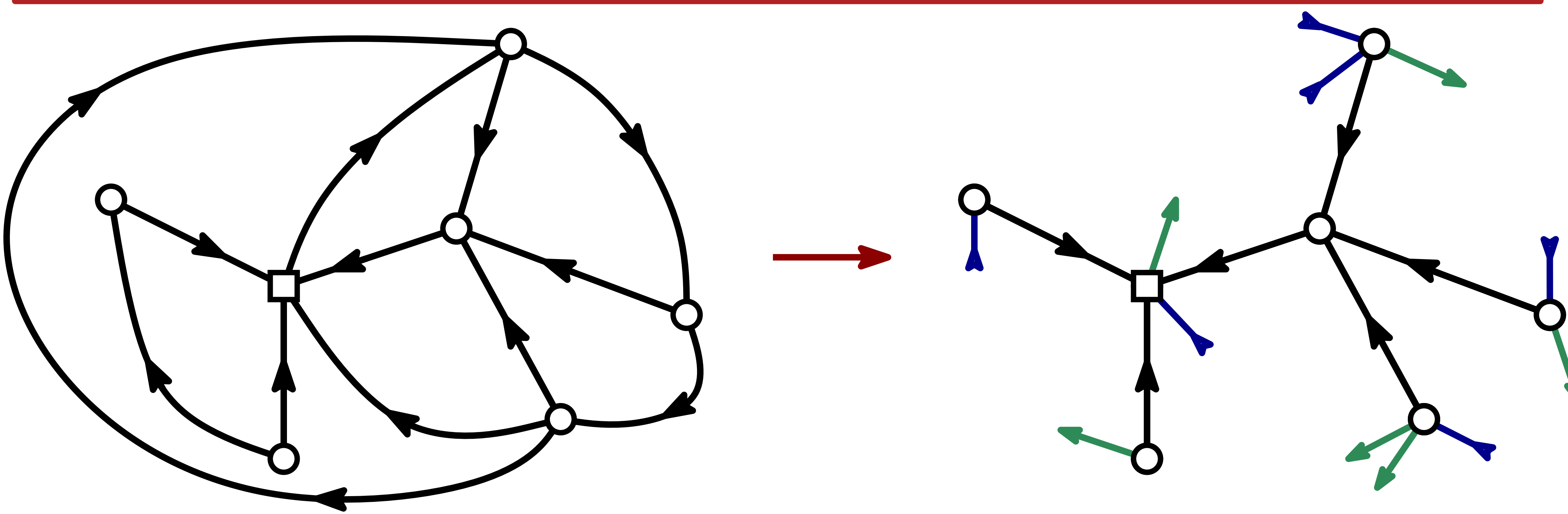
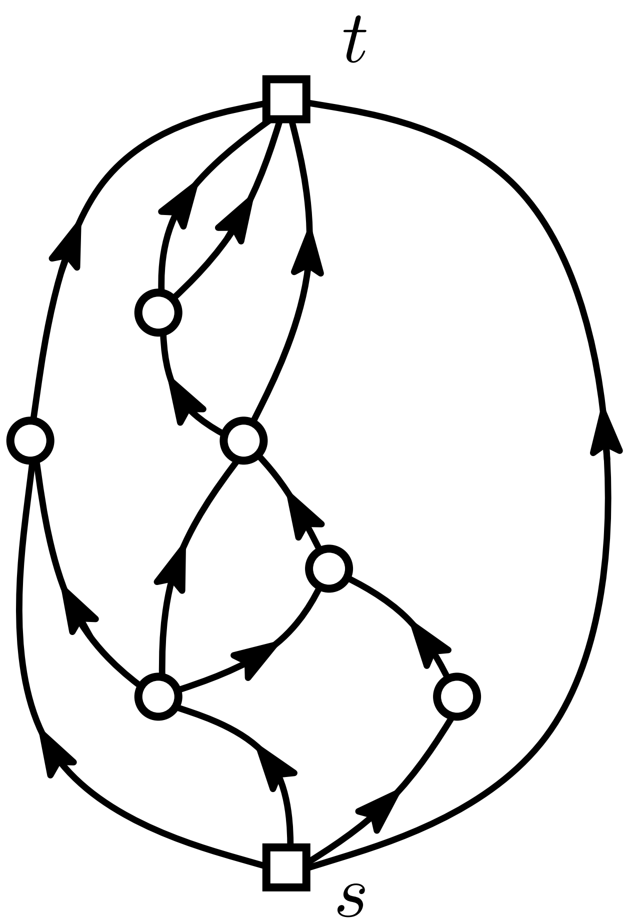
Proof by induction on the number of faces + identification of closure edges ....



Blossoming trees and bipolar orientations



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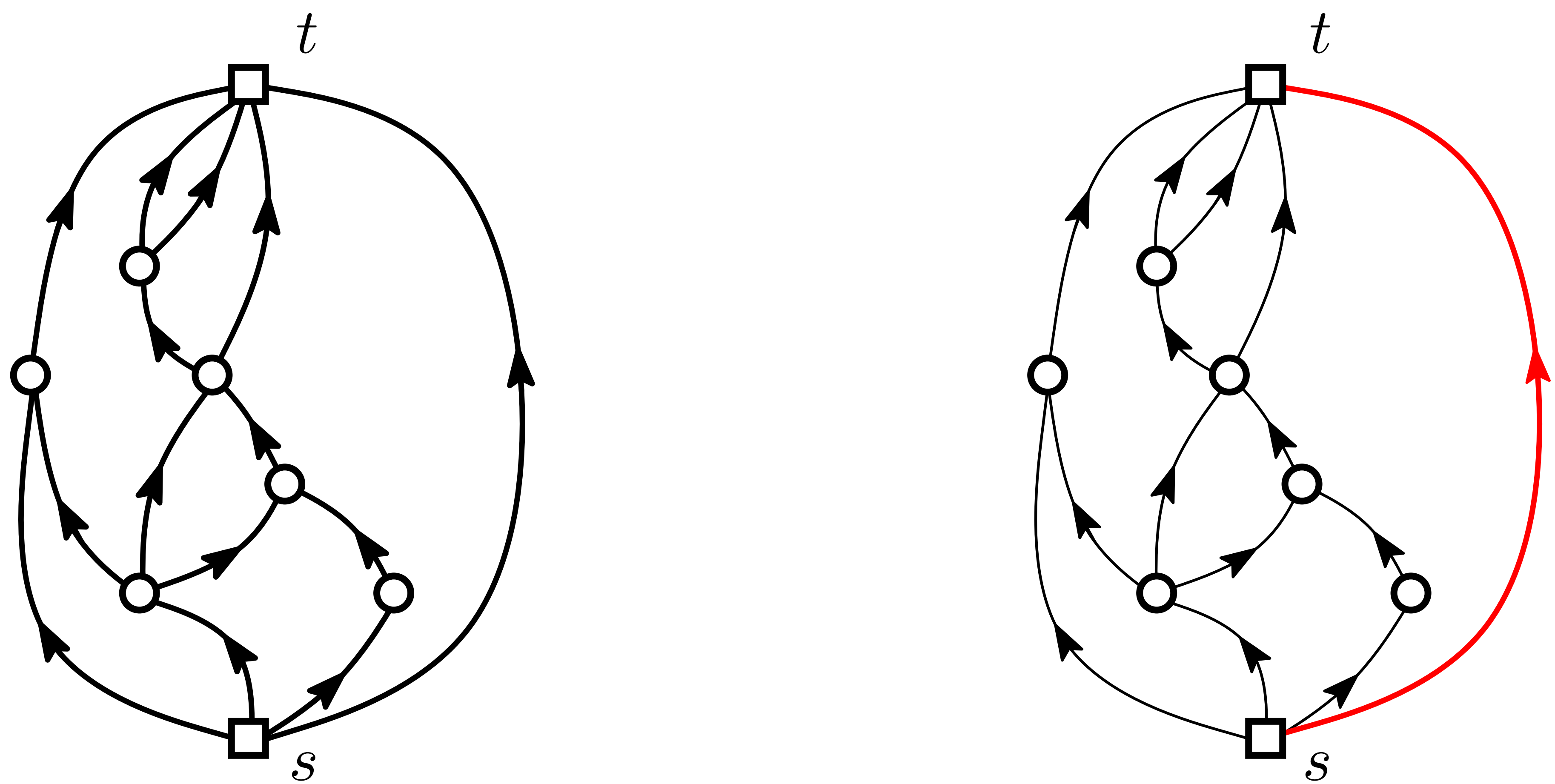

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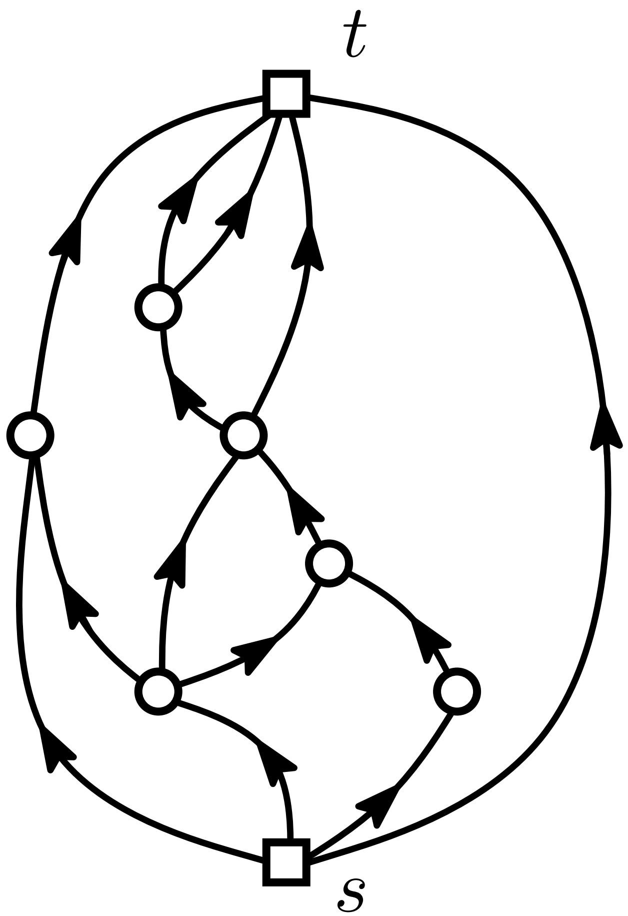

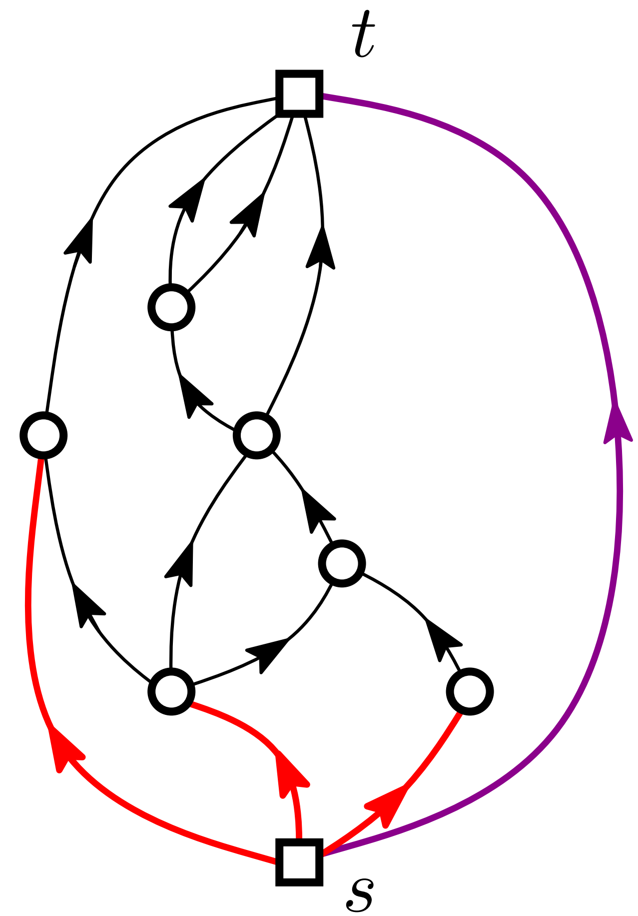

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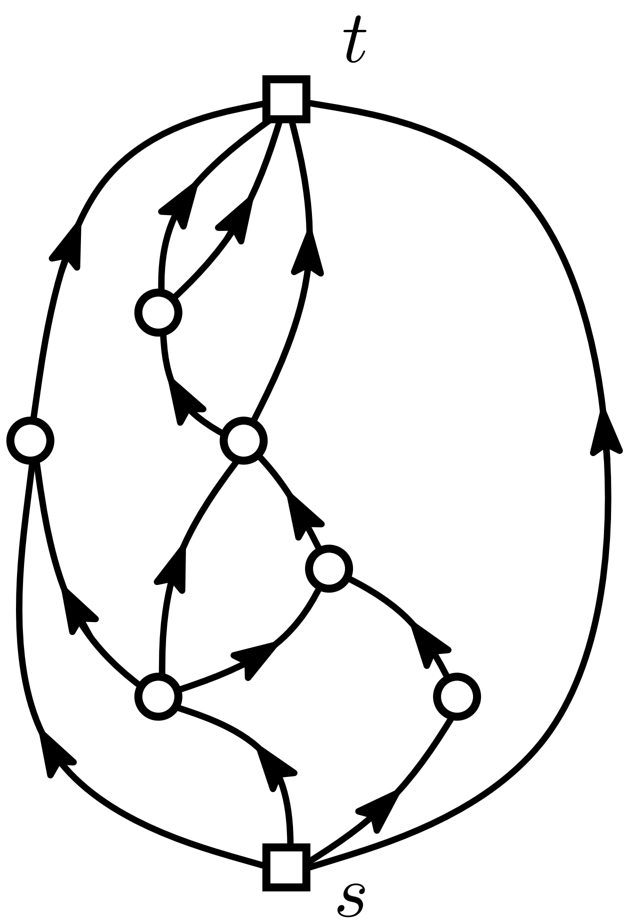

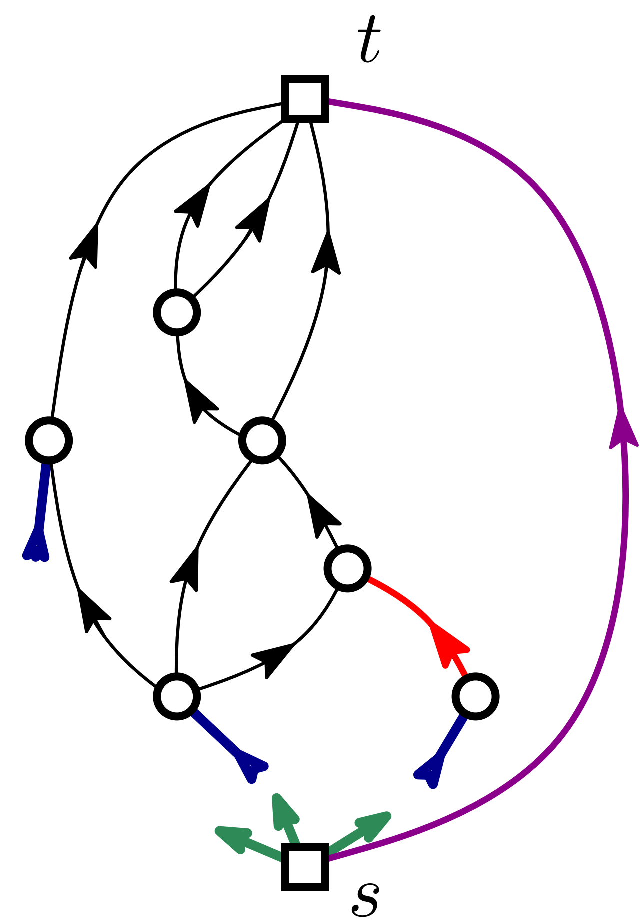

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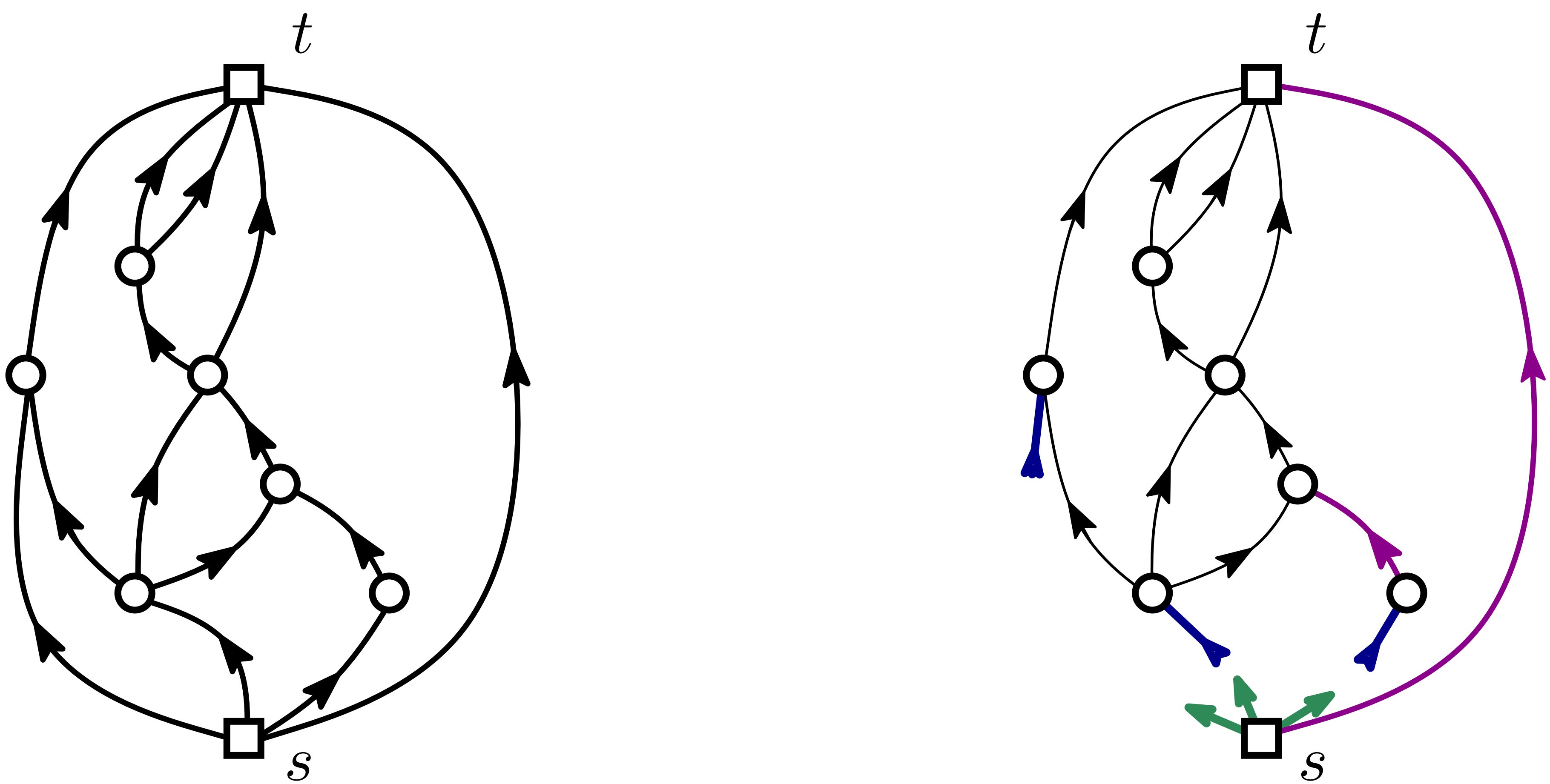

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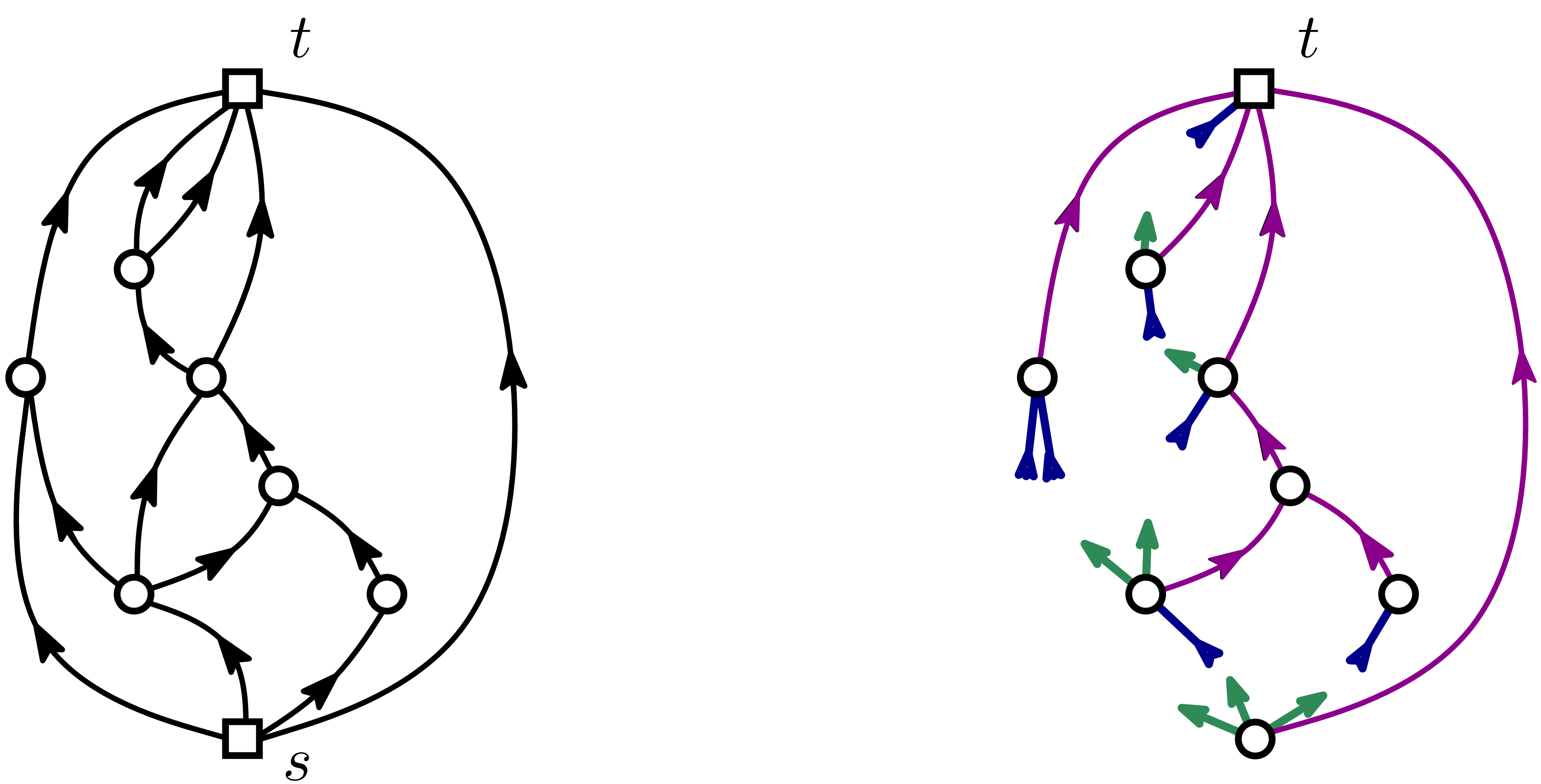

marked vertex $\in$ outer face $\Rightarrow$ easy to compute the blossoming tree

[Bernardi '07]

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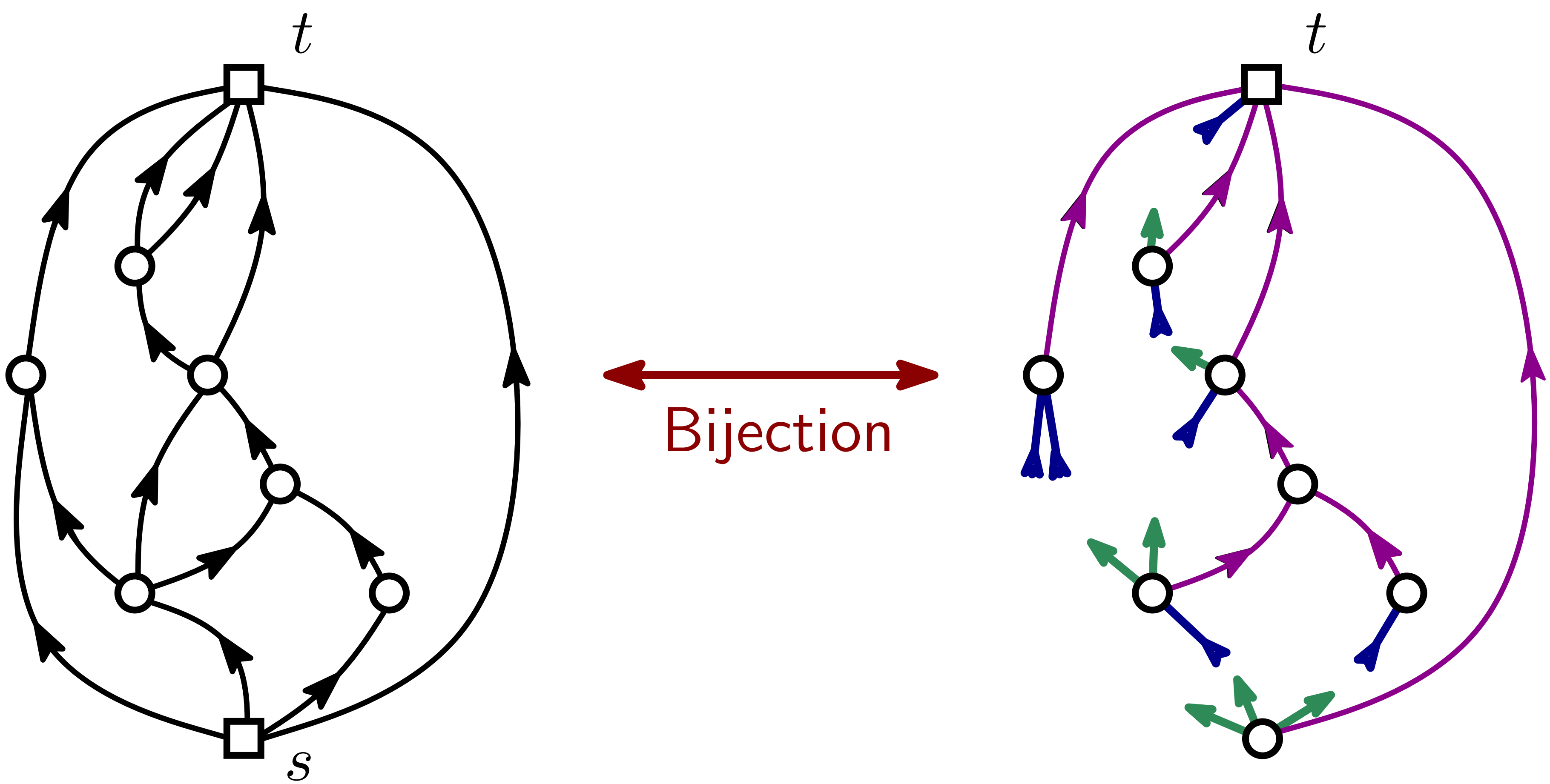

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Description/enumeration of these trees ?

Blossoming trees and triplet of paths

$T_{\textbf{bip}}(i, j) =$ blossoming trees obtained after opening a bipolar orientation with $i + 2$ vertices and $j + 1$ faces

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Theorem $=$ blossoming trees which closes into a bipolar orientation with $i + 2$ vertices and $j + 1$ faces

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Proposition: [A., Poulalhon]

There exists a one-to-one correspondence between :

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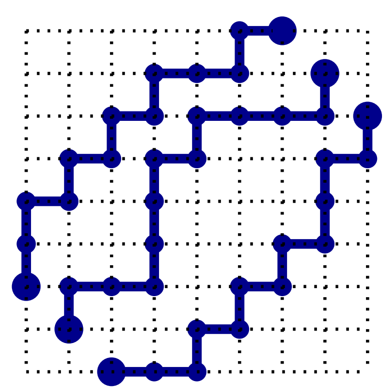

triplet of non-intersecting paths with i (vertical) and j (horizontal) and fixed first and final points

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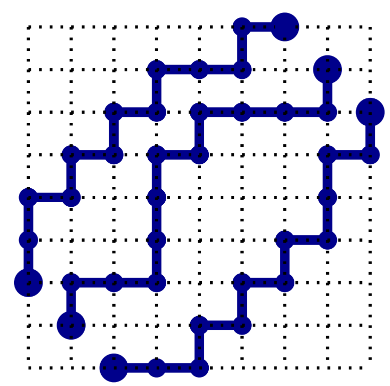
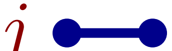
$T_{\textbf{bip}}(i,j)$ and  triplet of non-intersecting paths with i  and j  and fixed first and final points

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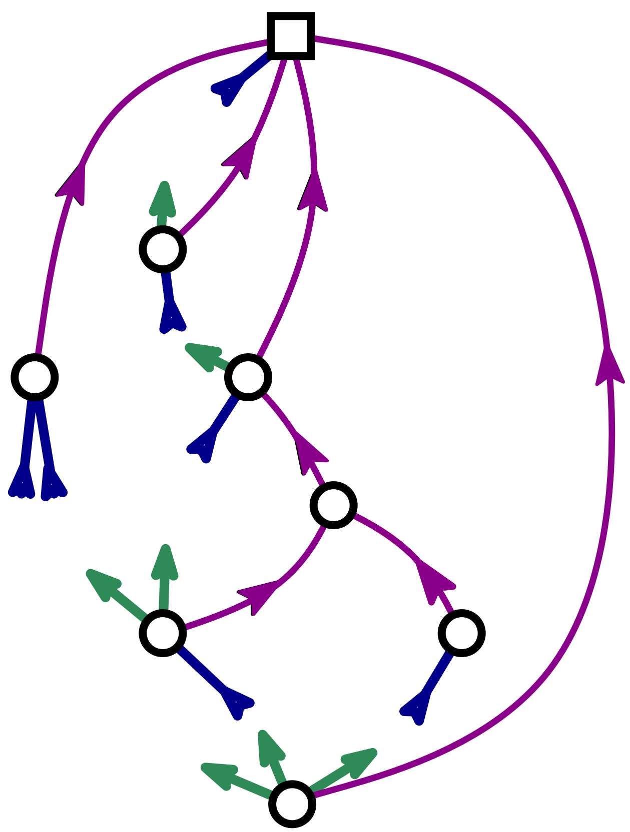
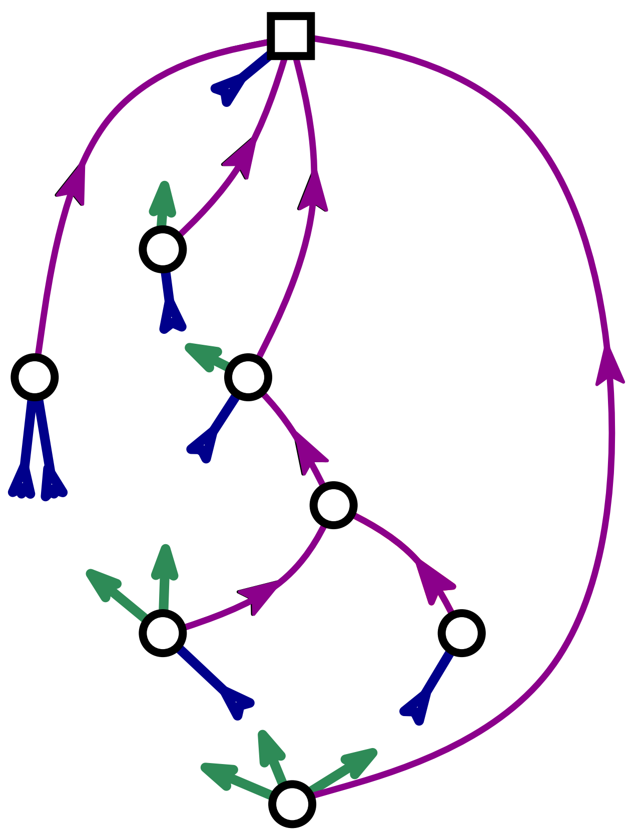
Lindström-Gessel-Viennot Lemma

Trees of $T_{\textbf{bip}}$



Trees of T_{bip}



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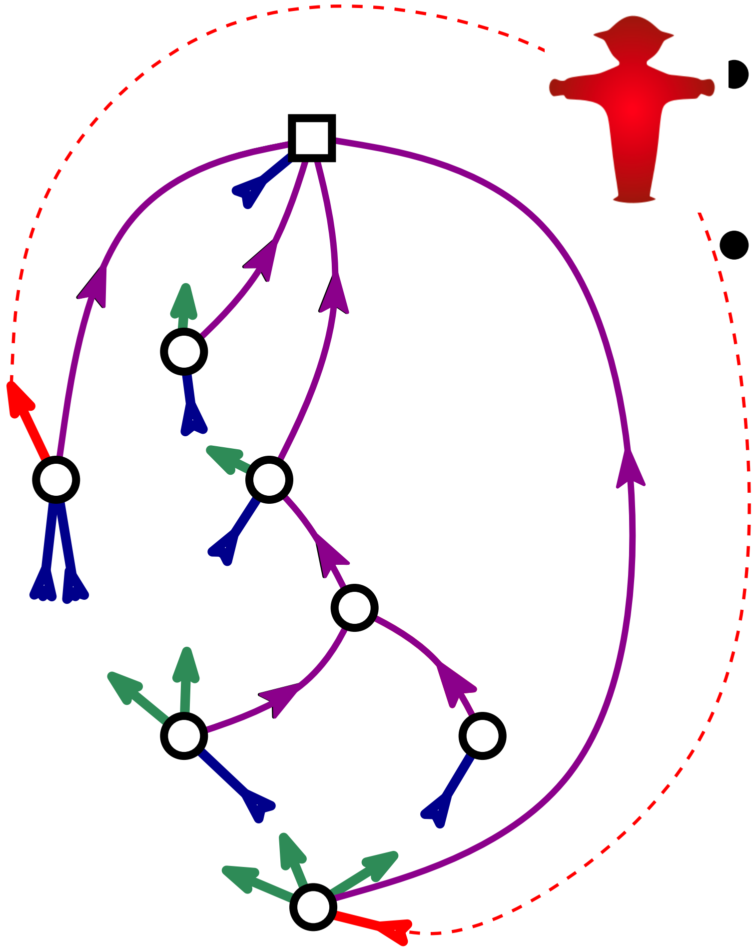

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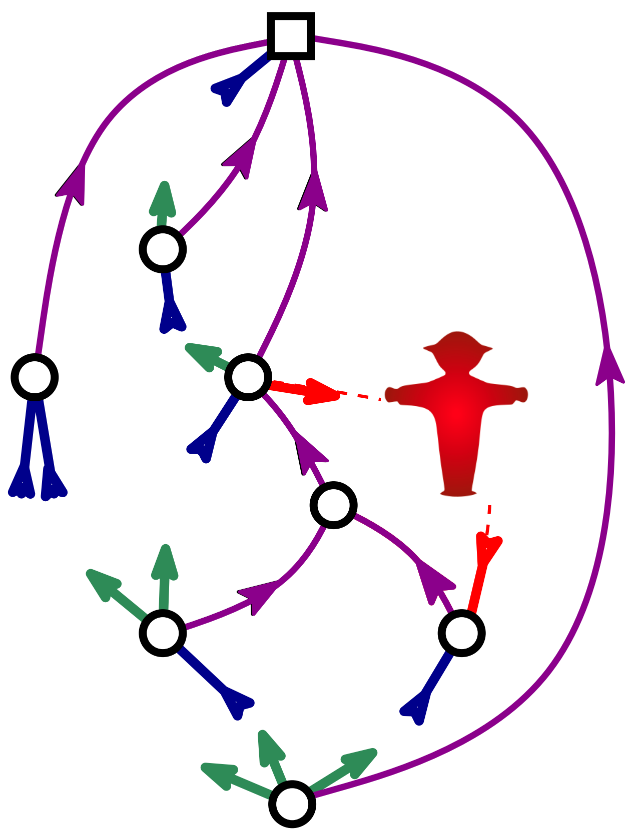

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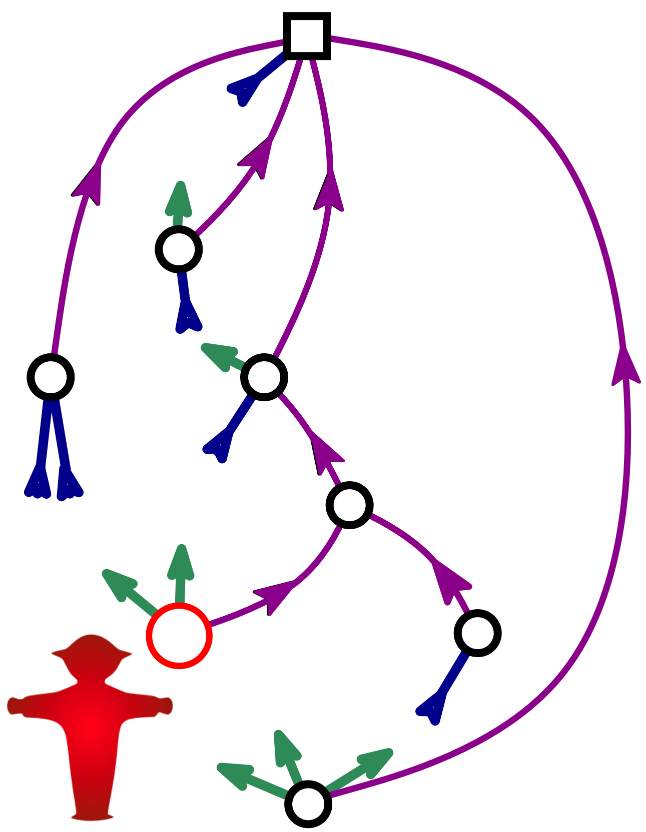

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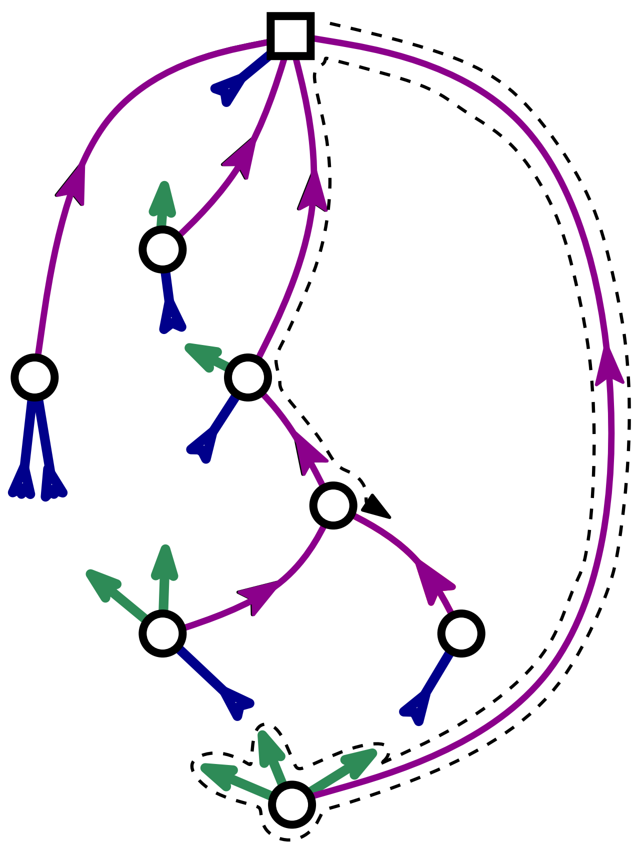

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Encoding of the blossoming tree = **contour word** = word on $\{e, \bar{e}, b, \bar{b}\}$ s.t.:

e, $\bar{e}$: first time, second time we see an edge

b, $\bar{b}$: opening stem, closing stem.

$$w = e\,b\,b\,b\,\bar{e}\,e\,e\,e\,\bar{b}\,\bar{e}\,e\,\bar{b}\,b\,b\,\bar{e}\,\bar{e}\,\bar{b}\,b\,\bar{e}\,e\,\bar{b}\,b\,\bar{e}\,\bar{b}\,e\,\bar{b}\,\bar{b}\,\bar{e}$$

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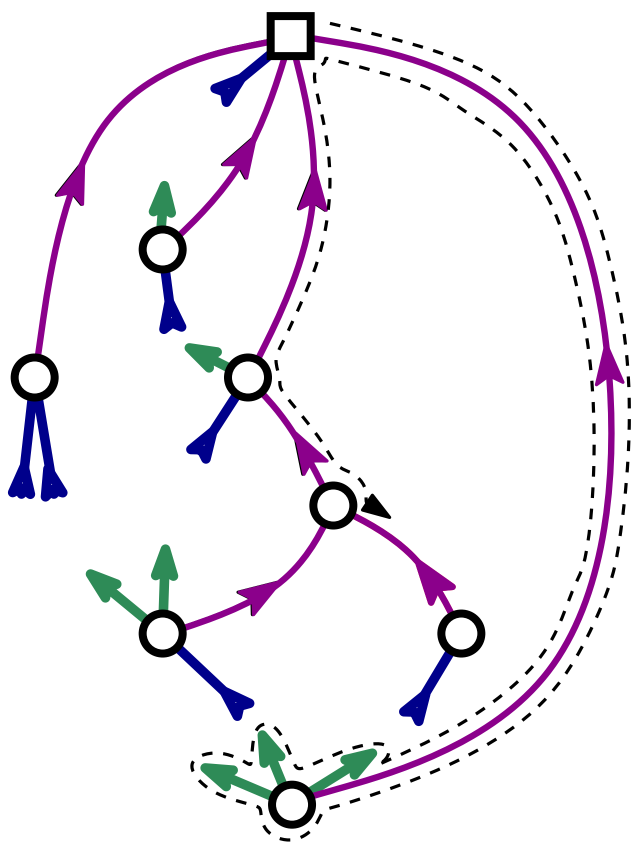

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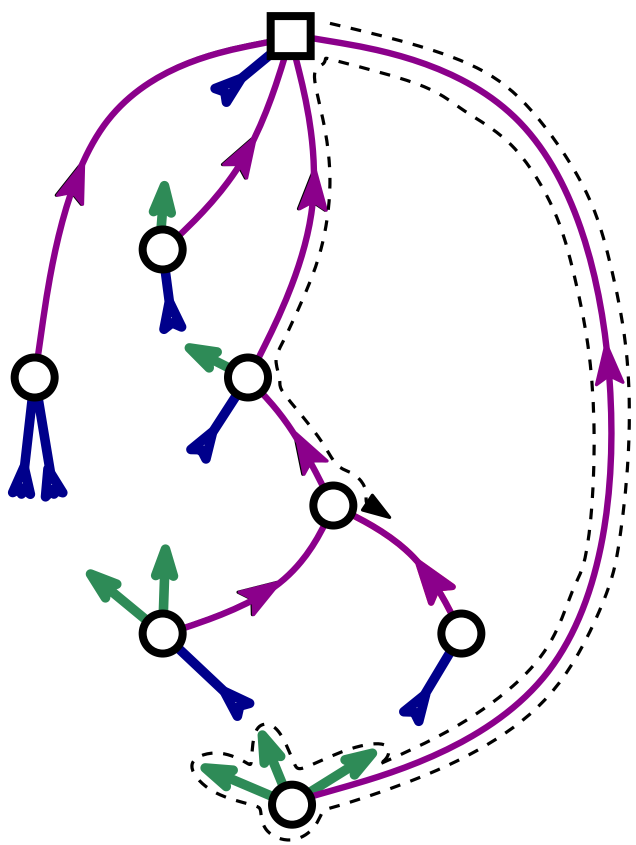

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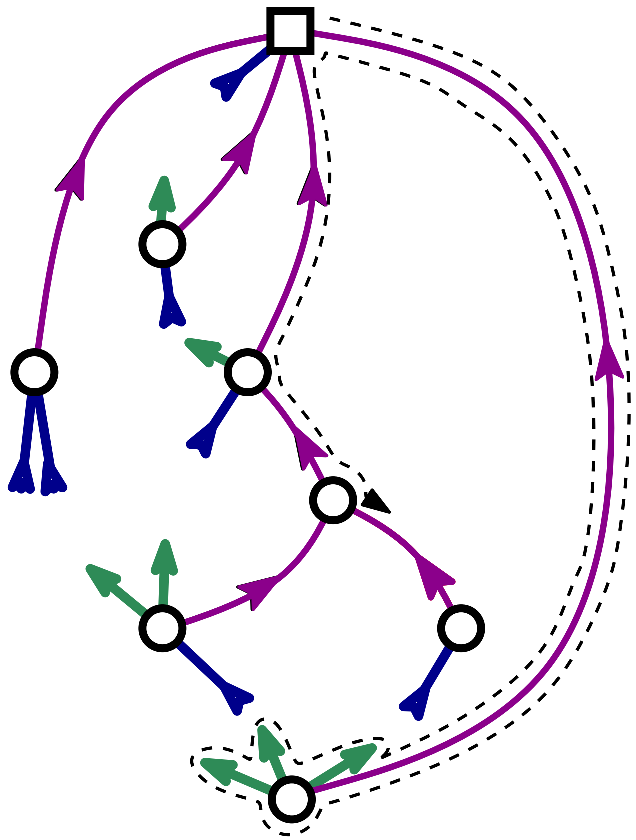

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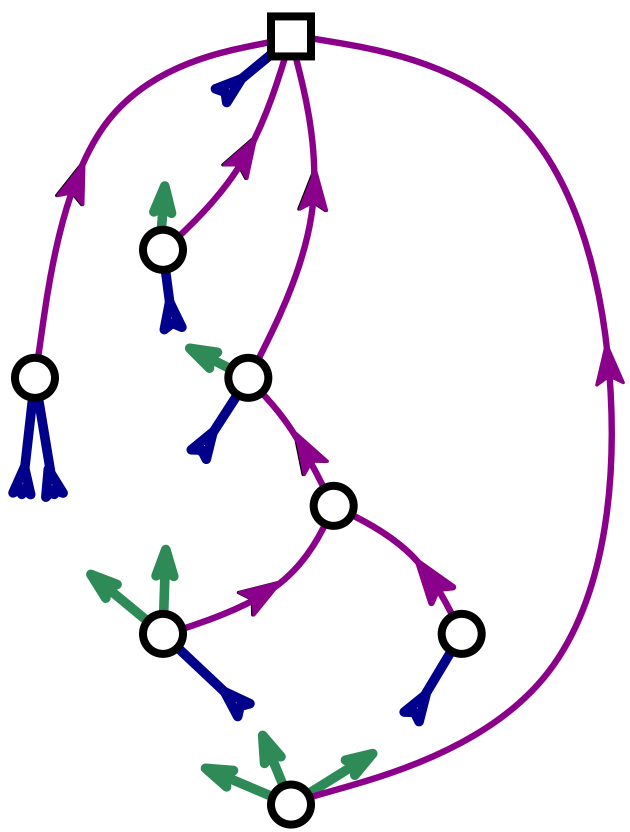
$$w = e\,b\,b\,b\,\bar{e}\,e\,e\,e\,\bar{b}\,\bar{e}\,e\,\bar{b}\,b\,b\,\bar{e}\,\bar{e}\,\bar{b}\,b\,\bar{e}\,e\,\bar{b}\,b\,\bar{e}\,\bar{b}\,e\,\bar{b}\,\bar{b}\,\bar{e}$$

Trees of $T_{\textbf{bip}}$ and triple of paths



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$w = \ldots b \ldots \bar{e} \ldots \bar{b} \ldots,\qquad w = e \ldots e \ldots \bar{b} \ldots \bar{e} \ldots$

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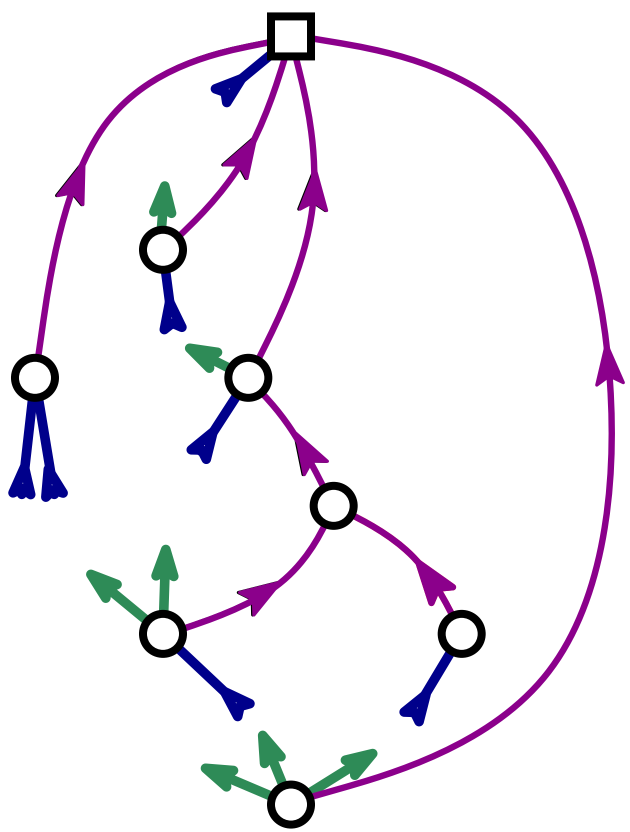

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$w = \dots b \dots \bar{e} \dots \bar{b} \dots,$ $w = e \dots e \dots \bar{b} \dots \bar{e} \dots$

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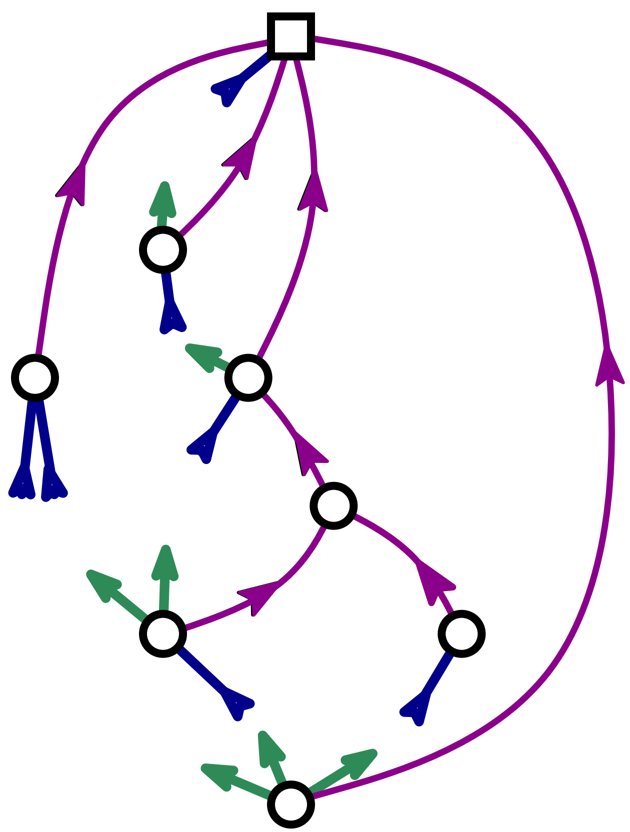
Trees of T_{bip} and triple of paths



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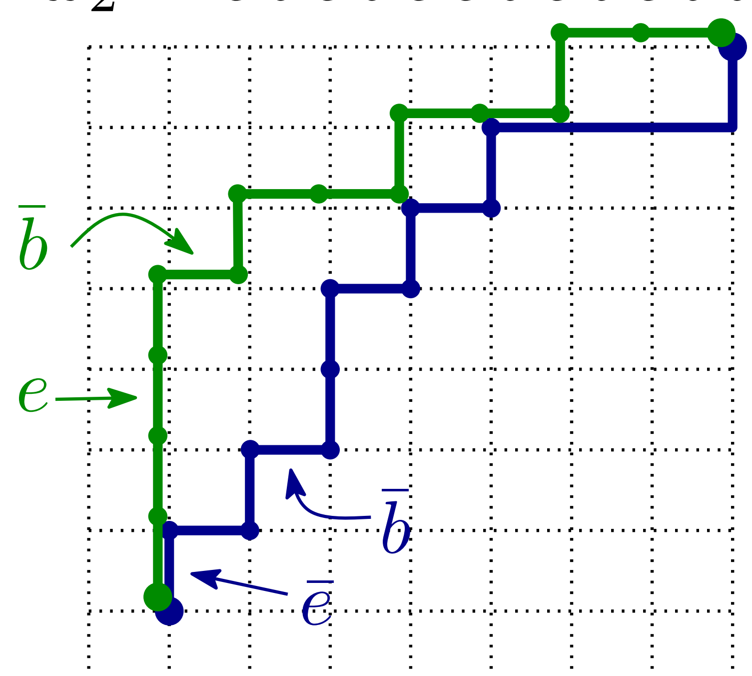

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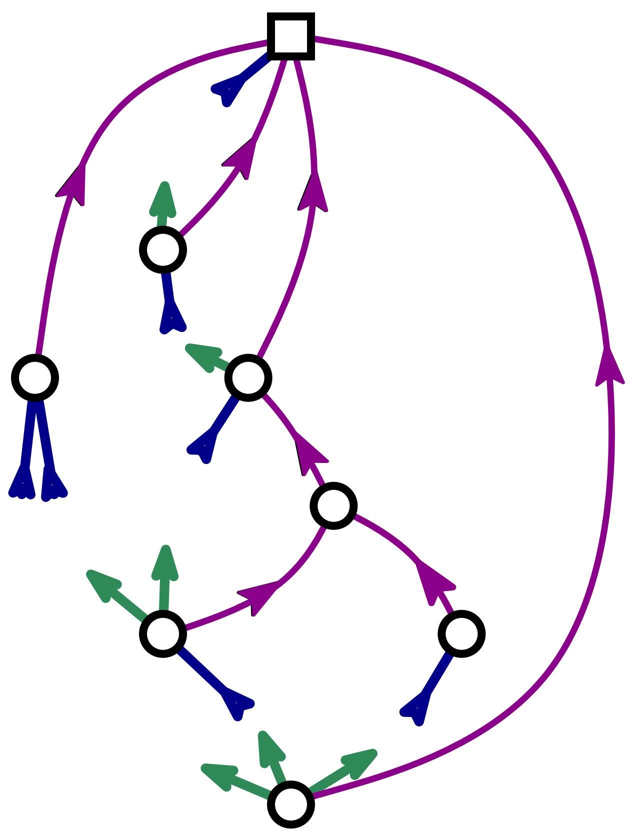

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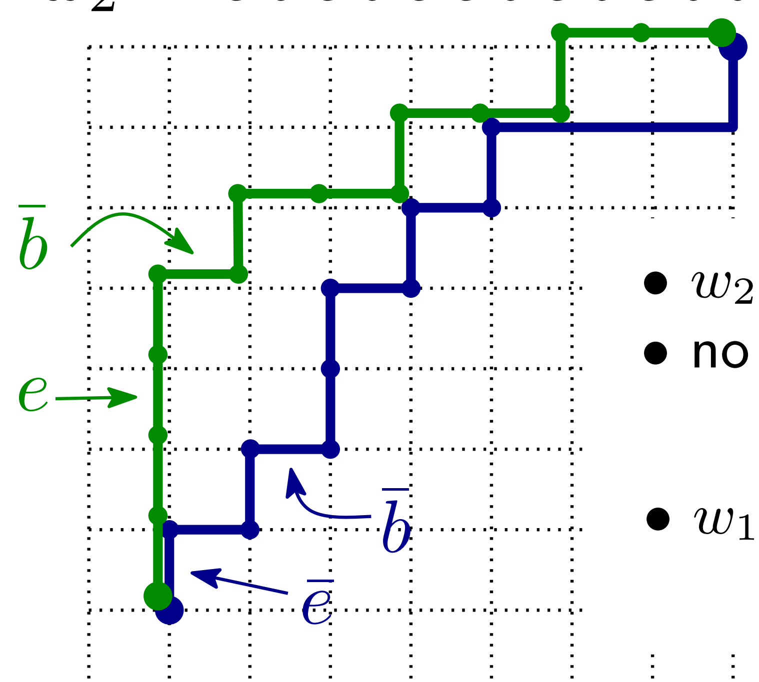

- w_2 = on the right/bottom of w_1
- no vertical edge in common.

$+$

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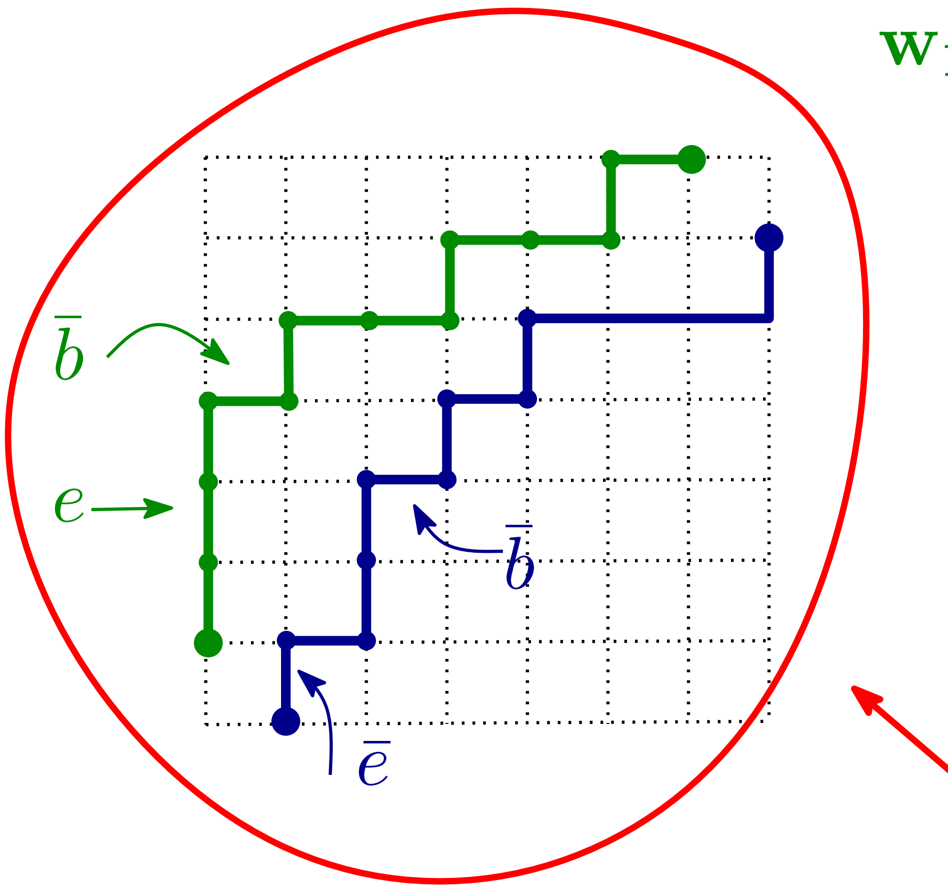

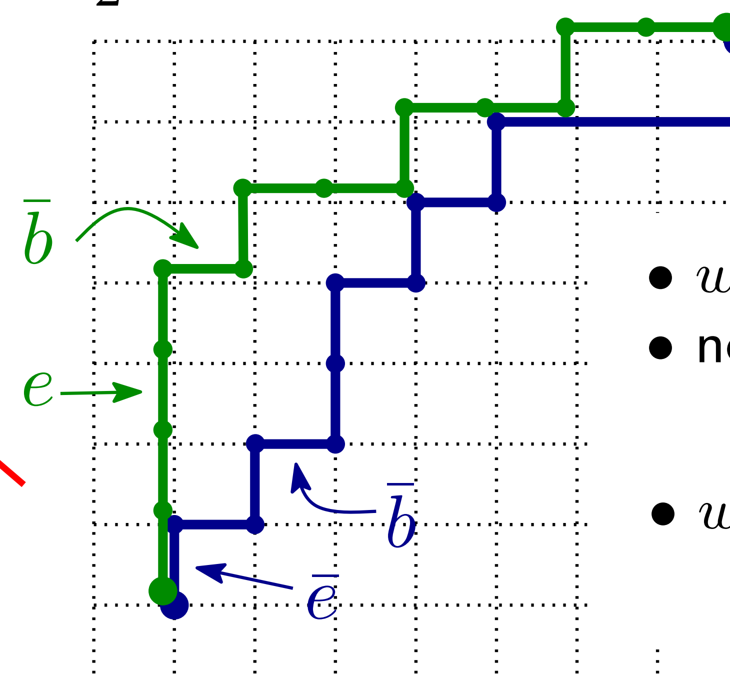

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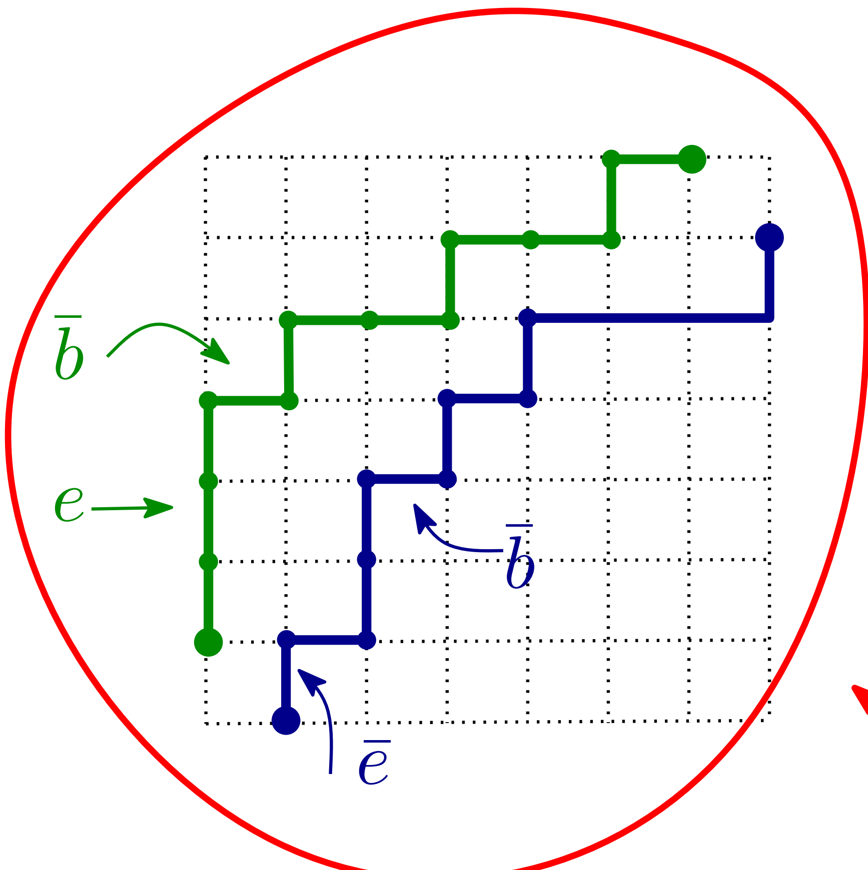

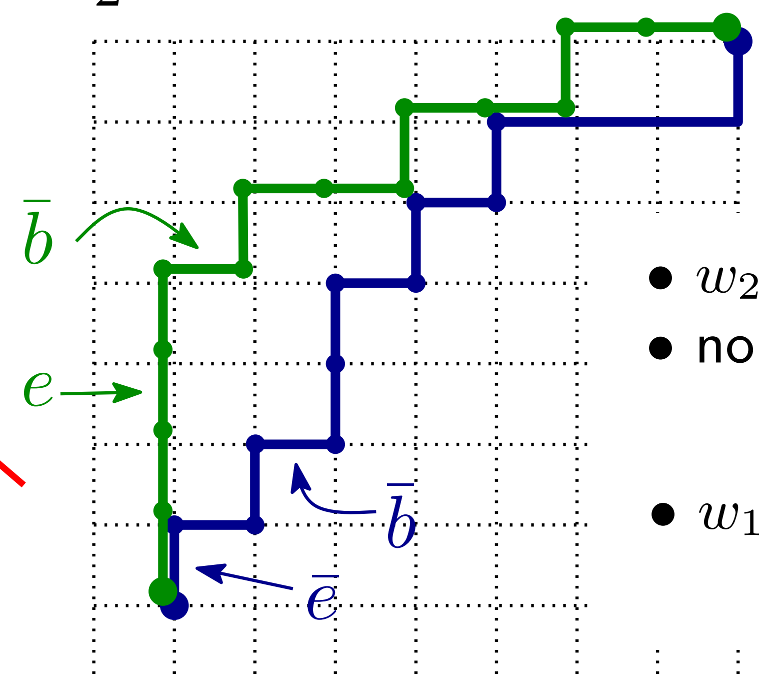

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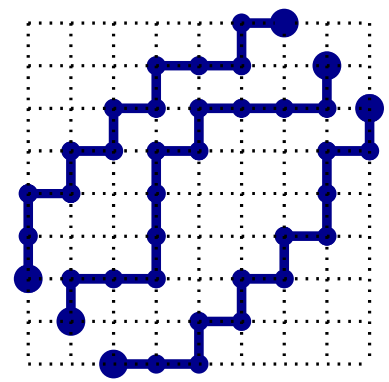
+ $w_3 = w[\bar{e}, b]$ = triple of paths !

Summary

Proposition: [A., Poulalhon]
There exists a one-to-one correspondence between :

$T_{\textbf{bip}}(i, j)$ and  triplet of non-intersecting paths with i | and j — and fixed first and final points

Corollary : The number Θ_{ij} of bipolar orientations with $i+2$ vertices and $j+1$ faces is equal to:

$$\Theta_{ij} = \frac{2(i+j)!(i+j+1)!(i+j+2)!}{i!(i+1)!(i+2)!j!(j+1)!(j+2)!}.$$

General framework ?

Theorem requires accessible orientation without ccw cycles :

Too much too ask ? **NO !**

Map M fixed + function $\alpha : V(M) \to \mathbb{N}$,

α-orientation = orientation of the edges such that $\forall v \in V(M)$, out$(v) = \alpha(v)$.

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If a map M admits an α-orientation, then there exists a unique α-orientation without ccw cycles.
If there exists one accessible α-orientation, all of them are accessible.

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- Eulerian and general maps
- Non-separable maps
- Constellations

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- d-angulations of girth d
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(Simple triangulations and quadrangulations; Eulerian and general maps; Non-separable maps; Constellations) } simpler proofs of known bijections

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Thank you !

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- ...

Simple triangulations through constellations: simpler proofs of known bijections

Plane bipolar orientations, d-angulations of girth d: New bijections