On the Delone property of $(-\beta)$-integers

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The $(-\beta)$-integers are natural generalisations of the β-integers, and thus of the integers, for negative real bases. They can be described by infinite words which are fixed points of anti-morphisms. We show that they are not necessarily uniformly discrete and relatively dense in the real numbers.

1   Introduction

We study the set of $(-\beta)$-*integers* for a real number $\beta > 1$. This set is defined by

$$\mathbb{Z}_{-\beta} = \bigcup_{n \ge 0} (-\beta)^n \, T_{-\beta}^{-n}(0) \,,$$

where $T_{-\beta}$ is the $(-\beta)$-*transformation*, defined by Ito and Sadahiro [5] as

$$T_{-\beta} : \left[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\right), \quad x \mapsto -\beta x - \left\lfloor \tfrac{\beta}{\beta+1} - \beta x \right\rfloor .$$

Equivalently, a $(-\beta)$-integer is a real number of the form

$$\sum_{k=0}^{n-1} d_k (-\beta)^k, \quad \text{with} \quad \frac{-\beta}{\beta+1} \le \sum_{k=0}^{m-1} d_k (-\beta)^{k-m} < \frac{1}{\beta+1} \quad \text{for all } 1 \le m \le n,$$

where $d_0, d_1, \ldots, d_{k-1}$ are integers. Examples of $(-\beta)$-transformations are depicted in Figure 1.

Recall that the set of β-*integers* is defined by

$$\mathbb{Z}_\beta = \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \quad \text{with} \quad \mathbb{Z}_\beta^+ = \bigcup_{n \ge 0} \beta^n \, T_\beta^{-n}(0) \,,$$

where T_β is the β-*transformation*,

$$T_\beta : [0,1) \to [0,1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor .$$

These sets were introduced in the domain of quasicrystallography, see e.g. [2].

It is not difficult to see that $\mathbb{Z}_{-\beta} = \mathbb{Z}$ when $\beta \in \mathbb{Z}$, and that $\mathbb{Z}_{-\beta} = \{0\}$ when $\beta < \frac{1+\sqrt{5}}{2}$. For $\beta \ge \frac{1+\sqrt{5}}{2}$, Ambrož et al. [1] showed that $\mathbb{Z}_{-\beta}$ can be described by the fixed point of an anti-morphism on a possibly infinite alphabet. They also calculated explicitly the set of distances between consecutive $(-\beta)$-integers when $T_{-\beta}^n \left(\frac{-\beta}{\beta+1}\right) \le 0$ and $T_{-\beta}^{2n-1}\left(\frac{-\beta}{\beta+1}\right) \ge \frac{1-\lfloor\beta\rfloor}{\beta}$ for all $n \ge 1$. It seems to be difficult to extend their methods to the general case. For the case when β is an *Yrrap number*, i.e., when $\left\{T_{-\beta}^n \left(\frac{-\beta}{\beta+1}\right) \mid n \ge 0\right\}$ is a finite set, a different approach can be found in [8]. The approach in Section 2 resembles that in [8], but it is simpler and works for general β. In Section 3, we discuss the Delone property for sets $\mathbb{Z}_{-\beta}$.

2 Fixed point of an anti-morphism

By Lemma 1, we can consider the set of $(-\beta)$-integers, $\beta > 1$, as a special instance of the preimage of a point in $\left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$ of the map

$$\iota_\beta : \mathbb{R} \to \left[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\right), \quad x \mapsto T_{-\beta}^n\left((-\beta)^{-n}x\right), \text{ with } n \geq 0 \text{ such that } (-\beta)^{-n}x \in \left(\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\right).$$

Since $T_{-\beta}\left((-\beta)^{-1}x\right) = x$ for all $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, the map ι_β is well defined.

Lemma 1. *For any $\beta > 1$, we have*

$$\mathbb{Z}_{-\beta} = \iota_\beta^{-1}(0).$$

Proof. If $x \in \iota_\beta^{-1}(0)$, then $T_{-\beta}^n\left((-\beta)^{-n}x\right) = 0$ for some $n \geq 0$, thus $x \in (-\beta)^n\, T_{-\beta}^{-n}(0)$, i.e., $x \in \mathbb{Z}_{-\beta}$. On the other hand, $x \in \mathbb{Z}_{-\beta}$ implies that $x \in (-\beta)^n\, T_{-\beta}^{-n}(0)$ for some $n \geq 0$. If $x(-\beta)^{-n} \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, this immediately implies that $\iota_\beta(x) = 0$. If $(-\beta)^{-n}x = \frac{-\beta}{\beta+1}$, then

$$T_{-\beta}^{n+2}\left((-\beta)^{-(n+2)}x\right) = T_{-\beta}^{n+2}\left(\tfrac{-1/\beta}{\beta+1}\right) = T_{-\beta}^{n+1}\left(\tfrac{-\beta}{\beta+1}\right) = T_{-\beta}^{n+1}\left((-\beta)^{-n}x\right) = T_{-\beta}(0) = 0,$$

thus $\iota_\beta(x) = 0$ as well. $\square$

Note that $\iota_\beta(x) = x$ for all $x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. For other x, we use the following relation.

Lemma 2. *For any $\beta > 1$, $x \in \mathbb{R}$, we have*

$$\iota_\beta(-\beta x) = T_{-\beta}\left(\iota_\beta(x)\right).$$

Proof. Let $x \in \mathbb{R}$ with $(-\beta)^{-n}x \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, $n \geq 0$. Then we have

$$\iota_\beta(-\beta x) = T_{-\beta}^{n+1}\left((-\beta)^{-(n+1)}(-\beta x)\right) = T_{-\beta}\, T_{-\beta}^n\left((-\beta)^{-n}x\right) = T_{-\beta}\left(\iota_\beta(x)\right). \qquad \square$$

An important role in the study of the $(-\beta)$-transformation is played by the orbit of the left endpoint $\frac{-\beta}{\beta+1}$. In the following, fix $\beta > 1$, and let

$$t_n = T_{-\beta}^n\left(\tfrac{-\beta}{\beta+1}\right) \quad (n \geq 0), \quad a_n = \lfloor -t_0 - \beta t_{n-1} \rfloor \quad (n \geq 1).$$

(As usual, $\lfloor x \rfloor$ denotes the largest integer $\leq x$, and $\lceil x \rceil$ denotes the smallest integer $\geq x$.) Then $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$, i.e.,

$$t_n = \sum_{k=1}^{\infty} \frac{a_{n+k}}{(-\beta)^k} \quad \text{for all } n \geq 0,$$

see [5]. Setting $t_{-1} = \frac{1}{\beta+1}$, $t_\infty = 0$, $\mathbb{N}_\infty = \{0, 1, 2, \ldots\} \cup \{\infty\}$, we consider open intervals

$$J_{(i,j)} = (t_{2i}, t_{2j-1}) \quad \text{with } i, j \in \mathbb{N}_\infty, \ 0 \leq t_{2i} < t_{2j-1} \text{ or } t_{2i} < t_{2j-1} \leq 0$$

(where $2\infty = \infty$ and $\infty - 1 = \infty$). We also set $a_0 = a_\infty = 0$, and

$$\mathscr{A} = \{(i,j) \mid i, j \in \mathbb{N}_\infty, 0 \leq t_{2i} < t_{2j-1} \text{ or } t_{2i} < t_{2j-1} \leq 0\}.$$

(Here, (i, j) is a pair of elements in $\mathbb{N}_\infty$, and not an open interval.) Let

$$L_\beta\big((i, j)\big) = t_{2j-1} - t_{2i} \quad \big((i, j) \in \mathscr{A}\big)$$

be the length of the interval $J_{(i,j)}$, and set

$$L_\beta(v_1 \cdots v_k) = L_\beta(v_1) + \cdots + L_\beta(v_k), \quad |v_1 \cdots v_k| = k,$$

for any word $v_1 \cdots v_k \in \mathscr{A}^*$, where $\mathscr{A}^*$ denotes the free monoid over $\mathscr{A}$.

Let $\psi_\beta : \mathscr{A}^* \to \mathscr{A}^*$ be an anti-morphism, which is defined on $(i, j) \in \mathscr{A}$ by

$$\psi_\beta\big((i, j)\big) = (j, i+1) \quad \text{if } a_{2i+1} = a_{2j}, \, t_{2i+1}t_{2j} \geq 0,$$

and otherwise by

$$\psi_\beta\big((i, j)\big) = \begin{cases} (j, \infty)\big((\infty, 0)(0, \infty)\big)^{a_{2i+1}-a_{2j}}(\infty, i+1) & \text{if } t_{2i+1} > 0, \, t_{2j} < 0, \\ (j, 0)\big((0, \infty)(\infty, 0)\big)^{a_{2i+1}-a_{2j}-1}(0, \infty)(\infty, i+1) & \text{if } t_{2i+1} > 0, \, t_{2j} \geq 0, \\ (j, 0)\big((0, \infty)(\infty, 0)\big)^{a_{2i+1}-a_{2j}-1}(0, i+1) & \text{if } t_0 < t_{2i+1} \leq 0, \, t_{2j} \geq 0, \\ (j, \infty)(\infty, 0)\big((0, \infty)(\infty, 0)\big)^{a_{2i+1}-a_{2j}-1}(0, i+1) & \text{if } t_0 < t_{2i+1} \leq 0, \, t_{2j} < 0, \\ (j, 0)\big((0, \infty)(\infty, 0)\big)^{a_{2i+1}-a_{2j}-1} & \text{if } t_{2i+1} = t_0, \, t_{2j} \geq 0, \\ (j, \infty)(\infty, 0)\big((0, \infty)(\infty, 0)\big)^{a_{2i+1}-a_{2j}-1} & \text{if } t_{2i+1} = t_0, \, t_{2j} < 0. \end{cases}$$

Here, anti-morphism means that $\psi_\beta(vw) = \psi_\beta(w)\psi_\beta(v)$ for all $v, w \in \mathscr{A}^*$. The anti-morphism ψ_β is naturally extended to infinite words over $\mathscr{A}$. (Right infinite words are mapped to left infinite words and vice versa.)

Lemma 3. *Let $\beta > 1$. For any $u \in \mathscr{A}$, we have*

$$L_\beta\big(\psi_\beta(u)\big) = \beta L_\beta(u).$$

Moreover, for any $1 \leq \ell \leq |\psi_\beta(u)|$, $0 < x < L_\beta(v_\ell)$, with $\psi_\beta(u) = v_1 \cdots v_{|\psi_\beta(u)|}$, we have

$$T_{-\beta}\big(t_{2j-1} - \beta^{-1}L_\beta(v_1 \cdots v_{\ell-1}) - \beta^{-1}x\big) = t_{2i'} + x, \quad \text{where } u = (i, j), \, v_\ell = (i', j').$$

Proof. This follows from the definitions of $T_{-\beta}$ and ψ_β. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\cdots u_{-1}u_0u_1 \cdots \in \mathscr{A}^\mathbb{Z}$ be the fixed point of ψ_β such that $u_0 = (\infty, 0)$, $u_0u_1 \cdots$ is a fixed point of ψ_β^2 and $\cdots u_{-2}u_{-1} = \psi_\beta(u_0u_1 \cdots)$, in particular $u_{-1} = (0, \infty)$. Let

$$Y_\beta = \{y_k \mid k \in \mathbb{Z}\} \quad \text{with} \quad y_k = \begin{cases} L_\beta(u_0 \cdots u_{k-1}) & \text{if } k \geq 0, \\ -L_\beta(u_k \cdots u_{-1}) & \text{if } k < 0. \end{cases}$$

Proposition 1. *Let $\beta > 1$. On every interval (y_k, y_{k+1}), $k \in \mathbb{Z}$, the map $\imath_\beta$ is a translation, with*

$$\imath_\beta\big((y_k, y_{k+1})\big) = J_{u_k}.$$

Proof. We have $\iota_\beta(x) = x$ on the intervals $(y_0, y_1) = (t_\infty, t_{-1}) = (0, \frac{1}{\beta+1})$ and $(y_{-1}, y_0) = (t_0, t_\infty) = (\frac{-\beta}{\beta+1}, 0)$, thus the statement of the proposition holds for $(y_k, y_{k+1}) \subset (\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$.

Assume that the statement holds for $k \in \mathbb{Z}$. By Lemma 3, we have $(-\beta)(y_k, y_{k+1}) = (y_{k'}, y_{k'+|\psi_\beta(u_k)|})$ and $\psi_\beta(u_k) = u_{k'} \cdots u_{k'+|\psi_\beta(u_k)|-1}$, with $k' = -|\psi_\beta(u_0 \cdots u_k)|$ if $k \geq 0$, $k' = |\psi_\beta(u_{k+1} \cdots u_{-1})|$ if $k < 0$. Then Lemma 2, the assumption $\iota_\beta((y_k, y_{k+1})) = J_{u_k}$, and Lemma 3 yield that

$$
\begin{aligned}
\iota_\beta\big((y_{k'+\ell}, y_{k'+\ell+1})\big) &= T_{-\beta}\big(\iota_\beta((-\beta)^{-1}(y_{k'+\ell}, y_{k'+\ell+1}))\big) \\
&= T_{-\beta}\big(\iota_\beta\big((y_{k+1} - \beta^{-1}L_\beta(u_{k'} \cdots u_{k'+\ell}), y_{k+1} - \beta^{-1}L_\beta(u_{k'} \cdots u_{k'+\ell-1}))\big)\big) \\
&= T_{-\beta}\big((t_{2j-1} - \beta^{-1}L_\beta(u_{k'} \cdots u_{k'+\ell}), t_{2j-1} - \beta^{-1}L_\beta(u_{k'} \cdots u_{k'+\ell-1}))\big) \\
&= J_{u_{k'+\ell}}
\end{aligned}
$$

for all $0 \leq \ell < |\psi_\beta(u_k)|$, where $u_k = (i, j)$.

By induction on n, we obtain for every $n \geq 0$ that the statement of the proposition holds for all $k \in \mathbb{Z}$ with $(y_k, y_{k+1}) \subset (-\beta)^n(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1})$, thus it holds for all $k \in \mathbb{Z}$. $\qquad\square$

Now we describe the set $Y_\beta = \{y_k \mid k \in \mathbb{Z}\}$, which is left out by the intervals (y_k, y_{k+1}).

Lemma 4. *For any $\beta > 1$, we have*

$$
Y_\beta = \mathbb{Z}_{-\beta} \cup \bigcup_{m,n \geq 0} (-\beta)^{m+n} T_{-\beta}^{-n}\big(\tfrac{-\beta}{\beta+1}\big).
$$

Proof. First note that

$$
\bigcup_{m,n \geq 0} (-\beta)^{m+n} T_{-\beta}^{-n}\big(\tfrac{-\beta}{\beta+1}\big) = \bigcup_{m \geq 0} (-\beta)^m \iota_\beta^{-1}\big(\tfrac{-\beta}{\beta+1}\big),
$$

similarly to Lemma 1. Indeed, $x \in \iota_\beta^{-1}(t_0)$ is equivalent to $(-\beta)^{-n}x \in T_{-\beta}^{-n}(t_0) \cap (t_0, t_{-1})$ for some $n \geq 0$. In the remaining case $(-\beta)^{-n}x = t_0 \in T_{-\beta}^{-n}(t_0)$, we have $x \in (-\beta)\iota_\beta^{-1}(t_0)$ since $T_{-\beta}^{n+1}((-\beta)^{-(n+2)}x) = T_{-\beta}^n(t_0) = t_0$. Note also that $(-\beta)^m \iota_\beta^{-1}(t_0) \subseteq \iota_\beta^{-1}(t_m)$.

Since $t_0 \notin J_{u_k}$ and $0 \notin J_{u_k}$ for all $k \in \mathbb{Z}$ by the definition of $\mathscr{A}$, Proposition 1 implies that $\iota_\beta^{-1}(0) \cup \iota_\beta^{-1}(t_0) \subseteq Y_\beta$, thus $\mathbb{Z}_{-\beta} \subseteq Y_\beta$ by Lemma 1. Since $(-\beta)Y_\beta \subseteq Y_\beta$ by Lemma 3, we obtain $(-\beta)^m \iota_\beta^{-1}(t_0) \subseteq Y_\beta$ for every $m \geq 0$ as well.

Let now $x \in Y_\beta \setminus \{0\}$. Then there exists some $m \geq 0$ such that $(-\beta)^{-m}x \in Y_\beta$ and $(-\beta)^{-m-1}x \notin Y_\beta$, i.e., $(-\beta)^{-m-1}x \in (y_k, y_{k+1})$ for some $k \in \mathbb{Z}$. By Lemma 2 and the definition of ψ_β, we obtain that $\iota_\beta((-\beta)^{-m}x) \in \{t_0, 0\}$, i.e., $x \in (-\beta)^m \iota_\beta^{-1}(t_0)$ or $x \in (-\beta)^m \mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$. Since $0 \in \mathbb{Z}_{-\beta}$, this proves the lemma. $\qquad\square$

Theorem 1. *Let $\beta > 1$, $x \in \mathbb{R}$. Then $x \in \mathbb{Z}_{-\beta}$ if and only if $x = y_k$ for some $k \in \mathbb{Z}$ with $t_{2j-1} = 0$ or $t_{2i} = 0$, where $u_{k-1} = (i', j)$, $u_k = (i, j')$.*

Proof. Since $\mathbb{Z}_{-\beta} \subseteq Y_\beta$ by Lemma 4, it is sufficient to consider $x = y_k$, $k \in \mathbb{Z}$. As in the proof of Lemma 4, let $m \geq 0$ be such that $(-\beta)^{-m}x \in Y_\beta$ and $(-\beta)^{-m-1}x \notin Y_\beta$. Then we have $\iota_\beta((-\beta)^{-m}x) \in \{t_0, 0\}$.

If $\iota_\beta((-\beta)^{-m}x) = 0$, then $\iota_\beta(x) = T_{-\beta}^m(0) = 0$. Moreover, ι_β is continuous at $(-\beta)^{-m}x$ in this case. Together with the continuity of $T_{-\beta}^m$ at 0, this implies that ι_β is continuous at x, i.e., $u_{k-1} = (i', j)$ with $t_{2j-1} = 0$ and $u_k = (i, j')$ with $t_{2i} = 0$.

If $\iota_\beta((-\beta)^{-m}x) = t_0$, then ι_β is right continuous at $(-\beta)^{-m}x$, and its limit from the left is t_{-1}. We obtain that $\iota_\beta(x) = t_m$, $u_{k-1} = (i', \lceil m/2 \rceil)$ and $u_k = (\lfloor m/2 \rfloor, j')$ for some $i', j' \in \mathbb{N}_\infty$. Let $i = \lfloor m/2 \rfloor$,

$j = \lceil m/2 \rceil$. Since $2i = m$ if m is even and $2j - 1 = m$ if m is odd, $x \in \mathbb{Z}_{-\beta}$ implies that $t_{2i} = 0$ or $t_{2j-1} = 0$. On the other hand, $t_{2i} = 0$ or $t_{2j-1} = 0$ yields that $t_{m-1} = 0$ or $t_m = 0$. Since $t_{m-1} = 0$ implies $t_m = 0$, we must have $x \in \mathbb{Z}_{-\beta}$. $\qquad\square$

By Theorem 1, the study of $\mathbb{Z}_{-\beta}$ is reduced to the study of the fixed point of ψ_β. Note that (i, j) and (i', j') can be identified when $t_{2i} = t_{2i'}$ and $t_{2j-1} = t_{2j'-1}$. After identification, $\mathscr{A}$ is finite if and only if $\{t_n \mid n \geq 0\}$ is a finite set, i.e., if β is an Yrrap number.

With the help of ψ_β, we can construct an anti-morphism describing the structure of $\mathbb{Z}_{-\beta}$ (for $\beta \geq \frac{1+\sqrt{5}}{2}$), similarly to [8]. First note that $\beta \geq \frac{1+\sqrt{5}}{2}$ implies $a_1 = 1$, $t_1 \geq 0$, or $a_1 \geq 2$, thus $\psi_\beta^2((\infty, 0)) = \psi_\beta((0, \infty))$ starts with $(\infty, 0)\,(0, \infty)$. Therefore, $L_\beta((\infty, 0)\,(0, \infty)) = t_{-1} - t_0 = 1$ is the smallest positive element of $\mathbb{Z}_{-\beta}$. Using Theorem 1 and $(-\beta)\mathbb{Z}_{-\beta} \subseteq \mathbb{Z}_{-\beta}$, the word $\psi_\beta((\infty, 0)\,(0, \infty))$ determines the set $\mathbb{Z}_{-\beta} \cap [-\beta, 0]$. Splitting up $\psi_\beta((\infty, 0)\,(0, \infty))$ according to Theorem 1 and applying ψ_β on each of the factors yields the set $\mathbb{Z}_{-\beta} \cap [0, \beta^2]$, etc. Consider all these factors, i.e., all words $u_k \cdots u_{k'-1}$ between consecutive elements $y_k, y_{k'} \in \mathbb{Z}_{-\beta}$, as letters. Using the described strategy, we define an anti-morphism $\varphi_{-\beta}$ on words consisting of these letters. Then the fixed point of $\varphi_{-\beta}$ codes the distances between the elements of $\mathbb{Z}_{-\beta}$, see the two simple examples below. For more complicated examples, we refer to [8]. By [1, 8], the alphabet is finite if and only if β is an Yrrap number.
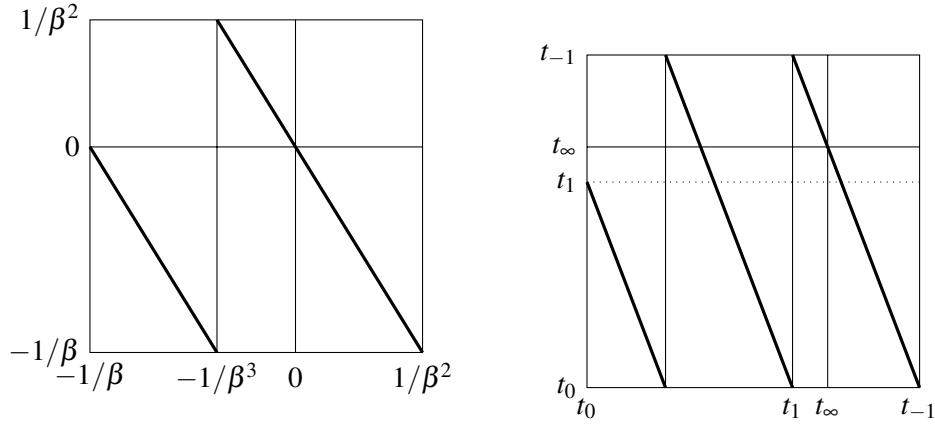

Figure 1: The $(-\beta)$-transformation for $\beta = \frac{1+\sqrt{5}}{2}$ (left) and $\beta = \frac{3+\sqrt{5}}{2}$ (right).

Example 1. Let $\beta = \frac{3+\sqrt{5}}{2} \approx 2.618$, i.e., $\beta^2 = 3\beta - 1$. Then we have $t_1 = \frac{\beta^2}{\beta+1} - 2 = \frac{-1/\beta}{\beta+1}$ and $t_2 = \frac{1}{\beta+1} - 1 = t_0$. Therefore, we can identify $(0, \infty)$ and $(1, \infty)$, and obtain

$$\psi_\beta: \quad (\infty, 0) \mapsto (0, \infty),$$
$$(0, \infty) \mapsto (\infty, 0)\,(0, \infty)\,(\infty, 0)\,(0, 1),$$
$$(0, 1) \mapsto (0, \infty)\,(\infty, 0)\,(0, 1).$$

The two-sided fixed point $\cdots u_{-2}u_{-1}\dot{}u_0 u_1 \cdots$ of ψ_β is equal to

$$\cdots (0, \infty)(\infty, 0)(0, 1)\,(0, \infty)\,(\infty, 0)(0, \infty)(\infty, 0)\,(0, 1)\,(0, \infty)\dot{}(\infty, 0)(0, \infty)(\infty, 0)(0, 1) \cdots.$$

Applying ψ_β to $(\infty, 0)\,(0, \infty)$ and to the factors described by Theorem 1 yields

$$(\infty, 0)\,(0, \infty) \mapsto (\infty, 0)\,(0, \infty)\,(\infty, 0)\,(0, 1)\,(0, \infty),$$
$$(\infty, 0)\,(0, 1)\,(0, \infty) \mapsto (\infty, 0)\,(0, \infty)\,(\infty, 0)\,(0, 1)\,(0, \infty)\,(\infty, 0)\,(0, 1)\,(0, \infty).$$

Therefore, setting $A = (\infty, 0)\,(0, \infty)$ and $B = (\infty, 0)\,(0, 1)\,(0, \infty)$, the fixed point

$$\cdots ABBABABBABBAB\,{}^{\boldsymbol{\cdot}}ABBABABBABBAB \cdots$$

of the anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB, \quad B \mapsto ABB,$$

describes the set of $(-\beta)$-integers, see Figure 2. The distances between consecutive elements of $\mathbb{Z}_{-\beta}$ are $L_\beta(A) = 1$ and $L_\beta(B) = \beta - 1 > 1$.
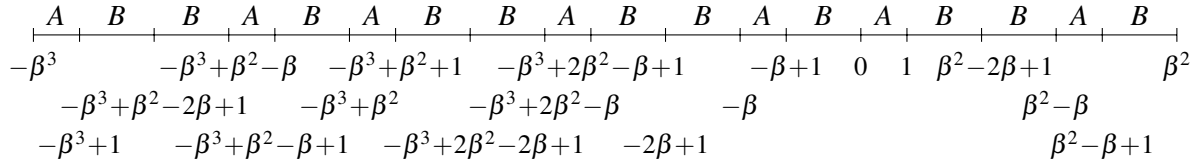

| | A | B | | B | A | B | A | B | | B | A | B | | B | A | B | | B | A | B |

$$
\begin{array}{l}
-\beta^3 \qquad\qquad -\beta^3+\beta^2-\beta \quad -\beta^3+\beta^2+1 \quad -\beta^3+2\beta^2-\beta+1 \qquad -\beta+1 \quad 0 \quad 1 \quad \beta^2-2\beta+1 \qquad\qquad \beta^2 \\
\quad -\beta^3+\beta^2-2\beta+1 \quad -\beta^3+\beta^2 \qquad -\beta^3+2\beta^2-\beta \qquad -\beta \qquad\qquad\qquad \beta^2-\beta \\
-\beta^3+1 \qquad -\beta^3+\beta^2-\beta+1 \quad -\beta^3+2\beta^2-2\beta+1 \quad -2\beta+1 \qquad\qquad\qquad\qquad \beta^2-\beta+1
\end{array}
$$

Figure 2: The set $\mathbb{Z}_{-\beta} \cap [-\beta^3, \beta^2]$, $\beta = (3+\sqrt{5})/2$.

Example 2. Let $\beta = \frac{1+\sqrt{5}}{2} \approx 1.618$, i.e., $\beta^2 = \beta + 1$. Then we have $t_1 = \frac{\beta^2}{\beta+1} - 1 = 0 = t_\infty$. Identifying $(0, \infty)$ and $(0, 1)$, we obtain

$$\psi_\beta: \quad (\infty, 0) \mapsto (0, \infty), \quad (0, \infty) \mapsto (\infty, 0)\,(0, \infty).$$

The two-sided fixed point $\cdots u_{-2} u_{-1} {}^{\boldsymbol{\cdot}} u_0 u_1 \cdots$ of ψ_β is equal to

$$\cdots (\infty, 0)(0, \infty)\,(0, \infty)\,{}^{\boldsymbol{\cdot}}(\infty, 0)(0, \infty)\,(\infty, 0)(0, \infty)\,(0, \infty) \cdots.$$

The words $u_k \cdots u_{k'-1}$ between consecutive elements $y_k, y_{k'} \in \mathbb{Z}_{-\beta}$ are $A = (\infty, 0)\,(0, \infty)$ and $B = (0, \infty)$, since

$$(\infty, 0)\,(0, \infty) \mapsto (\infty, 0)\,(0, \infty)\,(0, \infty), \quad (0, \infty) \mapsto (\infty, 0)\,(0, \infty).$$

Note that B does not start with a letter (i, j) with $t_{2i} = 0$, thus ι_β is discontinuous at the corresponding points $y_k \in \mathbb{Z}_{-\beta}$. The fixed point

$$\cdots AABAABABAABAB\,{}^{\boldsymbol{\cdot}}AABAABABAABAABABAABAB \cdots$$

of the anti-morphism

$$\varphi_{-\beta}: \quad A \mapsto AB, \quad B \mapsto A,$$

describes the set of $(-\beta)$-integers, with $L_\beta(A) = 1$ and $L_\beta(B) = \beta - 1 < 1$, see Figure 3. Note that $(-\beta)^n$ can also be represented as $(-\beta)^{n+2} + (-\beta)^{n+1}$.
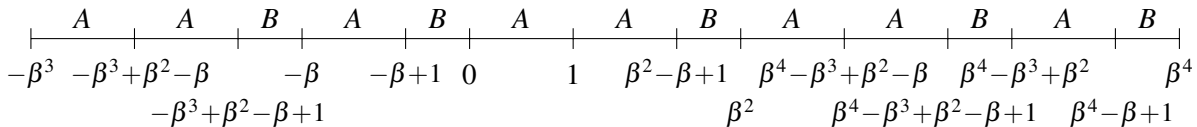

| | A | | A | B | A | B | A | | A | B | A | | A | B | A | B |

$$
\begin{array}{l}
-\beta^3 \quad -\beta^3+\beta^2-\beta \qquad -\beta \qquad -\beta+1 \quad 0 \qquad 1 \qquad \beta^2-\beta+1 \quad \beta^4-\beta^3+\beta^2-\beta \quad \beta^4-\beta^3+\beta^2 \qquad \beta^4 \\
\qquad\qquad -\beta^3+\beta^2-\beta+1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \beta^2 \qquad\quad \beta^4-\beta^3+\beta^2-\beta+1 \quad \beta^4-\beta+1
\end{array}
$$

Figure 3: The $(-\beta)$-integers in $[-\beta^3, \beta^4]$, $\beta = (1+\sqrt{5})/2$.

3 Delone property

A set $S \subset \mathbb{R}$ is called *Delone set* (or Delaunay set) if it is *uniformly discrete* and *relatively dense*; i.e., if there are numbers $R > r > 0$, such that each interval of length r contains at most one point of S, and every interval of length R contains at least one point of S.

If β is an Yrrap number, then the set of distances between consecutive $(-\beta)$-integers is finite by [1, 8], thus $\mathbb{Z}_{-\beta}$ is a Delone set. For general β, we show in this section that $\mathbb{Z}_{-\beta}$ need neither be uniformly discrete nor relatively dense.

3.1 Uniform discreteness

Since every point $x \in Y_\beta$ is separated from $Y_\beta \setminus \{x\}$ by an interval around x, Y_β is discrete, and the same holds for $\mathbb{Z}_{-\beta}$. It is well known that $\mathbb{Z}_\beta$ is uniformly discrete if and only if 0 is not an accumulation point of $\{T_\beta^n(1) \mid n \geq 0\}$ (where $T_\beta(1) = \beta - \lfloor \beta \rfloor$). For $(-\beta)$-integers, the situation is more complicated.

Proposition 2. *Let $\beta > 1$. If 0 is not an accumulation point of $\left\{T_{-\beta}^{2n-1}\left(\frac{-\beta}{\beta+1}\right) > 0 \mid n \geq 1\right\}$, then the set $\mathbb{Z}_{-\beta}$ is uniformly discrete.*

Proof. When β is an Yrrap number, then $\{L_\beta(u_k) \mid k \in \mathbb{Z}\}$ is finite, thus $\mathbb{Z}_{-\beta}$ is uniformly discrete. Therefore, assume that β is not an Yrrap number, in particular that $t_n \neq 0$ for all (finite) $n \geq 0$, with the notation of Section 2. Then ι_β is continuous at every point $y_k \in \mathbb{Z}_{-\beta}$, $k \in \mathbb{Z}$, thus $u_k = (i, j)$ with $t_{2i} = 0$. Since $L_\beta(u_k) \geq \inf\{t_{2n-1} > 0 \mid n \geq 0\}$, the set $\mathbb{Z}_{-\beta}$ is uniformly discrete if $\inf\{t_{2n-1} > 0 \mid n \geq 1\} > 0$. $\square$

In order to give examples of β where $\mathbb{Z}_{-\beta}$ is not uniformly discrete or not relatively dense, we need to know which sequences are possible $(-\beta)$-expansions of $\frac{-\beta}{\beta+1}$. The corresponding problem for β-expansions was solved by Parry [7]. Góra [4, Theorem 25] gave an answer to a more general question, but his theorem is incorrect, as noticed in [3]. However, Góra proved the following result, where $\leq_{\text{alt}}$ denotes the alternate order on words, i.e., $x_1 x_2 \cdots <_{\text{alt}} y_1 y_2 \cdots$ if $(-1)^n(x_{n+1} - y_{n+1}) < 0$, $x_1 \cdots x_n = y_1 \cdots y_n$ for some $n \geq 0$.

Lemma 5. *Let $a_1 a_2 \cdots$ be a sequence of non-negative integers satisfying $a_{n+1} a_{n+2} \cdots \leq_{\text{alt}} a_1 a_2 \cdots$ for all $n \geq 1$, with $a_1 \geq 2$. Then there exists a unique $a_1 \leq \beta \leq a_1 + 1$ such that*

$$\sum_{k=1}^{\infty} \frac{a_k}{(-\beta)^k} = \frac{-\beta}{\beta+1}, \quad \text{and} \quad \frac{-\beta}{\beta+1} \leq \sum_{k=1}^{\infty} \frac{a_{n+k}}{(-\beta)^k} \leq \frac{1}{\beta+1} \quad \text{for all } n \geq 1.$$

Proof. As the generalised β-transformations in [4] with $E = (1, \ldots, 1)$ are intimately related to our $T_{-\beta}$ (see the introduction of [6]), the lemma follows from the proof of Theorem 25 in [4]. $\square$

The flaw in [4, Theorem 25] is that $\sum_{k=1}^{\infty} \frac{a_{n+k}}{(-\beta)^k} = \frac{1}{\beta+1}$ cannot be excluded, which would be necessary for $a_1 a_2 \cdots$ to be the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$. Indeed, the sequence $a_1 a_2 \cdots = 2(10)^\omega$ satisfies the conditions of Lemma 5 and yields $\beta = 2$, but $\sum_{k=1}^{\infty} \frac{a_{k+2}}{(-2)^k} = \frac{1}{3}$; the (-2)-expansion of $\frac{-2}{3}$ is 2^ω. In order to avoid this problem, we define a relation $<'_{\text{alt}}$ by $x_1 x_2 \cdots <'_{\text{alt}} y_1 y_2 \cdots$ if $(-1)^n(x_{n+1} - y_{n+1}) < -1$, $x_1 \cdots x_n = y_1 \cdots y_n$ for some $n \geq 0$.

Lemma 6. *Let $a_1 a_2 \cdots$ and β be as in Lemma 5. If $a_{n+1} a_{n+2} \cdots <'_{\text{alt}} a_1 a_2 \cdots$ for all $n \geq 2$ such that $a_n = 0$, then $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$.*

Proof. We have to show that $\sum_{k=1}^{\infty}\frac{a_{n-1+k}}{(-\beta)^k} < \frac{1}{\beta+1}$ for all $n \geq 2$. If $a_n > 0$, then this inequality follows from $\sum_{k=1}^{\infty}\frac{a_{n-1+k}}{(-\beta)^k} \leq \frac{-1}{\beta} + \frac{1}{\beta+1} < 0$. If $a_n = 0$, then we have some $m \geq 0$ such that $(-1)^m(a_{n+m+1} - a_{m+1}) \leq -2$, $a_{n+1}\cdots a_{n+m} = a_1 \cdots a_m$. This implies that

$$\sum_{k=1}^{\infty}\frac{a_{n+k}}{(-\beta)^k} = \sum_{k=1}^{m}\frac{a_k}{(-\beta)^k} + \frac{1}{(-\beta)^m}\sum_{k=1}^{\infty}\frac{a_{n+m+k}}{(-\beta)^k}$$

$$\geq \sum_{k=1}^{m}\frac{a_k}{(-\beta)^k} + \frac{a_{m+1}}{(-\beta)^{m+1}} + \frac{2 - \frac{\beta}{\beta+1}}{\beta^{m+1}} > \sum_{k=1}^{m+1}\frac{a_k}{(-\beta)^k} + \frac{\frac{\beta}{\beta+1}}{\beta^{m+1}} \geq \sum_{k=1}^{\infty}\frac{a_k}{(-\beta)^k} = \frac{-\beta}{\beta+1},$$

thus $\sum_{k=1}^{\infty}\frac{a_{n-1+k}}{(-\beta)^k} = \frac{1}{-\beta}\sum_{k=1}^{\infty}\frac{a_{n+k}}{(-\beta)^k} < \frac{1}{\beta+1}$. $\qquad\square$

Proposition 3. *Let* $a_1 a_2 \cdots = 3010^3 10^5 1 \cdots$. *Then* $a_1 a_2 \cdots$ *is the* $(-\beta)$*-expansion of* $\frac{-\beta}{\beta+1}$ *for some* $\beta > 1$, *and* $\mathbb{Z}_{-\beta}$ *is not uniformly discrete.*

Proof. By Lemmas 5 and 6, there exists a $\beta > 1$ such that $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$. Since $a_{2n} a_{2n+1} \cdots$ starts with an odd number of zeros for all $n \geq 1$, we have $t_{2n-1} > 0$ for all $n \geq 0$. Therefore, induction on n yields that $\psi_{\beta}^{2n}((\infty,0))$ ends with (∞,n), and $\psi_{\beta}^{2n+1}((\infty,0))$ starts with (n,∞) for all $n \geq 0$. This implies that $\psi_{\beta}^{2n+1}((\infty,0)(0,\infty))$ contains the factor $(\infty,n)(n,\infty)$ for all $n \geq 0$, and $L_{\beta}((\infty,n)(n,\infty)) = t_{2n-1} - t_{2n}$ is a distance between consecutive $(-\beta)$-integers. For any $k \geq 1$, we have $a_{k(k-1)+2}\cdots a_{k(k+1)+1} = 0^{2k-1}1$, thus $0 < t_{k(k-1)+1} < \beta^{-2k}\frac{\beta^2}{\beta+1}$ and $-\beta^{-2k}\frac{\beta^3}{\beta+1} < t_{k(k-1)+2} < 0$. Therefore, the distance between consecutive elements of $\mathbb{Z}_{-\beta}$ can be arbitrarily small. $\qquad\square$

The following proposition shows that the converse of Proposition 2 is not true.

Proposition 4. *Let* $a_1 a_2 \cdots = 30032\,00\,00\,30032\,2\,00\,00\,00\,00\,30032\,00\,00\,30032\,22\cdots$ *be a fixed point of the morphism*

$$\sigma_1 : 3 \mapsto 30032,\ 2 \mapsto 2,\ 0 \mapsto 00.$$

Then $a_1 a_2 \cdots$ *is the* $(-\beta)$*-expansion of* $\frac{-\beta}{\beta+1}$ *for a* $\beta > 1$, $\inf\left\{T_{-\beta}^{2n-1}\left(\frac{-\beta}{\beta+1}\right) > 0 \mid n \geq 1\right\} = 0$ *and* $\mathbb{Z}_{-\beta}$ *is uniformly discrete.*

Proof. When $a_{n+1} = 3$, $n \geq 1$, then $a_{n+1} a_{n+2} \cdots$ starts with $\sigma_1^k(3)2$ for some $k \geq 0$. Since $|\sigma_1^k(3)|$ is odd for all $k \geq 0$, we have $\sigma_1^k(3)2 <'_{\mathrm{alt}} \sigma_1^k(3)0$, thus $a_{n+1}a_{n+2}\cdots <'_{\mathrm{alt}} a_1 a_2 \cdots$. Since $a_n = 0$ implies $a_{n+1} \in \{0,3\}$, the conditions of Lemmas 5 and 6 are satisfied, i.e., there exists a $\beta > 1$ such that $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$.

Let $n_k = |\sigma_1^k(300)| + |\sigma_1^{k-1}(3)| + 1$, $k \geq 1$. Then $a_{n_k+1}\cdots a_{n_k+2^k} = 0^{2^k-1}3$, thus we have $t_{n_k} > 0$ and $\lim_{k\to\infty} t_{n_k} = 0$. Since n_k is odd, this yields that $\inf\{t_{2n-1} > 0 \mid n \geq 1\} = 0$.

Let $n \geq 1$ with $t_{2n-1} > 0$. Then $a_{2n} a_{2n+1} \cdots$ starts with an odd number of zeros, thus $a_1 \cdots a_{2n-1}$ ends with $\sigma_1^k(3)0^{2j-1}$ for some $k \geq 0$, $1 \leq j \leq 2^{k-1}$. Let $2m = |\sigma_1^k(3)| + 2j - 1$. Recall that $\cdots u_{-1} u_0 u_1 \cdots$ is a fixed point of ψ_{β}. Any letter $u_k = (i,n)$, $i \in \mathbb{N}_{\infty}$, occurs only in $\psi_{\beta}^{2m}((i',n-m))$, $i' \in \mathbb{N}_{\infty}$ with $t_0 \leq t_{2i'} < t_{2n-2m-1}$. Since $a_1 \cdots a_{2m} = a_{2n-2m} \cdots a_{2n-1}$, we obtain that $t_{2m} \leq t_{2i} < t_{2n-1}$. Moreover, $a_{2m+1} a_{2m+2} \cdots$ starts with $0^{2^k-2j+1}3$, thus $t_{2m} > 0$. Now, the continuity of ι_{β} at every point in $\mathbb{Z}_{-\beta}$ yields that $y_k \notin \mathbb{Z}_{-\beta}$ if $u_k = (i,n)$, $n \geq 1$. Therefore, $y_k \in \mathbb{Z}_{-\beta}$ implies that $u_k = (\infty,0)$, hence $\mathbb{Z}_{-\beta}$ is uniformly discrete. $\qquad\square$

3.2 Relative denseness

Since the distance between consecutive β-integers is at most 1, $\mathbb{Z}_\beta$ is always relatively dense. We show that this is not always true for $\mathbb{Z}_{-\beta}$.

Proposition 5. *Let* $a_1 a_2 \cdots = 3123212312322123123212312312 3222 \cdots$ *be a fixed point of the morphism*

$$\sigma_2 : \; 3 \mapsto 31232, \; 2 \mapsto 2, \; 1 \mapsto 1.$$

Then $a_1 a_2 \cdots$ *is the* $(-\beta)$-*expansion of* $\frac{-\beta}{\beta+1}$ *for a* $\beta > 1$, *and* $\mathbb{Z}_{-\beta}$ *is not relatively dense.*

Proof. Since $|\sigma_2^k(3)|$ is odd for all $k \geq 0$, we have $\sigma_2^k(3) 2 <_{\mathrm{alt}} \sigma_2^k(3) 1$, thus $a_1 a_2 \cdots$ satisfies the conditions of Lemma 5. Since $a_n > 0$ for all $n \geq 2$, the condition of Lemma 6 holds trivially. Therefore, there exists a $\beta > 1$ such that $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$.

Next we show that, for any $k \geq 0$, $\psi_\beta^{|\sigma_2^k(3)|}((\infty, 0))$ starts with

$$\left(\lfloor |\sigma_2^k(3)|/2\rfloor, \lceil |\sigma_2^{k-1}(3)|/2\rceil\right) \cdots \left(\lfloor |\sigma_2^1(3)|/2\rfloor, \lceil |\sigma_2^0(3)|/2\rceil\right) (0, \infty). \tag{1}$$

We have $\psi_\beta((\infty, 0)) = (0, \infty)$ and

$$\psi_\beta^4((0, \infty)) = \cdots \psi_\beta^3((0, 1)) = \psi_\beta^2((1, \infty)) \cdots = \cdots \psi_\beta((\infty, 0)(0, 2)) = (2, 1)(0, \infty)\cdots,$$

where we have used that $a_1 > a_2$, $a_4 = a_1$, and $a_n > 0$ for all $n \geq 1$, which implies $t_n < 0$ for all $n \geq 1$. This yields the statement for $k = 0$ and $k = 1$. Supose that $\psi_\beta^{|\sigma_2^k(3)|}((\infty, 0))$ starts with (1) for some $k \geq 1$. Then we have

$$\psi_\beta^{|\sigma_2^{k+1}(3)|}((\infty, 0)) = \psi_\beta^{|\sigma_2^k(3)|+3}\left(\left(\lfloor |\sigma_2^k(3)|/2\rfloor, \lceil |\sigma_2^{k-1}(3)|/2\rceil\right)\right)\cdots$$

$$= \cdots \psi_\beta^{|\sigma_2^k(3)|+2}\left(\left(0, \lceil |\sigma_2^k(3)|/2\rceil\right)\right)$$

$$= \psi_\beta^{|\sigma_2^k(3)|+1}\left(\left(\lceil |\sigma_2^k(3)|/2\rceil, \infty\right)\right)\cdots$$

$$= \cdots \psi_\beta^{|\sigma_2^k(3)|}\left((\infty, 0)\left(0, \lceil |\sigma_2^k(3)|/2\rceil + 1\right)\right)$$

$$= \left(2\lceil |\sigma_2^k(3)|/2\rceil, \lceil |\sigma_2^k(3)|/2\rceil\right)$$
$$\left(\lfloor |\sigma_2^k(3)|/2\rfloor, \lceil |\sigma_2^{k-1}(3)|/2\rceil\right) \cdots \left(\lfloor |\sigma_2^1(3)|/2\rfloor, \lceil |\sigma_2^0(3)|/2\rceil\right)(0, \infty)\cdots,$$

where we have used the relations $a_{|\sigma_2^k(3)|} = 2 > 1 = a_{|\sigma_2^{k-1}(3)|+1}$, $a_1 > a_{|\sigma_2^k(3)|+1}$, and $a_1 \cdots a_{|\sigma_2^k(3)|} = a_{|\sigma_2^k(3)|+3} \cdots a_{2|\sigma_2^k(3)|+2}$. Since $2\lceil |\sigma_2^k(3)|/2\rceil = |\sigma_2^k(3)| + 1 = \lfloor |\sigma_2^{k+1}(3)|/2\rfloor$, we obtain inductively that $\psi_\beta^{|\sigma_2^k(3)|}((\infty, 0))$ starts with (1) for all $k \geq 0$.

For any $k \geq 0$, we have $a_{|\sigma_2^k(3)|} \geq 2$, $a_{|\sigma_2^k(3)|+1} = 1$, $a_{|\sigma_2^k(3)|+2} = 2$, and $t_k < 0$. For any $k \geq 1$, this yields that

$$L_\beta\left(\left(\lfloor |\sigma_2^k(3)|/2\rfloor, \lceil |\sigma_2^{k-1}(3)|/2\rceil\right)\right) = t_{|\sigma_2^{k-1}(3)|} - t_{|\sigma_2^k(3)|-1} > -\frac{1}{\beta} + \frac{2}{\beta^2} + \frac{t_0}{\beta^2} + \frac{2}{\beta} - \frac{1}{\beta^2} > \frac{1}{\beta}.$$

Let $k' = -|\psi_\beta^{|\sigma_2^k(3)|}|$. Then we have $u_{k'+j} = \left(\lfloor |\sigma_2^{k-j}(3)|/2\rfloor, \lceil |\sigma_2^{k-j-1}(3)|/2\rceil\right)$ for $0 \leq j < k$, thus $(y_{k'}, y_{k'+k}) \cap \mathbb{Z}_{-\beta} = \emptyset$ and $y_{k'+k} - y_{k'} > k/\beta$. Since $k \geq 1$ was chosen arbitrary, the distances between consecutive $(-\beta)$-integers are unbounded. □

Many other examples of sets $\mathbb{Z}_{-\beta}$ which are not relatively dense can be found by setting $a_1 a_2 \cdots = \sigma_3 \sigma_2^\infty(3)$ with a morphism σ_3 such that $\sigma_3(2) = 2$, $\sigma_3(1) = 1$, and $\sigma_3(3)$ is a suitable word of odd length.

We conclude the paper by stating the following three open problems, for which partial solutions are given in this section. Note that all the corresponding problems for positive bases have well-known, simple solutions, as mentioned above.

1. Characterise the sequences $a_1 a_2 \cdots$ which are possible $(-\beta)$-expansions of $\frac{-\beta}{\beta+1}$.

2. Characterise the numbers $\beta > 1$ such that $\mathbb{Z}_{-\beta}$ is uniformly discrete.

3. Characterise the numbers $\beta > 1$ such that $\mathbb{Z}_{-\beta}$ is relatively dense.

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