Journal de Théorie des Nombres
de Bordeaux 00 (XXXX), 000–000

Abstract β-expansions and ultimately periodic representations

par Michel Rigo et Wolfgang Steiner

Résumé. Pour les systèmes de numération abstraits construits sur des langages réguliers exponentiels (comme par exemple, ceux provenant des substitutions), nous montrons que l'ensemble des nombres réels possédant une représentation ultimement périodique est $\mathbb{Q}(\beta)$ lorsque la valeur propre dominante $\beta > 1$ de l'automate acceptant le langage est un nombre de Pisot. De plus, si β n'est ni un nombre de Pisot, ni un nombre de Salem, alors il existe des points de $\mathbb{Q}(\beta)$ n'ayant aucune représentation ultimement périodique.

Abstract. For abstract numeration systems built on exponential regular languages (including those coming from substitutions), we show that the set of real numbers having an ultimately periodic representation is $\mathbb{Q}(\beta)$ if the dominating eigenvalue $\beta > 1$ of the automaton accepting the language is a Pisot number. Moreover, if β is neither a Pisot nor a Salem number, then there exist points in $\mathbb{Q}(\beta)$ which do not have any ultimately periodic representation.

1. Introduction

In [7], abstract numeration systems on regular languages have been introduced. They generalize in a natural way a large variety of classical positional systems like the q-ary system or the Fibonacci system: each nonnegative integer n is represented by the nth word of an ordered infinite regular language L. For instance, considering the natural ordering of the digits, the genealogical enumeration of the words belonging to the language $L = \{0\} \cup \{1, \ldots, q-1\}\{0, \ldots, q-1\}^*$ (resp. $L = \{0, 1\} \cup \{10\}\{10, 0\}^*\{\lambda, 1\}$) leads back to the q-ary (resp. the Fibonacci) system. Later on, this setting has been extended to allow the representation of real numbers as well as of integers.

Various notions appearing in number theory, in formal languages theory or in the analysis of algorithms depend on how numbers are represented. So these abstract systems have led to new nontrivial applications. To cite just a few: the characterization of the so-called recognizable sets of integers, the investigation of the dynamical and topological properties of the "odometers" or the study of the asymptotic behavior of the corresponding "sum-of-digits" function.

As we will see, it turns out that this way of representing real numbers is a quite natural generalization of Rényi's β-expansions [11]. More precisely, the primitive automata lead to the representations of real numbers based on substitutions introduced by Dumont and Thomas [3]. The nonprimitive automata provide new numeration systems.

Real numbers having an ultimately periodic representation deserve a special attention. Indeed, for the q-ary system, these numbers are exactly the rational numbers. More generally, the set of ultimately periodic representations is dense in the set of all the admissible representations and therefore this rises number-theoretic questions like the approximation of real numbers by numbers having ultimately periodic expansions.

On the one hand, for Rényi's classical β-expansions it is well-known that the set of real numbers with ultimately periodic representation is $\mathbb{Q}(\beta)$ whenever β is a Pisot number [1, 12]. On the other hand, for abstract numeration systems, the algebraic structure of this set was unknown. A first attempt to solve this problem is done in [9] where it is shown that a real number has an ultimately periodic representation if it is the fixed point of the composition of some affine functions depending only on the automaton of the abstract system. Even if this latter result gives some insights about those generalized β-expansions, the algebraic structure of the set was still to determine.

To any regular language, is associated in a canonical way its minimal automaton and consequently some adjacency matrix. We can therefore speak of the eigenvalues of an automaton, the corresponding adjacency matrix being considered. The present paper studies abstract numeration systems having the following property: the dominating eigenvalue $\beta > 1$ of the minimal (trimmed) automaton of the regular language on which the system is built, is a Pisot number. Thus we settle the problem of describing the set of real numbers with ultimately periodic representation by obtaining an analogue of a theorem found independently by Bertrand [1] and Schmidt [12]. Note that our result restricted to classical β-expansions gives back a new short and intuitive proof for this case. Moreover, we show that if β is neither a Pisot nor a Salem number then there exists at least one point in $\mathbb{Q}(\beta)$ which does not have any ultimately periodic representation.

This paper is organized as follows. In Section 2, we recall all the necessary material about abstract numeration systems. In Section 3, we state precisely the results which are proved in Section 4. Finally, Section 5 is devoted to some examples which we hope could provide a better understanding of the key algorithm involved in the proof of our main result.

2. Preliminaries

Let $(\Sigma, <)$ be a finite and totally ordered alphabet. We denote by Σ^* the free monoid generated by Σ for the concatenation product. The neutral element is λ and the length of a word $w \in \Sigma^*$ is denoted $|w|$. Recall that if u and v are two words over $(\Sigma, <)$, then u is *genealogically less* than v if either $|u| < |v|$ or $|u| = |v|$ and there exist $p, u', v' \in \Sigma^*$, $s, t \in \Sigma$ such that $u = psu'$, $v = ptv'$ and $s < t$. In this case, we write $u <_{\text{gen}} v$ or simply $u < v$. In the literature, one also finds the term military ordering. This ordering is naturally extended to the set Σ^ω of all the infinite words over Σ by the lexicographical ordering.

Let L be an infinite regular language over $(\Sigma, <)$. The words of L can be enumerated by increasing genealogical ordering leading to a one-to-one correspondence between $\mathbb{N}$ and L. We say that $S = (L, \Sigma, <)$ is an *abstract numeration system*. If w is the nth word of the genealogically ordered language L for some $n \in \mathbb{N}$ (positions inside L are counted from 0), then we write $\text{val}(w) = n$ and we say that w is the *representation* of n or that n is the *numerical value* of w (the abstract numeration system S being understood). This way of representing nonnegative integers has been first introduced in [7] and generalizes classical numeration systems like the positional systems built over linear recurrent sequences of integers whose characteristic polynomial is the minimal polynomial of a Pisot number [2].

Under some natural assumptions on L, not only integers but also real numbers can be represented using infinite words [8]. We briefly present notation used throughout this paper. The minimal automaton of L is $\mathcal{M}_L = (Q, q_0, \Sigma, \tau, F)$ where Q is the set of states, $q_0 \in Q$ the initial state, $\tau : Q \times \Sigma \to Q$ the transition function and $F \subseteq Q$ the set of final states. The function τ is naturally extended to $Q \times \Sigma^*$ by $\tau(q, \lambda) = q$ and $\tau(q, sw) = \tau(\tau(q, s), w)$ where $q \in Q$, $s \in \Sigma$ and $w \in \Sigma^*$. We refer the reader to [4] for more about automata theory. For $q \in Q$ and $n \in \mathbb{N}$, we denote by $\mathbf{u}_q(n)$ the number of words of length n accepted from q in $\mathcal{M}_L$, i.e.,

$$\mathbf{u}_q(n) = \#\{w \in \Sigma^n \mid \tau(q, w) \in F\}$$

and by $\mathbf{v}_q(n)$ the number of words of length at most n accepted from q, $\mathbf{v}_q(n) = \sum_{i=0}^{n} \mathbf{u}_q(i)$. Observe in particular that $\mathbf{u}_{q_0}(n) = \#(L \cap \Sigma^n)$ is the *growth function* of L.

In this paper, we assume that L has the following properties (again we refer to [8] for details). There exist $\beta > 1$ and $P \in \mathbb{R}[x]$ such that for all states $q \in Q$, there exist some nonnegative real number a_q such that

$$\lim_{n\to\infty} \frac{\mathbf{u}_q(n)}{P(n)\beta^n} = a_q.$$

Moreover, w.l.o.g., we assume that $a_{q_0} = 1 - \frac{1}{\beta}$ (indeed, if a_{q_0} differs from $1-\frac{1}{\beta}$ then we replace the polynomial P with $\frac{a_{q_0}}{1-\frac{1}{\beta}}P$). Clearly $\beta > 1$ is the dominant eigenvalue of the automaton $\mathcal{M}_L$. (In order to relate β to the growth of L, $\mathcal{M}_L$ is assumed to be trim, i.e., it is accessible and coaccessible, and in particular τ could possibly be a partial function). We denote by χ_β the minimal polynomial of β,

$$\chi_\beta(x) = x^d - b_1x^{d-1} - b_2x^{d-2} - \cdots - b_{d-1}x - b_d \in \mathbb{Z}[x].$$

The set $\mathcal{L}_\infty$ is defined as the set of infinite words which are limit of the converging sequences of words in L (we use the usual infinite product topology on Σ^ω). If $(p_j)_{j\geq 0} \in L^{\mathbb{N}}$ converges to an infinite word $w \in \mathcal{L}_\infty$ then it is well-known that the limit

$$\lim_{j\to\infty} \frac{\mathrm{val}(p_j)}{\mathbf{v}_{q_0}(|p_j|)}$$

exists. Its value will be denoted $\mathrm{val}_\infty(w)$ and we say that w is a *representation* of $\mathrm{val}_\infty(w)$ or conversely that $\mathrm{val}_\infty(w)$ is the *numerical value* of w. In this setting, we are able to represent all the numbers lying in the interval $[1/\beta, 1]$. Moreover, the representation of a real number $x \in [1/\beta, 1]$ is not necessarily unique; we denote by $\mathrm{rep}(x)$ the set of words in $\mathcal{L}_\infty$ representing x, i.e.,

$$\mathrm{rep}(x) = \{w \in \mathcal{L}_\infty \mid \mathrm{val}_\infty(w) = x\}.$$

Note that this situation even occurs for classical numeration systems. For instance in base ten, $\mathrm{rep}(2/10) = \{.2(0)^\omega, .1(9)^\omega\}$. Denote by $\mathcal{W}_\ell$ the set of words of length ℓ which are prefix of an infinite number of words in L — they are prefix of at least one element in $\mathcal{L}_\infty$. If $u \in \mathcal{W}_\ell$, then we denote

$$X_u := \{w \in \mathcal{L}_\infty \mid \exists v \in \Sigma^\omega : w = uv\}.$$

If $|t| = 1$, i.e., if t is a letter in $\mathcal{W}_1$ then the set of real numbers having a representation starting with t is

$$\mathrm{val}_\infty(X_t) = \left[\frac{1}{\beta} + \sum_{z<t, z\in\mathcal{W}_1} \frac{a_{\tau(q_0,z)}}{\beta}, \frac{1}{\beta} + \sum_{z\leq t, z\in\mathcal{W}_1} \frac{a_{\tau(q_0,z)}}{\beta}\right] =: I_t \tag{2.1}$$

which is an interval of length $a_{\tau(q_0,t)}/\beta$. Note that

$$\mathbf{u}_q(n) = \sum_{t\in\Sigma} \mathbf{u}_{\tau(q,t)}(n-1)$$

(where the sum runs over those t for which $\tau(q,t)$ exists) and therefore

$$a_q = \sum_{\substack{t\in\Sigma,\\ (q,t)\in\mathrm{dom}\,\tau}} \frac{a_{\tau(q,t)}}{\beta}. \tag{2.2}$$

When looking at the real numbers represented by a word which has a prefix in $\mathcal{W}_2$ starting with the letter t, the interval I_t is then divided into smaller intervals: one interval for each letter s such that $ts \in \mathcal{W}_2$. This procedure of dividing intervals is repeated and one can obtain the numerical value of an infinite word $w = (w_j)_{j\geq 1} \in \mathcal{L}_\infty$ as

$$\mathrm{val}_\infty(w) = \frac{1}{\beta} + \sum_{j=1}^{\infty}\sum_{q\in Q} a_q \epsilon_{q,j}\beta^{-j} \tag{2.3}$$

with

$$\epsilon_{q,j} := \#\{s < w_j \mid \tau(q_0, w_1\cdots w_{j-1}s) = q\}. \tag{2.4}$$

A detailed proof of this formula can be found in [8, Corollary 7] (where the notation is slightly different and a_{q_0} is assumed to be 1).

The longer the known prefix of a representation of a real number is, the more accurate the approximation of this number is. Precisely, if u is a word of length $\ell \geq 1$ then it can be shown that $\mathrm{val}_\infty(X_u)$ is equal to

$$\left[\frac{1}{\beta} + \sum_{z<u, z\in\mathcal{W}_\ell} \frac{a_{\tau(q_0,z)}}{\beta^\ell}, \frac{1}{\beta} + \sum_{z\leq u, z\in\mathcal{W}_\ell} \frac{a_{\tau(q_0,z)}}{\beta^\ell}\right] =: I_u \tag{2.5}$$

since, for $u = u_1\cdots u_\ell \in \mathcal{W}_\ell$, we have

$$\sum_{z<u, z\in\mathcal{W}_\ell} \frac{a_{\tau(q_0,z)}}{\beta^\ell} = \sum_{j=1}^{\ell} \sum_{\substack{t<u_j,\\ u_1\cdots u_{j-1}t\in\mathcal{W}_j}} \frac{a_{\tau(q_0,u_1\cdots u_{j-1}t)}}{\beta^j}.$$

Finally, if $M \subseteq \Sigma^\omega$ we denote by $\mathrm{uper}(M)$ the set of words in M which are ultimately periodic. This means that $w \in \mathrm{uper}(M)$ if and only if there exist $u, v \in \Sigma^*$, $v \neq \lambda$ such that $w = u(v)^\omega$.

As usual, we denote by $\mathbb{Q}(\beta)$ the smallest field containing $\mathbb{Q}$ and β. Since β is algebraic and of degree d, we have $\mathbb{Q}(\beta) = \mathbb{Q}[\beta]$ and every element of $\mathbb{Q}(\beta)$ can be decomposed as $x = \sum_{i=1}^{d} x_i\beta^{-i}$ with $x_i \in \mathbb{Q}$. We write $x = .x_1\ldots x_d$. We assume that the reader is familiar with classical β-expansions, see for instance [6, 10, 11]. Note that we always refer to the β-expansions computed through the greedy algorithm. Recall that the greedy β-expansion of a number is the maximal one for the lexicographical order. In this way, if β is a Pisot number, then the usual β-expansions can be seen as a special case of the more abstract representations considered here (see [9, Section 9])

We denote by L_β the set of infinite words which are the β-expansions of the real numbers in $[0,1)$, therefore $\mathrm{uper}(L_\beta)$ is the set of real numbers with ultimately periodic β-expansion.

3. Results

In this paper, we will show the following results.

Proposition 3.1. *Let $S=(L,\Sigma,<)$ be an abstract numeration system satisfying the assumptions given in Section 2. For every $w \in \mathrm{uper}(\mathcal{L}_\infty)$, we have*

$$\mathrm{val}_\infty(w) \in \mathbb{Q}(\beta) \cap [1/\beta, 1].$$

The converse holds when β is a Pisot number.

Theorem 3.1. *Let $S=(L,\Sigma,<)$ be an abstract numeration system satisfying the assumptions given in Section 2. If β is a Pisot number, then every $x \in \mathbb{Q}(\beta) \cap [1/\beta,1]$ is the numerical value of some ultimately periodic word w, i.e.,*

$$\mathrm{rep}(x) \cap \mathrm{uper}(\mathcal{L}_\infty) \neq \emptyset \quad \textit{for all} \quad x \in \mathbb{Q}(\beta) \cap [1/\beta, 1].$$

In particular, the lexicographically maximal word $w \in \mathrm{rep}(x)$ is ultimately periodic.

Proposition 3.2. *Let $S=(L,\Sigma,<)$ be an abstract numeration system satisfying the assumptions given in Section 2. If β is neither a Pisot nor a Salem number, then we have some x in $\mathbb{Q}(\beta) \cap [1/\beta,1]$ with $\mathrm{rep}(x) \cap \mathrm{uper}(\mathcal{L}_\infty) = \emptyset$.*

For classical β-numeration systems, we have the following.

Corollary 3.1 (Bertrand [1], Schmidt [12])**.** *If $\beta > 1$ is a Pisot number, then*

$$\mathrm{uper}(L_\beta) = \mathbb{Q}(\beta) \cap [0,1).$$

If β is neither a Pisot nor a Salem number, then

$$\mathbb{Q}(\beta) \cap [0,1) \not\subseteq \mathrm{uper}(L_\beta).$$

Remark. In Corollary 3.1, the interval is different from $[\frac{1}{\beta},1]$, in order to state the result in the usual way. Indeed, the β-expansion of $x \in [\beta^{-m-1}, \beta^{-m})$ for some $m \geq 1$ is obtained by placing m zeroes in front of the expansion of $\beta^m x$ (and the β-expansion of 0 is $.0^\omega$).

Similarly, we can represent each number $x \in \mathbb{R}^+$ in our system by shifting the representations of $[1/\beta, 1]$. For instance, we can define

$$\mathrm{val}'_\infty(w_1 \cdots w_m . w_{m+1} w_{m+2} \cdots) := \beta^m \, \mathrm{val}_\infty(w_1 w_2 \cdots)$$

and

$$\mathrm{val}'_\infty(.0^m w_1 w_2 \cdots) := \beta^{-m} \, \mathrm{val}_\infty(w_1 w_2 \cdots),$$

where, in case 0 is a letter of the alphabet, 0 must not be accepted from q_0.

4. Proofs of the results

We will need a small lemma.

Lemma 4.1. *We have $a_q \in \mathbb{Q}(\beta)$ for all $q \in Q$.*

Proof. Applying (2.2) for all $q \in Q$ provides a system of linear equations for the a_q's. It is easily seen that the solutions of this system are exactly the eigenvectors of the automaton $\mathcal{M}_L$ to the eigenvalue β. If the eigenspace for β has dimension 1, then the solution is entirely determined by $a_{q_0} = 1 - \frac{1}{\beta}$ and has clearly elements in $\mathbb{Q}(\beta)$.

If the eigenspace has larger dimension, then observe that the number of paths from q to q' of length n, $m_{q,q'}(n)$, is the element (q, q') of A^n, where A denotes the adjacency matrix of the automaton. Hence $m_{q,q'}(n)$ satisfies a linear recurrence with characteristic polynomial equal to that of A and is therefore of the form $m_{q,q'}(n) = P_{q,q'}(n)\beta^n + \cdots$ for some polynomial $P_{q,q'} \in \mathbb{Q}(\beta)[x]$ (indeed, since $m_{q,q'}(n) \in \mathbb{Z}$, one can easily show using some reasoning about generating functions that $P_{q,q'} \in \mathbb{Q}(\beta)[x]$). We clearly have

$$a_q = \lim_{n\to\infty} \frac{\sum_{q'\in F} m_{q,q'}(n)}{P(n)\beta^n} = \lim_{n\to\infty} \left(1 - \frac{1}{\beta}\right) \frac{\sum_{q'\in F} P_{q,q'}(n) + \cdots}{\sum_{q'\in F} P_{q_0,q'}(n) + \cdots},$$

where the other terms can be neglected even if we have other eigenvalues of modulus β because we have assumed that this limit exists. Hence we have $a_q \in \mathbb{Q}(\beta)$. □

Proof of Proposition 3.1. Let $w = (w_j)_{j\geq 1} \in \mathrm{uper}(\mathcal{L}_\infty)$. By (2.3), we have

$$\mathrm{val}_\infty(w) = \frac{1}{\beta} + \sum_{j=1}^{\infty} \sum_{q\in Q} a_q \epsilon_{q,j} \beta^{-j}$$

with ultimately periodic sequences $(\epsilon_{q,j})_{j\geq 1}$ and, with Lemma 4.1, the first statement is proved. More details on the periodicity of $(\epsilon_{q,j})_{j\geq 1}$ are given in [9]. □

Proof of Theorem 3.1. The sketch of the proof is the following.

- (**A**) We show that representing a real number in an abstract system can be viewed as a generalization of the classical β-transformation.
- (**B**) We derive an algorithm to compute the representation of a given real number.
- (**C**) Using this algorithm, we obtain the expected result. Note that we use the fact that β is a Pisot number only in this last part. The first two parts are independent of the algebraic properties of β.

(**A**) Let $x \in \mathbb{Q}(\beta) \cap [1/\beta, 1]$ and $w = (w_j)_{j\geq 1}$ be the lexicographically maximal word in $\mathrm{rep}(x)$.

For $q \in Q$ and $t \in \Sigma$, set

$$\alpha_q(t) := \sum_{q' \in Q} \left(a_{q'} \cdot \#\{s < t \mid \tau(q,s) = q'\}\right)$$

Then from (2.3), (2.4) and the definition of the $\alpha_q(t)$'s and $\epsilon_{q,j}$'s, we have

$$\mathrm{val}_\infty(w) = (1 + \alpha_{q_0}(w_1))\beta^{-1} + \sum_{j=2}^{\infty} \alpha_{\tau(q_0, w_1 \cdots w_{j-1})}(w_j)\beta^{-j}. \tag{4.1}$$

Remark. Let us make a small digression about the classical β-numeration systems. If the β-expansion of 1 is finite or ultimately periodic (which in particular is true when β is a Pisot number) then the β-shift is sofic. The set of factors appearing in L_β is a regular language and the deterministic finite automaton $\mathcal{M}_\beta$ recognizing this language has a very special form and is depicted in Figure 1:

Let $t_1 \cdots t_m$ or $t_1 \cdots t_{m-p}(t_{m-p+1} \cdots t_m)^\omega$ be the expansion of 1. Then the set of states of $\mathcal{M}_\beta$ is $\{\mathbf{1}, \ldots, \mathbf{m}\}$, $\mathbf{1}$ is the initial state, all states are final and the alphabet of digits is $\Sigma = \{0, \ldots, \lfloor \beta \rfloor\}$. For every j, $1 \le j \le m$, we have t_j edges $\mathbf{j} \to \mathbf{1}$ labelled by $0, \ldots, t_j - 1$ and, for $j < m$, one edge $\mathbf{j} \to \mathbf{j+1}$ labelled by t_j. If the expansion of 1 is ultimately periodic, then we have an additional edge $\mathbf{m} \to \mathbf{m-p+1}$ labelled by t_m. (See for instance [5].)

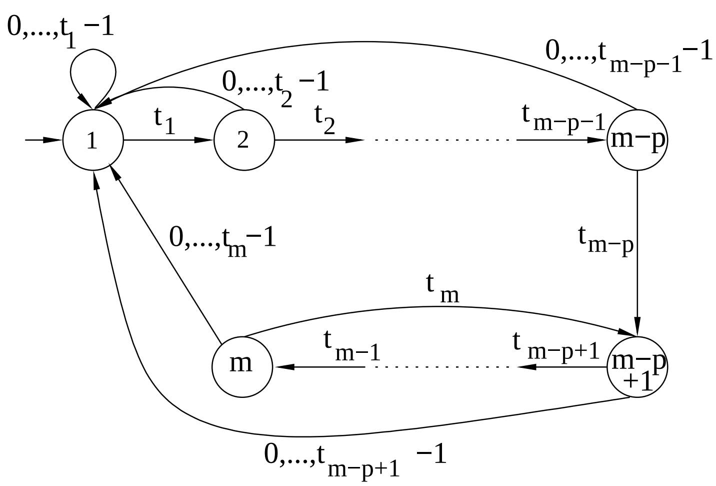

FIGURE 1. The automaton $\mathcal{M}_\beta$ in the ultimately periodic case.

Since, for abstract systems, leading zeroes may change the value of the words in the language (indeed for the genealogical ordering $v < 0^n v$ and therefore these two words lead to different numerical values), we need an automaton $\mathcal{M}'_\beta$ which differs slightly from $\mathcal{M}_\beta$ by adding a state q_0 which is dedicated to be the initial state of this new automaton and forbids the reading of an initial zero. More precisely, we add $t_1 - 1$ edges $q_0 \to \mathbf{1}$ labelled by $1, \dots, t_1 - 1$ and one edge $q_0 \to \mathbf{2}$ labelled by t_1. (See [9] and Example 5.)

For the automaton $\mathcal{M}'_\beta$, we have $\epsilon_{q,j} = 0$ for all $q \neq \mathbf{1}$ and $a_{\mathbf{1}} = 1$. So (4.1) has the form

$$x = (1 + \epsilon_{\mathbf{1},1})\beta^{-1} + \sum_{j=2}^{\infty} \epsilon_{\mathbf{1},j}\beta^{-j}.$$

The digits $\epsilon_{\mathbf{1},j}$ are obtained by the *β-transformation*

$$T_\beta : [0,1) \to [0,1) : y \mapsto \beta y - \lfloor \beta y \rfloor.$$

We have $\epsilon_{\mathbf{1},j} = \lfloor \beta T_\beta^{j-1}(x - \beta^{-1}) \rfloor$ for all $j \geq 1$.

Now, we go back to the general case. Let $\ell > 0$. If $v \in \mathcal{W}_\ell$, the interval I_v given in (2.5) can be split into intervals I_{vt}, $t \in \Sigma$. Clearly, if $|I|$ denotes the length of I then $|I_{vt}|/|I_v| = a_{\tau(q_0,vt)}/(\beta a_{\tau(q_0,v)})$. Roughly speaking, this is the reason why we will multiply all quantities by β. Therefore if a real number has a representation beginning with v then it is quite easy to determine the next letter t in the representation by determining to which interval I_{vt} it belongs. To that end, we compare with the $a_{\tau(q_0,vt)}$'s, $t \in \Sigma$, after multiplication by β. In a more precise way, to obtain a generalization of the β-transformation, we set

$$\lfloor y \rfloor_q = \max\{\alpha_q(s) \mid s \in \Sigma, \alpha_q(s) \leq y\}$$

and

$$T_{S,q} : [0, a_q] \to [0, \max\{a_{q'} \mid q' = \tau(q,s), s \in \Sigma\}] : \ y \mapsto \beta y - \lfloor \beta y \rfloor_q.$$

(**B**) For $x \in \mathbb{Q}(\beta)$, we have

$$x = .x_1 \dots x_d = \frac{x_1}{\beta} + \dots + \frac{x_d}{\beta^d} \text{ with } x_k \in \mathbb{Q}.$$

Starting with this expansion of x, we will calculate iteratively the sequence $(w_j)_{j\geq 1}$. During those computations we denote by q_j the state of $\mathcal{M}_\mathcal{L}$ obtained at the jth step of the procedure: $q_j = \tau(q_{j-1}, w_j)$. For $j = 1$, we start with the initial state q_0.

As a first step, we set $x_1^{(d)} = x_1, x_2^{(d-1)} = x_2, \ldots, x_d^{(1)} = x_d$. Then let inductively for $j \geq 1$,

$$\begin{array}{rcl} x_j^{(d+1)} & = & x_j^{(d)} + z_j \\ x_{j+1}^{(d)} & = & x_{j+1}^{(d-1)} - z_{j,1} \\ x_{j+2}^{(d-1)} & = & x_{j+2}^{(d-2)} - z_{j,2} \\ & \vdots & \\ x_{j+d-1}^{(2)} & = & x_{j+d-1}^{(1)} - z_{j,d-1} \\ x_{j+d}^{(1)} & = & -z_{j,d} \end{array} \tag{4.2}$$

with

$$z_1 = \lfloor \beta x - 1 \rfloor_{q_0} - (x_1^{(d)} - 1) = .z_{1,1} \cdots z_{1,d} \in \mathbb{Q}(\beta),$$
$$w_1 = \max\{s \in \Sigma \mid \alpha_{q_0}(s) = x_1^{(d+1)} - 1\}$$

and for $j \geq 2$

$$z_j = \lfloor x_j^{(d)} .x_{j+1}^{(d-1)} \cdots x_{j+d-1}^{(1)} \rfloor_{q_{j-1}} - x_j^{(d)} = .z_{j,1} \cdots z_{j,d} \in \mathbb{Q}(\beta),$$
$$w_j = \max\{s \in \Sigma \mid \alpha_{q_{j-1}}(s) = x_j^{(d+1)}\}. \tag{4.3}$$

It is easy to check that

$$.x_{j+1}^{(d)} \cdots x_{j+d}^{(1)} = T_{S,q_{j-1}}(.x_j^{(d)} \cdots x_{j+d-1}^{(1)})$$

Indeed, as explained earlier, we have first to multiply $.x_j^{(d)} \cdots x_{j+d-1}^{(1)}$ (which determines the position of x in the interval $I_{w_0 \cdots w_{j-1}}$) by β and therefore we consider $x_j^{(d)} .x_{j+1}^{(d-1)} \cdots x_{j+d-1}^{(1)}$. In order to retrieve the next letter w_j in the representation, this latter quantity has to be compared with the $\alpha_{q_j}(t)$'s, $t \in \Sigma$. Since we are looking for the lexicographically maximal word in $\mathrm{rep}(x)$, we consider (4.3).

We clearly have

$$\begin{aligned} x &= .x_1^{(d)} \cdots x_d^{(1)} = .x_1^{(d+1)} x_2^{(d)} \cdots x_{d+1}^{(1)} = \cdots \\ &= .x_1^{(d+1)} \cdots x_j^{(d+1)} x_{j+1}^{(d)} \cdots x_{j+d}^{(1)} = \cdots = .x_1^{(d+1)} x_2^{(d+1)} \cdots \end{aligned}$$

With

$$\begin{aligned} z_j =& \lfloor x_j^{(d)} .x_{j+1}^{(d-1)} \cdots x_{j+d-1}^{(1)} \rfloor - x_j^{(d)} \\ &+ \lfloor x_j^{(d)} .x_{j+1}^{(d-1)} \cdots x_{j+d-1}^{(1)} \rfloor_{q_{j-1}} - \lfloor x_j^{(d)} .x_{j+1}^{(d-1)} \cdots x_{j+d-1}^{(1)} \rfloor \end{aligned}$$

and

$$\Delta_q(y) = \lfloor y \rfloor_q - \lfloor y \rfloor = .\Delta_{q,1}(y) \cdots \Delta_{q,d}(y) \in \mathbb{Q}(\beta),$$

the $z_{j,i}$'s are given by

$$\begin{aligned} z_{j,i} &= \big(\lfloor x_j^{(d)}.x_{j+1}^{(d-1)}\cdots x_{j+d-1}^{(1)}\rfloor - x_j^{(d)}\big)b_i + \Delta_{q_{j-1},i}(x_j^{(d)}.x_{j+1}^{(d-1)}\cdots x_{j+d-1}^{(1)}) \\ &= \frac{b_i}{\beta}x_{j+1}^{(d-1)} + \cdots + \frac{b_i}{\beta^{d-1}}x_{j+d-1}^{(1)} - \{x_j^{(d)}.x_{j+1}^{(d-1)}\cdots x_{j+d-1}^{(1)}\}b_i \\ &\quad + \Delta_{q_{j-1},i}(x_j^{(d)}.x_{j+1}^{(d-1)}\cdots x_{j+d-1}^{(1)}), \end{aligned} \tag{4.4}$$

where we have used $y = .(yb_1)\cdots(yb_d)$ in the first line and $\lfloor y\rfloor = y - \{y\}$ in the second one.

Remark. In the special case of classical β-numeration systems, we have $\lfloor y\rfloor_q = \lfloor y\rfloor$, hence $\Delta_q(y) = 0$, the z_j's are rational numbers and $z_{j,i} = z_j b_i$.

(**C**) Thus from (4.2) and (4.4), we get

$$\begin{pmatrix} x_{j+1}^{(d)} \\ x_{j+2}^{(d-1)} \\ \vdots \\ x_{j+d-1}^{(2)} \\ x_{j+d}^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & 1-\frac{b_1}{\beta} & -\frac{b_1}{\beta^2} & \cdots & -\frac{b_1}{\beta^{d-1}} \\ 0 & -\frac{b_2}{\beta} & 1-\frac{b_2}{\beta^2} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\frac{b_{d-2}}{\beta^{d-1}} \\ \vdots & \vdots & & \ddots & 1-\frac{b_{d-1}}{\beta^{d-1}} \\ 0 & -\frac{b_d}{\beta} & \cdots & \cdots & -\frac{b_d}{\beta^{d-1}} \end{pmatrix} \begin{pmatrix} x_j^{(d)} \\ x_{j+1}^{(d-1)} \\ \vdots \\ x_{j+d-2}^{(2)} \\ x_{j+d-1}^{(1)} \end{pmatrix}$$
$$+ \{x_j^{(d)}.x_{j+1}^{(d-1)}\ldots x_{j+d-1}^{(1)}\} \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} - \begin{pmatrix} \Delta_{q_{j-1},1}(x_j^{(d)}.x_{j+1}^{(d-1)}\cdots x_{j+d-1}^{(1)}) \\ \vdots \\ \Delta_{q_{j-1},d}(x_j^{(d)}.x_{j+1}^{(d-1)}\ldots x_{j+d-1}^{(1)}) \end{pmatrix}$$

i.e., the $x_j^{(i)}$'s satisfy a linear recurrence up to two terms which are bounded because $\Delta_q(y)$ takes only finitely many values and $\{x_j^{(d)}.x_{j+1}^{(d-1)}\ldots x_{j+d-1}^{(1)}\} < 1$.

Denote the above matrix by M. We claim that its eigenvalues are 0 and the conjugates of β, which we denote by $\beta_2, \ldots, \beta_d$. To prove this, we show that $\mathbf{v}_1 = (1, 0, \ldots, 0)^t$ and

$$\mathbf{v}_i = (\beta_i^{d-1}, b_2\beta_i^{d-2} + b_3\beta_i^{d-3} + \cdots + b_d, \ldots, b_{d-1}\beta_i^{d-2} + b_d\beta_i^{d-3}, b_d\beta_i^{d-2})^t,$$

$2 \le i \le d$, are right eigenvectors.

For $\mathbf{v}_1$, this is obvious. For $2 \le i \le d$, note that the β_i's are roots of

$$\frac{\chi_\beta(x)}{x-\beta} = x^{d-1} + (\beta - b_1)x^{d-2} + \cdots + (\beta^{d-1} - b_1\beta^{d-2} - \cdots - b_{d-1}). \tag{4.5}$$

The k-th element of $M\mathbf{v}_i$, $2 \le i \le d$, is given by

$$(M\mathbf{v}_i)_k = -\, b_k \left(\left(\frac{b_2}{\beta} + \cdots + \frac{b_d}{\beta^{d-1}} \right) \beta_i^{d-2} + \cdots + \frac{b_d}{\beta} \right) + \left(b_{k+1}\beta_i^{d-2} + \cdots + b_d \beta_i^{k-1} \right).$$

Since $1 = .b_1 \ldots b_d$, we have $\frac{b_\ell}{\beta} + \cdots + \frac{b_d}{\beta^{d-\ell+1}} = \beta^{\ell-1} - b_1\beta^{\ell-2} - \cdots - b_{\ell-1}$, $2 \le \ell \le d$, and

$$(M\mathbf{v}_i)_k = -\, b_k \Big((\beta - b_1)\beta_i^{d-2} + \cdots + (\beta^{d-1} - b_1\beta^{d-2} - \cdots - b_{d-1}) \Big) + b_{k+1}\beta_i^{d-2} + \cdots + b_d\beta_i^{k-1}.$$

And finally, since β_i is a root of (4.5), we have

$$(M\mathbf{v}_i)_k = b_k\beta_i^{d-1} + b_{k+1}\beta_i^{d-2} + \cdots + b_d\beta_i^{k-1} = \beta_i(\mathbf{v}_i)_k$$

Now, if β is a Pisot number, then the eigenvalues β_i have modulus smaller than some $\rho < 1$. We clearly have

$$\begin{pmatrix} x_{j+1}^{(d)} \\ \vdots \\ x_{j+d}^{(1)} \end{pmatrix} = M^j \begin{pmatrix} x_1^{(d)} \\ \vdots \\ x_d^{(1)} \end{pmatrix} + \sum_{k=1}^{j} \{x_k^{(d)}.x_{k+1}^{(d-1)} \ldots x_{k+d-1}^{(1)}\} M^{j-k} \begin{pmatrix} b_1 \\ \vdots \\ b_d \end{pmatrix} - \sum_{k=1}^{j} M^{j-k} \begin{pmatrix} \Delta_{q_{j-1},1}(x_k^{(d)}.x_{k+1}^{(d-1)} \ldots x_{k+d-1}^{(1)}) \\ \vdots \\ \Delta_{q_{j-1},d}(x_k^{(d)}.x_{k+1}^{(d-1)} \ldots x_{k+d-1}^{(1)}) \end{pmatrix} = \sum_{i=1}^{d} \gamma_{i,j}\mathbf{v}_i$$

with $|\gamma_{1,j}| \le c$ for $j > 1$ and some constant c and

$$|\gamma_{i,j}| \le |\gamma_{i,0}||\beta_i|^j + c_i(|\beta_i|^{j-1} + \cdots + 1) < c_i \frac{1}{1-\rho} + \varepsilon$$

for $2 \le i \le d$, some constants c_i and all $j > J(\varepsilon)$. These bounds are obtained by considering the decomposition as sum of $\mathbf{v}_i$'s of the various vectors appearing in the above formula.

Hence $(x_{j+1}^{(d)}, \ldots, x_{j+d}^{(1)})$ is bounded. By the first line of (4.4), we see that the $z_{j,i}$'s are rational numbers and the denominator of $z_{j,i}$ divides the least common multiple of the denominator of $x_j^{(d)}$ and that of all $\Delta_{q,i}(y)$'s (which are only finitely many). So we get inductively that the $z_{j,i}$'s and the $x_j^{(i)}$'s with $i \le d$ are rational numbers with bounded denominator. Thus we have only finitely many possibilities for $(x_{j+1}^{(d)}, \ldots, x_{j+d}^{(1)}, q_{j-1})$ and this sequence is ultimately periodic. Since w_j is determined by this vector (see formula (4.3)), the sequence $(w_j)_{j\ge 1}$ is ultimately periodic too. $\square$

Remark. In the statement of Theorem 3.1, the lexicographically maximal word $w \in \operatorname{rep}(x)$ is said to be ultimately periodic. Note that we have more than one representation of x if and only if x is in $I_{w_1 \cdots w_{j-1} w_j}$ and $I_{w_1 \cdots w_{j-1} s}$ for some j and $s < w_j$. This means that x is the left boundary of $I_{w_1 \cdots w_{j-1} w_j}$ and the right boundary of $I_{w_1 \cdots w_{j-1} s}$. It is easy to see that the lexicographically minimal representation of each boundary point is ultimately periodic. Hence the lexicographically lexicographically minimal word in $w \in \operatorname{rep}(x)$ is ultimately periodic too.

If $a_q \neq 0$ for all $q \in Q$, then the lexicographically minimal and maximal words are the only elements in $\operatorname{rep}(x)$. If $a_q = 0$ for some $q \in Q$, then we may have uncountably many representations of x and some of them can be aperiodic.

Proof of Proposition 3.2. We use the same notation as in the proof of Theorem 3.1. Since β is neither Pisot nor Salem, let, w.l.o.g., $|\beta_2| > 1$. We have to find some x such that $\gamma_{2,1}$ satisfies

$$|\gamma_{2,1}| < |\gamma_{2,1}||\beta_2| - c_2,$$

because this implies $|\gamma_{2,2}| > |\gamma_{2,1}|$, hence $|\gamma_{2,2}| < |\gamma_{2,2}||\beta_2| - c_2$ and, inductively,

$$|\gamma_{2,1}| < |\gamma_{2,2}| < |\gamma_{2,3}| < \cdots$$

Then the sequence $(x_{j+1}^{(d)}, \ldots, x_{j+d}^{(1)})$ is aperiodic and the sequence $(w_j)_{j\geq 1}$ as well. Furthermore, $(x_{j+1}^{(d)}, \ldots, x_{j+d}^{(1)}) \neq \mathbf{0}$ for all $j \geq 0$ implies that $(w_j)_{j\geq 1}$ is the only representation of x. Indeed, by Remark 4, x is the left boundary point of an interval $I_{w_1 \cdots w_j}$ for some $j \geq 1$ if x has more than one representation, but this implies $.x_{j+1}^{(d)} \cdots x_{j+d}^{(1)} = 0$ and thus $(x_{j+1}^{(d)}, \ldots, x_{j+d}^{(1)}) = \mathbf{0}$.

To show that an x with sufficiently large $\gamma_{2,1}$ exists, we observe first that, if we change the value of $x_1^{(d)}$, then we only change $\gamma_{1,1}$, but not the other $\gamma_{j,1}$. Clearly we have for every choice of $(x_2^{(d-1)}, \ldots, x_d^{(1)})$ an $x_1^{(d)} \in \mathbb{Q}$ such that $.x_1^{(d)} \cdots x_d^{(1)} \in [1/\beta, 1]$. Finally, since the $\mathbf{v}_1, \ldots, \mathbf{v}_d$ (respectively the real and imaginary parts) form a basis of $\mathbb{R}^d$, we have points $(x_1^{(d)}, \ldots, x_d^{(1)}) \in \mathbb{Q}^d$ with arbitrarily large $\gamma_{2,1}$. $\square$

5. Examples

In this section, we consider two examples. The first one shows a run of the algorithm introduced in the proof of Theorem 3.1 for a numeration system built upon an arbitrary regular language. In the second one, we just present the Fibonacci system in this general setting.

Example. Consider the alphabet $\{a < b < c\}$ and the language accepted by the automaton depicted in Fig. 2. The states are denoted $\mathbf{1}$, $\mathbf{2}$ and $\mathbf{3}$. The initial state is $\mathbf{1}$ and the set of final states is $\{\mathbf{2}, \mathbf{3}\}$.

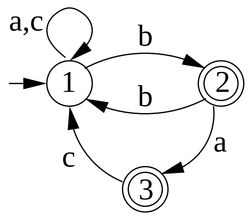

FIGURE 2. A trim minimal automaton.

The adjacency matrix of $\mathcal{M}_L$ is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

and we denote by β its dominating eigenvalue which is the real root of $\chi_\beta(x) = x^3 - 2x^2 - x - 1$ and a Pisot number ($\beta \simeq 2.5468$). By definition, we have $a_{\mathbf{1}} = 1 - 1/\beta$ ($\simeq 0.6074$) and it is easy to see that

$$a_{\mathbf{3}} = \frac{a_{\mathbf{1}}}{\beta} = \frac{1}{\beta} - \frac{1}{\beta^2} \ (\simeq 0.2385),$$
$$a_{\mathbf{2}} = \frac{a_{\mathbf{1}} + a_{\mathbf{3}}}{\beta} = \frac{1}{\beta} - \frac{1}{\beta^3} \ (\simeq 0.3321).$$

The interval $[1/\beta, 1]$ is split into three intervals I_a, I_b and I_c of respective lengths $a_{\mathbf{1}}/\beta$, $a_{\mathbf{2}}/\beta$ and $a_{\mathbf{1}}/\beta$. Let $x = \frac{9}{5}/\beta - \frac{6}{5}/\beta^2 - \frac{2}{5}/\beta^3$ ($\simeq 0.4975$). We denote by $(w_j)_{j\geq 1}$ the maximal representation of x. Since

$$(0.3926 \simeq) \ \frac{1}{\beta} < x < \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta} \ (\simeq 0.6311),$$

we have $w_1 = a$ and $q_1 = \mathbf{1}$. The interval I_a is divided into I_{aa}, I_{ab} and I_{ac} of length respectively $a_{\mathbf{1}}/\beta^2$, $a_{\mathbf{2}}/\beta^2$ and $a_{\mathbf{1}}/\beta^2$. Since

$$(0.4863 \simeq) \ \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta^2} < x < \frac{1}{\beta} + \frac{a_{\mathbf{1}} + a_{\mathbf{2}}}{\beta^2} \ (\simeq 0.5375),$$

we have $w_2 = b$ and $q_2 = \mathbf{2}$. Now, $I_{ab} = I_{aba} \cup I_{abb}$ and these latter intervals are of length $a_{\mathbf{3}}/\beta^3$ and $a_{\mathbf{1}}/\beta^3$. Here

$$(0.4863 \simeq) \ \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta^2} < x < \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta^2} + \frac{a_{\mathbf{3}}}{\beta^3} \ (\simeq 0.5007),$$

thus $w_3 = a$ and $q_3 = \mathbf{3}$. Since there is only one edge from state $\mathbf{3}$, we have $I_{aba} = I_{abac}$, $w_4 = c$ and $q_4 = \mathbf{1}$. As a last step, $I_{abac} = I_{abaca} \cup I_{abacb} \cup I_{abacc}$ and the corresponding lengths are $a_{\mathbf{1}}/\beta^5$, $a_{\mathbf{2}}/\beta^5$ and $a_{\mathbf{1}}/\beta^5$ respectively. Here,

$$(0.4951 \simeq) \ \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta^2} + \frac{a_{\mathbf{1}} + a_{\mathbf{2}}}{\beta^5} < x < \frac{1}{\beta} + \frac{a_{\mathbf{1}}}{\beta^2} + \frac{a_{\mathbf{1}} + a_{\mathbf{2}} + a_{\mathbf{1}}}{\beta^5} \ (\simeq 0.5007),$$

so $w_5 = c$ and $q_5 = \mathbf{1}$. To show the periodicity of the representation of x, one has to observe that $q_1 = q_5$ and the relative position of x inside I_a is the same as the position of x inside I_{abacc}:

$$\frac{x - 1/\beta}{a_{\mathbf{1}}/\beta} = \frac{x - 1/\beta - a_{\mathbf{1}}/\beta^2 - (a_{\mathbf{1}} + a_{\mathbf{2}})/\beta^5}{a_{\mathbf{1}}/\beta^5}$$

Hence $x = \mathrm{val}_\infty(a(bacc)^\omega)$.

Now we consider the algorithm and notation of the proof of Theorem 3.1: $d = 3$, $(b_1, b_2, b_3) = (2, 1, 1)$, $x = .x_1x_2x_3 = .\frac{9}{5}(-\frac{6}{5})(-\frac{2}{5})$. First, by definition of the $\alpha_q(t)$'s, we have

$$\alpha_{\mathbf{1}}(a) = 0,\ \alpha_{\mathbf{1}}(b) = a_{\mathbf{1}},\ \alpha_{\mathbf{1}}(c) = a_{\mathbf{1}} + a_{\mathbf{2}},\ \alpha_{\mathbf{2}}(a) = 0,\ \alpha_{\mathbf{2}}(b) = a_{\mathbf{3}},\ \alpha_{\mathbf{3}}(c) = 0.$$

As initialization step we set $x_1^{(3)} = \frac{9}{5}, x_2^{(2)} = -\frac{6}{5}, x_3^{(1)} = -\frac{2}{5}$ and $q_0 = 1$.

For $j = 1$, we have

$$z_1 = \lfloor \beta x - 1 \rfloor_{\mathbf{1}} - (x_1^{(3)} - 1) = \alpha_{\mathbf{1}}(a) - \frac{4}{5} = -\frac{4}{5} = .\left(-\frac{8}{5}\right)\left(-\frac{4}{5}\right)\left(-\frac{4}{5}\right)$$

because of $0 = \alpha_{\mathbf{1}}(a) \le \beta x - 1 < \alpha_{\mathbf{1}}(b) = \beta - 1$ and the first step gives

$$x_1^{(4)} = \frac{9}{5} - \frac{4}{5} = 1, \quad x_2^{(3)} = -\frac{6}{5} + \frac{8}{5} = \frac{2}{5}, \quad x_3^{(2)} = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}, \quad x_4^{(1)} = \frac{4}{5},$$

$w_1 = a$ and $q_1 = \tau(\mathbf{1}, a) = \mathbf{1}$.

For $j = 2$, we have

$$z_2 = \lfloor x_2^{(3)}.x_3^{(2)}x_4^{(1)} \rfloor_{\mathbf{1}} - x_2^{(3)} = \alpha_{\mathbf{1}}(b) - \frac{2}{5} = \frac{3}{5} - \frac{1}{\beta} = .\frac{1}{5}\frac{3}{5}\frac{3}{5}$$

because of $1 - 1/\beta = \alpha_{\mathbf{1}}(b) \le x_2^{(3)}.x_3^{(2)}x_4^{(1)} = \frac{2}{5}.\frac{2}{5}\frac{4}{5} < \alpha_{\mathbf{1}}(c) = 1 - 1/\beta^3$, hence

$$x_2^{(4)} = \frac{2}{5} + \frac{3}{5} - \frac{1}{\beta} = a_{\mathbf{1}},\ x_3^{(3)} = \frac{2}{5} - \frac{1}{5} = \frac{1}{5},\ x_4^{(2)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5},\ x_5^{(1)} = -\frac{3}{5},$$

$w_2 = b$ and $q_2 = \tau(\mathbf{1}, b) = \mathbf{2}$.

For $j = 3$, we have

$$z_3 = \lfloor x_3^{(3)}.x_4^{(2)}x_5^{(1)} \rfloor_{\mathbf{2}} - x_3^{(3)} = \alpha_{\mathbf{2}}(a) - \frac{1}{5} = -\frac{1}{5} = .\left(-\frac{2}{5}\right)\left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right)$$

because of $0 = \alpha_{\mathbf{2}}(a) \le x_3^{(3)}.x_4^{(2)}x_5^{(1)} = \frac{1}{5}.\frac{1}{5}(-\frac{3}{5}) < \alpha_{\mathbf{2}}(b) = 1/\beta - 1/\beta^3$ and

$$x_3^{(4)} = \frac{1}{5} - \frac{1}{5} = 0,\ x_4^{(3)} = \frac{1}{5} + \frac{2}{5} = \frac{3}{5},\ x_5^{(2)} = -\frac{3}{5} + \frac{1}{5} = -\frac{2}{5},\ x_6^{(1)} = \frac{1}{5},$$

$w_3 = a$ and $q_3 = \tau(\mathbf{2}, a) = \mathbf{3}$.

For $j = 4$, we only have the possibility $w_4 = c$, $q_4 = \mathbf{1}$, because of $q_3 = \mathbf{3}$. Thus $z_3 = -x_4^{(3)} = -\frac{3}{5} = .(-\frac{6}{5})(-\frac{3}{5})(-\frac{3}{5})$ and

$$x_4^{(4)} = 0,\ x_5^{(3)} = -\frac{2}{5} + \frac{6}{5} = \frac{4}{5},\ x_6^{(2)} = \frac{1}{5} + \frac{3}{5} = \frac{4}{5},\ x_7^{(1)} = \frac{3}{5}.$$

Finally for $j = 5$, we have

$$z_5 = \lfloor x_5^{(3)}.x_6^{(2)}x_7^{(1)} \rfloor_{\mathbf{2}} - x_5^{(3)} = \alpha_{\mathbf{1}}(c) - \frac{4}{5} = \frac{1}{5} - \frac{1}{\beta^3} = .\frac{2}{5}\frac{1}{5}\left(-\frac{4}{5}\right)$$

because of $1 - 1/\beta^3 = \alpha_{\mathbf{2}}(c) \leq x_5^{(3)}.x_6^{(2)}x_7^{(1)} = \frac{4}{5}.\frac{4}{5}\frac{3}{5} < \beta a_{\mathbf{1}} = \beta - 1$ and

$$x_5^{(4)} = \frac{4}{5} + \frac{1}{5} - \frac{1}{\beta^3} = a_1 + a_2,\ x_6^{(3)} = \frac{4}{5} - \frac{2}{5} = \frac{2}{5},\ x_7^{(2)} = \frac{3}{5} - \frac{1}{5} = \frac{2}{5},\ x_8^{(1)} = \frac{4}{5},$$

$w_5 = c$ and $q_5 = \tau(\mathbf{1}, c) = \mathbf{1}$.

Hence we have $(x_2^{(3)}, x_3^{(2)}, x_4^{(1)}) = (x_6^{(3)}, x_7^{(2)}, x_8^{(1)})$ and $q_1 = q_5$. This clearly implies that the sequence is ultimately periodic, $w_1 w_2 \cdots = a(bacc)^\omega$ and $x = .1(a_{\mathbf{1}}00(a_{\mathbf{1}} + a_{\mathbf{2}}))^\omega$.

Example. Here we consider the classical Fibonacci system. If β is the golden ratio $\frac{1+\sqrt{5}}{2}$, then we get the automaton depicted in Fig. 3. This

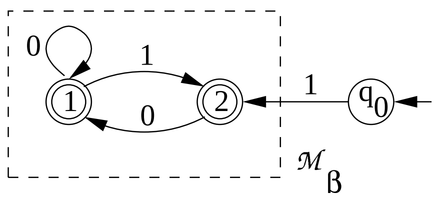

Figure 3. The automaton $\mathcal{M}'_\beta$.

automaton has two parts: an initial state q_0 where the digit 0 is not accepted and the usual automaton given by the states $\mathbf{1}$ and $\mathbf{2}$ where the factor 11 is not accepted. In this setting, $a_{q_0} = 1 - 1/\beta = 1/\beta^2$ implies

$$a_{\mathbf{2}} = a_{q_0}\beta = \frac{1}{\beta}$$
$$a_{\mathbf{1}} = a_{\mathbf{2}}\beta = 1$$

The reader can check that the usual Rényi's β-expansion of a real number $x \in [1/\beta, 1)$ coincides with the representation computed by the algorithm given in the proof of Theorem 3.1.

Acknowledgments

This work was initiated when the first author was visiting the Vienna University of Technology. He would like to thank W. Steiner and M. Drmota for their kind hospitality.

References

[1] A. Bertrand, Développements en base de Pisot et répartition modulo 1, *C. R. Acad. Sc. Paris* **285** (1977), 419–421.

[2] V. Bruyère, G. Hansel, Bertrand numeration systems and recognizability, Latin American Theoretical INformatics (Valparaíso, 1995), *Theoret. Comput. Sci.* **181** (1997), 17–43.

[3] J.-M. Dumont, A. Thomas, Systèmes de numération et fonctions fractales relatifs aux substitutions, *J. Theoret. Comput. Sci.* **65** (1989), 153–169.

[4] S. Eilenberg, *Automata, languages, and machines*, Vol. A, Pure and Applied Mathematics, Vol. 58, Academic Press , New York (1974).

[5] C. Frougny, B. Solomyak, On representation of integers in linear numeration systems, in Ergodic theory of Z_d actions (Warwick, 1993–1994), 345–368, *London Math. Soc. Lecture Note Ser.* **228**, Cambridge Univ. Press, Cambridge (1996).

[6] C. Frougny, Numeration systems, in *M. Lothaire, Algebraic combinatorics on words*, Encyclopedia of Mathematics and its Applications, 90. Cambridge University Press, Cambridge (2002).

[7] P. B. A. Lecomte, M. Rigo, Numeration systems on a regular language, *Theory Comput. Syst.* **34** (2001), 27–44.

[8] P. Lecomte, M. Rigo, On the representation of real numbers using regular languages, *Theory Comput. Syst.* **35** (2002), 13–38.

[9] P. Lecomte, M. Rigo, Real numbers having ultimately periodic representations in abstract numeration systems, to appear in *Inform. and Comput.*

[10] W. Parry, On the β-expansions of real numbers, *Acta Math. Acad. Sci. Hungar.* **11** (1960), 401–416.

[11] A. Rényi, Representation for real numbers and their ergodic properties, *Acta Math. Acad. Sci. Hungar.* **8** (1957), 477–493.

[12] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, *Bull. London Math. Soc.* **12** (1980), 269–278.

Michel Rigo
Université de Liège,
Institut de Mathématiques,
Grande Traverse 12 (B 37),
B-4000 Liège, Belgium.
E-mail : `M.Rigo@ulg.ac.be`
URL: `http://www.discmath.ulg.ac.be/`

Wolfgang Steiner
TU Wien, Institut für Diskrete Mathematik und Geometrie,
Wiedner Hauptstrasse 8-10/104,
A-1040 Wien, Austria

Universität Wien, Institut für Mathematik,
Strudlhofgasse 4,
A-1090 Wien, Austria.
E-mail : `steiner@dmg.tuwien.ac.at`
URL: `http://dmg.tuwien.ac.at/steiner/`