THUE–MORSE–STURMIAN WORDS AND CRITICAL BASES FOR TERNARY ALPHABETS

by

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Abstract. — The set of unique β-expansions over the alphabet $\{0,1\}$ is trivial for β below the golden ratio and uncountable above the Komornik–Loreti constant. Generalisations of these thresholds for three-letter alphabets were studied by Komornik, Lai and Pedicini (2011, 2017). We use a class of S-adic words including the Thue–Morse sequence (which defines the Komornik–Loreti constant) and Sturmian words (which characterise generalised golden ratios) to determine the value of a certain generalisation of the Komornik–Loreti constant to three-letter alphabets.

Résumé **(Mots de Thue–Morse–Sturm et bases critiques pour les alphabets ternaires)**

L'ensemble des β-développements uniques avec l'alphabet $\{0,1\}$ est trivial pour β au-dessous du nombre d'or et non dénombrable au-dessus de la constante de Komornik–Loreti. Des généralisations de ces seuils pour les alphabets de trois lettres furent étudiées par Komornik, Lai et Pedicini (2011, 2017). Nous utilisons une classe de mots S-adiques comprenant la suite de Thue–Morse (qui définit la constante de Komornik–Loreti) et les mots sturmiens (qui caractérisent les nombres d'or généralisés) pour déterminer la valeur d'une certaine généralisation de la constante de Komornik–Loreti aux alphabets de trois lettres.

1. Introduction and main results

For a base $\beta > 1$ and a sequence of digits $u_1 u_2 \cdots \in A^\infty$, with $A \subset \mathbb{R}$, let

$$\pi_\beta(u_1 u_2 \cdots) = \sum_{k=1}^{\infty} \frac{u_k}{\beta^k};$$

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we say that $u_1u_2\cdots$ is a β-expansion of this number. This paper deals with *unique β-expansions over A*, that is with

$$U_\beta(A) = \{\mathbf{u} \in A^\infty :\, \pi_\beta(\mathbf{u}) \neq \pi_\beta(\mathbf{v}) \text{ for all } \mathbf{v} \in A^\infty \setminus \{\mathbf{u}\}\}.$$

We know from [**DK93**] that $U_\beta(\{0,1\})$ is trivial if and only if $\beta \le \frac{1+\sqrt{5}}{2}$, where trivial means that $U_\beta(\{0,1\}) = \{\overline{0}, \overline{1}\}$, $\overline{a}$ being the infinite repetition of a. Therefore,

$$\mathcal{G}(A) = \inf\{\beta > 1 :\, |U_\beta(A)| > 2\}$$

is called *generalised golden ratio* of A. By [**GS01**], the set $U_\beta(\{0,1\})$ is uncountable if and only if β is larger than or equal to the Komornik–Loreti constant $\beta_{\mathrm{KL}} \approx 1.787$; we call

$$\mathcal{K}(A) = \inf\{\beta > 1 :\, U_\beta(A) \text{ is uncountable}\}$$

generalised Komornik–Loreti constant of A. (We can replace *uncountable* throughout the paper by *has the cardinality of the continuum.*) The precise structure of $U_\beta(\{0,1\})$ was described in [**KKL17**]. For integers $M \ge 2$, $\mathcal{G}(\{0,1,\dots,M\})$ was determined by [**Bak14**], and $U_\beta(\{0,1,\dots,M\})$ was described in [**KLLdV17, ABBK19**].

For $x, y \in \mathbb{R}$, $x \neq 0$, we have $(xu_1+y_1)(xu_2+y_2)\cdots \in U_\beta(xA+y)$ if and only if $u_1u_2\cdots \in U_\beta(A)$, thus $\mathcal{G}(xA+y) = \mathcal{G}(A)$ and $\mathcal{K}(xA+y) = \mathcal{K}(A)$. Hence, the only two-letter alphabet to consider is $\{0,1\}$. A three-letter alphabet $\{a_1,a_2,a_3\}$ with $a_1 < a_2 < a_3$ can be replaced by $\{0,1,\frac{a_3-a_1}{a_2-a_1}\}$ or $\{0,1,\frac{a_3-a_1}{a_3-a_2}\}$. Since $\frac{a_3-a_1}{a_2-a_1}$ and $\frac{a_3-a_1}{a_3-a_2}$ are on opposite sides of 2 (or both equal to 2), we can restrict to alphabets $\{0,1,m\}$, $m \in (1,2]$. Of course, it is also possible to restrict to $m \ge 2$ as in [**KLP11**] (note that the alphabet $\{0,1,m\}$ can be replaced by $\{0,1,\frac{m}{m-1}\}$), but we find it easier to work with $m \le 2$. We write

$$U_\beta(m) = U_\beta(\{0,1,m\}), \quad \mathcal{G}(m) = \mathcal{G}(\{0,1,m\}), \quad \mathcal{K}(m) = \mathcal{K}(\{0,1,m\}).$$

It was established in [**KLP11, Lai11, BS17**] that the generalised golden ratio $\mathcal{G}(m)$ is given by mechanical words, i.e., Sturmian words and their periodic counterparts; in particular, we can restrict to sequences $\mathbf{u} \in \{0,1\}^\infty$. Calculating $\mathcal{K}(m)$ seems to be much harder since this restriction is not possible. Therefore, we study

$$\mathcal{L}(m) = \inf\{\beta > 1 :\, U_\beta(m) \cap \{0,1\}^\infty \text{ is uncountable}\},$$

following [**KP17**], where this quantity was determined for certain intervals. We give a complete characterisation in Theorem 1 below.

To this end, we use the substitutions (or morphisms)

$$L: \begin{array}{l} 0 \mapsto 0, \\ 1 \mapsto 01, \end{array} \qquad M: \begin{array}{l} 0 \mapsto 01, \\ 1 \mapsto 10, \end{array} \qquad R: \begin{array}{l} 0 \mapsto 01, \\ 1 \mapsto 1, \end{array}$$

which act on finite and infinite words by $\sigma(u_1u_2\cdots) = \sigma(u_1)\sigma(u_2)\cdots$. The monoid generated by a set of substitutions S (with the usual product of substitutions) is denoted by S^*. An infinite word $\mathbf{u}$ is a *limit word* of a sequence of substitutions $(\sigma_n)_{n\ge1}$ (or an *S-adic word* if $\sigma_n \in S$ for all $n \ge 1$) if there is a sequence of words $(\mathbf{u}^{(n)})_{n\ge1}$ with $\mathbf{u}^{(1)} = \mathbf{u}$, $\mathbf{u}^{(n)} = \sigma_n(\mathbf{u}^{(n+1)})$ for all $n \ge 1$. The sequence $(\sigma_n)_{n\ge1}$ is called *primitive* if for each $k \ge 1$ there is an $n \ge k$ such that both words $\sigma_k\sigma_{k+1}\cdots\sigma_n(0)$ and $\sigma_k\sigma_{k+1}\cdots\sigma_n(1)$ contain both letters 0 and 1. For $S = \{L, M, R\}$, this means that there is no $k \geq 1$ such that $\sigma_n = L$ for all $n \geq k$ or $\sigma_n = R$ for all $n \geq k$. Let $\mathcal{S}_S$ be the set of limit words of primitive sequences of substitutions in S^∞. Then $\mathcal{S}_{\{L,R\}}$ consists of *Sturmian words*, and $\mathcal{S}_{\{M\}}$ consists of the *Thue-Morse word* $0\mathbf{u} = 0110100110010110\cdots$, which defines the Komornik–Loreti constant by $\pi_{\beta_{\mathrm{KL}}}(\mathbf{u}) = 1$, and its reflection by $0 \leftrightarrow 1$. We call the elements of $\mathcal{S}_{\{L,M,R\}}$, which to our knowledge have not been studied yet, *Thue–Morse–Sturmian words*. For details on S-adic and other words, we refer to [**Lot02, BD14**].

For $\mathbf{u} \in \{0,1\}^\infty$ and $m \in (1,2]$, define $f_{\mathbf{u}}(m)$ (if $\mathbf{u}$ contains at least two ones) and $g_{\mathbf{u}}(m)$ as the unique positive solutions of

$$f_{\mathbf{u}}(m)\, \pi_{f_{\mathbf{u}}(m)}(\sup O(\mathbf{u})) = m \quad \text{and} \quad (g_{\mathbf{u}}(m) - 1)(1 + \pi_{g_{\mathbf{u}}(m)}(\inf O(\mathbf{u}))) = m$$

respectively, where $O(u_1u_2\cdots) = \{u_k u_{k+1} \cdots : k \geq 1\}$ denotes the shift orbit and infinite words are ordered by the *lexicographic order*. For the existence and monotonicity properties of $f_{\mathbf{u}}(m)$ and $g_{\mathbf{u}}(m)$, see [**BS17**, Lemmas 3.11 and 3.12] and Lemma 1 below. We define $\mu_{\mathbf{u}}$ by

$$f_{\mathbf{u}}(\mu_{\mathbf{u}}) = g_{\mathbf{u}}(\mu_{\mathbf{u}}),$$

i.e., $f_{\mathbf{u}}(\mu_{\mathbf{u}}) = g_{\mathbf{u}}(\mu_{\mathbf{u}}) = \beta$ with $\beta\, \pi_\beta(\sup O(\mathbf{u})) = (\beta - 1)(1 + \pi_\beta(\inf O(\mathbf{u})))$.

The main result of [**KLP11**] on generalised golden ratios of three-letter alphabets can be written as

$$\mathcal{G}(m) = \begin{cases} f_{\sigma(\overline{0})}(m) & \text{if } m \in [\mu_{\sigma(1\overline{0})}, \mu_{\sigma(\overline{0})}],\ \sigma \in \{L,R\}^*M, \\ g_{\sigma(\overline{0})}(m) & \text{if } m \in [\mu_{\sigma(\overline{0})}, \mu_{\sigma(0\overline{1})}],\ \sigma \in \{L,R\}^*M, \\ f_{\overline{1}}(m) & \text{if } m \in [\mu_{0\overline{1}}, 2], \\ 1 + \sqrt{m} & \text{if } m = \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,R\}}; \end{cases}$$

cf. [**BS17**, Proposition 3.18], where substitutions $\tau_h = L^h R$ are used and $f, g, \mu, \mathcal{S}$ are defined slightly differently. Our main theorem looks similar, but we need $\{L,M,R\}$ instead of $\{L,R\}$, and the roles of f and g are exchanged.

Theorem 1. — *The function $\mathcal{L}(m) = \inf\{\beta > 1 : U_\beta(m) \cap \{0,1\}^\infty$ is uncountable$\}$ is given for $1 < m \leq 2$ by*

$$\mathcal{L}(m) = \begin{cases} g_{\sigma(1\overline{0})}(m) & \textit{if } m \in [\mu_{\sigma(1\overline{0})}, \mu_{\sigma(01\overline{0})}],\ \sigma \in \{L,M,R\}^*M, \\ f_{\sigma(0\overline{1})}(m) & \textit{if } m \in [\mu_{\sigma(10\overline{1})}, \mu_{\sigma(0\overline{1})}],\ \sigma \in \{L,M,R\}^*M, \\ g_{0\overline{1}}(m) & \textit{if } m \in [\mu_{0\overline{1}}, 2], \\ f_{\mathbf{u}}(m) & \textit{if } m = \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}. \end{cases}$$

The Hausdorff dimension of $\pi_\beta(U_\beta(m))$ is positive for all $\beta > \mathcal{L}(m)$.

The graphs of $\mathcal{G}(m)$ and $\mathcal{L}(m)$ are drawn in Figure 1. For example, $\sigma = M$ gives

$$\mathcal{L}(m) = \begin{cases} g_{0\overline{01}}(m) & \text{if } m \in [\mu_{0\overline{01}}, \mu_{110\overline{01}}] \approx [1.281972, 1.46811], \\ f_{1\overline{10}}(m) & \text{if } m \in [\mu_{001\overline{10}}, \mu_{1\overline{10}}] \approx [1.516574, 1.55496]. \end{cases}$$

Taking $\sigma = M^2$, we have $\sigma(0) = 0110$, $\sigma(1) = 1001$, and

$$\mathcal{L}(m) = \begin{cases} g_{0010\overline{0110}}(m) & \text{if } m \in [\mu_{0010\overline{0110}}, \mu_{11010010\overline{0110}}] \approx [1.47571, 1.503114], \\ f_{1101\overline{1001}}(m) & \text{if } m \in [\mu_{00101101\overline{1001}}, \mu_{1101\overline{1001}}] \approx [1.504152, 1.509304]. \end{cases}$$

Subintervals of the first three intervals were also given by [**KP17**].

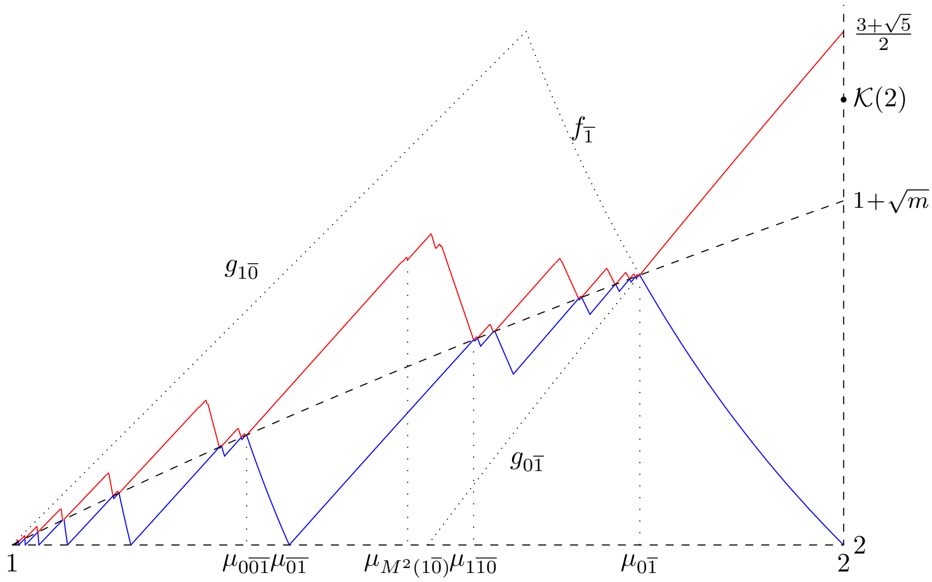

FIGURE 1. The critical bases $\mathcal{G}(m)$ (below $1+\sqrt{m}$, blue) and $\mathcal{L}(m)$ (above $1+\sqrt{m}$, red).

By [**KLP11, KP17**], we have, for all $m \in (1, 2]$,

$$2 \leq \mathcal{G}(m) \leq 1+\sqrt{m} \leq \mathcal{K}(m) \leq \mathcal{L}(m) \leq g_{1\bar{0}}(m) = 1+m,$$

with $\mathcal{G}(m) = \mathcal{L}(m)$ if and only if $m \in \{\mu_{\sigma(1\bar{0})}, \mu_{\sigma(0\bar{1})}\}$, $\sigma \in \{L,R\}^*M$, or $m = \mu_{\mathbf{u}}$, $\mathbf{u} \in \mathcal{S}_{\{L,R\}}$. Besides those m, we only know the value of $\mathcal{K}(m)$ for $m = 2$ from [**KL02**]: $\pi_{\mathcal{K}(2)}(2102012101202102\cdots) = 1$, thus $\mathcal{K}(2) \approx 2.536 < \frac{3+\sqrt{5}}{2} = \mathcal{L}(2)$. The functions $\mathcal{G}(m)$, $\mathcal{K}(m)$ and $\mathcal{L}(m)$ are continuous for $m > 1$ by [**KLP11, KP17**]; at least for the generalised golden ratio, this also holds for larger alphabets by [**BS17**].

2. Proof of the main theorem

We first prove that $f_{\mathbf{u}}(m)$, $g_{\mathbf{u}}(m)$ and $\mu_{\mathbf{u}}$ are well defined, and we determine monotonicity properties. For convenience, we write $\inf(\mathbf{u})$ for $\inf O(\mathbf{u})$ and $\sup(\mathbf{u})$ for $\sup O(\mathbf{u})$ in the following.

Lemma 1. — *Let $m \in (1,2]$, $\mathbf{u}, \mathbf{u}' \in \{0,1\}^\infty$. Then $g_{\mathbf{u}}(m)$ is well defined. If $\mathbf{u}$ contains at least two ones, then $f_{\mathbf{u}}(m)$ and $\mu_{\mathbf{u}}$ are well defined, and we have*

$$\max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m)) \ge 2,$$
$$\beta > 1,\ \beta\,\pi_\beta(\sup(\mathbf{u})) < m \quad \textit{if and only if} \quad \beta > f_{\mathbf{u}}(m),$$
$$\beta > 1,\ (\beta-1)(1+\pi_\beta(\inf(\mathbf{u})) > m \quad \textit{if and only if} \quad \beta > g_{\mathbf{u}}(m),$$
$$f_{\mathbf{u}}(m) > f_{\mathbf{u}}(m') \quad \textit{and} \quad g_{\mathbf{u}}(m) < g_{\mathbf{u}}(m') \quad \textit{if } m < m',$$
$$f_{\mathbf{u}}(m) < f_{\mathbf{u}'}(m) \quad \textit{if } \sup(\mathbf{u}) < \sup(\mathbf{u}') \textit{ and } f_{\mathbf{u}}(m) \ge 2,$$
$$g_{\mathbf{u}}(m) > g_{\mathbf{u}'}(m) \quad \textit{if } \inf(\mathbf{u}) < \inf(\mathbf{u}') \textit{ and } g_{\mathbf{u}'}(m) \ge 2.$$

Proof. — Set $h_{\mathbf{v}}(x,m) = x\pi_x(\mathbf{v}) - m$ with $\mathbf{v} = \sup(\mathbf{u})$. Then $h_{\mathbf{v}}(x,m)$ is strictly decreasing in x (for $x > 1$) and m. If $\mathbf{u}$ contains at least two ones, then $\mathbf{v}$ also contains at least two ones, thus $\lim_{x\to 1} h_{\mathbf{v}}(x,m) \ge 2-m$ and $\lim_{x\to\infty} h_{\mathbf{v}}(x,m) = 1-m$. Therefore, there is, for each $m \in (1,2]$, a unique $x_{m,\mathbf{v}} \ge 1$ such that $h_{\mathbf{v}}(x_{m,\mathbf{v}}, m) = 0$, i.e., $f_{\mathbf{u}}(m) = x_{m,\mathbf{v}}$, and we have $\beta\,\pi_\beta(\sup(\mathbf{u})) < m$ for $\beta > 1$ if and only if $\beta > f_{\mathbf{u}}(m)$. If $m < m'$, then we have $x_{m,\mathbf{v}} > x_{m',\mathbf{v}}$, thus $f_{\mathbf{u}}(m) > f_{\mathbf{u}}(m')$. If $\mathbf{v} < \mathbf{v}'$ and $x \ge 2$, then we have $h_{\mathbf{v}}(x,m) < h_{\mathbf{v}'}(x,m)$, thus $x_{m,\mathbf{v}} < x_{m,\mathbf{v}'}$ if $x_{m,\mathbf{v}} \ge 2$, hence $f_{\mathbf{u}}(m) < f_{\mathbf{u}'}(m)$ if $\sup(\mathbf{u}) < \sup(\mathbf{u}')$ and $f_{\mathbf{u}}(m) \ge 2$.

Let now $h_{\mathbf{v}}(x,m) = \frac{m}{x-1} - \pi_x(\mathbf{v}) - 1$ with $\mathbf{v} = \inf(\mathbf{u})$. Since $\frac{m}{x-1} = \pi_x(\overline{m})$, $h_{\mathbf{v}}(x,m)$ is strictly decreasing in x (for $x > 1$) and strictly increasing in m. Again, there is, for each $m \in (1,2]$, a unique $x_{m,\mathbf{v}} > 1$ such that $h_{\mathbf{v}}(x_{m,\mathbf{v}}, m) = 0$, i.e., $g_{\mathbf{u}}(m) = x_{m,\mathbf{v}}$. We have $h_{\mathbf{v}}(x,m) < 0$ for $x > 1$ if and only if $x > x_{m,\mathbf{v}}$, $x_{m,\mathbf{v}} < x_{m',\mathbf{v}}$ if $m < m'$, and $h_{\mathbf{v}}(x,m) > h_{\mathbf{v}'}(x,m)$ if $\mathbf{v} < \mathbf{v}'$, $x \ge 2$, thus $x_{m,\mathbf{v}} > x_{m,\mathbf{v}'}$ if $x_{m,\mathbf{v}'} \ge 2$. This proves the monotonicity properties of g.

Since $f_{\mathbf{u}}(m)$ is strictly decreasing, $g_{\mathbf{u}}(m)$ is strictly increasing, $\lim_{m\to 1} f_{\mathbf{u}}(m) = \infty$, $f_{\mathbf{u}}(2) \le 2$, and $g_{\mathbf{u}}(2) \ge 2$, we have $f_{\mathbf{u}}(m) = g_{\mathbf{u}}(m)$ for a unique $m \in (1,2]$.

Let $\beta = f_{\mathbf{u}}(\mu_{\mathbf{u}}) = g_{\mathbf{u}}(\mu_{\mathbf{u}})$, i.e., $\beta\,\pi_\beta(\sup(\mathbf{u})) = (\beta-1)(1+\pi_\beta(\inf(\mathbf{u})))$. We have $\sup(\mathbf{u}) \ge 1\inf(\mathbf{u})$. If equality holds, then $\beta = 2$. Otherwise, $\sup(\mathbf{u})$ starts with $1v_1\cdots v_{k-1}1$ and $\inf(\mathbf{u})$ starts with $v_1\cdots v_{k-1}0$ for some $v_1\cdots v_{k-1}$, $k \ge 1$. Then

$$\beta\pi_\beta(\sup(\mathbf{u})) \ge 1 + \sum_{i=1}^{k-1}\frac{v_i}{\beta^i} + \frac{1}{\beta^k},\ (\beta-1)(1+\pi_\beta(\inf(\mathbf{u}))) \le (\beta-1)\left(1+\sum_{i=1}^{k-1}\frac{v_i}{\beta^i}\right) + \frac{1}{\beta^k},$$

thus $\beta \ge 2$. By the monotonicity properties that are proved above, this implies that $\max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m)) \ge 2$ for all $m \in (1,2]$. □

Next we establish relations between $f_{\mathbf{u}}(m)$, $g_{\mathbf{u}}(m)$ and $\mathbf{u} \in U_\beta(m)$.

Lemma 2. — *Let $m \in (1,2]$, $\beta \in (1, 1+m]$. For $\mathbf{u} \in \{0,1\}^\infty$, we have $\mathbf{u} \in U_\beta(m)$ if and only if $0\mathbf{u} \in U_\beta(m)$. For $\mathbf{u} \in 1\{0,1\}^\infty \setminus \{1\overline{0}\}$, $\mathbf{u} \in U_\beta(m)$ implies that $\beta \ge \max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m))$, and $\beta > \max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m))$ implies that $\mathbf{u} \in U_\beta(m)$.*

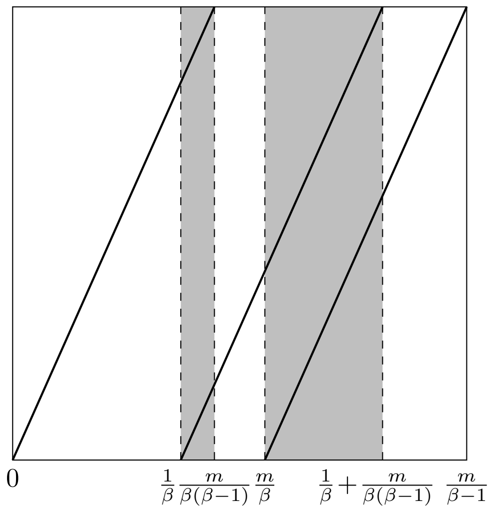

FIGURE 2. The branching β-transformation T for $\beta = 9/4$, $m = 3/2$.

Proof. — For $\beta \in (1, 1+m]$, $\mathbf{u} = u_1u_2\cdots \in \{0,1,m\}^\infty$, $x \in [0, \frac{m}{\beta-1}]$, we have $\pi_\beta(\mathbf{u}) = x$ if and only if $u_k = d(T^{k-1}(x))$ for all $k \ge 1$, with the branching β-transformation

$$T:\, [0, \tfrac{m}{\beta-1}] \to [0, \tfrac{m}{\beta-1}],\ x \mapsto \beta x - d(x),\ d(x) = \begin{cases} 0 & \text{if } x < \frac{1}{\beta}, \\ 0 \text{ or } 1 & \text{if } \frac{1}{\beta} \le x \le \frac{m}{\beta(\beta-1)}, \\ 1 & \text{if } \frac{m}{\beta(\beta-1)} < x < \frac{m}{\beta}, \\ 1 \text{ or } m & \text{if } \frac{m}{\beta} \le x \le \frac{1}{\beta} + \frac{m}{\beta(\beta-1)}, \\ m & \text{if } x > \frac{1}{\beta} + \frac{m}{\beta(\beta-1)}, \end{cases}$$

see Figure 2. We thus have

$$\mathbf{u} \in U_\beta(m) \iff \pi_\beta(u_ku_{k+1}\cdots) \notin [\tfrac{1}{\beta}, \tfrac{m}{\beta(\beta-1)}] \cup [\tfrac{m}{\beta}, \tfrac{1}{\beta} + \tfrac{m}{\beta(\beta-1)}] \text{ for all } k \ge 1.$$

For $\mathbf{u} \in \{0,1\}^\infty \setminus \{\overline{0}\}$, this means that $\beta > 2$ and

$$\pi_\beta(u_ku_{k+1}\cdots) < \tfrac{m}{\beta},\ \pi_\beta(u_{k+1}u_{k+2}\cdots) > \tfrac{m}{\beta-1} - 1 \text{ for all } k \ge 1 \text{ such that } u_k = 1,$$

see [**BS17**, Lemma 3.9], i.e.,

$$\beta\, \pi_\beta(\sup(\mathbf{u})) \le m \le (\beta-1)(1 + \pi_\beta(\inf\nolimits_1(\mathbf{u}))),$$

where $\inf_1(u_1u_2\cdots) = \inf\{u_{k+1}u_{k+2}\cdots :\, k \ge 1, u_k = 1\}$, with strict equalities if the supremum and infimum are attained. This shows that $\mathbf{u} \in U_\beta(m)$ if and only if $0\mathbf{u} \in U_\beta(m)$. Note that $\inf_1(\mathbf{u}) \ne \inf(\mathbf{u})$ implies that $\inf(\mathbf{u}) = \mathbf{u}$, hence we have $\inf_1(\mathbf{u}) = \inf(\mathbf{u})$ when $\mathbf{u}$ starts with 1. Then, by Lemma 1, $\mathbf{u} \in U_\beta(m)$ implies that $\beta \ge \max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m))$, and $\beta > \max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m))$ implies that $\mathbf{u} \in U_\beta(m)$. $\square$

To calculate $f_{\mathbf{u}}(m)$ and $g_{\mathbf{u}}(m)$, it is crucial to determine $\inf(\mathbf{u})$ and $\sup(\mathbf{u})$. Similarly to $\inf_1(u_1u_2\cdots) = \inf\{u_{k+1}u_{k+2}\cdots : k \ge 1, u_k = 1\}$, set

$$\sup\nolimits_0(u_1u_2\cdots) = \sup\{u_{k+1}u_{k+2}\cdots : k \ge 1, u_k = 0\}.$$

Lemma 3. — *For all* $\mathbf{u} \in \{0,1\}^\infty$*, we have*

$$\inf(L(\mathbf{u})) = L(\inf(\mathbf{u})), \quad \inf(R(\mathbf{u})) = R(\inf(\mathbf{u})), \quad 0\sup(L(\mathbf{u})) = L(\sup(\mathbf{u})).$$

If $\inf(\mathbf{u}) = \inf_1(\mathbf{u})$*, then* $\inf(M(\mathbf{u})) = 0M(\inf(\mathbf{u}))$*. If* $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$*, then*

$$\sup(R(\mathbf{u})) = 1R(\sup(\mathbf{u})), \quad \sup(M(\mathbf{u})) = 1M(\sup(\mathbf{u})).$$

For each $\sigma \in \{L,M,R\}^*$*, there is a suffix* w *of* $\sigma(1)$ *such that* $\inf_1(\sigma(\mathbf{u})) = \inf(\sigma(\mathbf{u})) = w\sigma(\inf(\mathbf{u}))$ *for all* $\mathbf{u} \in \{0,1\}^\infty$ *with* $\inf(\mathbf{u}) = \inf_1(\mathbf{u})$*.*

For each $\sigma \in \{L,M,R\}^*M \cup \{L,M,R\}^*R$*, there is a suffix* w *of* $\sigma(0)$ *such that* $\sup_0(\sigma(\mathbf{u})) = \sup(\sigma(\mathbf{u})) = w\sigma(\sup(\mathbf{u}))$ *for all* $\mathbf{u} \in \{0,1\}^\infty$ *with* $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$*.*

For each $\sigma \in \{L,M,R\}^*L$*, there is a prefix* w *of* $\sigma(\overline{0})$ *such that* $w\sup_0(\sigma(\mathbf{u})) = w\sup(\sigma(\mathbf{u})) = \sigma(\sup(\mathbf{u}))$ *for all* $\mathbf{u} \in \{0,1\}^\infty$ *with* $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$*.*

Proof. — The first statements follow from the facts that L, M, R are order-preserving on infinite words and that $\inf(\mathbf{u}) = \inf_1(\mathbf{u})$, $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$ mean that $1\inf(\mathbf{u})$, $0\sup(\mathbf{u})$ are in the closure of $O(\mathbf{u})$.

We claim that, for each $\sigma \in \{L,M,R\}^*$, there is a suffix $1w$ of $\sigma(1)$ such that $\inf_1(\sigma(\mathbf{u})) = \inf(\sigma(\mathbf{u})) = w\sigma(\inf(\mathbf{u}))$ for all $\mathbf{u} \in \{0,1\}^\infty$ with $\inf(\mathbf{u}) = \inf_1(\mathbf{u})$. If $1w$ is a suffix of $\sigma(1)$, then $1L(w)$, $10M(w)$ and $1R(w)$ are suffixes of $L\sigma(1)$, $M\sigma(1)$ and $R\sigma(1)$ respectively. Therefore, this claim holds for $L\sigma$, $M\sigma$ and $R\sigma$ when it holds for σ. Since it holds for $\sigma = \mathrm{id}$, it holds for all $\sigma \in \{L,M,R\}^*$.

Next we claim that, for each $\sigma \in \{L,M,R\}^*\{M,R\}$, there is a suffix $01w$ of $\sigma(0)$ such that $\sup_0(\sigma(\mathbf{u})) = \sup(\sigma(\mathbf{u})) = 1w\sigma(\sup(\mathbf{u}))$ for all $\mathbf{u} \in \{0,1\}^\infty$ with $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$. This holds for $\sigma \in \{M,R\}$. If $01w$ is a suffix of $\sigma(0)$, then $01L(w)$, $01M(1w)$ and $01R(1w)$ are suffixes of $L\sigma(0)$, $M\sigma(0)$ and $R\sigma(0)$ respectively. Therefore, this claim holds for all $\sigma \in \{L,M,R\}^*\{M,R\}$.

Finally we claim that, for each $\sigma \in \{L,M,R\}^*L$, there is a prefix $w0$ of $\sigma(\overline{0})$ such that $w0\sup_0(\sigma(\mathbf{u})) = w0\sup(\sigma(\mathbf{u})) = \sigma(\sup(\mathbf{u}))$ for all $\mathbf{u} \in \{0,1\}^\infty$ with $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$. This holds for $\sigma = L$. If $w0$ is a prefix of $\sigma(\overline{0})$, then $L(w0)0$, $M(w)0$ and $R(w)0$ are prefixes of $L\sigma(\overline{0})$, $M\sigma(\overline{0})$ and $R\sigma(\overline{0})$ respectively. Therefore, this claim holds for all $\sigma \in \{L,M,R\}^*L$. □

Now we can prove that Theorem 1 gives an upper bound for $\mathcal{L}(m)$, cf. Figure 3.

Proposition 1. — *Let* $m \in (1,2]$*. We have*

$$\mathcal{L}(m) \le \begin{cases} g_{\sigma(1\overline{0})}(m) & \text{if } m \ge \mu_{\sigma(1\overline{0})},\ \sigma \in \{L,M,R\}^*M, \\ f_{\sigma(0\overline{1})}(m) & \text{if } m \le \mu_{\sigma(0\overline{1})},\ \sigma \in \{L,M,R\}^*M, \\ g_{0\overline{1}}(m) & \text{if } m \ge \mu_{0\overline{1}}, \\ g_{\mathbf{u}}(m) & \text{if } m \ge \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}, \\ f_{\mathbf{u}}(m) & \text{if } m \le \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}. \end{cases}$$

If β *is above this bound, then the Hausdorff dimension of* $\pi_\beta(U_\beta(m))$ *is positive.*

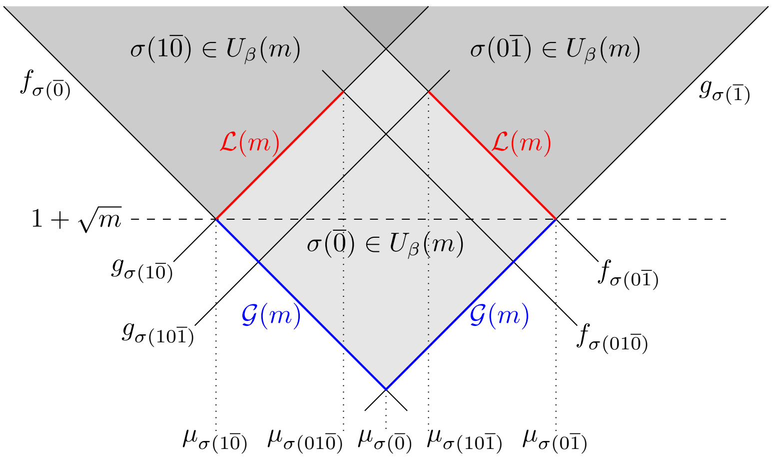

FIGURE 3. A schematic picture for $\sigma \in \{L,R\}^*M$. For $\sigma \in \{L,M,R\}^*M$, the situation is similar, except for $\mathcal{G}(m)$ and $1+\sqrt{m}$.

Proof. — Let $\sigma \in \{L,M,R\}^*$. For all $h \geq 1$, $\mathbf{v} \in 1\{0(01)^h, 0(01)^{h+1}\}^\infty$, we have

$$\inf(\sigma(\mathbf{v})) \geq \inf(\sigma(\overline{10(01)^{h-1}0})) \quad \text{and} \quad \sup(\sigma(\mathbf{v})) \leq \sup(\sigma(\overline{(01)^{h+1}0}))$$

by Lemma 3, with

$$\inf(\sigma(\overline{10(01)^{h-1}0})) \to \inf(\sigma M(1\bar{0})),\ \sup(\sigma(\overline{(01)^{h+1}0})) \to \sup(\sigma M(\bar{0})) \quad (h \to \infty).$$

Therefore, we have for each $\beta > \max(f_{\sigma M(\bar{0})}(m), g_{\sigma M(1\bar{0})}(m))$ some $h \geq 1$ such that $\sigma(\{0(01)^h, 0(01)^{h+1}\}^\infty) \subseteq U_\beta(m)$. If $m \geq \mu_{\sigma M(1\bar{0})}$, then $f_{\sigma M(\bar{0})}(m) = f_{\sigma M(1\bar{0})}(m) \leq g_{\sigma M(1\bar{0})}(m)$, thus $U_\beta(m) \cap \{0,1\}^\infty$ is uncountable (and has the cardinality of the continuum) for all $\beta > g_{\sigma M(1\bar{0})}(m)$, i.e., $\mathcal{L}(m) \leq g_{\sigma M(1\bar{0})}(m)$. By symmetry, sequences in $\sigma(\{1(10)^h, 1(10)^{h+1}\}^\infty)$ give that $\mathcal{L}(m) \leq f_{\sigma M(0\bar{1})}(m)$ for $m \leq \mu_{\sigma M(0\bar{1})}$. Similarly, sequences in $1\{01^h, 01^{h+1}\}^\infty$ give that $\mathcal{L}(m) \leq g_{0\bar{1}}(m)$ for $m \geq \mu_{0\bar{1}}$.

Let now $\mathbf{u}$ be a limit word of a primitive sequence $(\sigma_n)_{n\geq 1} \in \{L,M,R\}^\infty$, and set $\sigma'_n = \sigma_1\sigma_2\cdots\sigma_n$. Then $\inf(\sigma'_n(1\bar{0})) \leq \inf(\mathbf{u}) \leq \inf(\sigma'_n(10\bar{1}))$ for all $n \geq 1$, thus $\inf(\sigma'_n(1\bar{0})) \to \inf(\mathbf{u})$ and (by symmety) $\sup(\sigma'_n(0\bar{1})) \to \sup(\mathbf{u})$ as $n \to \infty$. Therefore, for each $\beta > \max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m))$ there is $n \geq 1$ such that $\sigma'_n(\mathbf{v}) \in U_\beta(m)$ for all $\mathbf{v} \in \{0,1\}^\infty \setminus \{\bar{0},\bar{1}\}$, hence $\mathcal{L}(m) \leq g_{\mathbf{u}}(m)$ for $m \geq \mu_{\mathbf{u}}$ and $\mathcal{L}(m) \leq f_{\mathbf{u}}(m)$ for $m \leq \mu_{\mathbf{u}}$.

If $\{v,w\}^\infty \subseteq U_\beta(m)$, then by [**Hut81**] we have $\dim_H(\pi_\beta(U_\beta(m))) \geq r$, with $r > 0$ such that $\beta^{-|v|r} + \beta^{-|w|r} = 1$, where $|v|$ and $|w|$ denote the lengths of v and w. □

For the lower bound, we use Lemma 5 below, which tells us that, if the orbit of a sequence satisfies inequalities that hold for all non-trivial images of $\sigma \in \{L,M,R\}^*$, then it is eventually in the image of σ. In particular, with $\sigma = M^n$, $n \geq 0$, this yields that $U_\beta(\{0,1\})$ is countable for all β less than the Komornik–Loreti constant; cf. [**GS01**]. First we show that the conditions of Lemma 3 are satisfied for a suffix.

Lemma 4. — *Let* $\mathbf{u} \in \{0,1\}^\infty$ *with* $\mathbf{u} \neq 0^k\overline{1}$ *and* $\mathbf{u} \neq 1^k\overline{0}$ *for all* $k \geq 0$. *There is a suffix* $\mathbf{v}$ *of* $\mathbf{u}$ *such that* $\inf(\mathbf{v}) = \inf_1(\mathbf{v}) = \inf_1(\mathbf{u})$ *and* $\sup(\mathbf{v}) = \sup_0(\mathbf{v}) = \sup_0(\mathbf{u})$.

Proof. — If $\inf(\mathbf{u}) = \inf_1(\mathbf{u})$ and $\sup(\mathbf{u}) = \sup_0(\mathbf{u})$, then we can take $\mathbf{v} = \mathbf{u}$. Otherwise, assume that $\inf(\mathbf{u}) \neq \inf_1(\mathbf{u})$, the case $\sup(\mathbf{u}) \neq \sup_0(\mathbf{u})$ being symmetric. Then we have $\inf(\mathbf{u}) = \mathbf{u} = 0^k01\mathbf{u}'$ for some $k \geq 0$, $\mathbf{u}' \in \{0,1\}^\infty \setminus \{\overline{1}\}$,

$$\sup_0(\mathbf{u}) = \sup_0(01\mathbf{u}') = \sup(01\mathbf{u}'),\ \inf_1(\mathbf{u}) = \inf_1(01\mathbf{u}') = \inf_1(1\mathbf{u}') = \inf(1\mathbf{u}').$$

If $\inf_1(01\mathbf{u}') \neq \inf(01\mathbf{u}')$, then $\mathbf{u}' = 1^n01\mathbf{u}''$ with $n \geq 0$, $\mathbf{u}'' > \mathbf{u}'$, which implies that $\sup_0(\mathbf{u}) = \sup_0(1\mathbf{u}') = \sup(1\mathbf{u}')$. Hence, we can take $\mathbf{v} = 01\mathbf{u}'$ or $\mathbf{v} = 1\mathbf{u}'$. □

Lemma 5. — *Let* $\mathbf{u} \in \{0,1\}^\infty$, $\sigma \in \{L,M,R\}^*$, *with* $\inf(\mathbf{u}) \geq \inf(\sigma(1\overline{0}))$, $\sup(\mathbf{u}) \leq \sup(\sigma(0\overline{1}))$. *Then* $\mathbf{u}$ *ends with* $\sigma(\mathbf{v})$ *for some* $\mathbf{v} \in \{0,1\}^\infty$ *or with* $\sigma'(\overline{0})$, $\sigma' \in \{L,M,R\}^*M$, $\sigma \in \sigma'\{L,M,R\}^*$.

Proof. — The statement is trivially true when σ is the identity. Suppose that it holds for some $\sigma \in \{L,M,R\}^*$, let $\varphi \in \{L,M,R\}$ and $\mathbf{u} \in \{0,1\}^\infty$ with $\inf(\mathbf{u}) \geq \inf(\varphi\sigma(1\overline{0}))$, $\sup(\mathbf{u}) \leq \sup(\varphi\sigma(0\overline{1}))$.

If $\varphi = L$, then $\sup(\mathbf{u}) \leq \overline{10}$, thus every 1 in $\mathbf{u}$ is followed by a 0, hence $\mathbf{u} = L(\mathbf{v})$ or $\mathbf{u} = 1L(\mathbf{v})$ for some $\mathbf{v} \in \{0,1\}^\infty$. Similary, if $\varphi = R$, then $\inf(\mathbf{u}) \geq \overline{01}$, hence $\mathbf{u} = R(\mathbf{v})$ or $\mathbf{u} = 0R(\mathbf{v})$ for some $\mathbf{v} \in \{0,1\}^\infty$. If $\varphi = M$, then $\inf(\mathbf{u}) \geq 0\overline{01}$ and $\sup(\mathbf{u}) \leq 1\overline{10}$. Hence, for all $k \geq 1$, $0(01)^k$ as well as $1(10)^k$ is always followed in $\mathbf{u}$ by 01 or 10. Since $\mathbf{u}$ contains 001 or 110 if $\mathbf{u} \notin \{M(\overline{0}), M(\overline{1})\}$, we obtain that $\mathbf{u}$ ends with $M(\mathbf{v})$ for some $\mathbf{v} \in \{0,1\}^\infty$.

We can assume that $\mathbf{v} \in \{\overline{0}, \overline{1}\}$ or $\inf_1(\mathbf{v}) = \inf(\mathbf{v})$ and $\sup_0(\mathbf{v}) = \sup(\mathbf{v})$, by Lemma 4. If $\mathbf{v} \neq \overline{0}$, then we cannot have $\inf(\mathbf{v}) < \inf(\sigma(1\overline{0}))$ because this would imply that $\inf(\varphi(\mathbf{v})) < \inf(\varphi\sigma(1\overline{0}))$ by Lemma 3. Similarly, we obtain that $\sup(\mathbf{v}) \leq \sup(\sigma(1\overline{0}))$ if $\mathbf{v} \neq \overline{1}$. If $\mathbf{v} = \overline{0}$, $\varphi \in \{L,R\}$, then $\inf(\varphi(\overline{0})) \geq \inf(\varphi\sigma(1\overline{0}))$ implies that $\inf(\sigma(1\overline{0})) = \overline{0}$, thus $\mathbf{v} = \sigma(\overline{0})$. Similarly, if $\mathbf{v} = \overline{1}$ and $\varphi \in \{L,R\}$, then $\sup(\varphi(\overline{1})) \leq \sup(\varphi\sigma(0\overline{1}))$ implies that $\sup(\sigma(0\overline{1})) = \overline{1}$, thus $\mathbf{v} = \sigma(\overline{1})$. If $\mathbf{v} \in \{\overline{0}, \overline{1}\}$, $\varphi = M$, then $\mathbf{u}$ ends with $M(\overline{0})$ since $M(\overline{1}) = 1M(\overline{0})$. Therefore, $\mathbf{u}$ ends with $\varphi\sigma(\mathbf{v})$ or with $\sigma'(\overline{0})$, $\sigma' \in \{L,M,R\}^*M$, $\varphi\sigma \in \sigma'\{L,M,R\}^*$. □

We obtain the following lower bound for $\mathcal{L}(m)$, cf. Figure 3.

Proposition 2. — *Let* $m \in (1,2]$. *We have* $\mathcal{L}(m) \geq g_{0\overline{1}}(m)$ *and*

$$\mathcal{L}(m) \geq \begin{cases} g_{\sigma(1\overline{0})}(m) & \text{if } m \leq \mu_{\sigma(01\overline{0})},\ \sigma \in \{L,M,R\}^*, \\ f_{\sigma(0\overline{1})}(m) & \text{if } m \geq \mu_{\sigma(10\overline{1})},\ \sigma \in \{L,M,R\}^*, \\ g_{\mathbf{u}}(m) & \text{if } m \leq \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}, \\ f_{\mathbf{u}}(m) & \text{if } m \geq \mu_{\mathbf{u}},\ \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}. \end{cases}$$

Proof. — For all $\mathbf{v} \in 1\{0,1\}^\infty \setminus \{\overline{1}\}$, we have $\inf(\mathbf{v}) \leq 0\overline{1}$. Then $\mathbf{v} \in U_\beta(m)$ implies that $\beta \geq g_{0\overline{1}}(m)$ by Lemma 2, hence $\mathcal{L}(m) \geq g_{0\overline{1}}(m)$.

Suppose that $U_\beta(m) \cap \{0,1\}^\infty$ is uncountable for $\beta < g_{\sigma(1\overline{0})}(m)$, $m \leq \mu_{\sigma(01\overline{0})}$, $\sigma \in \{L,M,R\}^*M$, thus $\beta < g_{\sigma(01\overline{0})}(m) \leq f_{\sigma(01\overline{0})}(m)$. Then $U_\beta(m)$ contains an aperiodic sequence $\mathbf{v} \in 1\{0,1\}^\infty$, with $f_{\mathbf{v}}(m) < f_{\sigma(01\overline{0})}(m)$ and $g_{\mathbf{v}}(m) < g_{\sigma(1\overline{0})}(m)$

by Lemma 2, thus $\inf(\mathbf{v}) > \inf(\sigma(1\overline{0}))$ and $\sup(\mathbf{v}) < \sup(\sigma(01\overline{0}))$ by Lemma 1. By Lemma 5, $\mathbf{v}$ ends with $\sigma(\mathbf{v}')$ for some (aperiodic) $\mathbf{v}' \in \{0,1\}^\infty$, contradicting that $\sup(\mathbf{v}) < \sup(\sigma(01\overline{0}))$. Symetrically, we get that $\mathcal{L}(m) \ge f_{\sigma(0\overline{1})}(m)$ for $m \ge \mu_{\sigma(10\overline{1})}$.

If $\mathbf{u}$ is a limit word of a primitive sequence $(\sigma_n)_{n\ge1} \in \{L,M,R\}^\infty$, then we have $\mu_{\sigma'_n(01\overline{0})} \to \mu_{\mathbf{u}}$ for $\sigma'_n = \sigma_1\sigma_2\cdots\sigma_n$ as $n \to \infty$, thus $\beta < g_{\mathbf{u}}(m)$, $m \le \mu_{\mathbf{u}}$ implies that $\beta < \min(g_{\sigma'_n(01\overline{0})}(m), f_{\sigma'_n(01\overline{0})}(m))$ for some $n \ge 1$, and we obtain as in the previous paragraph that $U_\beta(m) \cap \{0,1\}^\infty$ is at most countable. Therefore, we have $\mathcal{L}(m) \ge g_{\mathbf{u}}(m)$ and, similarly, $\mathcal{L}(m) \ge f_{\mathbf{u}}(m)$ for $m \ge \mu_{\mathbf{u}}$. □

Propositions 1 and 2 prove the formula for $\mathcal{L}(m)$ in Theorem 1. It remains to show that this covers all $m \in (1,2]$.

For the characterisation of $\mathcal{G}(m)$, in [**BS17**, Proposition 3.3] the partition

$$(\overline{0}, 0\overline{1}) = \mathcal{S}_{\{L,R\}} \cup \bigcup_{\sigma\in\{L,R\}^*} [\sigma(0\overline{01}), \sigma(\overline{01})]$$

for intervals of sequences in $\{0,1\}^\infty$ is used, which is a consequence of the partition

$$(\overline{0}, 0\overline{1}) = L((\overline{0}, 0\overline{1})) \cup [0\overline{01}, \overline{01}] \cup R((\overline{0}, 0\overline{1})).$$

We have to refine these partitions. For $\boldsymbol{\sigma} = (\sigma_n)_{n\ge1} \in \{L,M,R\}^\infty$, set

$$I_{\boldsymbol{\sigma}} = \begin{cases} \{\inf(\mathbf{u}) : \mathbf{u} \text{ is a limit word of } \boldsymbol{\sigma}\} & \text{if } \boldsymbol{\sigma} \text{ is primitive,} \\ \{\inf(\sigma_1\sigma_2\cdots\sigma_n(1\overline{0}))\} & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{L},\ n\ge1, \\ [\inf(\sigma_1\sigma_2\cdots\sigma_n(10\overline{1})), \inf(\sigma_1\sigma_2\cdots\sigma_n(\overline{1}))] & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{R},\ n\ge1, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$J_{\boldsymbol{\sigma}} = \begin{cases} \{\sup(\mathbf{u}) : \mathbf{u} \text{ is a limit word of } \boldsymbol{\sigma}\} & \text{if } \boldsymbol{\sigma} \text{ is primitive,} \\ [\sup(\sigma_1\sigma_2\cdots\sigma_n(\overline{0})), \sup(\sigma_1\sigma_2\cdots\sigma_n(01\overline{0}))] & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{L}, n\ge1, \\ \{\sup(\sigma_1\sigma_2\cdots\sigma_n(0\overline{1}))\} & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{R},\ n\ge1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that, for a primitive sequence $\boldsymbol{\sigma}$, $\inf(\mathbf{u})$ as well as $\sup(\mathbf{u})$ does not depend on the limit word $\mathbf{u}$. We order sequences in $\{L,M,R\}^\infty$ lexicographically.

Lemma 6. — *In $\{0,1\}^\infty$, we have*

$$(\overline{0}, 0\overline{1}) = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty} I_{\boldsymbol{\sigma}} \quad \textit{and} \quad (1\overline{0}, \overline{1}) = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty} J_{\boldsymbol{\sigma}}.$$

If $\boldsymbol{\sigma} < \boldsymbol{\sigma}'$, then $\mathbf{v} < \mathbf{v}'$ for all $\mathbf{v} \in I_{\boldsymbol{\sigma}}$, $\mathbf{v}' \in I_{\boldsymbol{\sigma}'}$, and for all $\mathbf{v} \in J_{\boldsymbol{\sigma}}$, $\mathbf{v}' \in J_{\boldsymbol{\sigma}'}$.

Proof. — We clearly have $I_{\boldsymbol{\sigma}} \subset (\overline{0}, 0\overline{1})$ for all $\boldsymbol{\sigma} \in \{L,M,R\}^\infty$. For all $\sigma \in \{L,M,R\}^*$, Lemma 3 gives that $\inf(\sigma(1\overline{0})) = \inf(\sigma L(1\overline{0}))$, $\inf(\sigma L(10\overline{1})) = \inf(\sigma M(1\overline{0}))$, and we have $M(\overline{1}) = R(1\overline{0})$, $R(10\overline{1}) = 10\overline{1}$, thus

$$\begin{aligned}(\inf(\sigma(1\overline{0})), \inf(\sigma(10\overline{1}))) = (&\inf(\sigma L(1\overline{0})), \inf(\sigma L(10\overline{1}))) \\ &\cup \{\inf(\sigma M(1\overline{0}))\} \cup (\inf(\sigma M(1\overline{0})), \inf(\sigma M(10\overline{1}))) \\ &\cup [\inf(\sigma M(10\overline{1})), \inf(\sigma M(\overline{1}))] \cup (\inf(\sigma R(1\overline{0})), \inf(\sigma R(10\overline{1})))\end{aligned}$$

(in this order). Inductively, we obtain that the sets $I_{\boldsymbol{\sigma}}$ are ordered by the lexicographical order on $\{L,M,R\}^\infty$. Moreover, the union of sets $I_{\boldsymbol{\sigma}}$ with $\boldsymbol{\sigma}$ ending in $M\overline{L}$ or $M\overline{R}$ covers $(\inf(1\overline{0}), \inf(10\overline{1})) = (\overline{0}, 0\overline{1})$, except for points lying in the intersection of nested intervals $\bigcap_{n\ge1}(\inf(\sigma_1\cdots\sigma_n(1\overline{0})), \inf(\sigma_1\cdots\sigma_n(10\overline{1})))$ for some $\boldsymbol{\sigma} = (\sigma_n)_{n\ge1} \in \{L,M,R\}^\infty$. Since $\sigma_1\cdots\sigma_n(\overline{0})$ is close to $\sigma_1\cdots\sigma_n(0\overline{1})$ for large n, these intervals tend to some $\mathbf{v} \in \{0,1\}^\infty$. If $\boldsymbol{\sigma}$ is primitive, then $I_{\boldsymbol{\sigma}} = \{\mathbf{v}\}$. If $\sigma_{n+1}\sigma_{n+2}\cdots$ is $\overline{L}$ or $\overline{R}$, then we have $\mathbf{v} = \inf(\sigma_1\cdots\sigma_n(1\overline{0}))$ or $\mathbf{v} = \inf(\sigma_1\cdots\sigma_n(10\overline{1}))$, which are not in the intersection.

The proof for $(1\overline{0}, \overline{1}) = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty} J_{\boldsymbol{\sigma}}$ is similar, with

$$\begin{aligned}(\sup(\sigma(01\overline{0})), \sup(\sigma(0\overline{1}))) &= (\sup(\sigma L(01\overline{0})), \sup(\sigma L(0\overline{1})))\\ &\cup [\sup(\sigma M(\overline{0})), \sup(\sigma M(01\overline{0}))] \cup (\sup(\sigma M(01\overline{0})), \sup(\sigma M(0\overline{1})))\\ &\cup \{\sup(\sigma M(0\overline{1}))\} \cup (\sup(\sigma R(01\overline{0})), \sup(\sigma R(0\overline{1}))).\end{aligned}$$

Hence, the $J_{\boldsymbol{\sigma}}$ are also ordered by the lexicographical order on $\{L,M,R\}^\infty$. □

Proposition 3. — *We have the partition*

$$(1, \mu_{0\overline{1}}) = \{\mu_{\mathbf{u}} : \mathbf{u} \in \mathcal{S}_{\{L,M,R\}}\} \cup \bigcup_{\sigma\in\{L,M,R\}^* M} \big([\mu_{\sigma(1\overline{0})}, \mu_{\sigma(01\overline{0})}] \cup [\mu_{\sigma(10\overline{1})}, \mu_{\sigma(0\overline{1})}]\big).$$

Proof. — For $m \in (1, \mu_{0\overline{1}})$, $\boldsymbol{\sigma} \in \{L,M,R\}^\infty$, let

$$I'_{\boldsymbol{\sigma}}(m) = \begin{cases}\{g_{\mathbf{u}}(m) : \mathbf{u} \text{ is a limit word of } \boldsymbol{\sigma}\} & \text{if } \boldsymbol{\sigma} \text{ is primitive,}\\ \{g_{\sigma_1\sigma_2\cdots\sigma_n(1\overline{0})}(m)\} & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{L},\ n\ge1,\\ [g_{\sigma_1\sigma_2\cdots\sigma_n(\overline{1})}(m), g_{\sigma_1\sigma_2\cdots\sigma_n(10\overline{1})}(m)] & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{R},\ n\ge1,\\ \emptyset & \text{otherwise,}\end{cases}$$

$$J'_{\boldsymbol{\sigma}}(m) = \begin{cases}\{f_{\mathbf{u}}(m) : \mathbf{u} \text{ is a limit word of } \boldsymbol{\sigma}\} & \text{if } \boldsymbol{\sigma} \text{ is primitive,}\\ [f_{\sigma_1\sigma_2\cdots\sigma_n(\overline{0})}(m), f_{\sigma_1\sigma_2\cdots\sigma_n(01\overline{0})}(m)] & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{L}, n\ge1,\\ \{f_{\sigma_1\sigma_2\cdots\sigma_n(0\overline{1})}(m)\} & \text{if } \sigma_n\sigma_{n+1}\cdots = M\overline{R},\ n\ge1,\\ \emptyset & \text{otherwise.}\end{cases}$$

By Lemmas 1 and 6, we have

$$(1, g_{1\overline{0}}(m)) = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty} I'_{\boldsymbol{\sigma}}(m) \quad \text{and} \quad (1, f_{0\overline{1}}(m)) = \bigcup_{\boldsymbol{\sigma}\in\{L,M,R\}^\infty} J'_{\boldsymbol{\sigma}}(m).$$

(Note that $f_{\mathbf{u}}(m)$ is close to $f_{\mathbf{u}'}(m)$ if $\sup(\mathbf{u})$ is close to $\sup(\mathbf{u}')$, $g_{\mathbf{u}}(m)$ is close to $g_{\mathbf{u}'}(m)$ if $\inf(\mathbf{u})$ is close to $\inf(\mathbf{u}')$.) If $\boldsymbol{\sigma} < \boldsymbol{\sigma}'$, then we have $\beta > \beta'$ if $\beta \in I'_{\boldsymbol{\sigma}}(m)$, $2 \le \beta' \in I'_{\boldsymbol{\sigma}'}(m)$, and $\beta < \beta'$ if $2 \le \beta \in J'_{\boldsymbol{\sigma}}(m)$, $\beta' \in J'_{\boldsymbol{\sigma}'}(m)$, by Lemmas 1 and 6. Since $\max(f_{\mathbf{u}}(m), g_{\mathbf{u}}(m)) \ge 2$ for all $\mathbf{u} \in \{0,1\}^\infty$ and $\inf(\sigma M(1\overline{0})) \le \inf(\sigma M(\overline{0}))$, $\sup(\sigma M(0\overline{1})) \ge \sup(\sigma M(\overline{1}))$ for all $\sigma \in \{L,M,R\}^*$, we have $I'_{\boldsymbol{\sigma}}(m) \subset [2,\infty)$ or $J'_{\boldsymbol{\sigma}}(m) \subset [2,\infty)$ for all $\boldsymbol{\sigma} \in \{L,M,R\}^\infty$. Therefore, we have $I'_{\boldsymbol{\sigma}}(m) \cap J'_{\boldsymbol{\sigma}}(m) \ne \emptyset$ for some $\boldsymbol{\sigma} \in \{L,M,R\}^\infty$. If $\boldsymbol{\sigma}$ is primitive, this means that $m = \mu_{\mathbf{u}}$. If $\sigma_n\sigma_{n+1}\cdots = M\overline{L}$, then we have $g_{\sigma_1\cdots\sigma_n(1\overline{0})}(m) \in [f_{\sigma_1\cdots\sigma_n(\overline{0})}(m), f_{\sigma_1\cdots\sigma_n(01\overline{0})}(m)]$, which means that

$m \in [\mu_{\sigma_1\cdots\sigma_n(1\overline{0})}, \mu_{\sigma_1\cdots\sigma_n(01\overline{0})}]$, see Figure 3. Similarly, if $\sigma_n\sigma_{n+1}\cdots = M\overline{R}$, then we have that $m \in [\mu_{\sigma_1\cdots\sigma_n(10\overline{1})}, \mu_{\sigma_1\cdots\sigma_n(0\overline{1})}]$. □

Proof of Theorem 1. — This is a direct consequence of Propositions 1, 2 and 3. □

3. Final remarks and open questions

By [**KLP11, BS17, Kwo18**], there are simple formulas for $\mu_{\sigma(1\overline{0})}$, $\mu_{\sigma(\overline{0})}$ and $\mu_{\sigma(0\overline{1})}$, $\sigma \in \{L,R\}^*M$, and for $\mu_{\mathbf{u}}$, $\mathbf{u} \in \mathcal{S}_{L,R}$. This is because, for $\mathbf{u} \in \{\sigma(1\overline{0}), \sigma(0\overline{1})\}$, $\sigma \in \{L,R\}^*M$, or $\mathbf{u} \in \mathcal{S}_{L,R}$, we have $\inf(\mathbf{u}) = 0\mathbf{v}$, $\sup(\mathbf{u}) = 1\mathbf{v}$ for some $\mathbf{v}$, thus $(\beta-1)(1+\pi_\beta(0\mathbf{v})) = (\beta-1)^2 = \beta\pi_\beta(1\mathbf{v})$, where $\beta > 1$ is defined by $\pi_\beta(20\mathbf{v}) = 1$, which gives that $\mu_{\mathbf{u}} = (\beta-1)^2$. For $\mathbf{u} = \sigma(\overline{0})$, we have $\inf(\mathbf{u}) = \overline{0w1}$, $\sup(\mathbf{u}) = \overline{1w0}$, with $\sigma(0) = 0w1$, and

$$(\beta-1)(1+\pi_\beta(\overline{0w1})) = (\beta-1)\beta\pi_\beta(\overline{10w}) = \frac{(\beta-1)^2\beta^{|\sigma(0)|}}{\beta^{|\sigma(0)|}-1} = \beta\pi_\beta(\overline{1w0}),$$

where $\beta > 1$ is defined by $\pi_\beta(20w\overline{0}) = 1$ and $|\sigma(0)|$ is the length of $\sigma(0)$, hence $\mu_{\sigma(\overline{0})} = (\beta-1)^2\beta^{|\sigma(0)|}/(\beta^{|\sigma(0)|}-1)$. Are there similar formulas for $\sigma \in \{L,M,R\}^*M$?

In [**BS17, Kwo18**], it was proved that the Hausdorff dimension of $\{\mu_{\mathbf{u}} : \mathbf{u} \in \mathcal{S}_{L,R}\}$ is 0, using that the number of balanced words grows polynomially. What is the complexity of $\mathcal{S}_{L,M,R}$?

As mentioned in the Introduction, we know the generalised Komornik–Loreti constant $\mathcal{K}(m)$ only for $m = 2$ and when $\mathcal{G}(m) = 1 + \sqrt{m} = \mathcal{K}(m) = \mathcal{L}(m)$. This is due to the fact that it is usually difficult to study maps with two holes; see Figure 2. (For $m = 2$, we can use the symmetry of the map T, and for $\mathcal{L}(m) = 1 + \sqrt{m}$, we can restrict to sequences in $\{0,1\}^\infty$.) New ideas are needed for the general case.

Finally, *Sturmian holes* are key ingredients in [**Sid14**], where supercritical holes for the doubling map are studied. Do our Thue–Morse–Sturmian sequences also play a role in this context?

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