# DIGITAL EXPANSIONS WITH NEGATIVE REAL BASES

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ABSTRACT. Similarly to Parry's characterization of $\beta$-expansions of real numbers in real bases $\beta > 1$, Ito and Sadahiro characterized digital expansions in negative bases, by the expansions of the endpoints of the fundamental interval. Parry also described the possible expansions of 1 in base $\beta > 1$. In the same vein, we characterize the sequences that occur as $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta > 1$. These sequences also describe the itineraries of 1 by linear mod one transformations with negative slope.

## 1. INTRODUCTION

Digital expansions in real bases $\beta > 1$ were introduced by Rényi [Rén57]: The (greedy) *$\beta$-expansion* of a real number $x \in [0, 1)$ is

$$x = \frac{\varepsilon_1(x)}{\beta} + \frac{\varepsilon_2(x)}{\beta^2} + \cdots \quad \text{with} \quad \varepsilon_n(x) = \lfloor \beta\, T_\beta^{n-1}(x) \rfloor,$$

where $\lfloor\cdot\rfloor$ denotes the floor function and $T_\beta$ is the *$\beta$-transformation*

$$T_\beta : [0, 1) \to [0, 1), \quad x \mapsto \beta x - \lfloor \beta x \rfloor .$$

Rényi suggested representing arbitrary $x \in \mathbb{R}$ by

$$x = \lfloor x \rfloor + \frac{\varepsilon_1(\lfloor x \rfloor)}{\beta} + \frac{\varepsilon_2(\lfloor x \rfloor)}{\beta^2} + \cdots ,$$

whereas nowadays it is more usual (for $x \ge 0$) to multiply the $\beta$-expansion of $x\beta^{-k}$ by $\beta^k$, with $k$ an arbitrary integer satisfying $x\beta^{-k} \in [0, 1)$. Anyway, the possible expansions can be described by those of $x \in [0, 1)$. A sequence $b_1 b_2 \cdots$ is called *$\beta$-admissible* if and only if it is (the digit sequence of) the $\beta$-expansion of a number $x \in [0, 1)$, i.e., $b_n = \varepsilon_n(x)$ for all $n \ge 1$. Parry [Par60] showed that an integer sequence $b_1 b_2 \cdots$ is $\beta$-admissible if and only if

$$00 \cdots \le_{\mathrm{lex}} b_k b_{k+1} \cdots <_{\mathrm{lex}} a_1 a_2 \cdots \quad \text{for all } k \ge 1,$$

where $<_{\mathrm{lex}}$ denotes the lexicographic order and $a_1 a_2 \cdots$ is the (quasi-greedy) *$\beta$-expansion of* 1, i.e., $a_n = \lim_{x \to 1-} \varepsilon_n(x)$. Moreover, a sequence of integers $a_1 a_2 \cdots$ is the (quasi-greedy) $\beta$-expansion of 1 for some $\beta > 1$ if and only if

$$00 \cdots <_{\mathrm{lex}} a_k a_{k+1} \cdots \le_{\mathrm{lex}} a_1 a_2 \cdots \quad \text{for all } k \ge 2.$$

Part of this research was conducted while the author was visiting academic at the Department of Computing of the Macquarie University, Sydney.

(These results are stated in a slightly different way in [Par60]; quasi-greedy $\beta$-expansions were introduced by Daróczy and Kátai [DK95] under the name "quasi-regular" and got the present name from Komornik and Loreti [KL07].)

Following [Rén57] and [Par60], a lot of papers were dedicated to the study of $\beta$-expansions and $\beta$-transformations, but surprisingly little attention was given to digital expansions in negative bases. This changed only in recent years, after Ito and Sadahiro [IS09] considered $(-\beta)$-*expansions*, $\beta > 1$, defined for $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$ by

$$x = \frac{\varepsilon_1(x)}{-\beta} + \frac{\varepsilon_2(x)}{(-\beta)^2} + \cdots \quad \text{with} \quad \varepsilon_n(x) = \big\lfloor \tfrac{\beta}{\beta+1} - \beta\, T_{-\beta}^{n-1}(x) \big\rfloor, \tag{1.1}$$

where the $(-\beta)$-*transformation* is defined by

$$T_{-\beta} : \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big) \to \big[\tfrac{-\beta}{\beta+1}, \tfrac{1}{\beta+1}\big), \quad x \mapsto -\beta x - \big\lfloor \tfrac{\beta}{\beta+1} - \beta x \big\rfloor.$$

A sequence $b_1 b_2 \cdots$ is $(-\beta)$-*admissible* if and only if it is the $(-\beta)$-expansion of some $x \in \big[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\big)$, i.e., $b_n = \varepsilon_n(x)$ for all $n \geq 1$. Since the map $x \mapsto -\beta x$ is order-reversing, the $(-\beta)$-admissible sequences are characterized using the alternating lexicographic order. By [IS09], a sequence $b_1 b_2 \cdots$ is $(-\beta)$-admissible if and only if

$$a_1 a_2 \cdots \geq_{\text{alt}} b_k b_{k+1} \cdots >_{\text{alt}} 0 a_1 a_2 \cdots \quad \text{for all } k \geq 1, \tag{1.2}$$

where $a_1 a_2 \cdots$ is the $(-\beta)$-expansion of the left endpoint $\frac{-\beta}{\beta+1}$, i.e., $a_n = \varepsilon_n\big(\frac{-\beta}{\beta+1}\big)$, which is supposed not to be periodic with odd period length. If $a_1 a_2 \cdots = \overline{a_1 a_2 \cdots a_{2\ell+1}}$ for some $\ell \geq 0$, and $\ell$ is minimal with this property, then the condition (1.2) is replaced by

$$a_1 a_2 \cdots \geq_{\text{alt}} b_k b_{k+1} \cdots >_{\text{alt}} \overline{0 a_1 \cdots a_{2\ell} (a_{2\ell+1} - 1)} \quad \text{for all } k \geq 1. \tag{1.3}$$

Recall that the alternating lexicographic order is defined on sequences $x_1 x_2 \cdots$, $y_1 y_2 \cdots$ with $x_1 \cdots x_{k-1} = y_1 \cdots y_{k-1}$ and $x_k \neq y_k$ by

$$x_1 x_2 \cdots <_{\text{alt}} y_1 y_2 \cdots \quad \text{if and only if} \quad \begin{cases} x_k < y_k & \text{when } k \text{ is odd,} \\ y_k < x_k & \text{when } k \text{ is even.} \end{cases}$$

The main result of this paper is a characterization of the sequences $a_1 a_2 \cdots$ that are the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta > 1$. This turns out to be more complicated than the corresponding problem for $\beta$-expansions, and we will see that several proofs cannot be directly carried over from positive to negative bases. From (1.2) and (1.3), one deduces that

$$a_k a_{k+1} \cdots \leq_{\text{alt}} a_1 a_2 \cdots \quad \text{for all } k \geq 2. \tag{1.4}$$

The proof of Proposition 3.5 in [LS] (see also Theorem 3 below) shows that

$$a_1 a_2 \cdots >_{\text{alt}} u_1 u_2 \cdots = 1001110010010011110011 \cdots, \tag{1.5}$$

where $u_1 u_2 \cdots$ is the sequence starting with $\varphi^n(1)$ for all $n \geq 0$, with $\varphi$ being the morphism of words on the alphabet $\{0, 1\}$ defined by $\varphi(1) = 100$, $\varphi(0) = 1$. (See the remarks following Theorem 3 and note that the alphabet is shifted by 1 in [LS].) Our first result states that a sequence satisfying (1.4) and (1.5) is "almost" the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta > 1$.

**Theorem 1.** *Let $a_1a_2\cdots$ be a sequence of non-negative integers satisfying* (1.4) *and* (1.5). *Then there exists a unique $\beta>1$ such that*

$$\sum_{j=1}^{\infty}\frac{a_j}{(-\beta)^j}=\frac{-\beta}{\beta+1}\quad\text{and}\quad\sum_{j=1}^{\infty}\frac{a_{k+j}}{(-\beta)^j}\in\left[\frac{-\beta}{\beta+1},\frac{1}{\beta+1}\right]\quad\text{for all } k\ge 1. \tag{1.6}$$

For a $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$, we have to exclude the possibility that $\sum_{j=1}^{\infty}\frac{a_{k+j}}{(-\beta)^j}=\frac{1}{\beta+1}$ for some $k\ge1$. If $\overline{a_1\cdots a_k}>_{\mathrm{alt}}u_1u_2\cdots$, then out of $\{a_1\cdots a_k,\,a_1\cdots a_{k-1}(a_k-1)0\}^\omega$, which is the set of infinite sequences composed of blocks $a_1\cdots a_k$ and $a_1\cdots a_{k-1}(a_k-1)0$, only the periodic sequence $\overline{a_1\cdots a_k}$ is possibly the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta>1$, see Section 4. This implies that

$$a_1a_2\cdots\notin\{a_1\cdots a_k,\,a_1\cdots a_{k-1}(a_k-1)0\}^\omega\setminus\{\overline{a_1\cdots a_k}\}\qquad\text{for all } k\ge1 \text{ with } \overline{a_1\cdots a_k}\succ u_1u_2\cdots, \tag{1.7}$$

$$a_1a_2\cdots\notin\{a_1\cdots a_k0,\,a_1\cdots a_{k-1}(a_k+1)\}^\omega\qquad\text{for all } k\ge1 \text{ with } \overline{a_1\cdots a_{k-1}(a_k+1)}\succ u_1u_2\cdots. \tag{1.8}$$

The main result states that there are no other conditions on $a_1a_2\cdots$.

**Theorem 2.** *A sequence of non-negative integers $a_1a_2\cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some (unique) $\beta>1$ if and only if it satisfies* (1.4), (1.5), (1.7), *and* (1.8).

It is easy to see that the natural order of bases $\beta>1$ is reflected by the lexicographical order of the (quasi-greedy) $\beta$-expansions of 1 [Par60]. For negative bases, a similar relation with the alternating lexicographic order holds, although it is a bit harder to prove.

**Theorem 3.** *Let $a_1a_2\cdots$ be the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ and $a_1'a_2'\cdots$ be the $(-\beta')$-expansion of $\frac{-\beta'}{\beta'+1}$, with $\beta,\beta'>1$. Then $\beta<\beta'$ if and only if $a_1a_2\cdots<_{\mathrm{alt}}a_1'a_2'\cdots$.*

It is often convenient to study a slightly different $(-\beta)$-transformation,

$$\widetilde{T}_{-\beta}:\,(0,1]\to(0,1],\quad x\mapsto-\beta x+\lfloor\beta x\rfloor+1.$$

As already noted in [LS], the transformations $T_{-\beta}$ and $\widetilde{T}_{-\beta}$ are conjugate via the involution $\phi(x)=\frac{1}{\beta+1}-x$, i.e.,

$$T_{-\beta}\circ\phi(x)=\phi\circ\widetilde{T}_{-\beta}(x)\quad\text{for all } x\in(0,1].$$

Setting $\tilde\varepsilon_n(x)=\lfloor\beta\,\widetilde{T}_{-\beta}^{n-1}(x)\rfloor$ for $x\in(0,1]$, we have $x=-\sum_{n=1}^{\infty}\frac{\tilde\varepsilon_n(x)+1}{(-\beta)^n}=\frac{1}{\beta+1}-\sum_{n=1}^{\infty}\frac{\tilde\varepsilon_n(x)}{(-\beta)^n}$, and $\tilde\varepsilon_n(x)=\varepsilon_n(\phi(x))$. Note that $\widetilde{T}_{-\beta}(x)=-\beta x-\lfloor-\beta x\rfloor$ except for finitely many points, hence $\widetilde{T}_{-\beta}$ is a natural generalization of the beta-transformation. The map $\widetilde{T}_{-\beta}$ was studied e.g. by Góra [Gór07], where it corresponds to the case $E=[1,1,\ldots,1]$, and in [LS]. The following corollary is an immediate consequence of Theorems 1 and 2.

**Corollary 1.** *Let $a_1a_2\cdots$ be a sequence of non-negative integers satisfying* (1.4) *and* (1.5). *Then there exists a unique $\beta > 1$ such that*

$$-\sum_{j=1}^{\infty}\frac{a_j+1}{(-\beta)^j}=1 \quad \text{and} \quad -\sum_{j=1}^{\infty}\frac{a_{k+j}+1}{(-\beta)^j}\in[0,1] \quad \text{for all } k\ge 1. \tag{1.9}$$

*Moreover, $\sum_{j=1}^{\infty}\frac{a_{k+j}+1}{(-\beta)^j}\neq 0$ for all $k\ge 1$ if and only if* (1.7) *and* (1.8) *hold.*

With the notation of [Gór07], this means, for $E=[1,1,\ldots,1]$, that $a_1a_2\cdots$ is the itinerary $\mathrm{It}_\beta(1)$ for some $\beta>1$ if and only if (1.4), (1.5), (1.7), and (1.8) hold. Note that Góra [Gór07, Theorems 25 and 28] claims that already (1.4) is sufficient when $a_1\ge 2$, and he has a less explicit statement for $a_1=1$. However, his proof deals only with the first part of the theorem, i.e., that there exists a unique $\beta>1$ satisfying (1.9). To see that this is not sufficient, consider the sequences $a_1a_2\cdots\in\{2,1\,0\}^\omega$. They all satisfy (1.9) with $\beta=2$, and there are uncountably many of them satisfying (1.4) and $a_1=2$. All these uncountably many sequences would have to be equal to $\mathrm{It}_2(1)$ by [Gór07, Theorem 25], which is of course not true. (See also [DMP11].) Moreover, Góra's proof of the existence of a unique $\beta>1$ satisfying (1.9) is incorrect when $\beta$ is small; see Remark 1.

## 2. Proof of Theorem 3

Let $\beta>1$. For a sequence of digits $b_1\cdots b_n$, set

$$I_{b_1\cdots b_n}=\left\{x\in\left[\tfrac{-\beta}{\beta+1},\tfrac{1}{\beta+1}\right):\,\varepsilon_1(x)\cdots\varepsilon_n(x)=b_1\cdots b_n\right\},$$

with $\varepsilon_j(x)$ as in (1.1). Let $L_{\beta,n}$ be the number of different sequences $b_1\cdots b_n$ such that $I_{b_1\cdots b_n}\neq\emptyset$, and let $L'_{\beta,n}$ be the number of different sequences $b_1\cdots b_n$ such that $I_{b_1\cdots b_n}$ is an interval of positive length. (The latter is called the lap number of $T_{-\beta}^n$.)

**Lemma 1.** *For any $\beta>1$, we have that $\lim_{n\to\infty}\frac{1}{n}\log L_{\beta,n}=\lim_{n\to\infty}\frac{1}{n}\log L'_{\beta,n}=\log\beta$.*

*Proof.* It is well known that the entropy of $T_{-\beta}$, which is a piecewise linear map of constant slope $-\beta$, is $\log\beta$. The lemma can be derived from this fact, see [FL11], but we prefer giving a short elementary proof, following Faller [Fal08, Proposition 3.6]. As $\left|\frac{d}{dx}T_{-\beta}^n(x)\right|=\beta^n$ at all points of continuity of $T_{-\beta}^n$, the length of any interval $I_{b_1\cdots b_n}$ is at most $\beta^{-n}$. Since the intervals $I_{b_1\cdots b_n}$ form a partition of an interval of length 1, we obtain that $L_{\beta,n}\ge L'_{\beta,n}\ge\beta^n$.

To get an upper bound for $L'_{\beta,n}$, let $m$ be the smallest positive integer such that $\beta^m>2$, and let $\delta$ be the minimal positive length of an interval $I_{b_1\cdots b_m}$. Consider an interval $I_{b_1\cdots b_n}$, $n>m$, such that $b_1\cdots b_n$ is neither the minimal nor the maximal sequence (with respect to the alternating lexicographic order) starting with $b_1\cdots b_{n-m}$ and satisfying $I_{b_1\cdots b_n}\neq\emptyset$. Then each prolongation $b_1b_2\cdots$ satisfies the inequalities in (1.2) and (1.3), respectively, for $1\le k\le n-m$. Therefore, $b_1b_2\cdots$ is $(-\beta)$-admissible if and only if $b_{n-m+1}b_{n-m+2}\cdots$ is $(-\beta)$-admissible. This implies that $T_{-\beta}^{n-m}(I_{b_1\cdots b_n})=I_{b_{n-m+1}\cdots b_m}$, and the length of $I_{b_1\cdots b_n}$ is $\beta^{m-n}$ times the length of $I_{b_{n-m+1}\cdots b_m}$, thus at least $\beta^{m-n}\delta$ when the length is positive. There are at least $L'_{\beta,n}-2L'_{\beta,n-m}$ sequences $b_1\cdots b_n$ such that $I_{b_1\cdots b_n}$ has positive length and $b_1 \cdots b_n$ is neither the minimal nor the maximal sequence starting with $b_1 \cdots b_{n-m}$ and satisfying $I_{b_1 \cdots b_n} \neq \emptyset$. This yields that $(L'_{\beta,n} - 2L'_{\beta,n-m})\beta^{m-n}\delta \leq 1$ for all $n > m$, thus

$$L'_{\beta,n} \leq \frac{\beta^{n-m}}{\delta} + 2L'_{\beta,n-m} \leq \frac{\beta^{n-m}}{\delta} + \frac{2\beta^{n-2m}}{\delta} + 4L'_{\beta,n-2m} \leq \cdots$$
$$\leq \frac{\beta^{n-m}}{\delta} \sum_{j=0}^{\lceil n/m \rceil -2} \left(\frac{2}{\beta^m}\right)^j + 2^{\lceil n/m \rceil -1} L'_{\beta,n-\lceil n/m \rceil m+m} < \frac{\beta^n}{\delta}\frac{1}{\beta^m - 2} + \beta^n L'_{\beta,m} \leq \frac{\beta^n}{\delta}\frac{\beta^m - 1}{\beta^m - 2}.$$

This shows that $\lim_{n\to\infty} \frac{1}{n} \log L'_{\beta,n} = \beta$.

An interval $I_{b_1 \cdots b_n}$ consists only of one point if and only if $I_{b_1 \cdots b_k} = \left\{\frac{-\beta}{\beta+1}\right\}$ and $b_{k+1} \cdots b_n = a_1 \cdots a_{n-k}$ for some $k \leq n$. (This can happen only in case that $a_1 a_2 \cdots$ is periodic with odd period length.) Therefore, we can estimate $L_{\beta,n} - L'_{\beta,n} \leq L'_{\beta,0} + L'_{\beta,1} + \cdots + L'_{\beta,n} \leq C\beta^n$ for some constant $C > 0$, thus $\lim_{n\to\infty} \frac{1}{n} \log L_{\beta,n} = \lim_{n\to\infty} \frac{1}{n} \log L'_{\beta,n}$. □

For the proof of Theorem 3, let $a_1 a_2 \cdots$ be the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ and $a'_1 a'_2 \cdots$ be the $(-\beta')$-expansion of $\frac{-\beta'}{\beta'+1}$, $\beta, \beta' > 1$. If $\beta = \beta'$, then we clearly have that $a_1 a_2 \cdots = a'_1 a'_2 \cdots$. If $a_1 a_2 \cdots = a'_1 a'_2 \cdots$, then the $(-\beta)$-admissible sequences are equal to the $(-\beta')$-admissible sequences, thus $L_{\beta,n} = L_{\beta',n}$ for all $n \geq 1$, and $\beta = \beta'$ by Lemma 1. Therefore, the equations $\beta = \beta'$ and $a_1 a_2 \cdots = a'_1 a'_2 \cdots$ are equivalent. Hence, it suffices to show that $a_1 a_2 \cdots <_{\mathrm{alt}} a'_1 a'_2 \cdots$ implies that $\beta < \beta'$, as the other direction follows by contraposition.

Assume that $a_1 a_2 \cdots <_{\mathrm{alt}} a'_1 a'_2 \cdots$, and let $b_1 b_2 \cdots$ be a $(-\beta)$-admissible sequence. By (1.2) and (1.3) respectively, we have that

$$b_k b_{k+1} \cdots \leq_{\mathrm{alt}} a_1 a_2 \cdots <_{\mathrm{alt}} a'_1 a'_2 \cdots . \tag{2.1}$$

Furthermore, as $\overline{0a_1 \cdots a_{2\ell}(a_{2\ell+1}-1)} >_{\mathrm{alt}} 0a_1 a_2 \cdots$ for all $\ell \geq 0$, we obtain that

$$b_k b_{k+1} \cdots >_{\mathrm{alt}} 0a_1 a_2 \cdots >_{\mathrm{alt}} 0a'_1 a'_2 \cdots . \tag{2.2}$$

If $a'_1 a'_2 \cdots$ is not periodic with odd period length, then (2.1) and (2.2) show that $b_1 b_2 \cdots$ is $(-\beta')$-admissible, thus $L_{\beta,n} \leq L_{\beta',n}$ for all $n \geq 1$, and $\beta \leq \beta'$ by Lemma 1. Since $a_1 a_2 \cdots \neq a'_1 a'_2 \cdots$, this yields that $\beta < \beta'$. In case $a'_1 a'_2 \cdots = \overline{a'_1 \cdots a'_{2\ell'+1}}$, we show that

$$a_1 a_2 \cdots \leq_{\mathrm{alt}} \overline{a'_1 \cdots a'_{2\ell'}(a'_{2\ell'+1}-1)0}. \tag{2.3}$$

This is clearly true when $a_1 \cdots a_{2\ell'+1} <_{\mathrm{alt}} a'_1 \cdots a'_{2\ell'}(a'_{2\ell'+1}-1)$. If $a_1 \cdots a_{2\ell'+1} = a'_1 \cdots a'_{2\ell'+1}$, then $a_{2\ell'+2} a_{2\ell'+3} \cdots >_{\mathrm{alt}} a'_{2\ell'+2} a'_{2\ell'+3} \cdots = a'_1 a'_2 \cdots >_{\mathrm{alt}} a_1 a_2 \cdots$, contradicting (1.4). It remains to consider the case that $a_1 \cdots a_{2\ell'+1} = a'_1 \cdots a'_{2\ell'}(a'_{2\ell'+1}-1)$. If $a_{2\ell'+1} > 0$, then (2.3) holds, otherwise $a_1 \cdots a_{2\ell'+2} = a'_1 \cdots a'_{2\ell'}(a'_{2\ell'+1}-1)0$. In the latter case, (1.4) implies that $a_{2\ell'+3} \cdots a_{4\ell'+4} \leq_{\mathrm{alt}} a_1 \cdots a_{2\ell'+2} = a'_1 \cdots a'_{2\ell'}(a'_{2\ell'+1}-1)0$, and we obtain inductively that (2.3) holds. Now, (2.1), (2.2), and (2.3) show that $b_1 b_2 \cdots$ is $(-\beta')$-admissible, which yields as above that $\beta < \beta'$.

## 3. Proof of Theorem 1

Let $a_1 a_2 \cdots$ be a sequence of non-negative integers satisfying (1.4) and (1.5). We show that there exists a unique $\beta > 1$ satisfying (1.9), which is equivalent to (1.6). For $n \ge 1$, set

$$P_n(x) = (-x)^n + \sum_{j=1}^{n} (a_j + 1)\,(-x)^{n-j},$$

$$J_n = \big\{x > 1 \mid P_j(x) \in [0,1] \text{ for all } 1 \le j \le n\big\}.$$

Then $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$, and $J_n$ is compact if and only if $\inf J_n \ne 1$.

First note that, for $\beta > 1$, (1.9) is equivalent to $\beta \in \bigcap_{n\ge1} J_n$. Indeed, if (1.9) holds, then $P_n(\beta) = -\sum_{j=1}^{\infty} \frac{a_{n+j}+1}{(-\beta)^j} \in [0,1]$ for all $n \ge 1$. On the other hand, if $P_n(\beta) \in [0,1]$ for all $n \ge 1$, then $\big|1 + \sum_{j=1}^{\infty} \frac{a_j+1}{(-\beta)^j}\big| = \lim_{n\to\infty} \frac{P_n(\beta)}{(-\beta)^n} = 0$, thus (1.9) holds.

Inductively for $n \ge 1$, we show the following statements, where we use the abbreviations $v_{[j,k]}$ for $v_j v_{j+1} \cdots v_k$ and $v_{[j,k)}$ for $v_j v_{j+1} \cdots v_{k-1}$:

(1) $J_n$ is a non-empty interval, with $\inf J_n = 1$ if and only if $a_{[1,n]} = u_{[1,n]}$.
If $P_n(\beta) = P_n(\beta') \in \{0,1\}$ with $\beta, \beta' \in J_n$, then $\beta = \beta'$.

(2) If $n$ is even, $a_{[1,n-2m+1]} = u_{[1,n-2m+1]}$ or $a_{[n-2m+2,n]} \ne a_{[1,2m)}$ for all $1 \le m \le n/2$, and $a_{[1,n]} \ne u_{[1,n]}$, then $P_n(\min J_n) = 0$.
If $n$ is odd and $a_{[n-2m+2,n]} \ne a_{[1,2m)}$ for all $1 \le m \le n/2$, then $P_n(\max J_n) = 0$.

(3) If $n$ is even, $a_{[1,n-2m+1]} \ne u_{[1,n-2m+1]}$ and $a_{[n-2m+2,n]} = a_{[1,2m)}$ for some $1 \le m \le n/2$, and $m$ is maximal with this property, then $P_n(\min J_n) = P_{2m-1}(\min J_n)$.
If $n$ is odd, $a_{[n-2m+2,n]} = a_{[1,2m)}$ for some $1 \le m \le n/2$, and $m$ is maximal with this property, then $P_n(\max J_n) = P_{2m-1}(\max J_n)$.

(4) If $n$ is even and $a_{[n-2m+1,n]} \ne a_{[1,2m]}$ for all $1 \le m < n/2$, then $P_n(\max J_n) = 1$.
If $n$ is odd, $a_{[1,n-2m]} = u_{[1,n-2m]}$ or $a_{[n-2m+1,n]} \ne a_{[1,2m]}$ for all $1 \le m < n/2$, and $a_{[1,n]} \ne u_{[1,n]}$, then $P_n(\min J_n) = 1$.

(5) If $n$ is even, $a_{[n-2m+1,n]} = a_{[1,2m]}$ for some $1 \le m < n/2$, and $m$ is maximal with this property, then $P_n(\max J_n) = P_{2m}(\max J_n)$.
If $n$ is odd, $a_{[1,n-2m]} \ne u_{[1,n-2m]}$ and $a_{[n-2m+1,n]} = a_{[1,2m]}$ for some $1 \le m < n/2$, and $m$ is maximal with this property, then $P_n(\min J_n) = P_{2m}(\min J_n)$.

We have that $P_1(x) = a_1 + 1 - x$, and $a_1 \ge 1$ by (1.5). If $a_1 \ge 2$, then $J_1 = [a_1, a_1+1]$, $P_1(a_1) = 1$ and $P_1(a_1+1) = 0$; if $a_1 = 1$, then $J_1 = (1,2]$ and $P_1(2) = 0$. Therefore, the statements hold for $n = 1$. Assume that they hold for $n-1$, and set

$$B = \big\{b \in \{0,1,\ldots,a_1\} : b + 1 - x P_{n-1}(x) \in [0,1] \text{ for some } x \in J_{n-1}\big\},$$

i.e., $J_n \ne \emptyset$ if and only if $a_n \in B$.

Assume first that $a_{[1,n)} \ne u_{[1,n)}$, i.e., $\inf J_{n-1} = \min J_{n-1} > 1$, and that $n$ is even.

(i) If $a_{[n-2m+1,n)} \ne a_{[1,2m)}$ for all $1 \le m < n/2$, then $P_{n-1}(\max J_{n-1}) = 0$, thus

$$1 - (\max J_{n-1})\, P_{n-1}(\max J_{n-1}) = 1.$$

This implies that $0 \in B$, and $P_n(\max J_n) = P_n(\max J_{n-1}) = 1$ if $a_n = 0$. Since the map $x \mapsto x P_{n-1}(x)$ is continuous and $J_{n-1}$ is an interval, we get that $P_n(\max J_n) = 1$

for $a_n > 0$ as well, when $J_n \neq \emptyset$. Moreover, we clearly have that $a_{[n-2m+1,n]} \neq a_{[1,2m]}$ for all $1 \leq m < n/2$, thus (4) holds when $a_n \in B$.

(ii) If $a_{[n-2m+1,n)} = a_{[1,2m)}$ for some $1 \leq m < n/2$, and $m$ is maximal with this property, then $P_{n-1}(\max J_{n-1}) = P_{2m-1}(\max J_{n-1})$, thus

$$a_{2m} + 1 - (\max J_{n-1})\, P_{n-1}(\max J_{n-1}) = P_{2m}(\max J_{n-1}) \in [0,1],$$

where we have used that $J_{n-1} \subseteq J_{2m}$ and $P_{2m}(J_{2m}) \subseteq [0,1]$. This gives $a_{2m} \in B$. If $a_n = a_{2m}$, then $\max J_n = \max J_{n-1}$ and $P_n(\max J_{n-1}) = P_{2m}(\max J_{n-1})$, thus $P_n(\max J_n) = P_{2m}(\max J_n)$ and $a_{[n-2m+1,n]} = a_{[1,2m]}$. By the maximality of $m$, we have that $a_{[n-2\ell+1,n]} \neq a_{[1,2\ell]}$ for all $m < \ell < n/2$, thus (5) holds.
If $a_n \neq a_{2m}$, then the equation $a_{[n-2m+1,n)} = a_{[1,2m)}$ and (1.4) yield that $a_n > a_{2m}$, thus $P_n(\max J_n) = 1$ when $J_n \neq \emptyset$, similarly to (i). If $a_{[1,2\ell)} = a_{[n-2\ell+1,n)}$, $1 \leq \ell < m$, then we also have that $a_{[1,2\ell)} = a_{[2m-2\ell+1,2m)}$, thus $a_{2\ell} \leq a_{2m} < a_n$. This implies that $a_{[n-2\ell+1,n]} \neq a_{[1,2\ell]}$ for all $1 \leq \ell < n/2$, thus (4) holds when $a_n \in B$.

(iii) If $a_{[1,n-2m)} = u_{[1,n-2m)}$ or $a_{[n-2m,n)} \neq a_{[1,2m]}$ for all $1 \leq m \leq n/2-1$, then we have that $P_{n-1}(\min J_{n-1}) = 1$, thus

$$a_1 + 1 - (\min J_{n-1})\, P_{n-1}(\min J_{n-1}) = P_1(\min J_{n-1}) \in [0,1],$$

and $a_1 \in B$. If $a_n = a_1$, then $\min J_n = \min J_{n-1}$ and $P_n(\min J_{n-1}) = P_1(\min J_{n-1})$, thus $P_n(\min J_n) = P_1(\min J_n)$, and $a_{[1,n-2m+1]} = u_{[1,n-2m+1]}$ or $a_{[n-2m+2,n]} \neq a_{[1,2m)}$ for all $2 \leq m \leq n/2$. Therefore, (3) holds. If $a_n < a_1$, then $P_n(\min J_n) = 0$ when $J_n \neq \emptyset$, $a_{[1,n-2m+1]} = u_{[1,n-2m+1]}$ or $a_{[n-2m+2,n]} \neq a_{[1,2m)}$ for all $1 \leq m \leq n/2$, thus (2) holds when $a_n \in B$.

(iv) If $a_{[1,n-2m)} \neq u_{[1,n-2m)}$ and $a_{[n-2m,n)} = a_{[1,2m]}$ for some $1 \leq m \leq n/2-1$, and $m$ is maximal with this property, then $P_{n-1}(\min J_{n-1}) = P_{2m}(\min J_{n-1})$, thus

$$a_{2m+1} + 1 - (\min J_{n-1})\, P_{n-1}(\min J_{n-1}) = P_{2m+1}(\min J_{n-1}) \in [0,1],$$

hence $a_{2m+1} \in B$. If $a_n = a_{2m+1}$, then $\min J_n = \min J_{n-1}$ and $P_n(\min J_{n-1}) = P_{2m+1}(\min J_{n-1})$, thus $P_n(\min J_n) = P_{2m+1}(\min J_n)$, and $a_{[n-2m,n]} = a_{[1,2m+1]}$. The maximality of $m$ yields that $a_{[1,n-2\ell+1]} = u_{[1,n-2\ell+1]}$ or $a_{[n-2\ell+2,n]} \neq a_{[1,2\ell)}$ for all $m+1 < \ell \leq n/2$, thus (3) holds. If $a_n \neq a_{2m+1}$, then $a_n < a_{2m+1}$ by (1.4). If moreover $a_{[1,2\ell-2]} = a_{[n-2\ell+2,n)}$, $1 \leq \ell \leq m$, then we have that $a_{[1,2\ell-2]} = a_{[2m-2\ell+3,2m]}$, thus $a_{2\ell-1} \geq a_{2m+1} > a_n$. Then we get that $P_n(\min J_n) = 0$ when $J_n \neq \emptyset$, $a_{[1,n-2\ell+1]} = u_{[1,n-2\ell+1]}$ and $a_{[n-2\ell+2,n]} \neq a_{[1,2\ell)}$ for all $1 \leq \ell \leq n/2$, thus (2) holds when $a_n \in B$.

Since $x \mapsto x P_{n-1}(x)$ is continuous and $J_{n-1}$ is an interval, the set $B$ is an interval of integers. The paragraphs (i) and (ii) show that $a_n$ is not smaller than the smallest element of $B$, (iii) and (iv) show that $a_n$ is not larger than the largest element of $B$, thus $a_n \in B$. We have therefore proved that $J_n \neq \emptyset$ and (2)–(5) hold, when $a_{[1,n)} \neq u_{[1,n)}$ and $n$ is even. For odd $n$, the proof runs along the same lines and is left to the reader.

If $a_{[1,n)} = u_{[1,n)}$, then $\inf J_{n-1} = 1$. From [LS, Proposition 3.5], we know that $u_n \in B$, that $\inf J_n = 1$ when $a_n = u_n$, and that $\min J_n > 1$ when $u_n \neq a_n \in B$. Let first $n$ be even, thus $a_n \leq u_n$ by (1.5). If $a_{[n-2m+1,n)} \neq a_{[1,2m)}$ for all $1 \leq m < n/2$, then we obtain as in (i) that $0 \in B$, thus $a_n \in B$, and (4) holds. If $a_{[n-2m+1,n)} = a_{[1,2m)}$ for some $1 \leq m < n/2$, and

$m$ is maximal with this property, then (ii) yields that $a_{2m} \in B$ and $a_{2m} \le a_n$, thus $a_n \in B$. If $a_n = a_{2m}$, then (5) holds; if $a_n > a_{2m}$, then (4) holds. Moreover, if $a_n < u_n$, then we get that $P_n(\min J_n) = 0$, thus (2) holds. Again, if $n$ is odd, then similar arguments apply. Hence, we have proved that $J_n \neq \emptyset$ and (2)–(5) hold for the case that $a_{[1,n)} = u_{[1,n)}$ too.

If $J_n$ is not an interval, then the continuity of $x \mapsto xP_{n-1}(x)$ on the interval $J_{n-1}$ implies that $P_n$ meets the lower bound 0 or the upper bound 1 at least twice within $J_n$. Therefore, suppose that $P_n(\beta) = P_n(\beta') \in \{0,1\}$ for $\beta, \beta' \in J_n$. If $P_j(\beta) \in (0,1]$ and $P_j(\beta') \in (0,1]$ for all $1 \le j < n$, then the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ and the $(-\beta')$-expansion of $\frac{-\beta'}{\beta'+1}$ are both $\overline{a_{[1,n]}}$ (if $P_n(\beta) = 1$) or $\overline{a_{[1,n)}(a_n{+}1)}$ (if $P_n(\beta) = 0$), thus $\beta = \beta'$ by Theorem 3.

Suppose in the following that $P_j(\beta') = 0$ for some $1 \le j < n$, and let $\ell \ge 1$ be minimal such that $P_\ell(\beta') \in \{0,1\}$. If $P_\ell(\beta') = 0$, then $a_{\ell+1} = 0$ and $P_{\ell+1}(\beta') = 1$, hence $a_{[1,n]}$ is a concatenation of blocks $a_{[1,\ell]}0$ and $a_{[1,\ell)}(a_\ell{+}1)$, except possibly for the last block, which is $a_{[1,\ell]}$ when $P_n(\beta') = 0$. If $P_\ell(\beta') = 1$, then $a_{[1,n]}$ is a concatenation of blocks $a_{[1,\ell]}$ and $a_{[1,\ell)}(a_\ell{-}1)0$, ending with $a_{[1,\ell)}(a_\ell{-}1)$ when $P_n(\beta') = 0$. We obtain that

$$P_n(x) = P_n(\beta') + \big(P_\ell(x) - P_\ell(\beta')\big)\, Q(x)$$

for some polynomial $Q(x) = \sum_{j=0}^{n-\ell} q_j\, (-x)^j$ with coefficients $q_j \in \{0,1\}$, and $q_{j-1} = q_{j-2} = \cdots = q_{j-\ell+1} = 0$ whenever $q_j = 1$. If $P_\ell(\beta) = P_\ell(\beta')$, then the induction hypotheses yield that $\beta = \beta'$. If $P_\ell(\beta) \neq P_\ell(\beta')$, then $Q(\beta) = 0$, which implies that $1 < \frac{1}{\beta^{\ell+1}} + \frac{1}{\beta^{2\ell+1}} + \cdots = \frac{1}{\beta^{\ell+1}-\beta}$ when $\ell$ is even, $1 < \frac{1}{\beta^{\ell}} + \frac{1}{\beta^{2\ell+1}} + \cdots = \frac{\beta}{\beta^{\ell+1}-1}$ when $\ell$ is odd, i.e., $\beta^{\ell+1} < \beta + 1$.

To exclude the latter case, suppose that $P_n(\beta) = P_n(\beta') \in \{0,1\}$ for $\beta, \beta' \in J_n$, $\beta \neq \beta'$, and that $\beta^{\ell+1} < \beta + 1$ for the minimal $\ell \ge 1$ such that $P_\ell(\beta') \in \{0,1\}$. Set $g_k = \lfloor 2^{k+1}/3 \rfloor$, and let, for $k \ge 1$, $\gamma_k$ and $\eta_k$ be the real numbers greater than 1 satisfying $\gamma_k^{g_k+1} = \gamma_k + 1$, $\eta_k^{g_k+1} = \eta_k^{g_{k-1}+1} {+} 1$ when $k$ is even, $\eta_k^{g_k} = \eta_k^{g_{k-1}} {+} 1$ when $k$ is odd, as in [LS]. For the positive integer $m$ satisfying $g_m \le \ell < g_{m+1}$, we have that $\beta < \gamma_m < \eta_m$. By Proposition 3.5 in [LS] and its proof, $\beta < \eta_m$ implies that the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ starts with $\varphi^m(1)$ and that $\widetilde{T}_{-\beta}^j(1) \notin \{0,1\}$ for all $1 \le j \le |\varphi^m(1)| = g_{m+1} + \frac{1-(-1)^m}{2}$, where $|w|$ denotes the length of the word $w$. Since $\beta \in J_n$ and $P_n(\beta) \in \{0,1\}$, we obtain that $a_1 a_2 \cdots$ starts with $\varphi^m(1)$ and that $n > |\varphi^m(1)|$. By equation (3.2) in [LS], we have that $P_{2^m}(x) > 1$ for all $x > \eta_m$ (note that $2^m = |\varphi^{m-1}(10)| < |\varphi^m(1)|$), thus $J_{2^m} = (1, \eta_m]$, and $\ell < g_{m+1}$ yields that $\beta' = \eta_m$, $\ell = 2^m$. As $\beta$ and $\beta'$ are in the interval $J_{n-1}$, we also have that $\gamma_m \in J_{n-1}$. The $(-\gamma_m)$-expansion of $\frac{-\gamma_m}{\gamma_m+1}$ is $\varphi^{m-1}(1)\, \overline{\varphi^{m-1}(0)}$ by [LS, Theorem 2.5]. Since $n \ge 2\ell$ by the above block decomposition of $a_{[1,n]}$, we obtain that $a_1 a_2 \cdots$ starts with $\varphi^{m-1}(1000)$ if $m \ge 2$, and with 100 if $m = 1$. In case $m = 1$, we get that $P_3(2) \notin J_3$, contradicting that $2 = \eta_1 = \beta' \in J_n$. For $m \ge 2$, we have that $P_{|\varphi^{m-1}(1000)|}(\eta_m) > P_{|\varphi^{m-1}(10)|}(\eta_m) = 1$ because $P_{|\varphi^{m-1}(100)|}(\eta_m) = P_{|\varphi^{m}(1)|}(\eta_m) < P_{|\varphi^{m-1}(1)|}(\eta_m)$ by equation (3.4) in [LS] and, using the notation of [LS], the function $f_{\gamma_m, \varphi^{m-1}(0)}$ is order-reversing. Again, this contradicts that $\eta_m = \beta' \in J_n$. Therefore, we have shown that $\beta = \beta'$ whenever $P_n(\beta) = P_n(\beta') \in \{0,1\}$, $\beta, \beta' \in J_n$. Hence, $J_n$ is an interval, and (1)–(5) hold for all $n \ge 1$.

As the $J_n$ form a sequence of nested non-empty intervals that are compact for sufficiently large $n$, we have that $\bigcap_{n\ge1} J_n \neq \emptyset$, thus there exists some $\beta > 1$ satisfying (1.9), which is equivalent to (1.6). To show that $\beta$ is unique, suppose that $\bigcap_{n\ge1} J_n$ is not a single point. Then $\bigcap_{n\ge1} J_n$ is an interval of positive length, thus there exist $\beta,\beta'\in\bigcap_{n\ge1} J_n$, $\beta\ne\beta'$, such that $P_n(\beta)\in(0,1]$ and $P_n(\beta')\in(0,1]$ for all $n\ge1$. This means that $a_1a_2\cdots$ is both the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ and the $(-\beta')$-expansion of $\frac{-\beta'}{\beta'+1}$, which contradicts that $\beta\ne\beta'$ by Theorem 3. This concludes the proof of Theorem 1.

*Remark* 1. Some parts of the proofs of Theorems 1 and 3 can be simplified when one is only interested in $\beta>1$ not too close to 1. Since $P_n(x)=a_n+1-xP_{n-1}(x)$ for $n\ge2$, and $P_1'(x)=-1$, the derivative of $P_n(x)$ is

$$P_n'(x)=(-1)\left(P_{n-1}(x)+xP_{n-1}'(x)\right)=\cdots=(-1)^n x^{n-1}\left(1+\sum_{j=1}^{n-1}\frac{P_j(x)}{(-x)^j}\right).$$

If $x\in J_{n-1}$, then $1+\sum_{j=1}^{n-1}\frac{P_j(x)}{(-x)^j}>1-\frac{1}{x}-\frac{1}{x^3}-\cdots=\frac{x^2-x-1}{x^2-1}$. If moreover $x\ge(1+\sqrt{5})/2$, then we get that $(-1)^nP_n'(x)>0$, hence $P_n$ is a strictly increasing (decreasing) function on $J_{n-1}\cap[(1+\sqrt{5})/2,\infty)$ when $n$ is even (odd). Moreover, $\lim_{n\to\infty}|P_n'(x)|=\infty$ if $x\ge(1+\sqrt{5})/2$ and $x\in J_n$ for all $n\ge1$.

However, it is not true that $P_n$ is always increasing (decreasing) on $J_{n-1}$ when $n$ is even (odd). For instance, if $a_1a_2\cdots$ starts with 1001, then $P_4(x)=x^4-2x^3+x^2-x+2$ and $J_3=(1,\beta]$ with $\beta^3=2\beta^2-\beta+1$ ($\beta\approx1.755$). The function $P_4$ decreases on $(1,\beta']$, with $\beta'\approx1.261$, and increases on $[\beta',\infty)$. Note that this is a major flaw in the proof of Theorem 28 of [Gór07] (besides the fact that the statement is incorrect, as explained in the Introduction). This lack of monotonicity is what makes Theorems 1 and 3 more difficult to prove than the corresponding statements for $\beta$-expansions.

## 4. Proof of Theorem 2

Let $a_1a_2\cdots$ be a sequence of non-negative integers satisfying (1.4) and (1.5). We have already seen in the Introduction that these conditions are necessary to be the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta>1$. Moreover, $\beta$ can only be the number given by Theorem 1. Then $a_1a_2\cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ if and only if $\sum_{j=1}^\infty\frac{a_{k+j}}{(-\beta)^j}\ne\frac{1}{\beta+1}$ for all $k\ge1$.

Suppose first that $\sum_{j=1}^\infty\frac{a_{k+j}}{(-\beta)^j}=\frac{1}{\beta+1}$ for some $k\ge1$, and let $\ell\ge1$ be minimal such that $\sum_{j=1}^\infty\frac{a_{\ell+j}}{(-\beta)^j}\in\left\{\frac{-\beta}{\beta+1},\frac{1}{\beta+1}\right\}$. If $\sum_{j=1}^\infty\frac{a_{\ell+j}}{(-\beta)^j}=\frac{-\beta}{\beta+1}$, then the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ is $\overline{a_{[1,\ell]}}$. Then $a_1a_2\cdots$ is composed of blocks $a_{[1,\ell]}$ and $a_{[1,\ell)}(a_\ell-1)0$. Since $\sum_{j=1}^\infty\frac{a_{k+j}}{(-\beta)^j}=\frac{1}{\beta+1}$ for some $k\ge1$, we have at least one block $a_{[1,\ell)}(a_\ell-1)0$, i.e., $a_1a_2\cdots\in\{a_{[1,\ell]},\,a_{[1,\ell)}(a_\ell-1)0\}^\omega\setminus\{\overline{a_{[1,\ell]}}\}$. As $\overline{a_{[1,\ell]}}$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$, we have that $\overline{a_{[1,\ell]}}>_{\mathrm{alt}}u_1u_2\cdots$, thus (1.7) does not hold. If $\sum_{j=1}^\infty\frac{a_{\ell+j}}{(-\beta)^j}=\frac{1}{\beta+1}$, then the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ is $\overline{a_{[1,\ell)}(a_\ell+1)}$, $a_1a_2\cdots$ is composed of blocks $a_{[1,\ell]}0$ and $a_{[1,\ell)}(a_\ell+1)$, and we have that $\overline{a_{[1,\ell)}(a_\ell+1)}>_{\mathrm{alt}}u_1u_2\cdots$, thus (1.8) does not hold. Therefore, (1.4), (1.5), (1.7), and (1.8) imply that $a_1a_2\cdots$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some (unique) $\beta>1$.

Suppose now that (1.7) does not hold, i.e., $a_1a_2\cdots\in\{a_{[1,k]},\,a_{[1,k)}(a_k-1)0\}^\omega\setminus\{\overline{a_{[1,k]}}\}$ for some $k\ge1$ with $\overline{a_{[1,k]}}>_{\mathrm{alt}}u_1u_2\cdots$. We show that the sequence $\overline{a_{[1,k]}}$ satisfies (1.4).

Suppose on the contrary that $a_{[j,k]}\,\overline{a_{[1,k]}} >_{\rm alt} \overline{a_{[1,k]}}$ for some $2 \le j \le k$. This implies that $a_{[j,k]}\,a_{[1,j)} >_{\rm alt} a_{[1,k]}$. Since $a_{[k+1,2k)} = a_{[1,k)}$, we obtain that $a_{[j,j+k)} = a_{[j,k]}\,a_{[1,j)} >_{\rm alt} a_{[1,k]}$, thus $a_j a_{j+1} \cdots >_{\rm alt} a_1 a_2 \cdots$, contradicting that $a_1 a_2 \cdots$ satisfies (1.4). Therefore, $\overline{a_{[1,k]}}$ satisfies (1.4) and (1.5), and we can apply Theorem 1 for this sequence. Let $\beta' > 1$ be the number satisfying (1.6) for the sequence $\overline{a_{[1,k]}}$. Then $\beta'$ also satisfies (1.6) for the original sequence $a_1 a_2 \cdots$, thus $\beta' = \beta$. Therefore, $a_1 a_2 \cdots$ is not the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$.

Suppose finally that (1.8) does not hold, i.e., $a_1 a_2 \cdots \in \{a_{[1,k]}0,\, a_{[1,k)}(a_k{+}1)\}^\omega$ for some $k \ge 1$ with $\overline{a_{[1,k)}(a_k{+}1)} >_{\rm alt} u_1 u_2 \cdots$. If $a_1 a_2 \cdots = \overline{a_{[1,k]}0}$, then $\sum_{j=1}^\infty \frac{a_{k+j}}{(-\beta)^j} = \frac{1}{\beta+1}$, thus $a_1 a_2 \cdots$ is not the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$. If $a_1 a_2 \cdots \ne \overline{a_{[1,k]}0}$, then we show that the sequence $\overline{a_{[1,k)}(a_k{+}1)}$ satisfies (1.4). Suppose that $a_{[j,k)}(a_k{+}1)\,\overline{a_{[1,k)}(a_k{+}1)} >_{\rm alt} \overline{a_{[1,k)}(a_k{+}1)}$ for some $2 \le j \le k$. This implies that $a_{[j,k)}(a_k{+}1)a_{[1,j)} >_{\rm alt} a_{[1,k]}$. Since $a_{[j,k)}(a_k{+}1)a_{[1,j)} = a_{[\ell,\ell+k)}$ for some $\ell \ge 2$, we have that $a_\ell a_{\ell+1} \cdots >_{\rm alt} a_1 a_2 \cdots$, contradicting that $a_1 a_2 \cdots$ satisfies (1.4). As in the preceding paragraph, the number given by Theorem 1 for the sequence $\overline{a_{[1,k)}(a_k{+}1)}$ is $\beta$, thus $a_1 a_2 \cdots$ is not the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$. Therefore, (1.7) and (1.8) are necessary for $a_1 a_2 \cdots$ to be the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta > 1$.


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