Pisot family substitutions and Meyer sets

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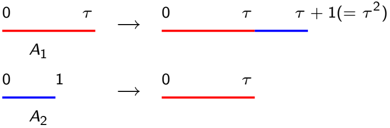
(Joint work with Boris Solomyak)

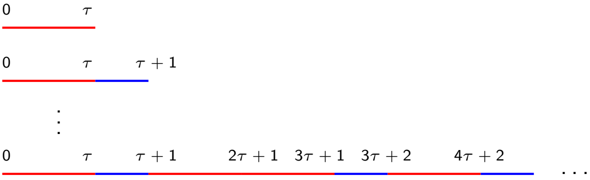
Outline

Fibonacci substitution tiling



$$\tau A_1 = A_1 \cup (A_2 + \tau)$$
$$\tau A_2 = A_1$$

where $\tau^2 - \tau - 1 = 0$.



Substitution tiling

- A tiling $\mathcal{T}$ in $\mathbb{R}^d$ is a set of tiles which cover $\mathbb{R}^d$ and distinct tiles have disjoint interiors.
- Let an expansion map Q, compact sets A_i's with $A_i = \overline{A_i^\circ} \neq \emptyset$ and finite sets $\mathcal{D}_{ij}$ satisfy

$$Q(A_j) = \bigcup_{i \le m} (A_i + \mathcal{D}_{ij}), \quad i \le m$$

  where all sets in the right-hand side have disjoint interiors. We can construct a substitution tiling $\mathcal{T}$.
- If the substitution tiling $\mathcal{T}$ is repetitive with finite local complexity(FLC), it is called a self-affine tiling. In addition, if the expansion map is a similarity map, then it is called a self-similar tiling.

Dynamical spectrum

Let X be the collection of tilings in $\mathbb{R}^d$ which are locally indistinguishable from $\mathcal{T}$.

Let $X_{\mathcal{T}} := \overline{\{x + \mathcal{T} : x \in \mathbb{R}^d\}}$ be with a metric on X. We consider a group action of $\mathbb{R}^d$ on $X_{\mathcal{T}}$ by translations and get a topological dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$.

We consider the spectrum of the unitary operators U_x arising from the translational action of $\mathbb{R}^d$ on $L_2(X_{\mathcal{T}}, \mu)$ with a unique invariant probability measure μ.

We say that $\mathcal{T}$ has pure discrete(or point) dynamical spectrum if the eigenfunctions for the $\mathbb{R}^d$-action span a dense subspace of $L_2(X_{\mathcal{T}}, \mu)$.

Meyer sets

Λ is a Meyer set *if* Λ is relatively dense & $\Lambda - \Lambda$ is uniformly discrete.

Example (Meyer sets)

- $\Lambda = (1 + 2\mathbb{Z}) \cup S$, for any subset $S \subset 2\mathbb{Z}$
- $\Lambda = \bigcup_{k=0}^{\infty} 2^k(1 + 2\mathbb{Z})$
- $\Lambda = \{a + b\tau \in \mathbb{Z}[\tau] : a + b\tau' \in [0, 1]\}$, where $\tau^2 - \tau - 1 = 0$, $\tau' = -\frac{1}{\tau}$

Example (Non-Meyer sets)

- $\Lambda = \{n + \frac{1}{n} : n \in \mathbb{Z}\backslash\{0\}\}$
- $\Lambda = \{n + \frac{1}{6} \sin \frac{2\pi n}{\sqrt{2}} : n \in \mathbb{Z}\}$

Meyer sets

Theorem (Meyer '72, Lagarias '95, Moody '97)

Let Λ be a Delone set. TFAE

1. Λ is a Meyer set.
2. $\Lambda - \Lambda \subset \Lambda + F$ for some finite set $F \subset \mathbb{R}^d$ (almost lattice).
3. Λ is a subset of a model set.
4. $[\Lambda]$ is finitely generated and Λ has the linear approximation property.
5. For each $\epsilon > 0$, there is a dual set Λ^ϵ in $\widehat{\mathbb{R}^d}$

$$\Lambda^\epsilon = \{\chi \in \widehat{\mathbb{R}^d} : |\chi(x) - 1| < \epsilon \text{ for all } x \in \Lambda\}$$

which is relatively dense.

Theorem (Strungaru '05)

If Λ is a Meyer set with uniform cluster frequencies(UCF), then all eigenvalues of $(X_\Lambda, \mathbb{R}^d, \mu)$ form a relatively dense set.

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If Λ is a Meyer set with uniform cluster frequencies(UCF), then all eigenvalues of $(X_\Lambda, \mathbb{R}^d, \mu)$ form a relatively dense set.

Theorem (Lee-Solomyak '08)

Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$. Then TFAE

1. all the eigenvalues of $(X_\mathcal{T}, \mathbb{R}^d, \mu)$ are relatively dense.
2. $\cup_{i=1}^m \Lambda_i$ is a Meyer set.

Corollary (Lee-Solomyak '08)

Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$.
If $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has pure discrete dynamical spectrum, then $\cup_{i=1}^m \Lambda_i$ is a Meyer set.

Lagarias's question

Let $\mathcal{T}$ be a tiling with FLC & repetitivity.
If $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ has pure discrete dynamical spectrum, should $\cup_{i=1}^m \Lambda_i$ be a Meyer set?

The corollary answers Lagarias's question on substitution tilings(or point sets).

Let $\Xi := \{x \in \mathbb{R}^d : T = x + T', T, T' \in \mathcal{T}\}$.

Theorem (Kenyon '94, Solomyak '06)

Let $\mathcal{T}$ be a self-similar tiling in $\mathbb{R}^d$ with a similarity map θ for which $|\theta| > 1$. Let $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$. Then

$$\Xi \subset \mathbb{Z}[\theta]\alpha_1 + \cdots + \mathbb{Z}[\theta]\alpha_d$$

for some basis $\{\alpha_1, \cdots, \alpha_d\}$ in $\mathbb{R}^d$.

Theorem (Lee-Solomyak '10)

Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$ with an expansion map ϕ. Suppose that ϕ is diagonalizable over $\mathbb{C}$ and all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity m.
Then

$$\Xi \subset \mathbb{Z}[\phi]\alpha_1 + \cdots + \mathbb{Z}[\phi]\alpha_K,$$

where $\{\alpha_1, \cdots, \phi^{m-1}\alpha_1, \cdots, \alpha_K, \cdots, \phi^{m-1}\alpha_K\}$ forms a basis of $\mathbb{R}^d$.

Let us only consider the simple case that all eigenvalues of ϕ are distinct and algebraic conjugates.

<u>SKETCH OF PROOF.</u>

Without loss of generality, we can assume that $\mathcal{C} := \cup_{i \le m} \Lambda_i$ satisfies $\phi\mathcal{C} \subset \mathcal{C}$. Since $\mathcal{T}$ has FLC,

$$\langle \mathcal{C} \rangle_{\mathbb{Z}} \subset \mathbb{Q}[\phi]\gamma_1 \oplus \cdots \oplus \mathbb{Q}[\phi]\gamma_L := \mathcal{D}$$

where $\mathcal{D}$ is a module over $\mathbb{Q}[\phi]$. **The goal is proving $\langle \mathcal{C} \rangle_{\mathbb{Z}} \subset \mathbb{Q}[\phi]\alpha$ for some $\alpha \in \mathbb{R}^d$ for which $\{\alpha, \phi\alpha, \cdots, \phi^{d-1}\alpha\}$ is a basis of $\mathbb{R}^d$.** Choose $\beta \in \mathcal{C}$ whose each entry is non-zero. Define a module homomorphism $\pi : \mathcal{D} \to \mathbb{Q}[\phi]\beta$ such that $\pi(\gamma_\ell) = \beta$ for each $1 \le \ell \le L$. Let $\mathcal{C}_\infty := \bigcup_{k=0}^{\infty} \phi^{-k}\mathcal{C}$. Define

$$\pi' : \mathcal{C}_\infty \to \mathbb{Q}[\phi]\beta$$

such that $\pi'(x) = \pi|_{\mathcal{C}_\infty}(x)$ for $x \in \mathcal{C}_\infty$.

We show the following steps.

1. π' is uniformly continuous on $\mathcal{C}_\infty$ by showing Hölder's continuity.
2. Extend π' on $\mathbb{R}^d$.
3. $\pi'|_{x+E_{\lambda_{min}}}$ is affine linear.
4. π' is affine linear by using the algebraic conjugacy of all eigenvalues.
5. $\beta, \phi\beta, \cdots, \phi^{d-1}\beta$ are linearly independent in $\mathbb{R}^d$ over $\mathbb{R}$.
   Thus π' is an isomorphism.
6. $\mathcal{C} \subset \mathbb{Q}[\phi]\beta$.
6. By FLC, $\mathcal{C} \subset \mathbb{Z}[\phi]\alpha$, where $\{\alpha, \phi\alpha, \cdots, \phi^{d-1}\alpha\}$ is a basis of $\mathbb{R}^d$ for some $\alpha \in \mathbb{R}^d$.

Pisot number and Pisot family

- A Pisot number(Pisot-Vijayaraghavan number) θ is an algebraic integer $\theta > 1$ whose algebraic conjugates θ' are all less than 1 in absolute value.
- A Pisot family is a set of algebraic integers $\Theta = \{\theta_1, \cdots, \theta_n\}$ such that for each $1 \leq i \leq n$, every algebraic conjugate γ of θ_i with $|\gamma| \geq 1$ is in Θ.

Theorem (Meyer '70, Lagarias '99, Solomyak '06)

Let $\mathcal{T}$ be a self-similar tiling in $\mathbb{R}^d$ and $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$ with a similarity factor θ for which $|\theta| > 1$. Then TFAE

1. $\cup_{i=1}^{m} \Lambda_i$ is a Meyer set.
2. θ is a Pisot number.

Theorem (Lee-Solomyak '10)

Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ and $\mathcal{T} = \{x_i + T_i : x_i \in \Lambda_i, i \leq m\}$ with an expansion map ϕ. Suppose that ϕ is diagonalizable over $\mathbb{C}$ and all the eigenvalues of ϕ are algebraically conjugate with the same multiplicity. Then TFAE

1. $\cup_{i=1}^m \Lambda_i$ is a Meyer set.
2. all the eigenvalues of ϕ form a Pisot family.
3. $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$ is not weakly mixing.

Questions

1. Let $\mathcal{T}$ be a self-affine tiling in $\mathbb{R}^d$ with an expansion map Q. If the substitution matrix is irreducible and the set of eigenvalues of Q forms a Pisot family, does $\mathcal{T}$ have pure discrete dynamical spectrum?

2. If $\mathcal{T}$ is a substitution tiling in $\mathbb{R}^d$ with an expansion map Q whose eigenvalues form a Pisot family, then should $\mathcal{T}$ necessarily have FLC?

3. If the expansion map Q is non-diagonalizable, what can we say about the module structure of the tiling $\mathcal{T}$. What about the relation between Pisot family condition and Meyer property?

THANK YOU!