Equivalence Between Model-Checking Flat Counter Systems and Presburger Arithmetic[*]

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Abstract. We show that model-checking flat counter systems over CTL*
(with arithmetical constraints on counter values) has the same complex-
ity as the satisfiability problem for Presburger arithmetic. The lower
bound already holds with the temporal operator EF only, no arithmeti-
cal constraints in the logical language and with guards on transitions
made of simple linear constraints. This complements our understanding
of model-checking flat counter systems with linear-time temporal log-
ics, such as LTL for which the problem is already known to be (only)
NP-complete with guards restricted to the linear fragment.

1 Introduction

Branching-time temporal logics for counter systems. At first glance, model-
checking counter systems with temporal logics seems a hopeless enterprise since
control-state reachability problem for Minsky machines is already known to be
undecidable [23]. Fortunately, many subclasses of counter systems admit a de-
cidable reachability problem and more importantly, sometime the reachability
sets are even definable in Presburger arithmetic (PA) [24]. That is why, model-
checking problems with temporal logics for one-counter automata [15,16], Petri
nets [17], reversal-bounded counter systems [2], flat counter systems [14] have
been considered. The previous list is certainly not exhaustive and a general ques-
tion is how to take advantage of the decidability of the reachability problem to
conclude the decidability of model-checking problems with temporal logics. This
can lead to endless studies, since the variety of subclasses of counter systems
and temporal logics is extremely rich. By way of example, reachability sets of
flat counter systems are known to be definable in (PA), see e.g. [3,6,14,4], but it
is unclear how this can be extended to model-checking problems with temporal
logics, which indeed, is done in [10] for flat counter systems. A complexity char-
acterization for model-checking linear-time properties is provided in [9]. In the
present paper, we study flat counter systems and branching-time temporal log-
ics, more specifically with a variant of CTL* [12], already known to be difficult
to mechanize in the propositional case with labelled transition systems.

Our motivations We have seen that reachability problems and the verification of linear-time properties for flat counter systems are nowadays well-studied (see e.g. [9,4]) and in this paper we wish to understand the computational complexity for branching-time temporal logics such as CTL or CTL* (see e.g. [12]). Branching-time extensions often lead to undecidability, see e.g. the case with Petri nets for which CTL is undecidable (with propositional variables only) whereas the reachability problem and model-checking for several LTL variants are known to be decidable [17]. For flat counter systems, we are on the safe side since decidability of model-checking CTL* formulae is established in [10] but no lower bound is provided in [10] and the translation into (PA) gives an exponential size formula, which is rather unsatisfactory. Our main motivation is therefore to understand the complexity of model-checking flat counter systems with branching-time logics so that optimal algorithms for model-checking can be eventually designed.

Our contribution. We show that the model-checking problem for flat counter systems over a version of CTL* with arithmetical constraints on counter values is equivalent to satisfiability for (PA), modulo logarithmic-space reductions.

  – For the complexity lower bound, we show that the satisfiability problem for (PA) can be reduced to the model-checking problem but there is no need for arithmetical constraints and for temporal operators other than EF.
  – For the complexity upper bound, we reduce the model-checking problem to satisfiability problem in (PA) by using the fact that runs in flat counter systems can be encoded by tuples of natural numbers and then the semantics for CTL* can be internalized in (PA). This very fact has been already observed in [10] but herein, we provide a logarithmic-space reduction which makes a substantial difference with [10]. Indeed, we are also able to quantify over path schemas (symbolic representation of potential infinite sets of runs), but concisely. This witnesses once more, that verification problems can be encoded efficiently to (PA), see e.g. [5].
  – As a consequence, we are able to get the equivalence with (PA) to known branching-time temporal logics stronger than CTL$_{EF}$ (such as CTL) and our proof technique can be applied to extensions with past-time operators with a minimal amount of change.

As far as proofs are concerned, for the lower bound, we take advantage of the observation that a quantification in (PA) over a variable z can be simulated by a loop that increments a corresponding counter and there is a correspondence between first-order quantifier $\exists$ [resp. $\forall$] and temporal connective EF [resp. AG]. For the upper bound, quantification over path schemas is done, directly followed by a quantification over the number of times loops are visited. However, we provide a new way to encode runs in flat counter systems, which is rewarding complexity-wise. Not only we provide a much better complexity characterization than [10] but also our reduction into (PA) is much simpler, and therefore this leaves some hope to use then some solvers for (PA), see e.g. [7,20].
Due to lack of space, omitted proofs can be found in [11].

2 Branching-time temporal logics on flat counter systems

Presburger arithmetic. (PA), i.e. the first-order theory of natural numbers with addition, is introduced by M. Presburger who has shown decidability by quantifier elimination. Let $\text{VAR} = \{z_1, z_2, z_3, \ldots\}$ be a countably infinite set of *variables*. *Terms* are expressions of the form $a_1 z_1 + \cdots + a_n z_n + k$ where a_1, $\ldots$, a_n, k are in $\mathbb{N}$. A *valuation* f is a map $\text{VAR} \to \mathbb{N}$ and it can be extended to the set of all terms as follows: $f(k) = k$, $f(az) = a \times f(z)$ and $f(t + t') = f(t) + f(t')$ for all terms t and t'. *Formulae* are defined by the grammar $\phi ::= \ t \leq t' \ | \ \neg\phi \ | \ \phi \wedge \phi \ | \ \exists\, z\, \phi$ where t and t' are terms and $z \in \text{VAR}$. A formula ϕ is in the *linear fragment* $\stackrel{\text{def}}{\Leftrightarrow} \phi$ is a Boolean combination of atomic formulae of the form $t \leq t'$. The semantics for formulae in (PA) is defined with the satisfaction relation $\models$: for instance $f \models t \leq t' \ \stackrel{\text{def}}{\Leftrightarrow} \ f(t) \leq f(t')$ and $f \models \exists\, z\, \phi \ \stackrel{\text{def}}{\Leftrightarrow}$ there is $n \in \mathbb{N}$ such that $f[z \mapsto n] \models \phi$. Any formula $\phi(z_1, \ldots, z_n)$ whose free variables are among $z_1, \ldots, z_n$, with $n \geq 1$, defines a set of n-tuples $[\![\phi(z_1, \ldots, z_n)]\!] \stackrel{\text{def}}{=} \{\langle f(z_1), \ldots, f(z_n)\rangle \in \mathbb{N}^n : f \models \phi\}$, in that case for a vector $\mathbf{v} \in \mathbb{N}^n$, we will also write $\mathbf{v} \models \phi$ for $\mathbf{v} \in [\![\phi(z_1, \ldots, z_n)]\!]$. For a given PA formula ϕ, the set of free variables of ϕ is denoted by $\texttt{free}(\phi)$. The *satisfiability problem* for (PA) is a decision problem that takes as input a formula ϕ and asks whether there is a valuation f such that $f \models \phi$. If such a valuation exists, we say that ϕ is *satisfiable*. Decidability of Presburger arithmetic has been shown in [24]. An exact complexity characterization is provided in [1].

Counter systems. Let $\texttt{C} = \{x_1, x_2, \ldots\}$ be a countably infinite set of *counters* with the finite subset $\{x_1, \ldots, x_n\}$ denoted as $\texttt{C}_n$ and $\text{AT} = \{p_1, p_2, \ldots\}$ be a countably infinite set of atomic propositional variables. A *counter system* is a tuple $\langle Q, \texttt{C}_n, \Delta, \ell \rangle$ where Q is a finite set of *control states*, $\ell : Q \to 2^{\text{AT}}$ is a *labelling function*, $\Delta \subseteq Q \times G_n \times \mathbb{Z}^n \times Q$ is a finite set of edges labelled by *guards* and *updates* of the counter values *(transitions)* where G_n is a finite set of Presburger formulae ϕ with $\texttt{free}(\phi) \subseteq \{x_1, \ldots, x_n\}$. Guards are quite general and we basically only need them in the linear fragment. However, since we provide a translation into (PA), we can be a bit more general, as in Presburger counter machines [10,19].

For each transition $\delta = \langle q, \mathbf{g}, \mathbf{u}, q'\rangle$ in Δ, we use the following notations: $source(\delta) = q$, $target(\delta) = q'$, $guard(\delta) = \mathbf{g}$ and $update(\delta) = \mathbf{u}$. As usual, to a counter system $S = \langle Q, \texttt{C}_n, \Delta, \ell \rangle$, we associate a labelled transition system $\mathfrak{T}(S) = \langle \mathcal{C}, \to \rangle$ where $\mathcal{C} = Q \times \mathbb{N}^n$ is the set of *configurations* and $\to \subseteq \mathcal{C} \times \Delta \times \mathcal{C}$ is the *transition relation* defined by: $\langle\langle q, \mathbf{v}\rangle, \delta, \langle q', \mathbf{v}'\rangle\rangle \in \to$ (also written $\langle q, \mathbf{v}\rangle \xrightarrow{\delta} \langle q', \mathbf{v}'\rangle$) iff $q = source(\delta)$, $q' = target(\delta)$, $\mathbf{v} \models guard(\delta)$ and $\mathbf{v}' = \mathbf{v} + update(\delta)$. We write $c \to c'$ iff there exists some edge δ, such that $c \xrightarrow{\delta} c'$.

Given $c_0 \in Q \times \mathbb{N}^n$, a *run* ρ starting from c_0 in S is a (possibly infinite) path in the associated transition system $\mathfrak{T}(S)$ denoted as $\rho := c_0 \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{m-1}} c_m \xrightarrow{\delta_m} \cdots$, where $c_i \in Q \times \mathbb{N}^n$ and $\delta_i \in \Delta$, for all $i \in \mathbb{N}$.

Let $trans(\rho)$ be the ω-word $\delta_0 \delta_1 \ldots$ of the sequence of transitions appearing in the run ρ. For every $i \geq 0$, we define $\rho(i) = c_i$ and $\rho_{\leq i} = c_0 \xrightarrow{\delta_0} c_1 \cdots \xrightarrow{\delta_{i-1}} c_i$. Also, we say $c \rightarrow^* c'$ iff there exist a run ρ and $i \in \mathbb{N}$ such that $\rho(0) = c$ and $\rho(i) = c'$. Note that a run in a counter system S is either finite or infinite. A run ρ is *maximal* iff either it is infinite, or it is finite and the last configuration is a deadlock (i.e. with no successor configurations). For a finite maximal run $\rho := c_0 \xrightarrow{\delta_0} \cdots \xrightarrow{\delta_{m-1}} c_m \xrightarrow{\delta_m} c_{m+1}$, we write $|\rho| = m$, otherwise for an infinite maximal run ρ, $|\rho| = \omega$.

A counter system is *flat* if every node in the underlying graph belongs to at most one simple cycle (a cycle being simple if no edge is repeated twice in it) [6,22]. In a flat counter system, simple cycles can be organized as a DAG where two simple cycles are in the relation whenever there is path between a node of the first cycle and a node of the second cycle. We denote by FlatCS the class of flat counter systems.

Logical specifications. The formulae for CTL* are defined as follows: $\phi ::= \texttt{p} \mid \psi(\texttt{x}_1, \ldots, \texttt{x}_n) \mid \phi \wedge \phi \mid \neg \phi \mid \texttt{X}\phi \mid \phi \texttt{U} \phi \mid \texttt{E}\phi$ where $\texttt{p} \in \text{AT}$ and $\psi(\texttt{x}_1, \ldots, \texttt{x}_n)$ is a Presburger formula. We write CTL$_{\text{EF}}$ to denote the fragment of CTL* in which the only (unary) temporal operator is EF (EF$\phi \stackrel{\text{def}}{=} \texttt{E}\ (\top\ \texttt{U}\ \phi)$ and $\top \stackrel{\text{def}}{=} (\texttt{x}_1 = \texttt{x}_1)$). Our version of CTL* is defined as the standard version, see e.g. [12], except that the Kripke structures are replaced by transition systems from counter systems and at the atomic level, arithmetical constraints are allowed. Let $S = \langle Q, \texttt{C}_n, \Delta, \ell \rangle$ be a counter system with transition system $\mathfrak{T}(S) = \langle \mathcal{C}, \rightarrow \rangle$. The satisfaction relation $\models$ is defined as follows (CTL* formula ϕ, maximal run ρ in $\mathfrak{T}(S)$, position $i < |\rho|$):

$$\rho, i \models \texttt{p} \qquad \stackrel{\text{def}}{\Leftrightarrow}\ \texttt{p} \in \ell(q), \text{ where } \rho(i) = \langle q, \mathbf{v} \rangle$$
$$\rho, i \models \psi(\texttt{x}_1, \ldots, \texttt{x}_n) \stackrel{\text{def}}{\Leftrightarrow}\ \mathbf{v} \models \psi(\texttt{x}_1, \ldots, \texttt{x}_n), \text{ where } \rho(i) = \langle q, \mathbf{v} \rangle$$
$$\rho, i \models \texttt{X}\psi \qquad \stackrel{\text{def}}{\Leftrightarrow}\ \rho, i+1 \models \psi \text{ and } i+1 < |\rho|$$
$$\rho, i \models \psi_1 \texttt{U} \psi_2 \qquad \stackrel{\text{def}}{\Leftrightarrow}\ \rho, j \models \psi_2 \text{ for some } i \leq j$$
$$\qquad\qquad\qquad\qquad \text{such that } j < |\rho| \text{ and } \rho, k \models \psi_1 \text{ for all } i \leq k < j$$
$$\rho, i \models \texttt{E}\phi \qquad \stackrel{\text{def}}{\Leftrightarrow}\ \text{there is a maximal run } \rho' \text{ s.t. } \rho'(0) = \rho(i) \text{ and } \rho', 0 \models \phi$$

Given a CTL* formula ϕ, a counter system S and a configuration c from S, we say that $S, c \models \phi$ iff there exists a maximal run ρ in the configuration graph $\mathfrak{T}(S)$ with $\rho(0) = c$ such that $\rho, 0 \models \phi$ (note that there is an overload for $\models$ in $S, c \models \phi$). A flat counter system S is called *non-blocking* if every maximal run ρ in S is infinite. Otherwise it is called a *blocking* flat counter system.

Lemma 1. *Let* L *be either* CTL* *or* CTL$_{\text{EF}}$. *Given a flat counter system S, a configuration c and a formula ϕ in* L*, there exist a non-blocking flat counter system S', a configuration c' and a formula ϕ' in* L *such that $S, c \models \phi$ iff $S', c' \models \phi'$. Such a reduction can be performed in logarithmic space.*

It is easy to see that we can add some dummy state to a blocking flat counter system to obtain a non-blocking one preserving the satisfiability of formulae.

The formula is also transformed by using a standard relativization method (a new propositional variable is introduced that holds true only on configurations that were not reachable in the original counter system). Due to Lemma 1, henceforth we consider only non-blocking flat counter systems. Since the reachability relation is definable in (PA) for flat counter systems [10], it is even possible to decide whether all maximal runs from a given configuration are infinite.

The model-checking problem for flat counter systems over CTL* is defined as follows (let us call it MC(CTL*, FlatCS)): given a flat counter system S, a configuration c and a formula ϕ in CTL*, determine whether $S, c \models \phi$. We know that MC(CTL*, FlatCS) is decidable [10] but its exact complexity is not fully characterized (actually, this is the purpose of the present paper). The restriction to LTL formulae is known to be NP-complete [8] when guards are restricted to the linear fragment. In Section 3, we show that the satisfiability problem for (PA) can be reduced to MC(CTL*, FlatCS) restricted to CTL$_{\text{EF}}$ without arithmetical constraints and to flat counter systems such that the guards are restricted to simple linear constraints. By contrast, model-checking flat finite Kripke structures over CTL* is Δ_2^P-complete [13,18].
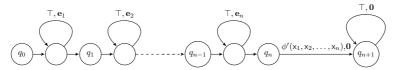
3   Reducing (PA) to a subproblem of MC(CTL*, FlatCS)

In a flat counter system with n counters, the guards on transitions are Presburger formulae with free variables in $\{x_1, \ldots, x_n\}$. That is why, it is easy to show that the satisfiability problem for (PA) can be reduced (in logarithmic space) to MC(CTL*, FlatCS). Clearly, this is not interesting and the generality of the guards in the paper is only considered because, when establishing the complexity upper bound, we can be quite liberal with the set of guards. However, a more reasonable set of guards is the linear fragment (i.e., without any first-order quantification). Below, we show that a very restricted fragment of MC(CTL*, FlatCS), simply called MC$^-$(CTL*, FlatCS), is already as hard as the satisfiability problem for (PA) and our reduction is based on a simple and nice correspondence between quantifiers in (PA) and the temporal operators EF and AG in CTL*. First, let us define MC$^-$(CTL*, FlatCS) as the subproblem of MC(CTL*, FlatCS) with the following restrictions: (a) atomic formulae are restricted to propositional variables and the only temporal connective is EF (and its dual AG, by closure under negation) and (b) the guards on the transitions are linear constraints $t \leq t'$ or their negations.

Theorem 2. *There is a logarithmic-space reduction from the satisfiability problem for (PA) to MC$^-$(CTL*, FlatCS).*

Proof. (sketch) Let ϕ be a formula in (PA). Without any loss of generality, we can assume that ϕ has the form $\mathcal{Q}_1 \, z_1 \, \mathcal{Q}_2 \, z_2 \, \cdots \, \mathcal{Q}_n \, z_n \, \phi'(z_1, z_2, \ldots, z_n)$ with $\mathcal{Q}_1, \mathcal{Q}_2, \ldots, \mathcal{Q}_n \in \{\exists, \forall\}$ and ϕ' is a quantifier-free formula. Note that given any formula in (PA), we can reduce it to an equisatisfiable formula of that form in logarithmic space (which is then fine for our main result since logarithmic-space reductions are closed under composition). This is essentially based on the construction of formulae in prenex normal form in first-order logic.

Let us consider S_ϕ defined below where $\mathbf{e}_i \in \mathbb{N}^n$ is the ith unit vector.

Observe that $\phi'(\mathsf{x}_1, \mathsf{x}_2, \ldots, \mathsf{x}_n)$ may contain Boolean connectives but we explain below how to get rid of them in S_ϕ. Below, we define ψ in $\mathrm{CTL}_{\mathsf{EF}}$ whose atomic formulae are among $q_1, \ldots, q_{n+1}$ (also abusively understood as control states) such that (†) $S_\phi, \langle q_0, \mathbf{0} \rangle \models \psi$ iff ϕ is satisfiable in (PA). Intuitively, the possible value associated to each variable z_i from ϕ is taken care of by the ith loop (that can only increment the ith counter). This is not enough, and additionally, the quantifications from ϕ are simulated in the formula ψ by using EF or AG, depending whether the first-order quantification is either existential or universal. Let us define below the formulae ψ_i with $i \in [1, n+1]$ so that $\psi \stackrel{\text{def}}{=} \psi_1$, $\psi_{n+1} \stackrel{\text{def}}{=}$ $\mathsf{EF}\ q_{n+1}$, and for every $i \in [1, n]$, if $\mathcal{Q}_i = \exists$ then $\psi_i \stackrel{\text{def}}{=} \mathsf{EF}(q_i \wedge \psi_{i+1})$, otherwise $\psi_i \stackrel{\text{def}}{=} \mathsf{AG}(q_i \Rightarrow \psi_{i+1})$. In order to establish (†) it is sufficient to show the property (††) below. Given a valuation $f : \mathrm{VAR} \to \mathbb{N}$, we write $\mathbf{v}_f \in \mathbb{N}^n$ to denote the vector such that $\mathbf{v}_f[i] \stackrel{\text{def}}{=} f(\mathsf{z}_i)$ for every $i \in [1, n]$. One can show that (††) for all f, we have $f \models \phi'(\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_n)$ iff $\langle q_n, \mathbf{v}_f \rangle \models \psi_{n+1}$ and for all $i \in [1, n]$ and for all valuations f such that $f(\mathsf{z}_i) = \cdots = f(\mathsf{z}_n) = 0$, we have $f \models \mathcal{Q}_i\ \mathsf{z}_i\ \cdots\ \mathcal{Q}_n\ \mathsf{z}_n\ \phi'(\mathsf{z}_1, \mathsf{z}_2, \ldots, \mathsf{z}_n)$ iff $\langle q_{i-1}, \mathbf{v}_f \rangle \models \psi_i$. We get the property (†) by applying (††) with $i = 1$.

To eliminate the Boolean connectives in the guard of the transition between q_n and q_{n+1}, we follow two simple rules, while preserving flatness (easy to check since that transition does not belong to a loop). W.l.o.g., we can assume that negations are only in front of linear constraints. A transition $q \xrightarrow{\psi_1 \wedge \psi_2, \mathbf{0}} q'$ is replaced by $q \xrightarrow{\psi_1, \mathbf{0}} q'' \xrightarrow{\psi_2, \mathbf{0}} q'$ where q'' is new. Similarly, a transition $q \xrightarrow{\psi_1 \vee \psi_2, \mathbf{0}} q'$ is replaced by $q \xrightarrow{\psi_1, \mathbf{0}} q'$ and $q \xrightarrow{\psi_2, \mathbf{0}} q'$, assuming that q does not belong to a loop. It is easy to show that S_ϕ can be transformed into a flat counter system S'_ϕ by applying the two rules above as much as possible so that eventually, S'_ϕ is a proper counter system for $\mathrm{MC}^-(\mathrm{CTL}^*, \mathtt{FlatCS})$. □

4 From MC(CTL*, FlatCS) to (PA)

In this section, we present a logarithmic-space reduction from $\mathrm{MC}(\mathrm{CTL}^*, \mathtt{FlatCS})$ to the satisfiability problem for (PA). In [10], a reduction is already presented to get decidability of $\mathrm{MC}(\mathrm{CTL}^*, \mathtt{FlatCS})$. Unfortunately, it requires exponential space and is quite difficult to parse. Following a similar idea, we propose here a simpler reduction that has the great advantage to be optimal complexity-wise. The idea of this reduction is based on the two following points:

1. encoding the runs in flat counter systems by tuples of natural numbers thanks to a symbolic representation for potential infinite sets of runs, see path schemas in [8],

2. internalizing CTL* semantics into (PA) by using the encoding of runs.

Below, we consider a fixed flat counter system $S = \langle Q, \mathsf{C}_n, \Delta, \ell \rangle$ and w.l.o.g, $Q = \{1, \ldots, \alpha\}$ for some $\alpha \geq 1$ and $\Delta = \{\delta_1, \ldots, \delta_\beta\}$. Since $Q \subseteq \mathbb{N}$, configurations of S are vectors in $\mathbb{N}^{n+1}$ where the first component represents the control state.

4.1 Minimal path schemas

In [8], following an idea from [21], *minimal path schemas* are introduced as a mean to symbolically represent all runs in flat counter systems. Path schemas can be defined as finite sequences made of transitions or simple loops (conditions apply). Formal definition is recalled below. A *simple loop l* of S is a non-empty finite sequence of transitions $\delta_1, \ldots, \delta_m$ such that $source(\delta_1) = target(\delta_m)$, $source(\delta_j) = target(\delta_{j+1})$ for all $j \in [1, m-1]$, and, for all $j, k \in [1, m]$, if $j \neq k$ then $\delta_j \neq \delta_k$. The length of l, written $length(l)$, is the value m and we denote by $source(l) = target(l)$ the control state $source(\delta_1)$. The number of simple loops is necessarily finite and we assume that the set of loops of S is $L = \{l_1, l_2, \ldots, l_\gamma\}$. Since S is flat, we have $\gamma \leq \alpha$. A *minimal path schema P* is a non-empty sequence $u_1, \ldots, u_N$ s.t. each $u_i \in \Delta \cup L$ and the following conditions are verified.

1. u_N is a loop,
2. $i \neq j$ implies $u_i \neq u_j$,
3. for all $i \in [1, N-1]$, we have $target(u_i) = source(u_{i+1})$.

The second condition guarantees minimality whereas the third condition ensures that P respects the control graph of S. The *size* of P, denoted by $size(P)$, is equal to N. For all $j \in [1, N]$, we write $P[j]$ to denote u_j. Here is an obvious result.

Lemma 3. *The size of a minimal path schema is bounded by $\beta + \gamma \leq \beta + \alpha$.*

In order to obtain concrete paths from a path schema P, we augment P with a vector specifying how many times each internal loop is visited. By definition, a loop in P is *internal* if it is not the last one. An *iterated path schema* is a pair $\langle P, \mathbf{m} \rangle$ where P is a minimal path schema and $\mathbf{m} \in \mathbb{N}^{size(P)}$ such that $\mathbf{m}[1] = size(P)$ and for all $i \in [1, size(P) - 1]$, $\mathbf{m}[i+1] > 0$ and if $P[i] \in \Delta$, then $\mathbf{m}[i+1] = 1$. From $\langle P, \mathbf{m} \rangle$, we define the ω-word

$$trans(P, \mathbf{m}) \overset{\text{def}}{=} P[1]^{\mathbf{m}[2]} \ldots P[j]^{\mathbf{m}[j+1]} \ldots P[size(P) - 1]^{\mathbf{m}[size(P)]} P[size(P)]^\omega$$

Lemma 4 below states that iterated path schemas encode all runs from flat counter systems by noting that infinite runs necessarily end by a simple loop (repeated infinitely) and the visit of loops is strictly ordered.

Lemma 4. *[8] Given an infinite run ρ in a flat counter system S, there exists an iterated path schema $\langle P, \boldsymbol{m} \rangle$ such that $trans(\rho) = trans(P, \boldsymbol{m})$.*

Encoding iterated path schemas. Thanks to Lemma 3, we show that it is possible to encode path schemas by vectors in $\mathbb{N}^K$ with $K = 1 + \beta + \gamma$. Intuitively, we encode a path schema P by two vectors $\mathbf{v}_p$ and $\mathbf{v}_t$ in $\mathbb{N}^K$ where the first element of each vector is equal to $size(P)$ and for all $i \in [2, size(P)+1]$, we have $\mathbf{v}_t[i] = 1$ if $P[i]$ is a loop and $\mathbf{v}_t[i] = 0$ otherwise. So, $\mathbf{v}_t$ encodes the *type* of each element (transition vs. loop) in the sequence defining P. Similarly, $\mathbf{v}_p[i]$ represents the number of the associated transition or loop; for instance, $\mathbf{v}_p[i] = 2$ and $\mathbf{v}_t[i] = 1$ encodes that $P[i]$ is the second loop, say l_2. Furthermore, we encode the vector $\mathbf{m}$ by a vector $\mathbf{v}_{it} \in \mathbb{N}^K$. Let us formalize this. First, we define the function $\tau : ((\{0\} \times [1, \beta]) \cup (\{1\} \times [1, \gamma])) \to \Delta \cup L$ such that $\tau(0, i) \overset{\text{def}}{=} \delta_i$ and $\tau(1, i) \overset{\text{def}}{=} l_i$. Now, we provide a set of conditions $\mathfrak{C}$ on the vectors $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^K$ which have to be respected so that, we can build from them an iterated path schema.

$\mathfrak{C}.1$ $\mathbf{v}_p[1] = \mathbf{v}_t[1] = \mathbf{v}_{it}[1]$ with $\mathbf{v}_p[1] \in [1, K-1]$; for all $i \in [\mathbf{v}_{it}[1] + 2, K]$, $\mathbf{v}_p[i] = \mathbf{v}_t[i] = \mathbf{v}_{it}[i] = 0$,

$\mathfrak{C}.2$ $\mathbf{v}_t[i] \in \{0, 1\}$ for all $i \in [2, K]$,

$\mathfrak{C}.3$ if $\mathbf{v}_t[i] = 0$ then $\mathbf{v}_p[i] \in [1, \beta]$, for all $i \in [2, \mathbf{v}_p[1] + 1]$,

$\mathfrak{C}.4$ if $\mathbf{v}_t[i] = 1$ then $\mathbf{v}_p[i] \in [1, \gamma]$, for all $i \in [2, \mathbf{v}_p[1] + 1]$,

$\mathfrak{C}.5$ $\mathbf{v}_t[\mathbf{v}_p[1] + 1] = 1$,

$\mathfrak{C}.6$ there are no $i, j \in [2, \mathbf{v}_p[1] + 1]$ such that $i \neq j$, $\mathbf{v}_t[i] = \mathbf{v}_t[j]$ and $\mathbf{v}_p[i] = \mathbf{v}_p[j]$,

$\mathfrak{C}.7$ $target(\tau(\mathbf{v}_t[i], \mathbf{v}_p[i])) = source(\tau(\mathbf{v}_t[i+1], \mathbf{v}_p[i+1]))$ for all $i \in [2, \mathbf{v}_p[1]]$,

$\mathfrak{C}.8$ for all $i \in [2, \mathbf{v}_p[1]]$, $\mathbf{v}_{it}[i] > 0$ and if $\mathbf{v}_t[i] = 0$ then $\mathbf{v}_{it}[i] = 1$.

The first four conditions ensure that the vectorial representation is coherent. The three next conditions guarantee that the encoding respects the structure of a minimal path schema, i.e. that the last element is a loop ($\mathfrak{C}.5$), that there are no two identical transitions or loops in the schema ($\mathfrak{C}.6$) and that the succession of elements effectively represents a path in the counter system ($\mathfrak{C}.7$). The last condition ensures that $\mathbf{v}_{it}$ matches the definition of the vector in an iterated path schema. It follows that given vectors $\mathbf{v}_p$, $\mathbf{v}_t$ and $\mathbf{v}_{it}$ in $\mathbb{N}^K$ that satisfy all the conditions ($\mathfrak{C}.i$), we can build a minimal path schema $P_{\mathbf{v}_t, \mathbf{v}_p}$ equal to
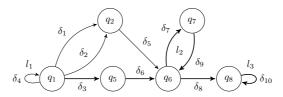
$$\tau(\mathbf{v}_t[2], \mathbf{v}_p[2]) \cdots \tau(\mathbf{v}_t[\mathbf{v}_p[1] + 1], \mathbf{v}_p[\mathbf{v}_p[1] + 1])$$

From the vector $\mathbf{v}_{it}$, we can define the vector $\mathbf{m}_{\mathbf{v}_{it}} \in \mathbb{N}^{\mathbf{v}_{it}[1]}$ such that for all $i \in [1, \mathbf{v}_{it}[1]]$, $\mathbf{m}_{\mathbf{v}_{it}}[i] \overset{\text{def}}{=} \mathbf{v}_{it}[i]$. We will see that there exists a Presburger formula $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ over the sets of variables $\mathsf{Z}_p = \{\mathsf{z}_p^1, \ldots, \mathsf{z}_p^K\}$, $\mathsf{Z}_t = \{\mathsf{z}_t^1, \ldots, \mathsf{z}_t^K\}$ and $\mathsf{Z}_{it} = \{\mathsf{z}_{it}^1, \ldots, \mathsf{z}_{it}^K\}$ to express the conditions $(\mathfrak{C}.i)_{i \in [1,8]}$.

Lemma 5.

1. *Let P be a finite non-empty sequence of length $N \leq \beta + \gamma$ over the alphabet $\Delta \cup L$ and $\boldsymbol{m} \in \mathbb{N}^N$. Then, $\langle P, \boldsymbol{m} \rangle$ is an iterated path schema iff there are $\boldsymbol{v}_t$, $\boldsymbol{v}_p$ and $\boldsymbol{v}_{it}$ in $\mathbb{N}^K$ respecting $\mathfrak{C}$ and such that $P = P_{\boldsymbol{v}_t, \boldsymbol{v}_p}$ and $\boldsymbol{m} = \boldsymbol{m}_{\boldsymbol{v}_{it}}$.*

2. *One can build a (PA) formula $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ of polynomial size in the size of the counter system S such that for all $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \in \mathbb{N}^K$, we have $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \models Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ iff $\boldsymbol{v}_t, \boldsymbol{v}_p$ and $\boldsymbol{v}_{it}$ satisfy $\mathfrak{C}$.*

Let us consider the following flat counter system.

There are 10 transitions and 3 simple loops. The enumeration of edges and loops is done as shown above. Let $\langle P, \mathbf{m}\rangle$ be such that $P = \delta_3 \cdot \delta_6 \cdot l_2 \cdot \delta_8 \cdot l_3$ and $\mathbf{m} = (5, 1, 1, 146, 1)$. So, we get the resulting ω-word as $\delta_3 \cdot \delta_6 \cdot (l_2)^{146} \cdot \delta_8 \cdot (l_3)^\omega$. From the previous encoding, the ω-word is encoded by vectors $\mathbf{v}_p = (5, 3, 6, 2, 8, 3, 0, 0, 0, 0, 0, 0, 0, 0)$, $\mathbf{v}_t = (5, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0)$ and $\mathbf{v}_{it} = (5, 1, 1, 146, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0)$.

4.2 Encoding runs using vectors

Lemma 4 states that any infinite run can be encoded by an iterated path schema. However, not every iterated path schema corresponds to a run in due form. We will see now how to check that an iterated path schema indeed represents a run starting from a given configuration c. First, we need to introduce a notion of *pseudo-run* in which only the updates are precise. Given $c \in Q \times \mathbb{Z}^n$, a *pseudo-run* ρ starting from c in S is an infinite sequence $\rho := c_0 \overset{\delta_1}{\leadsto} \cdots \overset{\delta_{m-1}}{\leadsto} c_m \overset{\delta_m}{\leadsto} \cdots$ where $c_0 = c$, $c_i = \langle q_i, \mathbf{v}_i\rangle \in Q \times \mathbb{Z}^n$ for all $i \geq 0$ and for all transitions $\delta \in \Delta$ we have $\langle q, \mathbf{v}\rangle \overset{\delta}{\leadsto} \langle q', \mathbf{v}'\rangle \overset{\text{def}}{\Leftrightarrow} q = source(\delta)$, $q' = target(\delta)$ and $\mathbf{v}' = \mathbf{v} + update(\delta)$. So, a pseudo-run $\rho = \langle q_0, \mathbf{v}_0\rangle \overset{\delta_1}{\leadsto} \cdots \overset{\delta_{m-1}}{\leadsto} \langle q_m, \mathbf{v}_m\rangle \cdots$ is a run iff for all $i \in \mathbb{N}$, $\mathbf{v}_i \models guard(\delta_i)$ and $\mathbf{v}_i \in \mathbb{N}^n$. Note also that for all configurations $\langle q, \mathbf{v}\rangle$, if $\langle P, \mathbf{m}\rangle$ is an iterated path schema such that $source(P[1]) = q$ then there exists a pseudo-run starting from c such that $trans(\rho) = trans(\langle P, \mathbf{m}\rangle)$.

From these observations, we conclude that an iterated path schema $\langle P, \mathbf{m}\rangle$ augmented with $c_0 = \langle q_0, \mathbf{v}_0\rangle$ verifying $source(P[1]) = q_0$, defines a unique pseudo-run that is denoted by $\rho(P, \mathbf{m}, c_0)$. Given a configuration $c_0 = \langle q_0, \mathbf{v}_0\rangle$, we say that $\rho(P, \mathbf{m}, c_0)$ is *well-defined* if $source(P[1]) = q_0$. For every $i \in [0, size(P) - 1]$ let $\mathfrak{p}_i \overset{\text{def}}{=} \sum_{j=1}^{j=i} \mathbf{m}[j] * length(P[j])$ with limit case $\mathfrak{p}_0 = 0$. The value $\mathfrak{p}_i$ is called the *position* of the *ith witness configuration* $\rho(P, \mathbf{m}, c_0)(\mathfrak{p}_i)$. Intuitively, we reach the ith witness configuration after going through the first i elements of the path schema P in $\rho(P, \mathbf{m}, c_0)$. We say that $\rho(P, \mathbf{m}, c_0)$ is *positive* if for all the witness configurations $\langle q, \mathbf{v}\rangle$, we have $\mathbf{v} \in \mathbb{N}^n$. Note that since for all path schemas P, we have $size(P) \leq \beta + \gamma$, the number of witness configurations for such a pseudo-run is bounded by $\beta + \gamma$.

Now, we show how to build a (PA) formula whose set of solutions corresponds to the witness configurations of a pseudo-run associated to an iterated path schema equipped with an initial configuration. Before defining the formula, we explain some further notions. In the sequel, we use the sets of variables $\mathsf{X}_0 = \{\mathsf{x}_0^1, \ldots, \mathsf{x}_0^{n+1}\}$ and $\mathsf{X} = \{\mathsf{x}^1, \ldots, \mathsf{x}^{n+1}\}$ to represent configurations and $\mathsf{W}_i =$

$\{w_i^1, \ldots, w_i^{n+1}\}$ for every $i \in [0, \beta + \gamma - 1]$ to represent pseudo-configurations and the variables $p_0, \ldots, p_{\beta+\gamma-1}$ and y to represent positions in (pseudo)-runs. Furthermore, given sets of variables X, W representing (pseudo)-configurations, a variable x and a vector $\mathbf{u} \in \mathbb{N}^n$, we use the shortcut $X = W + x.\mathbf{u}$ to denote the formula $\bigwedge_{i=2}^{n+1} x^i = w^i + x.\mathbf{u}[i-1]$. Let us define the formula *Witness* that states whether a given set of configurations and natural numbers represent the witness configurations and their respective positions in a pseudo-run associated to an iterated path schema. The main idea of the formula is to check at each step whether the control states of the witness configurations match with the states of the taken transitions or loops in the path schema and then to compute the effect of the corresponding element of the iterated path schema taking into account the number of iterations.

$$Witness(W_0, \ldots, W_{\beta+\gamma-1}, p_0, \ldots, p_{\beta+\gamma-1}, Z_t, Z_p, Z_{it}, X_0) \stackrel{\text{def}}{=}$$
$$(p_0 = 0 \wedge X_0 = W_0 \wedge \bigvee_{t=0}^{1} \bigvee_{j=1}^{\max(\beta,\gamma)} z_p^2 = j \wedge z_t^2 = t \wedge x_0^1 = source(\tau(t,j))) \wedge$$

$$\bigwedge_{i=1}^{\beta+\gamma-1} (i < z_t^1 \Rightarrow \bigvee_{t=0}^{1} \bigvee_{j=1}^{\max(\beta,\gamma)} (z_p^{i+1} = j \wedge z_t^{i+1} = t \wedge p_i = p_{i-1} + z_{it}^{i+1} * length(\tau(t,j)) \wedge$$
$$w_i^1 = target(\tau(t,j)) \wedge W_i = W_{i-1} + z_{it}^{i+1} * update(\tau(t,i)))$$

Lemma 6 below characterizes the formula *Witness*.

Lemma 6. *Let* $\boldsymbol{w}_0, \ldots, \boldsymbol{w}_{\beta+\gamma-1}, \boldsymbol{c}_0 \in \mathbb{N}^{n+1}$ *and* $\mathfrak{p}_0, \ldots, \mathfrak{p}_{\beta+\gamma-1} \in \mathbb{N}$ *and* $\boldsymbol{v}_t,$ $\boldsymbol{v}_p, \boldsymbol{v}_{it}$ *in* $\mathbb{N}^K$ *such that* $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \models Schema(Z_t, Z_p, Z_{it})$. *We have* $\boldsymbol{w}_0, \ldots,$ $\boldsymbol{w}_{\beta+\gamma-1}, \mathfrak{p}_0, \ldots, \mathfrak{p}_{\beta+\gamma-1}, \boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it}, \boldsymbol{c}_0 \models Witness$ *iff* $\rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0)$ *is well-defined and positive and for all* $j \in [0, \beta + \gamma - 1]$, *if* $j < \text{size}(P_{\boldsymbol{v}_t,\boldsymbol{v}_p})$, *then* $\boldsymbol{w}_j$ *represents the jth witness configuration of* $\rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0)$ *and* $\mathfrak{p}_j$ *its position.*

Using *Witness*, one can build in logarithmic space a formula to check whether a vector $\mathbf{c}$ is the ith configuration $\langle q_i, \mathbf{v}_i \rangle$ of a pseudo-run $\rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ with the property that $\mathbf{v}_i \models guard(trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle)[i+1])$ and $\mathbf{v}_i \in \mathbb{N}^n$ (here i is the number of transitions to reach that configuration) and then to construct a formula to check whether a pseudo-run is a run. In fact, as observed earlier, it is enough to check whether at each step the ith configuration satisfies the guard of the $(i+1)$th transition.

Lemma 7. *One can build in logarithmic-space in the size of flat counter system* S *two PA formulae* $Conf(Z_t, Z_p, Z_{it}, X_0, y, X)$ *and* $Run(Z_t, Z_p, Z_{it}, X_0)$ *such that for all* $\boldsymbol{c}_0, \boldsymbol{c} \in \mathbb{N}^{n+1}$, *for all* $i \in \mathbb{N}$ *and for all* $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \in \mathbb{N}^{\beta+\gamma+1}$, *we have the two following properties:*

1. $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it}, \boldsymbol{c}_0, i, \boldsymbol{c} \models Conf$ *iff* $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \models Schema$ *and* $\rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0)$ *is well defined and* $\boldsymbol{c} = \rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0)(i)$ *and* $\boldsymbol{c}[2], \cdots, \boldsymbol{c}[n+1] \models guard(trans(\langle P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}} \rangle)[i+1])$
2. $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it}, \boldsymbol{c}_0 \models Run$ *iff* $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \models Schema$ *and* $\rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0)$ *is well-defined and is a run.*

4.3 Encoding CTL* formulae using (PA)

We can encode path schemas and runs using vectors and check their validity using Presburger arithmetic formula, our next aim is to encode a given CTL* formula using a formula in (PA). The forthcoming encoding internalizes CTL* semantics and a similar idea has been already used in [10]. For each CTL* formula ϕ, we build a (PA) formula $Check_\phi(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y})$ where the variables $\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}$ and X_0 represent a run as in the formula Run and, y represents a position such that the formula checks whether the CTL* formula is satisfied at the current position. Formula $Check_\phi$ is defined recursively (Boolean clauses are omitted):

$$Check_\mathtt{p} \stackrel{\text{def}}{=} \exists\, \mathsf{X}\ (Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X}) \wedge \bigvee_{\{j\ \mid\ \mathtt{p}\ \in\ \ell(j)\}} \mathsf{x}^1 = j)$$

$$Check_{\psi(\mathsf{x}_1,\ldots,\mathsf{x}_n)} \stackrel{\text{def}}{=} \exists\, \mathsf{X}\ (Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X}) \wedge \psi(\mathsf{X}))$$

$$Check_{\mathtt{X}\phi} \stackrel{\text{def}}{=} \exists\, \mathsf{y}'\ (\mathsf{y}' = \mathsf{y} + 1 \wedge Check_\phi(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}'))$$

$$Check_{\phi\mathtt{U}\phi'} \stackrel{\text{def}}{=} \exists\, \mathsf{y}''\ (\mathsf{y} \leq \mathsf{y}'' \wedge Check_{\phi'}(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}'')\wedge$$
$$\forall\, \mathsf{y}'\ (\mathsf{y} \leq \mathsf{y}' < \mathsf{y}'' \Rightarrow Check_\phi(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}')))$$

$$Check_{\mathtt{E}\phi} \stackrel{\text{def}}{=} \exists\, \mathsf{Z}'_t\ \exists\, \mathsf{Z}'_p\ \exists\, \mathsf{Z}'_{it}\ \exists\, \mathsf{X}\ (Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X})\wedge$$
$$Run(\mathsf{Z}'_t, \mathsf{Z}'_p, \mathsf{Z}'_{it}, \mathsf{X}) \wedge\ \exists\, \mathsf{y}'\ (\mathsf{y}' = 0 \wedge Check_\phi(\mathsf{Z}'_t, \mathsf{Z}'_p, \mathsf{Z}'_{it}, \mathsf{X}, \mathsf{y}')))$$

Now, we can state the main property concerning the formulae $Check_\phi$ based on Lemmas 4, 5 and 7.

Lemma 8. *Let $\boldsymbol{c}_0 \in \mathbb{N}^{n+1}$, $i \in \mathbb{N}$ and $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it} \in \mathbb{N}^K$ be such that $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it}$, $\boldsymbol{c}_0 \models Run$. We have $\rho(P_{\boldsymbol{v}_t,\boldsymbol{v}_p}, \boldsymbol{m}_{\boldsymbol{v}_{it}}, \boldsymbol{c}_0), i \models \phi$ iff $\boldsymbol{v}_t, \boldsymbol{v}_p, \boldsymbol{v}_{it}, \boldsymbol{c}_0, i \models Check_\phi(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y})$.*

This allows us to conclude the main result of this section.

Theorem 9. *There is a logarithmic-space reduction from MC(CTL*, `FlatCS`) to the satisfiability problem for (PA).*

It is possible to extend the reduction by admitting linear past-time operators to the temporal language since we have seen that we can easily quantify over runs. However, in that case, finite prefixes in runs should not be reset.

5 Conclusion

We have been able to characterize the computational complexity for MC(CTL*, `FlatCS`) by showing that the problem is equivalent to the satisfiability problem for Presburger arithmetic (modulo logarithmic-space reductions). The lower bound is obtained by considering a quite strong restriction (no arithmetical constraints in formulae, the only temporal operator is `EF`, guards on transitions are simple linear constraints). By contrast, the restriction of the problem to LTL formulae is known to be NP-complete [8] when guards are in the linear fragment and the restriction of the problem to formulae in $CTL_\mathtt{EF}$ is also equivalent to

(PA). We have proposed a new way for encoding runs in flat counter systems using Presburger arithmetic formulae, but without any exponential blow up, which allows us to get a precise complexity characterization. It remains open to determine which extensions of CTL* on flat counter systems preserve decidability, if not an efficient translation into (PA).

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A Proof of Lemma 1

Proof. Let $S = \langle Q, \Delta, \mathtt{C}_n, \ell \rangle$ be a flat counter system. Let us build the non-blocking counter system as follows:

- $Q' \stackrel{\mathrm{def}}{=} Q \uplus \{q_\perp\}$.
- $\ell'(q_\perp) \stackrel{\mathrm{def}}{=} \{\mathtt{p}_\perp\}$ where $\mathtt{p}_\perp$ is a new propositional variable and for all $q \in Q$, $\ell'(q) \stackrel{\mathrm{def}}{=} \ell(q)$.
- $\Delta' \stackrel{\mathrm{def}}{=} \Delta \cup \{\langle q, \neg \mathtt{g}_q, \mathbf{0}, q_\perp \rangle | q \in Q\} \cup \{\langle q_\perp, \top, \mathbf{0}, q_\perp \rangle\}$ where for every state $q \in Q$, $\mathtt{g}_q = \bigvee\limits_{source(\delta)=q} guard(\delta)$.

Firstly, observe that since S is flat, the counter system S' is flat too as the construction does not introduce any new loops. For every maximal run ρ in S, we write $\mathtt{maximize}(\rho)$ to denote the maximal run in S' defined as follows:

- if ρ is infinite, then $\mathtt{maximize}(\rho) \stackrel{\mathrm{def}}{=} \rho$,
- otherwise, $\mathtt{maximize}(\rho)$ is defined from ρ as follows:

$$\mathtt{maximize}(\rho) = \rho' = \rho \xrightarrow{\delta_{\alpha+1}} \langle q_\perp, \mathbf{v}_\alpha \rangle \xrightarrow{\delta_\perp} \langle q_\perp, \mathbf{v}_\alpha \rangle \xrightarrow{\delta_\perp} \cdots$$

where $|\rho| = \alpha$, $\rho[\alpha] = \langle q_\alpha, \mathbf{v}_\alpha \rangle$, $\delta_\perp = \langle q_\perp, \top, \mathbf{0}, q_\perp \rangle$ and $\delta_{\alpha+1} = \langle q_\alpha, \neg(\bigvee\limits_{source(\delta)=q_\alpha} guard(\delta)), \mathbf{0}, q_\perp \rangle$.

Note that since $\langle q_\alpha, \mathbf{v}_\alpha \rangle$ is a deadlock configuration we have that $\mathbf{v}_\alpha \not\models guard(\delta)$ for every δ such that $source(\delta) = q_\alpha$. Thus, $\mathbf{v}_\alpha \models \neg(\bigvee\limits_{source(\delta)=q_\alpha} guard(\delta))$

and hence the only configuration possible after $\langle q, \mathbf{v} \rangle$ is $\langle q_\perp, \mathbf{v} \rangle$. So, for all such finite runs this is the only way to get an infinite maximal run extending ρ.

So, clearly, for any run ρ in S there exists an unique run ρ' in S' such that $\rho' = \mathtt{maximize}(\rho)$. Hence, is a bijection between the set of maximal runs in S and the set of maximal runs in S' that start by a state in Q. Also, every maximal run in S' is infinite. Below, we also use the inverse map $\mathtt{maximize}^{-1}(\cdot)$ that returns a maximal run in S.

Let us define a formula map $\mathfrak{t}(\cdot)$ for formulas in CTL* such that $\mathfrak{t}(\cdot)$ is the identity for the atomic formulae, homomorphic for Boolean connectives and for the path quantifier E, and it satisfies the following clauses:

- $\mathfrak{t}(\mathtt{X}\psi) \stackrel{\mathrm{def}}{=} \mathtt{X}(\neg \mathtt{p}_\perp \wedge \mathfrak{t}(\psi))$.
- $\mathfrak{t}(\psi \mathtt{U} \psi') \stackrel{\mathrm{def}}{=} \mathfrak{t}(\psi) \mathtt{U}(\neg \mathtt{p}_\perp \wedge \mathfrak{t}(\psi'))$.

When dealing the the fragment CTL$_{\mathtt{EF}}$, these two clauses are replaced by $\mathfrak{t}(\mathtt{EF}\psi) = \mathtt{EF}(\neg \mathtt{p}_\perp \wedge \mathfrak{t}(\psi))$.

By induction on the structure of the formula, we will show that there exists a run ρ from S starting with c such that $\rho, 0 \models \phi$ iff $\mathtt{maximize}(\rho), 0 \models \mathfrak{t}(\phi)$. As a consequence, for all configurations c in S and for all formulae ϕ in CTL* [resp. CTL$_{\mathtt{EF}}$], we have $S, c \models \phi$ iff $S', c \models \mathfrak{t}(\phi)$. Note that by construction, S' is non-blocking and, S' and ϕ' can be built in logarithmic space in the size of S, ϕ.

- **Base case:** Observe that, by construction, each configurations c in S is also a configuration in S'. Thus, for any run ρ in S and any $i \leq |\rho|$, we can have a run ρ' in S' such that $\rho[i] = \rho'[i]$. The fact that $\mathsf{t}(\cdot)$ is identity for atomic formulas p and $\psi(\mathsf{x}_1, \cdots, \mathsf{x}_n)$ and the fact that by definition of satisfiability relation of CTL^*, satisfaction of atomic formulas depend on the current configuration, allows us to deduce that $\rho, i \models \mathsf{p}$ [$\rho, i \models \psi(\mathsf{x}_1, \cdots, \mathsf{x}_n)$ resp.] iff $\rho', i \models \mathsf{t}(\mathsf{p})$ [$\rho', i \models \mathsf{t}(\psi(\mathsf{x}_1, \cdots, \mathsf{x}_n))$ resp.] for any $i \leq |\rho|$.
- **Induction step:** We assume that for all formula ϕ of size $k-1$ and configuration c from S, there exists a run ρ from S starting with c such that $\rho, 0 \models \phi$ iff $\mathtt{maximize}(\rho), 0 \models \mathsf{t}(\phi)$. We will prove the same for formulas of size k.
  - *Case $\mathsf{X}\psi$:* First, suppose that $\rho, i \models \mathsf{X}\psi$. By definition of $\models$, $i < |\rho|$ and $\rho, i+1 \models \psi$. By induction hypothesis, $\mathtt{maximize}(\rho), i+1 \models \mathsf{t}(\psi)$. Since $i < |\rho|$, $\mathtt{maximize}(\rho)(i+1) = \rho(i+1)$ and since $\mathtt{maximize}(\rho)(i+1) \models \neg\mathsf{p}_\perp$, we get that $\mathtt{maximize}(\rho), i \models \mathsf{X}(\neg\mathsf{p}_\perp \wedge \mathsf{t}(\psi))$. Assume that $\mathtt{maximize}(\rho), i \models \mathsf{X}(\neg\mathsf{p}_\perp \wedge \mathsf{t}(\psi))$. So, $\mathtt{maximize}(\rho), i+1 \models \mathsf{t}(\psi)$ and $\mathtt{maximize}(\rho), i+1 \models \neg\mathsf{p}_\perp$. By construction, for all the positions j in $\mathtt{maximize}(\rho)$ such that $\mathtt{maximize}(\rho), j \models \neg\mathsf{p}_\perp$, $\mathtt{maximize}(\rho)(j)$ is configuration in ρ. Thus, $|\rho| > i$. By induction hypothesis, $\mathtt{maximize}^{-1}(\rho), i+1 \models \psi$. Therefore $\rho, i \models \mathsf{X}\,\psi$.
  - *Case $\psi\mathsf{U}\psi'$:* First suppose that $\rho, i \models \psi\mathsf{U}\psi'$. There is $i \leq j < |\rho|$ such that $\rho, j \models \psi'$ and for all $k \in [i, j-1]$, we have $\rho, k \models \psi$. By induction hypothesis, $\mathtt{maximize}(\rho), j \models \mathsf{t}(\psi')$ and for all $k \in [i, j-1]$, $\mathtt{maximize}(\rho), k \models \mathsf{t}(\psi)$. Since $\mathtt{maximize}(\rho)(j) = \rho(j)$, we get that $\mathtt{maximize}(\rho), j \models \neg\mathsf{p}_\perp$. Consequently, there is $i \leq j$ such that $\mathtt{maximize}(\rho), j \models \neg\mathsf{p}_\perp \wedge \mathsf{t}(\psi')$ and for all $k \in [i, j-1]$, we have $\mathtt{maximize}(\rho), k \models \mathsf{t}(\psi)$. So, $\mathtt{maximize}(\rho), i \models \mathsf{t}(\psi)\mathsf{U}(\neg\mathsf{p}_\perp \wedge \mathsf{t}(\psi')))$.

    Now suppose that $\mathtt{maximize}(\rho), i \models \mathsf{t}(\psi\mathsf{U}\psi')$. By definition of $\mathsf{t}(\cdot)$, there is $i \leq j'$ such that $\mathtt{maximize}(\rho), j' \models \psi'$, $\mathtt{maximize}(\rho), j' \models \neg\mathsf{p}_\perp$ and for all $k' \in [i, j'-1]$, we have $\rho, k' \models \psi$. Since $\mathtt{maximize}(\rho), j' \models \neg\mathsf{p}_\perp$, we know that $j' < |\rho|$ and $\rho[k] = \mathtt{maximize}(\rho)[k]$ for all $k \in [i, j]$. Hence, by induction hypothesis, $\rho, j' \models \psi'$ and for all $k \in [i', j'-1]$, $\rho, k' \models \psi$. So, $\rho, i \models \psi\mathsf{U}\psi'$.
  - *Case $\mathsf{E}\psi$:* First, suppose that $\rho, i \models \mathsf{E}\psi$. By definition of $\models$, there is a maximal run ρ' such that $\rho'(0) = \rho(i)$ and $\rho', 0 \models \psi$. By induction hypothesis, $\mathtt{maximize}(\rho'), 0 \models \mathsf{t}(\psi)$. Since $i < |\rho|$, $\mathtt{maximize}(\rho)(i) = \rho(i)$ and since $\rho'(0) = \mathtt{maximize}(\rho')(0)$, we get that $\mathtt{maximize}(\rho), i \models \mathsf{E}\,\mathsf{t}(\psi)$. Assume that $\mathtt{maximize}(\rho), i \models \mathsf{E}\,\mathsf{t}(\psi)$. So, there is a run ρ' such that $\rho'(0) = \mathtt{maximize}(\rho)(i)$ and $\rho', 0 \models \mathsf{t}(\psi)$. By induction hypothesis, $\mathtt{maximize}^{-1}(\rho'), 0 \models \psi$ (indeed the control state of $\rho'(0)$ is in Q). But, $\mathtt{maximize}^{-1}(\rho')(0) = \mathtt{maximize}(\rho)(i) = \rho(i)$ and therefore $\rho, i \models \mathsf{E}\,\psi$.
  - *Case $\mathsf{EF}\psi$:* First, suppose that $\rho, i \models \mathsf{EF}\psi$. By definition of $\models$, there is a maximal run ρ' and a position $0 \leq j < |\rho'|$ such that $\rho'(0) = \rho(i)$ and $\rho', j \models \psi$. By induction hypothesis, $\mathtt{maximize}(\rho'), j \models \mathsf{t}(\psi)$. Since $i < |\rho|$, $\mathtt{maximize}(\rho)(i) = \rho(i)$ and since $\rho'(j) = \mathtt{maximize}(\rho')(j)$, we get that $\mathtt{maximize}(\rho), i \models \mathsf{EF}(\neg\mathsf{p}_\perp \wedge \mathsf{t}(\psi))$. Assume that $\mathtt{maximize}(\rho), i \models$

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$\mathtt{EF}(\neg \mathtt{p}_\perp \wedge \mathtt{t}(\psi))$. So, there is a run ρ' in S' and position $j \geq 0$ such that $\rho'(0) = \mathtt{maximize}(\rho)(i)$ and $\rho', j \models \mathtt{t}(\psi)$ and $\rho', j \models \neg \mathtt{p}_\perp$. By induction hypothesis, $\mathtt{maximize}^{-1}(\rho'), j \models \psi$ and $\mathtt{maximize}^{-1}(\rho'), j \models \neg \mathtt{p}_\perp$. By construction, $|\mathtt{maximize}^{-1}(\rho')| > j$. But, $\mathtt{maximize}^{-1}(\rho')(0) = \mathtt{maximize}(\rho)(i) = \rho(i)$ and therefore $\rho, i \models \mathtt{EF}\,\psi$.

<div align="right">□</div>

B   Proof of Theorem 2

It remains to prove (††). First let us recall (††):

(††).1 for all valuation f, we have $f \models \phi'(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$ iff $\langle q_n, \mathbf{v}_f \rangle \models \psi_{n+1}$,

(††).2 for all $i \in [1, n]$ and valuations f such that $f(\mathtt{z}_i) = \cdots f(\mathtt{z}_n) = 0$, we have
$f \models \mathcal{Q}\,\mathtt{z}_i\,\cdots\,\mathcal{Q}_n\,\mathtt{z}_n\,\phi'(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$ iff $\langle q_{i-1}, \mathbf{v}_f \rangle \models \psi_i$.

First, let us prove (††).1. Note that $\psi_{n+1} = \mathtt{EF}\,q_{n+1}$ and the only edge between q_n and q_{n+1} is the following:

$$q_n \xrightarrow{\phi'(\mathtt{x}_1, \ldots, \mathtt{x}_n), \mathbf{0}} q_{n+1}$$

Let f be a valuation. By the definition of $\models$, we have $\langle q_n, \mathbf{v}_f \rangle \models \mathtt{EF}\,q_{n+1}$ iff $\mathbf{v}_f \models \phi'(\mathtt{x}_1, \ldots, \mathtt{x}_n)$. By definition of $\mathbf{v}_f$, we get $\mathbf{v}_f \models \phi'(\mathtt{x}_1, \ldots, \mathtt{x}_n)$ iff $f \models \phi'(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$.

Now, let us prove (††).2 by induction, for which the base case $i = n + 1$ corresponds precisely to the satisfaction of the property (††).1. Suppose that (††).2 holds for $i + 1$, let us show that it holds for i too.

Case 1: $\mathcal{Q}_i = \exists$.
Let f be a valuation such that $f(\mathtt{z}_i) = \cdots f(\mathtt{z}_n) = 0$. We prove that $f \models \exists\,\mathtt{z}_i\,\cdots\,\mathcal{Q}_n\,\mathtt{z}_n\,\phi'(\mathtt{z}_1, \ldots, \mathtt{z}_n)$ iff $\langle q_{i-1}, \mathbf{v}_f \rangle \models \mathtt{EF}(q_i \wedge \psi_{i+1})$.

First, let us assume that $f \models \exists\,\mathtt{z}_i\,\cdots\,\mathcal{Q}_n\,\mathtt{z}_n\,\phi'(\mathtt{z}_1, \ldots, \mathtt{z}_n)$. By assumption, there exists $a \in \mathbb{N}$ such that $f[\mathtt{z}_i \mapsto a] \models \mathcal{Q}_{i+1}\,\mathtt{z}_{i+1}\,\cdots\,\mathcal{Q}_n\,\mathtt{z}_n\,\phi'(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$. By induction hypothesis, $\langle q_i, \mathbf{v}_{f'} \rangle \models \psi_{i+1}$ where $f' = f[\mathtt{z}_i \mapsto a]$. Now consider the transition system $\mathfrak{T}(S_\phi)$ starting at the configuration $\langle q_{i-1}, \mathbf{v}_f \rangle$ associated with S_ϕ. There is a run ρ in $\mathfrak{T}(S)$ starting from $\langle q_{i-1}, \mathbf{v}_f \rangle$ that reaches q_i and where the loop l_i is visited exactly a times. Since, the effect of the simple loop l_i is to increment the counter $\mathtt{x}_i$ each time, and $\mathbf{v}_f[i] = 0$, there is a finite run from $\langle q_{i-1}, \mathbf{v}_f \rangle$ to $\langle q_i, \mathbf{v}_{f'} \rangle$. So, $\langle q_{i-1}, \mathbf{v}_f \rangle \models \mathtt{EF}(q_i \wedge \psi_{i+1})$.

On the other hand, let us assume that $\langle q_{i-1}, \mathbf{v}_f \rangle \models \mathtt{EF}(q_i \wedge \psi_{i+1})$. By definition of $\models$, there exists a run ρ in $\mathfrak{T}(S_\phi)$ with $\rho(0) = \langle q_{i-1}, \mathbf{v}_f \rangle$ such that for some position $j < |\rho|$, $\rho(j) \models q_i \wedge \psi_{i+1}$. Thus, $\rho(j) = \langle q_i, \mathbf{v}' \rangle$ for some $\mathbf{v}' \in \mathbb{N}^n$. Since q_i is reached only after taking the simple loop l_i, let us assume that the loop l_i is taken $a \geq 0$ times. Since, the effect of the simple loop l_i is to increment the counter $\mathtt{x}_i$ each time, $\mathbf{v}'[i] = a$ and $\mathbf{v}_f$ and $\mathbf{v}'$ differ only on the ith component. So, $\mathbf{v}' = \mathbf{v}_{f'}$ where $f' = f[\mathtt{z}_i \mapsto a]$. Since $\langle q_i, \mathbf{v}_{f'} \rangle \models \psi_{i+1}$, by the induction hypothesis $f[\mathtt{z}_i \mapsto a] \models \mathcal{Q}_{i+1}\,\mathtt{z}_{i+1}\,\cdots\,\mathcal{Q}_n\,\mathtt{z}_n\,\phi'(\mathtt{z}_1, \mathtt{z}_2, \ldots, \mathtt{z}_n)$. By definition of

$\models$ in PA, we can conclude that $f \models \exists z_i \cdots \mathcal{Q}_n z_n \phi'(z_1, \ldots, z_n)$.

Case 2: $\mathcal{Q}_i = \forall$.
Let f be a valuation such that $f(z_i) = \cdots f(z_n) = 0$. We prove that $f \models \forall z_i \cdots \mathcal{Q}_n z_n \phi'(z_1, \ldots, z_n)$ iff $\langle q_{i-1}, \mathbf{v}_f \rangle \models \text{AG}(q_i \rightarrow \psi_{i+1})$.

First, let us assume that $f \models \forall z_i \cdots \mathcal{Q}_n z_n \phi'(z_1, \ldots, z_n)$. By assumption, for all $a \in \mathbb{N}$ such that $f[z_i \mapsto a] \models \mathcal{Q}_{i+1} z_{i+1} \cdots \mathcal{Q}_n z_n \phi'(z_1, z_2, \ldots, z_n)$. By induction hypothesis, for all such $\mathbf{v}_{f'}$, we have that $\langle q_i, \mathbf{v}_{f'} \rangle \models \psi_{i+1}$ where $f' = f[z_i \mapsto a]$. Now consider the transition system $\mathfrak{T}(S_\phi)$ starting at the configuration $\langle q_{i-1}, \mathbf{v}_f \rangle$. There are two types of run ρ in $\mathfrak{T}(S)$ starting from $\langle q_{i-1}, \mathbf{v}_f \rangle$:

1. ρ reaches q_i and the loop l_i is visited exactly a times for every $a \in \mathbb{N}$.
2. ρ loops l_i infinitely many times and never reaches q_i.

Due to the implication, $q_i \Rightarrow \psi_{i+1}$ we can easily ignore the runs of type 2 above. Hence, we will consider only the runs where l_i is taken only finitely many times. Since, the effect of the simple loop l_i is to increment the counter x_i each time, and $\mathbf{v}_f[i] = 0$, there are finite runs from $\langle q_{i-1}, \mathbf{v}_f \rangle$ to $\langle q_i, \mathbf{v}_{f'} \rangle$ for each of the valuations $f' = f[z_i \mapsto a]$ for all $a \in \mathbb{N}$. So, $\langle q_{i-1}, \mathbf{v}_f \rangle \models \text{AG}(q_i \Rightarrow \psi_{i+1})$.

On the other hand, let us assume that $\langle q_{i-1}, \mathbf{v}_f \rangle \models \text{AG}(q_i \Rightarrow \psi_{i+1})$. By definition of $\models$, there exist a set of runs X, in $\mathfrak{T}(S_\phi)$ such that for all $\rho \in X$ we have $\rho(0) = \langle q_{i-1}, \mathbf{v}_f \rangle$ and for some position $j < |\rho|$, if $\rho(j) \models q_i$ then $\rho(j) \models \psi_{i+1}$. Thus, for every run $\rho \in X$, $\rho(j) = \langle q_i, \mathbf{v}' \rangle$ for some $\mathbf{v}' \in \mathbb{N}^n$. Since q_i is reached only after taking the simple loop l_i a finite number of times in any run, let us assume that for each value of $a \geq 0$, there exists a run $\rho_a \in X$ where the loop l_i is taken a times. Since, the effect of the simple loop l_i is to increment the counter x_i each time, $\mathbf{v}'[i] = a$ and $\mathbf{v}_f$ and $\mathbf{v}'$ differ only on the ith component in any run ρ_a in X. So, $\mathbf{v}' = \mathbf{v}_{f'}$ where $f' = f[z_i \mapsto a]$. Since $\langle q_i, \mathbf{v}_{f'} \rangle \models \psi_{i+1}$ for all such valuations $\mathbf{v}_{f'}$, by the induction hypothesis $f[z_i \mapsto a] \models \mathcal{Q}_{i+1} z_{i+1} \cdots \mathcal{Q}_n z_n \phi'(z_1, z_2, \ldots, z_n)$ for all $a \in \mathbb{N}$. By definition of $\models$ in PA, we can conclude that $f \models \forall z_i \cdots \mathcal{Q}_n z_n \phi'(z_1, \ldots, z_n)$.

C   Proof of Lemma 5

Proof.

1. Recall that by definition, an iterated path schema is defined as $\langle P, \mathbf{m} \rangle$, such that P is a minimal path schema and $\mathbf{m} \in \mathbb{N}^{\text{size}(P)}$. Now assume that we have vectors $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^K$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}$ satisfies all the conditions of $\mathfrak{C}$. We perform the construction of $P_{\mathbf{v}_t, \mathbf{v}_p}$. By definition of $P_{\mathbf{v}_t, \mathbf{v}_p}$ and $\mathfrak{C}.1$ through $\mathfrak{C}.4$ ensure that $P_{\mathbf{v}_t, \mathbf{v}_p}$ is a sequence of loops and edges from S. $\mathfrak{C}.5$ ensures that $P_{\mathbf{v}_t, \mathbf{v}_p}$ ends with a loop. Alongwith $\mathfrak{C}.7$ the construction ensures that $P_{\mathbf{v}_t, \mathbf{v}_p}$ is a path schema from S. $\mathfrak{C}.6$ ensures that the path schema $P_{\mathbf{v}_t, \mathbf{v}_p}$ is a minimal path schema where none of the edges or loops are repeated. By construction $\mathbf{v}_p[1]$ is exactly equal to $\text{size}(P_{\mathbf{v}_t, \mathbf{v}_p})$. Thus, $\mathbf{m}_{\mathbf{v}_{it}}$ is defined to

be in $\mathbb{N}^{\text{size}(P_{\mathbf{v}_t, \mathbf{v}_p})}$. Thus, we can conclude that $\langle P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle$ is an iterated path schema.

For the other side, given an iterated schema $\langle P, \mathbf{m} \rangle$, we construct $\mathbf{v}_t$ and $\mathbf{v}_p$ as:

- $\mathbf{v}_t[1] = \mathbf{v}_p[1] = \text{size}(P)$.
- $\mathbf{v}_p[i] = j$ and $\mathbf{v}_t[i] = 0$ if $P[i-1] = \delta_j$ for all $i \in [2, \text{size}(P)+1]$ and some j.
- $\mathbf{v}_p[i] = j$ and $\mathbf{v}_t[i] = 1$ if $P[i-1] = l_j$ for all $i \in [2, \text{size}(P)+1]$ and some j.
- $\mathbf{v}_p[i] = \mathbf{v}_t[i] = 0$ for all $i \in [\text{size}(P)+2, K]$.

Note that the above construction follows exactly the reverse steps of the construction of $P_{\mathbf{v}_t, \mathbf{v}_p}$. In other words, given an iterated path schema $\langle P, \mathbf{m} \rangle$, the above construction gives us the vectors $\mathbf{v}_t$ and $\mathbf{v}_p$ such that $P = P_{\mathbf{v}_t, \mathbf{v}_p}$. Moreover, we can construct $\mathbf{v}_{it}$ from $\mathbf{m} \in \mathbb{N}^{\text{size}(P)}$ as :

- $\mathbf{v}_{it}[1] = \text{size}(P)$,
- $\mathbf{v}_{it}[i] = \mathbf{m}[i]$ for all $i \in [1, \text{size}(P)]$
- $\mathbf{v}_{it}[i] = 0$ for all $i \in [\text{size}(P)+1, K]$.

Clearly by the construction of $\mathbf{v}_t, \mathbf{v}_p$ and the fact that $\text{size}(P) > 0$ we can conclude that $\mathfrak{C}.1$ is satisfied. Similarly, by construction, and the fact that P is a sequence of edges and loops, $\mathfrak{C}.2$ to $\mathfrak{C}.5$ are satisfied. Due to the fact that P is a minimal path schema, $\mathfrak{C}.6$ and $\mathfrak{C}.7$ are also satisfied. Since, $\langle P, \mathbf{m} \rangle$ is an iterated path schema, $\mathfrak{C}.8$ is also satisfied.

2. To construct the formula $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$, we first define a formula ψ_i for each condition $\mathfrak{C}.i$.

$$
\begin{aligned}
\psi_1 &\overset{\text{def}}{=} (\mathsf{z}_t^1 = \mathsf{z}_p^1 \wedge \mathsf{z}_p^1 = \mathsf{z}_{it}^1 \wedge \mathsf{z}_p^1 > 0 \wedge \mathsf{z}_p^1 < K) \wedge \\
&\bigwedge_{i=2}^{K} \mathsf{z}_p^1 + 1 < i \Rightarrow (\mathsf{z}_t^i = \mathsf{z}_p^i \wedge \mathsf{z}_p^i = \mathsf{z}_{it}^i \wedge \mathsf{z}_{it}^i = 0)
\end{aligned}
$$

$$
\psi_2 \overset{\text{def}}{=} \bigwedge_{i=2}^{K} (\mathsf{z}_t^i \leq 1)
$$

$$
\psi_3 \overset{\text{def}}{=} \bigwedge_{i=2}^{K} i \leq \mathsf{z}_p^1 + 1 \Rightarrow (\mathsf{z}_t^i = 0 \Rightarrow \mathsf{z}_p^i \leq \beta \wedge \mathsf{z}_p^i \geq 1)
$$

$$
\psi_4 \overset{\text{def}}{=} \bigwedge_{i=2}^{K} i \leq \mathsf{z}_p^1 + 1 \Rightarrow (\mathsf{z}_t^i = 1 \Rightarrow \mathsf{z}_p^i \leq \gamma \wedge \mathsf{z}_p^i \geq 1)
$$

$$
\psi_5 \overset{\text{def}}{=} \bigwedge_{i=2}^{K} i = \mathsf{z}_p^1 + 1 \Rightarrow \mathsf{z}_t^i = 1
$$

$$
\begin{aligned}
\psi_6 \overset{\text{def}}{=} \bigwedge_{i=2}^{K} (i \leq \mathsf{z}_p^1 + 1 \Rightarrow (\bigwedge_{j=2, j \neq i}^{K} \\
j \leq \mathsf{z}_p^1 + 1 \Rightarrow (\mathsf{z}_p^i \neq \mathsf{z}_p^j \vee \mathsf{z}_t^i \neq \mathsf{z}_t^j)))
\end{aligned}
$$

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$$\psi_7 \stackrel{\text{def}}{=} \bigwedge_{i=2}^{K} i \leq \mathsf{z}_p^1 + 1 \Rightarrow$$

$$\bigvee_{\{(a,b,a',b') \mid target(\tau(a,b)) = source(\tau(a'.b'))\}} (\mathsf{z}_t^i = a \wedge \mathsf{z}_p^i = b \wedge \mathsf{z}_t^{i+1} = a' \wedge \mathsf{z}_p^{i+1} = b')$$

$$\psi_8 \stackrel{\text{def}}{=} \bigwedge_{i=2}^{K} i \leq \mathsf{z}_p^1 + 1 \Rightarrow (\mathsf{z}_{it}^i > 0 \wedge (\mathsf{z}_t^i = 0 \Rightarrow \mathsf{z}_{it}^i = 1))$$

We construct the formula $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ as $(\psi_1 \wedge \cdots \wedge \psi_8)$. For all $i \in [1,8]$, $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}$ satisfies $\mathfrak{C}.i$ iff $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models \psi_i$. Finally, the formula $Schema$ states that all formulae ψ_i hold for the vectors $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}$. Hence it is easy to see that, $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ iff $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}$ satisfies $\mathfrak{C}$.
$\square$

D Proof of Lemma 6

Proof. Let $\mathbf{w}_0, \ldots, \mathbf{w}_{\beta+\gamma-1}, \mathbf{c}_0 \in \mathbb{N}^{n+1}$ and $\mathfrak{p}_0, \ldots \mathfrak{p}_{\beta+\gamma-1} \in \mathbb{N}$ and $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}$ in $\mathbb{N}^K$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$. By Lemma 5, we know that $\mathbf{v}_t$, $\mathbf{v}_p$ and $\mathbf{v}_{it}$ respect the conditions in $\mathfrak{C}$ and consequently that $\langle P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle$ is an iterated path schema.

First we suppose that $\mathbf{w}_0, \ldots, \mathbf{w}_{\beta+\gamma-1}, \mathfrak{p}_0, \ldots, \mathfrak{p}_{\beta+\gamma-1}, \mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models Witness$.

 – The part of the formula $\bigvee_{t=0}^{1} \bigvee_{j=1}^{\max(\beta,\gamma)} \mathsf{z}_p^2 = j \wedge \mathsf{z}_t^2 = t \wedge \mathsf{x}_0^1 = source(\tau(t,j))$ ensures that $q_0 = source(P_{\mathbf{v}_t, \mathbf{v}_p}[1])$; in fact it tells that $\mathbf{c}_0[1] = source(\tau(\mathbf{v}_t[2], \mathbf{v}_p[2]))$. Consequently, we can define the pseudo-run $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$.
 – Let $N = size(P_{\mathbf{v}_t, \mathbf{v}_p})$. We recall that $N = \mathbf{v}_t[1]$. We will now show that $\mathbf{w}_j$ is the jth witness configuration of $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ and $\mathfrak{p}_j$ is its position for all $j < N$.
   • First thanks to the part of formula saying $\mathsf{p}_0 = 0 \wedge \mathsf{X}_0 = \mathsf{W}_0$, we now that $d_0 = 0$ and that $\mathbf{w}_0 = \mathbf{c}_0$, whence $\mathbf{w}_0$ is effectively the witness configuration at position 0. Now thanks to the part of the formula saying that $\bigwedge_{i=1}^{\beta+\gamma-1} (i < \mathsf{z}_t^1 \Rightarrow \bigvee_{t=0}^{1} \bigvee_{j=1}^{\max(\beta,\gamma)} (\mathsf{z}_p^{i+1} = j \wedge \mathsf{z}_t^{i+1} = t \wedge \mathsf{p}_i = \mathsf{p}_{i-1} + \mathsf{z}_{it}^{i+1} *$ $length(\tau(t,j))))$, we have that for all $1 < j < N$, $\mathfrak{p}_j = \sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * length(\tau(\mathbf{v}_t[k+1], \mathbf{v}_p[k+1]))$ and this allows us to deduce that $\mathfrak{p}_j$ is the position of the jth witness configuration (using the definition of witness configuration).
   • Similarly, the part of the formula saying $\bigwedge_{i=1}^{\beta+\gamma-1} (i < \mathsf{z}_t^1 \Rightarrow \bigvee_{t=0}^{1} \bigvee_{j=1}^{\max(\beta,\gamma)} (\mathsf{z}_p^{i+1} = j \wedge \mathsf{z}_t^{i+1} = t \wedge \mathsf{w}_i^1 = target(\tau(t,j)) \wedge \mathsf{W}_i = \mathsf{W}_{i-1} + \mathsf{z}_{it}^{i+1} * update(\tau(t,i))))$ ensures us that for all $j < N$, $\mathbf{w}_j[1] = target(\tau(\mathbf{v}_t[j+1], \mathbf{v}_p[j+1]))$ and for all $i \in [2, n+1]$, $\mathbf{w}_j[i] = \mathbf{c}_0[i] + \sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * update(\tau(\mathbf{v}_t[k+1], \mathbf{v}_p[k+1]))[i]$ and consequently $\mathbf{w}_j$ corresponds effectively to the jth witness configuration. Furthermore since for all $j < N$, $\mathbf{w}_j$ belongs to $\mathbb{N}^{n+1}$, then we deduce that the pseudo-run $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ is positive.

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We now assume that $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ is well-defined and positive and for all $j \in [0, \beta + \gamma - 1]$, if $j < \text{size}(P_{\mathbf{v}_t, \mathbf{v}_p})$, then $\mathbf{w}_j$ is the jth witness configuration of $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ and $\mathfrak{p}_j$ its position. Let $N = \text{size}(P_{\mathbf{v}_t, \mathbf{v}_p})$. We recall that $N = \mathbf{v}_t[1]$ by definition of $P_{\mathbf{v}_t, \mathbf{v}_p}$. By definition of the witness configurations, the following assertions hold:

- $\mathfrak{p}_0 = 0$ and $\mathbf{w}_0 = \mathbf{c}_0$ and $\mathbf{c}_0[1] = source(P_{\mathbf{v}_t, \mathbf{v}_p}[1])$,
  whence $\mathbf{w}_0[1] = source(P_{\mathbf{v}_t, \mathbf{v}_p}[1])$;
- for all $1 < j < N$, $\mathfrak{p}_j = \sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * length(P_{\mathbf{v}_t, \mathbf{v}_p}[k]) = \sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * length(\tau(\mathbf{v}_t[k+1], \mathbf{v}_p[k+1])) = \mathfrak{p}_{j-1} + \mathbf{v}_{it}[j+1] * length(\tau(\mathbf{v}_t[j+1], \mathbf{v}_p[j+1]))$;
- for all $1 < j < N$, $\mathbf{w}_j[1] = target(P_{\mathbf{v}_t, \mathbf{v}_p}[j]) = target(\tau(\mathbf{v}_t[j+1], \mathbf{v}_p[j+1]))$;
- for all $1 < j < N$, for all $i \in [2, n+1]$,

$$\mathbf{w}_j[i] = \sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * update(P_{\mathbf{v}_t, \mathbf{v}_p}[k])[i] =$$

$$\sum_{k=1}^{k=j} \mathbf{v}_{it}[k+1] * update(\tau(\mathbf{v}_t[k+1], \mathbf{v}_p[k+1]))[i] =$$

$$\mathbf{w}_{j-1}[i] + \mathbf{v}_{it}[j+1] * update(\tau(\mathbf{v}_t[j+1], \mathbf{v}_p[j+1]))[i]$$

These assertions allow us to deduce that we effectively have $\mathbf{w}_0, \ldots, \mathbf{w}_{\beta+\gamma-1}, \mathfrak{p}_0, \ldots, \mathfrak{p}_{\beta+\gamma-1}, \mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models Witness$.

□

E Proof of Lemma 7

Proof. We will first explain how to build the formula $Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X})$. First this formula will use $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$ to check whether the variables correspond to a path schema. Then thanks to existential quantifiers, the formula will guess the positions and the values of the witness configurations using the formula $Witness(\mathsf{W}_0, \ldots, \mathsf{W}_{\beta+\gamma-1}, \mathfrak{p}_0, \ldots, \mathfrak{p}_{\beta+\gamma-1}, \mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0)$ and the result of Lemma 6. We recall that the formula wants to store in X the configuration of the corresponding pseudo-run at the position stored in the variable y, to do this, it performs a case analysis by looking at where the value stored in y lays compare to the positions of the witness configurations, the cases are the following:

1. the value stored in y corresponds to a witness position from which the next step in the path schema is an edge δ, it is then easy to find the value of the associated configuration which corresponds to a witness configuration and also thanks to the encoding of the path schema provided by the variables in Z_t, Z_p and in Z_{it} to check that the guards of δ is satisfied;
2. the value stored in y corresponds to a position between two consecutive witness positions and lays inside a loop, then one has to check how many times the corresponding loop has been taken and thanks to this information, the formula can compute the value of reached configuration and find the corresponding guard to be checked for satisfiability.

3. the value stored in y lies in the last loop of the path schema, then the treatment is the same as for the second case.

Before to give the definition of the formula $Conf$, we introduce the following notations. First we recall that in PA we can use the modulo operator $\equiv$, the following formula $t \equiv_c t'$ with $c \in \mathbb{N}$ being a short hand for the expression $\exists x.(t = c.x + t') \vee (t' = c.x + t)$. Then for a simple loop l of the form $\delta_1, \dots, \delta_m$, for all $1 \leq i \leq m$, we will use the notation $l[i]$ to represent the transitions δ_i. We are now ready to define the formula $Conf$ as follows:

$$Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X}) \stackrel{\text{def}}{\Leftrightarrow} Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}) \wedge$$
$$\exists \mathsf{W}_0 \dots \exists \mathsf{W}_{\beta+\gamma-1} \exists \mathsf{p}_0 \dots \exists \mathsf{p}_{\beta+\gamma-1} \, Witness(\mathsf{W}_0, \dots, \mathsf{W}_{\beta+\gamma-1}, \mathsf{p}_0, \dots, \mathsf{p}_{\beta+\gamma-1}, \mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0) \wedge$$
$$\bigvee_{j=0}^{\beta+\gamma-1} [(j \leq \mathsf{z}_p^1 \wedge \mathsf{y} = \mathsf{p}_j \wedge \mathsf{z}_t^{j+2} = 0 \wedge \mathsf{X} = \mathsf{W}_j \wedge \bigvee_{k=1}^{\beta} (\mathsf{z}_p^{j+2} = k \wedge guard(\delta_k)(\mathsf{x}^2, \cdots, \mathsf{x}^{n+1}))) \wedge$$
$$(j \leq \mathsf{z}_p^1 \wedge \mathsf{p}_j \leq \mathsf{y} < \mathsf{p}_{j+1} \wedge \mathsf{z}_t^{j+2} = 1 \wedge \bigvee_{k=1}^{\gamma} (\mathsf{z}_p^{j+2} = k \wedge ConfLoop_{l_k}(\mathsf{W}_j, \mathsf{y} - \mathsf{p}_j, \mathsf{X}))) \wedge$$
$$(j = \mathsf{z}_p^1 \wedge \mathsf{p}_j \leq \mathsf{y} \wedge \bigvee_{k=1}^{\gamma} (\mathsf{z}_p^{j+2} = k \wedge ConfLoop_{l_k}(\mathsf{W}_j, \mathsf{y} - \mathsf{p}_j, \mathsf{X})))]$$

where for all loop l_k with $k \in [1, \gamma]$, the formula $ConfLoop_{l_k}(\mathsf{W}, \mathsf{y}, \mathsf{X})$ is used to find the configurations corresponding to X obtained when taking the loop l_k from the configuration W and taking y transitions in the loop.

$$ConfLoop_{l_k}(\mathsf{W}, \mathsf{y}, \mathsf{X}) \stackrel{\text{def}}{\Leftrightarrow}$$
$$\bigvee_{j=0}^{length(l_k)-1} [\mathsf{y} \equiv_{length(l_k)} j \wedge \mathsf{x}^1 = source(l_k[j+1]) \wedge guard(l_k[j+1])(\mathsf{x}^2, \cdots, \mathsf{x}^{n+1}) \wedge$$
$$\exists \mathsf{h}.(\mathsf{h} * length(l_k) \leq \mathsf{y} < \mathsf{h} * length(l_k) + 1 \wedge$$
$$\mathsf{X} = \mathsf{W} + \mathsf{h} * update(l_k) + \sum_{r=1}^{r=j} update(l_k[r]))]$$

Note that in the definition of the formula $Conf$ the second line correspond to the first case we have stated above, the third line to the second case and the fourth line to the last and third case. We are now ready to prove the properties satisfied by this formula.

We assume that there exist $\mathbf{c}_0, \mathbf{c} \in \mathbb{N}^{n+1}$ and all $i \in \mathbb{N}$ and $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^{\beta+\gamma+1}$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c} \models Conf$. Note that thanks to the conjunct $Schema(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it})$, we have that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models Schema$ and using Lemma 5 we know that $\langle P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle$ is an iterated path-schema. Furthermore we know that there exist $\mathbf{w}_0, \dots, \mathbf{w}_{\beta+\gamma-1} \in \mathbb{N}^{n+1}$ and $\mathfrak{p}_0, \dots, \mathfrak{p}_{\beta+\gamma-1} \in \mathbb{N}$ such that $\mathbf{w}_0, \dots, \mathbf{w}_{\beta+\gamma-1}, \mathfrak{p}_0, \dots, \mathfrak{p}_{\beta+\gamma-1}, \mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models Witness$, hence thanks to Lemma 6, we have that $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ is well defined and positive and for all $j \in [0, \beta+\gamma-1]$, if $j < size(P_{\mathbf{v}_t, \mathbf{v}_p})$, then $\mathbf{w}_j$ represents the jth witness configuration of $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ and $\mathfrak{p}_j$ its position. We know proceed on a case analysis on the value of i:

– First assume that there exists $j \leq \mathbf{v}_p[1] = size(P_{\mathbf{v}_t, \mathbf{v}_p})$ such that $i = \mathfrak{p}_j$ and $\mathbf{v}_t[j+2] = 0$, then we know from the formula that $\mathbf{c} = \mathbf{w}_j$ and hence

since $\mathbf{w}_j = \rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[\mathfrak{p}_j]$ (by definition of witness configuration), we deduce that $\mathbf{c} = \rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[i]$. Furthermore we have that $\mathbf{v}_t[j+2] = 0$, hence $P_{\mathbf{v}_t,\mathbf{v}_p}[j+1] \in \Delta$ and since we also have that $\mathbf{c}[2], \ldots, \mathbf{c}[n+1] \models guard(\delta_{\mathbf{v}_p[j+2]})$. Using the definition of iterated path schema we have that $\delta_{\mathbf{v}_p[j+2]} = P_{\mathbf{v}_t,\mathbf{v}_p}[j+1] = trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[\sum_{k=1}^{k=j} length(P_{\mathbf{v}_t,\mathbf{v}_p}[k]) * \mathbf{v}_{it}[k+1] + 1]$ and using the definition of the position of the j-th witness configuration, we deduce that $\delta_{\mathbf{v}_p[j+2]} = trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[\mathfrak{p}_j + 1] = trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[i+1]$. Consequently $\mathbf{c}[2], \ldots, \mathbf{c}[n+1] \models guard(trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[i+1])$.

- Now assume that there exists $j \leq \mathbf{v}_p[1] = size(P_{\mathbf{v}_t,\mathbf{v}_p})$ such that $\mathfrak{p}_j \leq j < \mathfrak{p}_{j+1}$ and $\mathbf{v}_t[j+2] = 1$ (hence $P_{\mathbf{v}_t,\mathbf{v}_p}[j+1]$ is a loop in L). We have then that $\mathbf{w}_j, i - \mathfrak{p}_j, \mathbf{c} \models ConfLoop_{l_k}$ with $k = \mathbf{v}_p[j+2]$. Then let $u \in [0, length(l_k) - 1]$ and $v \in \mathbb{N}$ such that $i - \mathfrak{p}_j = v * length(l_k) + u$. Note that since $\mathfrak{p}_j \leq i < \mathfrak{p}_{j+1}$, we know that the pseudo-run $\rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ at position i lies exactly in the loop l_k and that it has taken this loop v times exactly and also the u first transition and also that the next transition will still be one of the loop (otherwise we would have $i \geq \mathfrak{p}_{j+1}$). The formula tells us then that $\mathbf{c}[1] = source(l_k[u+1])$ and for all $2 \leq \ell \leq n+1$, we have $\mathbf{c}[\ell] = \mathbf{w}_j[\ell] + \sum_{r=1}^{r=u} update(l_k[r])[\ell]$, hence this allows us to deduce that $\mathbf{c} = \rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[i]$. Furthermore we also have that $\mathbf{c}[2], \ldots, \mathbf{c}[n+1] \models guard(l_k[u+1])$, which allows us to deduce that $\mathbf{c}[2], \ldots, \mathbf{c}[n+1] \models guard(trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[i+1])$, in fact we have $trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[i+1] = trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[\mathfrak{p}_j + (i - \mathfrak{p}_j) + 1] = trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[\mathfrak{p}_j + v * length(l_k) + u + 1] = l_k[u+1]$.

- The case where there exist $j = vectv_p[1] = size(P_{\mathbf{v}_t,\mathbf{v}_p})$ and $\mathfrak{p}_j \leq j$ is equivalent to the previous one, the only difference being that this time the position of the run is in the last loop of the path schema $P_{\mathbf{v}_t,\mathbf{v}_p}$.

The other direction which consists in proving that if $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models Schema$ and $\rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ is well defined and $\mathbf{c} = \rho(P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[i]$ and $\mathbf{c}[2], \cdots, \mathbf{c}[n+1] \models guard(trans(\langle P_{\mathbf{v}_t,\mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}\rangle)[i+1])$ then $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c} \models Conf$ can be done in a similar way.

As we have already stated, a pseudo-run $\rho = \langle q_0, \mathbf{v}_0\rangle \overset{\delta_1}{\rightsquigarrow} \cdots \overset{\delta_{m-1}}{\rightsquigarrow} \langle q_m, \mathbf{v}_m\rangle \cdots$ is a run iff for all $i \in \mathbb{N}$, $\mathbf{v}_i \models guard(\delta_i)$ and $\mathbf{v}_i \in \mathbb{N}^n$. Consequently to construct the formula $Run(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0)$, we only have to check that the values contains in the set of variables Z_t, Z_p, Z_{it} and X_0 correspond to a pseudo-run for which at each step the vector value of the configuration is in $\mathbb{N}^n$ and satisfies the guard of the next transition. According to the properties satisfied by the formula $Conf$, we build the formula Run as follows:

$$Run(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0) \overset{\text{def}}{=} \forall\, i \,\exists\, \mathsf{X} \; Conf(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, i, \mathsf{X})$$

We will now prove the properties of this formula. First we assume that there exist $\mathbf{c}_0 \in \mathbb{N}^{n+1}$ and $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^{\beta+\gamma+1}$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models Run$, then for all

$i \in \mathbb{N}$, there exists $\mathbf{c} \in \mathbb{N}^{n+1}$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c} \models \mathit{Conf}$. Thanks to the previous property we have that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models \mathit{Schema}$ and $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ is well-defined and positive and $c = \rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[i]$ is represented by $\mathbf{c}$ and $\mathbf{c}[2], \cdots, \mathbf{c}[n+1] \models \mathit{guard}(\mathit{trans}(\langle P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle)[i+1])$. Consequently the pseudo-run $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ is a run.

Now assume that there exist $\mathbf{c}_0 \in \mathbb{N}^{n+1}$ and $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^{\beta+\gamma+1}$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \models \mathit{Schema}$ and $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)$ is well-defined and is a run. Let $i \in \mathbb{N}$ and let $\mathbf{c} = \rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, c_0)[i]$, by definition of a run we have $\mathbf{c}[2], \cdots , \mathbf{c}[n+1] \models \mathit{guard}(\mathit{trans}(\langle P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}} \rangle)[i+1])$, consequently we deduce that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c} \models \mathit{Conf}$. This allows us to deduce that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models \mathit{Run}$.

Finally, if we consider that the size $|S|$ of the system S is equal to its number of transitions multiplied by the size of each transition, which correspond to the size of the guards (ie the number of subformulae multiplied by the size of the maximal value found in the formulae encoded in binary) to which is added the size of the maximal value in the updated vector (encoded in binary), we remark by construction that the size of the formula $\mathit{Witness}$ is in $O(|\beta+\gamma|^2 * |S|)$ and for what concerns the size of the formula Conf it is in $O((\beta+\gamma)*(\beta*|S|+(\gamma*|S|)^2))$. Hence these two formulae can be built in polynomial time in the size of S. $\quad\square$

F Proof of Lemma 8

Proof. We will prove the lemma using induction on the size of the formula. Let $\mathbf{c}_0 \in \mathbb{N}^{n+1}$ and $i \in \mathbb{N}$ and $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it} \in \mathbb{N}^K$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0 \models \mathit{Run}$. Thus, from Lemma 7, we know that, $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)$ is a run in S. We will show that for any i, $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0), i \models \phi$ iff $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i \models \mathit{Check}_\phi(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y})$.

Base $\phi = \mathbf{p}$. By definition, $\mathit{Check}_{\mathbf{p}}(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}) = \exists \mathsf{X}. \mathit{Conf}(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X}) \wedge \bigvee_{\{j | \mathbf{p} \in \ell(j)\}} \mathsf{x}^1 = j$. Let us assume that for some i, $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0), i \models \mathbf{p}$. Then $\mathbf{p} \in \ell(\mathbf{c}_i[1])$ with $\mathbf{c}_i = \rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)[i]$. From Lemma 7 we know that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c}_i \models \mathit{Conf}$ and $\mathbf{c}_i[1] \in \{j \mid \mathbf{p} \in \ell(j)\}$. Thus, $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i \models \mathit{Check}_{\mathbf{p}}$.
On the other hand let us assume that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i \models \mathit{Check}_{\mathbf{p}}$ for some i. Thus, there exists a configuration $\mathbf{c}_i$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c}_i \models \mathit{Conf}$ and $\mathbf{c}_i[1] \in \{j \mid \mathbf{p} \in \ell(j)\}$. In that case by Lemma 7, $\mathbf{c}_i = \rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)[i]$ and $\mathbf{p} \in \ell(\mathbf{c}_i[1])$. Hence, $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0), i \models \mathbf{p}$.
$\phi = \psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$. By definition, $\mathit{Check}_{\psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)}(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}) = \exists \mathsf{X}. \mathit{Conf}(\mathsf{Z}_t, \mathsf{Z}_p, \mathsf{Z}_{it}, \mathsf{X}_0, \mathsf{y}, \mathsf{X}) \wedge \psi(\mathsf{x}^1, \ldots, \mathsf{x}^{n+1})$. Let us assume that for some i, $\rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0), i \models \psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$. Then $(\mathbf{c}_i[2], \ldots \mathbf{c}_i[n+1] \models \psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)$ with $\mathbf{c}_i = \rho(P_{\mathbf{v}_t, \mathbf{v}_p}, \mathbf{m}_{\mathbf{v}_{it}}, \mathbf{c}_0)[i]$. From Lemma 7 we know that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c}_i \models \mathit{Conf}$ and $\psi(\mathbf{c}_i[2], \ldots, \mathbf{c}_i[n+1])$ is true. Thus, $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i \models \mathit{Check}_{\psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)}$.
On the other hand let us assume that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i \models \mathit{Check}_{\psi(\mathsf{x}_1, \ldots, \mathsf{x}_n)}$ for some i. Thus, there exists a configuration $\mathbf{c}_i$ such that $\mathbf{v}_t, \mathbf{v}_p, \mathbf{v}_{it}, \mathbf{c}_0, i, \mathbf{c}_i \models \mathit{Conf}$ and $\psi(\mathbf{c}_i[2], \ldots, \mathbf{c}_i[n+1])$. In that case by Lemma 7, $\mathbf{c}_i =$

$\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0)[i]$ and $\mathbf{c}_i[2],\ldots,\mathbf{c}_i[n+1] \models \psi(\mathsf{x}_1,\ldots,\mathsf{x}_n)$. Hence, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i$
$\models \psi(\mathsf{x}_1,\ldots,\mathsf{x}_n)$.

Induction Let us assume that for some i and k, we have $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \phi$ iff
$\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_\phi(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$ for all $j \leq k$ and $|\phi| = j$. We will
prove the same for the case when $|\phi| = k+1$.

- $\mathsf{X}\phi$: First we assume that $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \mathsf{X}\phi$ for some i. By sat-
  isfaction relation, we have that $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i+1 \models \phi$. Since, for
  both $|\phi| < k$, by induction hypothesis, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i+1 \models Check_\phi(\mathsf{Z}_t,\mathsf{Z}_p,$
  $\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$. We can hence deduce that $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\mathsf{X}\phi}(\mathsf{Z}_t,\mathsf{Z}_p$
  $,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$.
  On the other hand, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\mathsf{X}\phi}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$ for some
  i. By definition of $Check_{\mathsf{X}\phi}$, we have that $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i+1 \models Check_{\mathsf{X}\phi}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$.
  By induction hypothesis, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i+1 \models \phi$. From the satis-
  faction relation we get that, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \mathsf{X}\phi$.
- $\mathsf{E}\phi$. Let us assume that $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \mathsf{E}\phi$ for some i. By defi-
  nition of the satisfaction relation there exists a run ρ' such that $\rho'[0] =$
  $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0)[i] = \mathbf{c}_i$ and $\rho',0 \models \phi$. By Lemma 4, we know that
  there exists an iterated path schema $\langle P,\mathbf{m}\rangle$ such that $trans(\rho') = trans($
  $\langle P,\mathbf{m}\rangle)$. From Lemma 5, we get that there exists vectors $\mathbf{v}'_t,\mathbf{v}'_p,\mathbf{v}'_{it}$
  such that $trans(\rho') = trans(\langle P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}_{\mathbf{v}'_{it}}\rangle)$ and ρ' can be represented
  as $\rho'(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}_{\mathbf{v}'_{it}},\mathbf{c}_i)$. Thus, $\rho'(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}_{\mathbf{v}'_{it}},\mathbf{c}_i),0 \models \phi$. Since, $\phi \leq k$, by
  induction hypothesis, $\mathbf{v}'_t,\mathbf{v}'_p,\mathbf{v}'_{it},\mathbf{c}_i,0 \models \mathsf{y} = 0 \wedge Check_\phi(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$.
  Moreover, since $\rho'(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}'_{\mathbf{v}'_{it}},\mathbf{c}_i)$ is a run, $\mathbf{v}'_t,\mathbf{v}'_p,\mathbf{v}'_{it},\mathbf{c}_i \models Run$. Hence,
  we have that, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\mathsf{E}\phi}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$
  Now we assume that $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\mathsf{E}\phi}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$. Thus,
  by definition of $Check_{\mathsf{E}\phi}$, there exist $\mathbf{v}'_t,\mathbf{v}'_p,\mathbf{v}'_{it}$ such that $\rho(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}_{\mathbf{v}'_{it}},\mathbf{c}_i)$
  is a run and $\mathbf{v}'_t,\mathbf{v}'_p,\mathbf{v}'_{it},\mathbf{c}_i,0 \models Check_\phi$ where $\mathbf{c}_i = \rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0)[i]$.
  Again, by induction hypothesis, $\rho(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}'_{\mathbf{v}'_{it}},\mathbf{c}_i),0 \models \phi$. Since, there
  exists a run $\rho(P_{\mathbf{v}'_t,\mathbf{v}'_p},\mathbf{m}'_{\mathbf{v}'_{it}},\mathbf{c}_i)$ from c_i satisfying ϕ, by definition of the
  satisfaction relation, $\rho'(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \mathsf{E}\phi$.
- $\phi\mathsf{U}\phi'$. First we assume that $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \phi\mathsf{U}\phi'$ for some i. By
  satisfaction relation, there exists $j \geq i$ such that $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),j \models$
  ϕ' and for all $i \leq l < j$, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),l \models \phi$. Since, for both
  $|\phi|,|\phi'| < k$, by induction hypothesis, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,j \models Check_{\phi'}(\mathsf{Z}_t,\mathsf{Z}_p,$
  $\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$ and for all $i \leq l < j$, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,l \models Check_\phi(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$.
  We can hence deduce that $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\phi\mathsf{U}\phi'}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$.
  On the other hand, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,i \models Check_{\phi\mathsf{U}\phi'}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$ for
  some i. By definition of $Check_{\phi\mathsf{U}\phi'}$, we have that there exists $i \leq j$
  such that $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,j \models Check_{\phi'}(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$ and for all $i \leq j <$
  j, $\mathbf{v}_t,\mathbf{v}_p,\mathbf{v}_{it},\mathbf{c}_0,l \models Check_\phi(\mathsf{Z}_t,\mathsf{Z}_p,\mathsf{Z}_{it},\mathsf{X}_0,\mathsf{y})$. By induction hypothesis,
  $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),j \models \phi'$ and for all $i \leq l < j$, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),l \models \phi$.
  From the satisfaction relation we get that, $\rho(P_{\mathbf{v}_t,\mathbf{v}_p},\mathbf{m}_{\mathbf{v}_{it}},\mathbf{c}_0),i \models \phi\mathsf{U}\phi'$.
- The other cases with temporal operator X or Boolean connectives can be
  proved in a similar manner.

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