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When Model-Checking Freeze LTL over Counter Machines Becomes Decidable ⋆

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Abstract. We study the decidability status of model-checking freeze LTL over various subclasses of counter machines for which the reachability problem is known to be decidable (reversal-bounded counter machines, vector additions systems with states, flat counter machines, one-counter machines). In freeze LTL, a register can store a counter value and at some future position an equality test can be done between a register and a counter value. Herein, we complete an earlier work started on one-counter machines by considering other subclasses of counter machines, and especially the class of reversal-bounded counter machines. This gives us the opportuniy to provide a systematic classification that distinguishes determinism vs. nondeterminism and we consider subclasses of formulae by restricting the set of atomic formulae or/and the polarity of the occurrences of the freeze operators, leading to the flat fragment.

1 Introduction

Counter machines. Counter machines are ubiquitous computational models that provide a natural class of infinite-state transition systems, suitable for modeling various applications such as embedded systems [3], broadcast protocols [22], time granularities [14] and programs with pointer variables [8], to quote a few examples. They are also known to be closely related to data logics for which decision procedures can be designed relying on those for counter machines, see e.g. remarkable examples in [7,5]. When dealing with this class of models, most interesting reachability problems are undecidable but subclasses leading to decidability have been designed including reversal-bounded counter machines [32], one-counter machines [33], flat counter machines [23] and vector addition systems with states (see e.g. [45]).
Model-checking with Freeze LTL. In order to verify properties on counter machines, we aim at comparing counter values and we shall use the so-called *freeze* operator. The freeze quantifier in real-time logics has been introduced in the logic TPTL, see e.g. [1]. The formula $x \cdot \phi(x)$ binds the variable x to the time t of the current state: $x \cdot \phi(x)$ is semantically equivalent to $\phi(t)$. This variable-binding mechanism, quite natural when rephrased in first-order logic, is present in various logical formalisms including for example hybrid logics [28,2], freeze LTL [20] and predicate λ-abstraction [25,41]. Freeze

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LTL is a powerful extension of LTL that allows to store counter values in registers. Infinitary satisfiability restricted to one register is already undecidable [20] just as model-checking for nondeterministic one-counter machines [21], which is quite unexpected since one-counter machines seem to be harmless operational models. Moreover, there is some hope that model-checking happens to be more tractable than satisfiability since more constraints are requested on models viewed as runs.
Our contribution. We carry on with the quest started in [21] to determine which classes of counter machines admit decidable model-checking with freeze LTL. In the paper, we consider the above-mentioned classes of counter machines for which the reachability problem is decidable. We provide an exhaustive analysis completing [21]; some results are obtained by adequately adapting known results to our framework or by designing simple reductions. However, at each position, we may have to deal with more than one counter values. Our main technical contributions allow us to establish the following results with a special focus on reversal-bounded counter machines.

- Model-checking freeze LTL (written $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow})$) over deterministic vector addition systems with states and deterministic reversal-bounded counter machines is decidable. However, $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow})$ over reversal-bounded counter machines is undecidable, even when restricted to one register.
- $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow})$ restricted to flat formulae over reversal-bounded counter machines is decidable as well as the restriction to positively flat formulae over one-counter machines, partly by taking advantage of recent results about parameterized one-counter machines from [29].

A complete summary can be found in Section 8. As a nice by-product of the classification we made, we show a tight relationship between reachability problems for parameterized counter machines and model-checking counter machines over the flat fragment of freeze LTL (see Section 7.2). Besides, we believe that the principles underlying our undecidability proof for $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow})$ over reversal-bounded counter machines could be reused for other problems on such counter machines.

Plan of the paper. Section 2 and Section 3 are preliminary sections about counter machines, their subclasses, freeze LTL and their fragment. In Section 4, we establish preliminary results or restate known results from the literature recasted in our context. Undecidability results for VASS and reversal-bounded counter machines are shown in Section 5 whereas we show decidability for subclasses with deterministic counter machines in Section 6. Finally, Section 7 deals with the decidability of model-checking over reversal-bounded counter machines and one-counter machines with flat formulae. Section 8 contains a summary and concluding remarks.

2 Standard Classes of Counter Machines

In this section, we recall standard definitions about various classes of counter machines. We write $\mathbb{N}$ [resp. $\mathbb{Z}$] for the set of natural numbers [resp. integers]. Given a dimension $n \geq 1$ and $k \in \mathbb{Z}$, we write $\mathbf{k}$ to denote the vector with all values equal to k and $\mathbf{e}_i$ to denote the unit vector for $i \in \{1, \ldots, n\}$. We recall that a semilinear set of $\mathbb{N}^n$ is a finite union of linear sets. We often refer to Presburger arithmetic which consists of first-order logic over the structure $\langle \mathbb{N}, 0, \leq, + \rangle$ (and more generally over $\langle \mathbb{Z}, 0, \leq, + \rangle$), details can be found for instance in [44,11]. It is known that a subset of $\mathbb{N}^k$ is semilinear if and only if it is definable by a formula in Presburger arithmetic with k free variables [26].

2.1 Counter machines

In the rest of the paper, a *counter machine* M is defined as a tuple (n, Q, Δ, q_0) where:

- $n \geq 1$ is the *dimension* of M,
- Q is a finite set of *control states*,
- $\Delta \subseteq Q \times G \times A \times Q$ is a finite set of *transitions* where $G = \{\mathtt{zero}, \mathtt{true}\}^n$ is the finite set of *guards* and $A = \{-1, 0, 1\}^n$ is the finite set of *actions*,
- $q_0 \in Q$ is the *initial* control state.

Given a counter machine M, we define the *transition system* $TS(M) = (Q \times \mathbb{N}^n, \rightarrow)$ where $Q \times \mathbb{N}^n$ is the set of *configurations* and $\rightarrow \subseteq (Q \times \mathbb{N}^n) \times (Q \times \mathbb{N}^n)$ is the *transition relation*: for $\langle q, \mathbf{v} \rangle, \langle q', \mathbf{v}' \rangle \in Q \times \mathbb{N}^n$, we have $\langle q, \mathbf{v} \rangle \rightarrow \langle q', \mathbf{v}' \rangle \overset{\text{def}}{\Leftrightarrow}$ there exists a transition $t = (q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ such that:

1. $\mathbf{v}' = \mathbf{v} + \mathbf{a}$,
2. for $1 \leq c \leq n$, $\mathbf{g}(c) = \mathtt{zero}$ implies $\mathbf{v}(c) = 0$.

We write $\xrightarrow{*}$ to denote the reflexive and transitive closure of $\rightarrow$ and the reachability set of M is $\mathrm{Reach}(M) \overset{\text{def}}{=} \{\langle q, \mathbf{v} \rangle \mid \langle q_0, \mathbf{0} \rangle \xrightarrow{*} \langle q, \mathbf{v} \rangle\}$. Observe that this reachability set implicitly depends on the initial configuration $\langle q_0, \mathbf{0} \rangle$: this is all what we need in the sequel. A finite (resp. infinite) *run* in $TS(M)$ is a finite (resp. infinite) sequence $\rho = \langle q_0, \mathbf{0} \rangle \rightarrow \langle q_1, \mathbf{v}_1 \rangle \rightarrow \ldots$. A counter machine M is *deterministic* (also known as *single-path*) whenever for each $\langle q, \mathbf{v} \rangle \in \mathrm{Reach}(M)$, there is at most one configuration $\langle q', \mathbf{v}' \rangle$ such that $\langle q, \mathbf{v} \rangle \rightarrow \langle q', \mathbf{v}' \rangle$. In the sequel, we shall use Minsky machines [43] that form a special class of deterministic 2-counter machines.

We present below two types of decision problems when $\mathcal{C}$ is a class of counter machines. The *reachability problem* for the class $\mathcal{C}$ is defined as follows.

instance: a machine $M \in \mathcal{C}$ and a configuration $\langle q, \mathbf{v} \rangle$.
question: $\langle q_0, \mathbf{0} \rangle \xrightarrow{*} \langle q, \mathbf{v} \rangle$?

Similarly, the *generalized repeated control-state reachability problem* for the class $\mathcal{C}$ is defined as follows.

instance: a counter machine $M \in \mathcal{C}$, N sets $F_1, \ldots, F_N$ of control states.
question: Is there a run of M such that for $1 \leq i \leq N$, there is a control state in F_i that is repeated infinitely often?

1CM. One-counter machines are naturally defined as counter machines of dimension one: they can be used for the verification of cryptographic protocols [39] and to characterize subclasses of context-free languages [4]. They also have nice computational properties, see for instance complexity results about behavioural equivalences in [36].

Various logical formalisms have been introduced to specify the behavior of one-counter machines, including Freeze LTL [21], EF logic [27] and first-order logic with reachability predicate [49]. Moreover, since one-counter automata are equivalent to pushdown systems with a singleton stack alphabet, the results on these systems can help to refine some results about pushdown systems. For instance, the model-checking problem for one-counter automata with the modal μ-calculus has been shown to be in PSPACE [47] whereas the model-checking problems for pushdown automata over the modal μ-calculus and the linear μ-calculus are in EXPTIME. When one-counter machines are enriched by a finite alphabet (so that transitions are labelled), the universality problem is undecidable [33], witnessing that this simple operational model can lead to natural undecidable problems.
VASS. Vector addition systems with states (a.k.a. VASS) are known to be equivalent to Petri nets, see e.g. [45], and they correspond to counter machines without zero-tests, i.e. each guard has no component equal to `zero`. To be precise, we are a bit less liberal than the usual definition since we only consider actions in $\{-1, 0, 1\}^n$ (instead of $\mathbb{Z}^n$) but this does not make a real difference for all the developments made in this paper.
Flat counter machines. A directed graph $G = \langle V, E \rangle$ (with $V \subseteq E \times E$) is said to be *flat* whenever each vertex belongs to at most one simple cycle (path for which the initial and final vertices coincide and no edge is repeated). A counter machine (n, Q, Δ, q_0) is *flat* whenever (1) between two control states there is at most one transition and (2) the directed graph $\langle Q, \{\langle q, q' \rangle \in Q^2 : (q, \mathbf{g}, \mathbf{a}, q') \in \Delta\} \rangle$ is flat. Reachability problems have been considered for flat counter machines in [6,13,23]; for instance it is proved that flat counter machines have an effectively computable semilinear set [6,23], see also [10].

2.2 Reversal-bounded counter machines

The class of *reversal-bounded* counter machines has been introduced in [32] by considering the following restriction: each counter performs only a bounded number of alternations between increasing and decreasing mode. This class of counter machines is particularly interesting because it has been shown that each reversal-bounded counter machine has a semilinear reachability set that can be effectively computed. We present below a more general class, introduced in [24], for which bounding the number of alternations is only considered above a given bound. This is the notion we adopt in the rest of the paper. The members of subclass introduced in [32] are called *Ibarra* reversal-bounded counter machines.

Given a bound $b \in \mathbb{N}$, we consider the number of alternations between increasing and decreasing mode when a counter is above b. Given a counter machine $M = (n, Q, \Delta, q_0)$, let us define the *modal* transition system

$$TS_b(M) = (Q \times \mathbb{N}^n \times \{\mathrm{DEC}, \mathrm{INC}\}^n \times \mathbb{N}^n, \rightarrow_b).$$

Intuitively, a configuration $(q, \mathbf{v}, \mathbf{mode}, \sharp\mathbf{alt})$ records a standard configuration of $TS(M)$, **mode** stores the current mode (either decreasing or increasing) for each counter and $\sharp$**alt** stores the number of alternations above b for each counter. The transition relation $\rightarrow_b$ is defined as follows: $(q, \mathbf{v}, \mathbf{mode}, \sharp\mathbf{alt}) \rightarrow_b (q', \mathbf{v}', \mathbf{mode}', \sharp\mathbf{alt}') \stackrel{\text{def}}{\Leftrightarrow}$ the following conditions hold: $(q, \mathbf{v}) \rightarrow (q', \mathbf{v}')$ and for $1 \leq c \leq n$, the relation described by the following table is verified:

$\mathbf{v}(c) - \mathbf{v}'(c)$	$\mathbf{mode}(c)$	$\mathbf{mode}'(c)$	$\mathbf{v}(c)$	$\sharp\mathbf{alt}'(c)$
> 0	DEC	DEC	$-$	$\sharp\mathbf{alt}(c)$
> 0	INC	DEC	$\leq b$	$\sharp\mathbf{alt}(c)$
> 0	INC	DEC	$> b$	$\sharp\mathbf{alt}(c) + 1$
< 0	INC	INC	$-$	$\sharp\mathbf{alt}(c)$
< 0	DEC	INC	$\leq b$	$\sharp\mathbf{alt}(c)$
< 0	DEC	INC	$> b$	$\sharp\mathbf{alt}(c) + 1$
$= 0$	DEC	DEC	$-$	$\sharp\mathbf{alt}(c)$
$= 0$	INC	INC	$-$	$\sharp\mathbf{alt}(c)$

Definition 1. *Let $b, k \in \mathbb{N}$. A counter machine M is k-*reversal-b-bounded $\overset{def}{\Leftrightarrow}$ *whenever $(q_0, \mathbf{0}, \mathrm{INC}, \mathbf{0}) \xrightarrow{*}_b (q, \mathbf{v}, \mathbf{mode}, \sharp\mathbf{alt})$, we have $\sharp\mathbf{alt} \leq \mathbf{k}$.*

This definition can be slightly refined: a counter machine M is *reversal-bounded* if there exist $k, b \in \mathbb{N}$ such that M is k-reversal-b-bounded. In the sequel, when reversal-bounded counter machines are part of the instances of some decision problems, we assume that they come with their k and b. As mentioned in [24], the above-defined class of reversal-bounded counter machines contains those defined in [32] and it also contains the counter machines for which the set of reachable configurations is finite.

Theorem 2. *[24] Reversal-bounded counter machines have an effectively computable reachability set.*

In [16], it is proved that the generalized repeated control-state reachability problem is decidable when the instances are made of an Ibarra reversal-bounded counter machine and one set of control states. This result has been extended to reversal-bounded counter machines in [46]. Note that we can easily reduce the generalized reachability problem with $N \geq 1$ sets of control states to its restriction to only one set (in the same generalized Büchi automata can be reduced to Büchi automata).

Corollary 3. *The generalized repeated control-state reachability problem for reversal-bounded counter machines is decidable.*

3 LTL with the Freeze Operator

In this section, we present a variant of temporal logic LTL with registers (also known as Freeze LTL) in order to reason about runs from counter machines. In [21], LTL with registers is used to specify properties about one-counter machines. The datum stored in a register is the current counter value and equality tests are performed between a register value and the current counter value. When dealing with counter machines, a register can store the value of a counter c and test it later against the value of counter c' with possibly $c \neq c'$. Below, we present different ways to restrict the equality tests between registers and counters.

Given a finite set Q of control states (possibly empty) and $n \geq 1$, the formulae of the logic $\mathrm{LTL}^{\downarrow}[Q,n]$ are defined as follows:

$$\phi ::= q \mid \uparrow_r^c \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \mathtt{U} \phi \mid \phi \mathtt{R} \phi \mid \mathtt{X}\phi \mid \downarrow_r^c \phi$$

where $q \in Q$, $c \in \{1,\ldots,n\}$ and $r \in (\mathbb{N} \setminus \{0\})$. Intuitively, the modality $\downarrow_r^c$ is used to store the value of the counter c into the register r; the atomic formula $\uparrow_r^c$ holds true if the value stored in the register r is equal to the current value of the counter c. An occurrence of $\uparrow_r^c$ within the scope of some freeze quantifier $\downarrow_r^c$ is bound by it; otherwise it is free. A sentence is a formula with no free occurrence of any $\uparrow_r^c$.

Models of $\mathrm{LTL}^{\downarrow}[Q,n]$ are runs of transition systems from counter machines of dimension n and with a set of control states containing Q. Given a counter machine (n, Q', Δ, q_0) with $Q \subseteq Q'$ and a run ρ, we write $|\rho|$ to denote its *length* in $\omega + 1$ and the ith configuration ($0 \leq i < |\rho|$) is denoted by $\langle q_i, \mathbf{v}_i \rangle$. A *register valuation* f is a finite partial map from $\mathbb{N} \setminus \{0\}$ to $\mathbb{N}$. Note that whenever $f(r)$ is undefined, the atomic formula $\uparrow_r^c$ is interpreted as false. Given a run ρ and a position $0 \leq i < |\rho|$, the satisfaction relation $\models$ is defined as follows (Boolean clauses are omitted):

$$\begin{array}{rcl}
\rho, i \models_f q & \stackrel{\text{def}}{\Leftrightarrow} & q_i = q \\
\rho, i \models_f \uparrow_r^c & \stackrel{\text{def}}{\Leftrightarrow} & r \in \mathrm{dom}(f) \text{ and } f(r) = \mathbf{v}_i(c) \\
\rho, i \models_f \mathtt{X}\phi & \stackrel{\text{def}}{\Leftrightarrow} & i+1 < |\rho| \text{ and } \rho, i+1 \models_f \phi \\
\rho, i \models_f \phi_1 \mathtt{U} \phi_2 & \stackrel{\text{def}}{\Leftrightarrow} & \text{for some } i \leq j < |\rho|,\ \rho, j \models_f \phi_2 \\
 & & \text{and for all } i \leq j' < j, \text{ we have } \rho, j' \models_f \phi_1 \\
\rho, i \models_f \phi_1 \mathtt{R} \phi_2 & \stackrel{\text{def}}{\Leftrightarrow} & \text{for all } i \leq j < |\rho|,\ \rho, j \models_f \phi_2 \\
 & & \text{or for some } i \leq j < |\rho|,\ \rho, j \models_f \phi_1 \\
 & & \quad \text{and for all } i \leq k \leq j,\ \rho, k \models_f \phi_2 \\
\rho, i \models_f \downarrow_r^c \phi & \stackrel{\text{def}}{\Leftrightarrow} & \rho, i \models_{f[r \mapsto \mathbf{v}_i(c)]} \phi
\end{array}$$

$f[r \mapsto \mathbf{v}_i(c)]$ denotes the register valuation equal to f except that the register r is mapped to $\mathbf{v}_i(c)$. In the sequel, we omit the subscript "f" in $\models_f$ when sentences are involved. We use the standard abbreviations for the temporal operators ($\mathtt{G}$, $\mathtt{F}$, ...) and for the Boolean operators and constants ($\Rightarrow$, $\top$, $\perp$, ...).

We defined below fragments of $\mathrm{LTL}^{\downarrow}[Q,n]$ by restricting the use of the freeze operators. The *strict* fragment, written $\mathrm{LTL}^{\downarrow,s}[Q,n]$, consists in associating a unique counter to each register (to store and to test). More precisely, a formula ϕ in $\mathrm{LTL}^{\downarrow,s}[Q,n]$ verifies the following syntactic property: if $\downarrow_r^c \psi$ is a subformula of ϕ, then ϕ has not subformulae of the form either $\uparrow_r^{c'}$ or $\downarrow_r^{c'} \psi'$ with $c \neq c'$. We also write $\mathrm{LTL}[Q]$ to denote the fragment of $\mathrm{LTL}^{\downarrow}[Q,n]$ in which the atomic formulae of the form $\uparrow_r^c$ are forbidden (and therefore $\downarrow_r^c$ becomes also useless).

Model-checking problems. The infinitary (existential) model-checking problem over counter machines, written $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$, is defined as follows:

instance: A counter machine $M = (n, Q', \Delta, q_0)$ and a sentence $\phi \in \mathrm{LTL}^{\downarrow}[Q,n]$ with $Q \subseteq Q'$;

question: Is there an infinite run ρ such that $\rho, 0 \models \phi$? If the answer is "yes", we write $M \models^{\omega} \phi$.

The subproblem of $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ with formulae restricted to $\mathrm{LTL}^{\downarrow,s}[Q,n]$ is written $\mathrm{MC}^\omega(\mathrm{LTL}^{\downarrow,\mathrm{s}}[\cdot,\cdot])$. Given $n \geq 1$, we write $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\mathrm{n}])$ to denote the subproblem of $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ with counter machines of dimension at most n. Similarly, we write $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\emptyset,\cdot])$ to denote the subproblem of $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ with no atomic formula made of control states. Similar notations are used with other fragments of $\mathrm{LTL}^\downarrow[Q,n]$. In this existential version of model checking, this problem can be viewed as a variant of satisfiability in which satisfaction of a formula can be only witnessed within a specific class of data words, namely the runs of the counter machine. Note that results for the universal version of model checking will follow easily from those for the existential version when considering fragments closed under negation or deterministic counter machines.

Flat formulae. We say that the occurrence of a subformula in a formula is *positive* if it occurs under an even number of negations, otherwise it is *negative*. Let $\mathcal{L}$ be a fragment of $\mathrm{LTL}^\downarrow[Q,n]$. The *flat fragment of* $\mathcal{L}$, written flat-$\mathcal{L}$, is the restriction of $\mathcal{L}$ where, for any occurrence of $\phi_1 \mathtt{U} \phi_2$ [resp. $\phi_2 \mathtt{R} \phi_1$], if it is positive then the freeze operator $\downarrow$ does not occur in ϕ_1, and if it is negative then the freeze operator $\downarrow$ does not occur in ϕ_2. A formula is *positively flat* when it is flat and no occurrence of the freeze operator $\uparrow$ occurs in the scope of an odd number of negations. For example, the formula below belongs to the positively flat fragment and it states that sometimes there is a value of the counter 1 such that (1) infinitely often counter 2 takes that value if and only if infinitely often counter 3 takes that value and (2) from some future position, the counter 4 has always that value:

$$\mathtt{F} \downarrow_1^1 [(\mathtt{GF} \uparrow_1^2 \Leftrightarrow \mathtt{GF} \uparrow_1^3) \wedge \mathtt{FG} \uparrow_1^4]$$

Considering flat fragments remains a standard means to regain decidability: for instance flat fragments of LTL variants have been studied in [15,12] and in the presence of the freeze operator in [20,9] (see also in [34, Section 5] the design of a flat logical temporal language for model-checking pushdown machines). Section 7 shall illustrate that flatness can lead to decidability but this is not always the case.

4 Preliminary Results

In this section, we present preliminary results that will be helpful to strenghten forthcoming results and we present results for flat counter machines and one-counter machines based on existing works. We shall study the effects of restricting the set of atomic formulae, for instance by allowing only atomic formulae that are control states [resp. that are of the form $\uparrow_r^c$].

4.1 Purification, or how to get rid of control states

Control states can be viewed as an internal piece of information about the counter machines and therefore, it is interesting to understand whether the absence of control states among the set of atomic formulae (called herein *purification*) makes a difference. Lemma 4 below roughly shows that control states can be always encoded by patterns for various classes of counter machines.

Lemma 4.
Given a counter machine $M = (n, Q, \Delta, q_0)$ and a sentence ϕ in $\mathrm{LTL}^{\downarrow}[Q, n]$, one can build in logspace a counter machine $M_P = (n+1, Q_P, \Delta_P, q_0)$ and a formula $\phi_P \in \mathrm{LTL}^{\downarrow}[\emptyset, n+1]$ such that $M \models^{\omega} \phi$ iff $M_P \models^{\omega} \phi_P$. Moreover,

- *M is deterministic [resp. reversal-bounded, flat] iff M_P is deterministic [resp. reversal-bounded, flat].*
- *$\phi \in \mathrm{LTL}^{\downarrow,s}[Q, n]$ iff $\phi_P \in \mathrm{LTL}^{\downarrow,s}[\emptyset, n+1]$.*

Proof. Let $M = (n, Q, \Delta, q_0)$ with $Q = \{q_1, \ldots, q_t\}$ and ϕ be a formula in $\mathrm{LTL}^{\downarrow}[Q, n]$. We shall build (in logarithmic space) a counter machine $M_P = (n+1, Q_P, \Delta_P, q_0)$ and a formula $\phi_P \in \mathrm{LTL}^{\downarrow}[\emptyset, n+1]$ such that $M \models^{\omega} \phi$ iff $M_P \models^{\omega} \phi_P$.

Intuitively, the counter machine M_P is built from M by adding an extra counter whose behavior in M_P encodes the control states from M. More precisely, when we are in a control state q_i, the value of the counter $n+1$ is incremented once and then remains constant during the $(i+1)$ next transitions (without changing the other original counters) and then is again incremented twice. Figure 1 illustrates the behavior of the counter $n+1$ when encoding a transition of the form $(q_i, \mathbf{g}, \mathbf{a}, q_j)$. The use of the

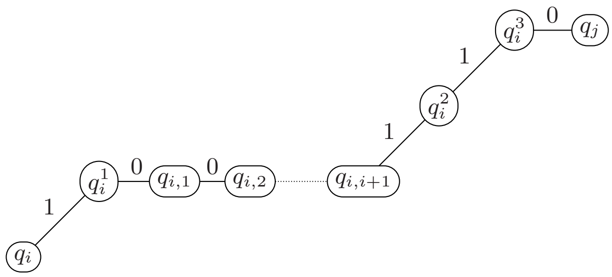

Fig. 1. Purification: projection on the $(n+1)$th counter

freeze quantifiers enables us to identify the control states q_i. Since the additional counter does only increase, this will guarantee that M is reversal-bounded iff M_P is reversal-bounded. Furthermore, it should be clear that M_P is deterministic [resp. flat] iff M is deterministic [resp. flat].

Let us define formally the machine M_P.

- $Q_P \stackrel{\text{def}}{=} Q \uplus Q'$ with:

  $Q' = \{q_i^1, q_i^2, q_i^3 \mid i \in \{1, \ldots, t\}\} \cup \{q_{i,j} \mid i \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, i+1\}\}$

- Δ_P is the smallest relation (with respect to set inclusion) satisfying the following properties:
  - for all $i \in \{1, \ldots, t\}$,
    - $(q_i, \mathbf{true}, \mathbf{e}_{n+1}, q_i^1), (q_i^1, \mathbf{true}, \mathbf{0}, q_{i,1}) \in \Delta_P$,

  * for all $j \in \{1, \ldots, i\}$, $(q_{i,j}, \mathbf{true}, \mathbf{0}, q_{i,j+1}) \in \Delta_P$,
  * $(q_{i,i+1}, \mathbf{true}, \mathbf{e}_{n+1}, q_i^2), (q_i^2, \mathbf{true}, \mathbf{e}_{n+1}, q_i^3) \in \Delta_P$,
- for each $(q_i, \mathbf{g}, \mathbf{a}, q_j) \in \Delta$, $(q_i^3, \mathbf{g}', \mathbf{a}', q_j) \in \Delta_P$ such that for all $c \in \{1, \ldots, n\}$, $\mathbf{a}'(c) = \mathbf{a}(c)$, $\mathbf{g}'(c) = \mathbf{g}(c)$ and, $\mathbf{a}'(n+1) = 0$ and $\mathbf{g}'(n+1) = \mathtt{true}$.

We are now in position to present a formula ϕ_{state} that holds true exactly on configurations belonging to some runs of M_P:

$$\phi_{state} = \downarrow_1^{n+1} \mathtt{X}(\neg \uparrow_1^{n+1} \wedge \downarrow_1^{n+1} \mathtt{X}(\uparrow_1^{n+1} \wedge \mathtt{X} \uparrow_1^{n+1}))$$

When ϕ belongs to a strict fragment and if we wish to preserve strictness, in the above formula we replace the register 1 by a new register not occurring in ϕ. Hence, for all runs ρ of M_P and $0 \leq j < |\rho|$, we have that $\rho, j \models \phi_{state}$ if and only if $\rho, j \models q$ for some $q \in Q$ and $j < |\rho| + 2$.

For $i \in \{1, \ldots, t\}$, let us define the formula ϕ_i as follows:

$$\phi_i = \mathtt{X} \downarrow_1^{n+1} ((\bigwedge_{k \in \{1, \ldots, i+1\}} \mathtt{X}^k \uparrow_1^{n+1}) \wedge \mathtt{X}^{i+2} \neg \uparrow_1^{n+1})$$

One can check that for all runs ρ of M_P and $0 \leq j < |\rho|$, we have that $\rho, j \models \phi_{state} \wedge \phi_i$ if and only if $\rho, j \models q_j$ and $j < |\rho| + 2$. As above, if we have further syntactic restrictions we may use $\downarrow_r^{n+1}$ where r is a new register.

Now, let define ϕ_P with the help of a translation $T(\cdot)$ such that $\phi_P = T(\phi)$ and, $T(\cdot)$ is homomorphic for Boolean operators and $\downarrow_r^c$. Basically, $T(\cdot)$ performs a simple relativization (we omit the clauses for Boolean connectives and for $\mathtt{R}$):

- $T(\uparrow_r^c) = \uparrow_r^c$; $T(q_i) = \phi_{state} \wedge \phi_i$; $T(\mathtt{X}\psi) = \mathtt{X}(\neg\phi_{state} \mathtt{U} (\phi_{state} \wedge T(\psi)))$,
- $T(\psi \mathtt{U} \psi') = (\phi_{state} \Rightarrow T(\psi)) \mathtt{U} (\phi_{state} \wedge T(\psi'))$.

□

The reduction in the proof of Lemma 4 does not preserve the number of counters; however, a purification lemma can be also established for the class of one-counter machines as shown in [21]. By the way, the construction in [21] could be also adapted to encode control states by patterns however, it does not preserve reversal-boundedness.

4.2 Restricting the atomic formulae to control states

Before considering decidability issues with the freeze operator, it is legitimate to wonder what happens when the atomic formulae are restricted to control states. We show below that for all subclasses of counter machines considered in this paper, this restriction leads to decidability (for flat counter machines, the proof is postponed to the next subsection). Basically, the proof is a consequence of the two following properties: LTL formulae can be translated into equivalent Büchi automata and repeated reachability problem is decidable for the concerned subclasses of counter machines.

Let $M = (n, Q, \Delta, q_0)$ be a counter machine and $\mathcal{A} = (Q', \delta, q_0, F)$ be a Büchi automaton over the alphabet Q ($\delta \subseteq Q' \times Q \times Q'$ and $F \subseteq Q'$). We write $M \otimes \mathcal{A}$ to denote the counter machine $(Q \times Q', \Delta', \langle q_0, q_0' \rangle)$ defined as follows: $(\langle q, q' \rangle, \mathbf{g}, \mathbf{a}, \langle q_1, q_1' \rangle) \in$

Δ' iff there exist $(q, \mathbf{g}, \mathbf{a}, q_1) \in \Delta$ and $(q', q, q'_1) \in \delta$. Observe that M is reversal-bounded [resp. one-counter, VASS] iff $M \otimes \mathcal{A}$ is reversal-bounded [resp. one-counter, VASS]. Given a formula $\phi \in \mathrm{LTL}[Q]$, one can effectively build a Büchi automaton $\mathcal{A}_\phi$ over the alphabet Q such that the language accepted by $\mathcal{A}_\phi = (Q', \delta, q_0, F_\phi)$ is precisely the sequence of ω-sequences satisfying ϕ, see e.g. [50].

Lemma 5. *Given a counter machine $M = (n, Q, \Delta, q_0)$ and a formula $\phi \in \mathrm{LTL}[Q]$, $M \models^\omega \phi$ iff there is a run of $M \times \mathcal{A}_\phi$ such that a control state in $Q \times F_\phi$ is repeated infinitely often.*

The proof is by an easy verification by using the properties of $\mathcal{A}_\phi$.

Theorem 6. $\mathrm{MC}^\omega(\mathrm{LTL}[\cdot])$ *restricted to one-counter machines, VASS, and reversal-bounded counter machines is decidable.*

Proof. Given a reversal-bounded counter machine M and a formula $\phi \in \mathrm{LTL}[Q]$, checking whether $(\star)$ $M \otimes \mathcal{A}_\phi$ has a run with a control state in $Q \times F_\phi$ repeated infinitely can be decided thanks to Corollary 3. Alternatively, when M is a one-counter machine, we can decide $(\star)$, see e.g. [16, Theorem 4] (the repeated reachability problem for one-counter machines being even in NLOGSPACE, see e.g. [18]). Finally, assuming that M is a VASS checking $(\star)$ can be decided thanks to [37]. It is sufficient to show that the repeated reachability problem for VASS is decidable, which is the case by [37, Theorem 7.27] and even in EXPSPACE by [30, Theorem 5.4]. □

4.3 Existing results for two subclasses

In this paper, we wish to provide a complete classification with respect to the above-mentioned subclasses. The two following results are known results recasted in our context. First, we observe that $\mathrm{LTL}^\downarrow[Q, n]$ can be viewed as a fragment of the temporal logic FOCTL*(Pr) [17] which extends the logic CTL* by allowing the use of Presburger formulae as atomic propositions to describe sets of configurations for a counter machine. Since model-checking FOCTL*(Pr) over flat counter machines is decidable [17], we establish the following theorem.

Theorem 7. $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot, \cdot])$ *restricted to flat counter machines is decidable.*

Proof. Let $M = (n, Q, \Delta, q_0)$ be a flat counter machine and $\phi \in \mathrm{LTL}^\downarrow[Q, n]$. The counter machine M is *admissible* in the sense of [17, Definition 5] plus the fact that Presburger formulae used for accelerations can be effectively computed thanks to [23]. Decidability of $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot, \cdot])$ can be then established by translating ϕ into a formula ϕ' of FOCTL*(Pr) and then using the decidability result in [17, Theorem 4]. The temporal logic FOCTL*(Pr) is a variant of CTL* with atomic formulae made of Presburger formulae on counters and with first-order quantification over counter values. The formula ϕ' is equal to $\mathtt{E}\,\mathbf{t}(\phi; (z_1, \ldots, z_N))$ where $\mathtt{E}$ quantifies existentially over runs and ϕ contains at most N registers. The map $\mathbf{t}(\cdot)$ is homomorphic for the Boolean and temporal operators:

– $\mathbf{t}(\uparrow^c_r, (z_1, \ldots, z_N)) \stackrel{\text{def}}{=} (z_r = x_c)$ where x_c is variable associated to counter c,

– $\mathtt{t}(\downarrow_r^c \psi; (z_1, \ldots, z_n)) \stackrel{\text{def}}{=} \exists\, z'_r\ (z'_r = x_c \wedge \mathtt{t}(\psi; (z_1, \ldots, z_{r-1}, z'_r, z_{r+1}, \ldots z_n))$.

One can show that $M \models^\omega \phi$ iff $M \models^\omega \phi'$. □

Moreover, we know the following results concerning the model-checking of LTL with registers over one-counter machines.

Theorem 8. *[21]*

(I) $\mathrm{MC}^{<\omega}(\mathrm{LTL}^{\downarrow}[\cdot, 1])$ *and* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot, 1])$ *are undecidable problems.*
(II) $\mathrm{MC}^{<\omega}(\mathrm{LTL}^{\downarrow}[\cdot, 1])$ *and* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot, 1])$ *restricted to deterministic one-counter machines are* PSPACE*-complete problems.*

5 Nondeterministic Counter Machines

Herein, we consider the model-checking problems over $\mathrm{LTL}^{\downarrow}[Q, n]$ for nondeterministic counter machines. We have seen that for the class of one-counter machines the problem is undecidable (see Theorem 8(I)) whereas it is decidable for flat counter machines (see Theorem 7).

5.1 VASS

First, we observe that zero-tests can be easily encoded in $\mathrm{LTL}^{\downarrow}[Q, n]$ by first storing the initial value of counters in some register r_0 and then performing a zero-test on counter c with the atomic formula $\uparrow_{r_0}^{c}$.

Theorem 9. $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot, \cdot])$ *restricted to VASS and to positively flat formulae with at most one register is undecidable.*

Proof. Let M be a deterministic Minsky machine (a special form of two-dimensional counter machine) with final control state q_f and no transition exiting from it. For each transition t, we write $\mathtt{zero}_t$ to denote the set of counters on which are performed the zero-test. Let M' be the VASS obtained from M by replacing systematically `zero` by `true` in guards and by adding a self-loop on q_f with guard **true** and action **0**. One can show that M can reach the control state q_f iff

$$M' \models^\omega \mathtt{F} q_f \wedge \downarrow_1^1 \bigwedge_{t=(q,\mathbf{g},\mathbf{a},q')\in\Delta} \mathtt{G}(q \wedge \mathtt{X} q' \Rightarrow \bigwedge_{c\in\mathtt{zero}_t} \uparrow_1^c).$$

□

5.2 Reversal-bounded counter machines

As far as reversal-bounded counter machines are concerned, we have the following result:

Theorem 10. $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot, 4])$ *restricted to reversal-bounded counter machines and to formulae with at most one register is undecidable.*

To prove this result, we present a reduction from the halting problem for Minsky machines; note that a similar reduction is used in [35] in order to prove that in reversal-bounded counter machines extended with equality tests between distinct counters, the reachability problem is undecidable. Indeed, assuming that guards of the form $c = c'$ are allowed, each counter c from the Minsky machine provides two increasing counters c^{inc} and c^{dec}, that counts the number of incrementations on c and the number of decrementations, respectively. Zero-test for c is simulated by a test $c^{inc} = c^{dec}$, that is logically equivalent to $\downarrow_1^{c^{dec}} \uparrow_1^{c^{inc}}$ in $\mathrm{LTL}^{\downarrow}[\cdot, \cdot]$.

Proof. Let $M = (2, Q, \Delta, q_0)$ be a Minsky machine (deterministic counter machine with two counters) and $q_F \in Q$ be a final control state with no transition from it. Without any loss of generality, we can assume that if $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ performs a decrementation, then the transition is of the form $(q, \mathtt{true}, -\mathbf{e}_c, q')$ for some $c \in \{1, 2\}$. Moreover, for $q, q' \in Q$, the set $\{\langle \mathbf{g}, \mathbf{a}\rangle : (q, \mathbf{g}, \mathbf{a}, q') \in \Delta\}$ contains at most one element. Let us build the reversal-bounded counter machine $M = (4, Q', \Delta', (q_0)_\emptyset)$ as follows:

- $Q' = \{q_X : q \in Q,\ X \subseteq \{1, 2\}\}$ (X records on which counter of M zero-test is needed next),
- Δ' is the smallest set of transitions satisfying the conditions below:
  - for $X \subseteq \{1, 2\}$, $((q_0)_\emptyset, \mathbf{true}, \mathbf{0}, (q_0)_X) \in \Delta'$,
  - for all $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$, we have $(q_1, \mathbf{true}, \mathbf{a}', q_1') \in \Delta'$ assuming that
    - $q_1 = q_X$ with $X = \{c \in \{1, 2\} : \mathbf{g}(c) = \mathtt{zero}\}$,
    - for $c \in \{1, 2\}$,
      - $\mathbf{a}(c) = 1$ implies $\mathbf{a}'(c) = 1$ and $\mathbf{a}'(c + 2) = 0$,
      - $\mathbf{a}(c) = -1$ implies $\mathbf{a}'(c) = 0$ and $\mathbf{a}'(c + 2) = 1$,
      - $\mathbf{a}(c) = 0$ implies $\mathbf{a}'(c) = \mathbf{a}'(c + 2) = 0$.
  - for $X \subseteq \{1, 2\}$, $((q_F)_X, \mathbf{true}, \mathbf{0}, (q_F)_X) \in \Delta$ (final loops).

By construction, the counter machine M' is reversal-bounded since the four counters only increase. The idea behind this construction is that the first [resp. second] and the third [resp. fourth] counters of M' respectively count the number of incrementations and decrementations of the first [resp. second] counter of M. No zero-test is performed in M'; in order to simulate a zero-test in M, we would need to test equality between two counters, which is not allowed in our models. Consequently, we encode these equality tests by formulae.

Let us build a formula ϕ in $\mathrm{LTL}^{\downarrow}[Q', 4]$ such that $M' \models^\omega \phi$ iff the control state q_F can be reached from the initial configuration of M. We consider the following auxiliary formulae ($c \in \{1, 2\}$):

$$\phi_c \stackrel{\text{def}}{=} \bigvee_{q \in Q} \bigvee_{\{c\} \subseteq X \subseteq \{1,2\}} q_X \quad \text{and} \quad \phi_q \stackrel{\text{def}}{=} \bigvee_{X \subseteq \{1,2\}} q_X .$$

We are now in position to define ϕ:

$$\phi \stackrel{\text{def}}{=} \mathtt{F}\phi_{q_F} \wedge \bigwedge_{c \in \{1,2\}} \mathtt{G}(\phi_c \Rightarrow \downarrow_1^c \uparrow_1^{c+2}) \wedge \bigwedge_{c \in \{1,2\}} \mathtt{G}(\bigwedge_{(q, \mathbf{true}, -\mathbf{e}_c, q') \in \Delta} q_\emptyset \wedge \mathtt{X}\phi_{q'} \Rightarrow \downarrow_1^c \neg \uparrow_1^{c+2})$$

One can show that $M' \models^\omega \phi$ iff the control state q_F can be reached in M. Actually, if there exists a run ρ of M' such that $\rho \models \phi$, then whenever a configuration of ρ satisfies ϕ_c, the value of the counter c is equal to the value of the counter $c+2$. This allows to build a corresponding finite run ρ' for M. Moreover, reaching a configuration satisfying ϕ_F in ρ, leads to a configuration satisfying q_F in ρ'.

Lemma 11. *$M' \models^\omega \phi$ iff the control state q_F can be reached in M.*

Proof. Let $\langle q_0, \mathbf{v}_0\rangle, \langle (q_0)_{X_0}, \mathbf{v}_0\rangle, \langle (q_1)_{X_1}, \mathbf{v}_1\rangle, \langle (q_2)_{X_2}, \mathbf{v}_2\rangle, \ldots$ be an infinite run of M' satisfying ϕ. We know that $\mathbf{v}_0 = \mathbf{0}$ and for some $I \geq 1$ and $X \subseteq \{1,2\}$, for all $j \geq I$, we have $q_j = (q_F)_X$. One can easily show that $\langle q_0, \mathbf{v}'_0\rangle, \langle q_1, \mathbf{v}'_1\rangle \ldots, \langle q_I, \mathbf{v}'_I\rangle$ is a finite run of M ending in q_F such that for $j \in \{1, \ldots, I\}$ and $c \in \{1,2\}$, $\mathbf{v}'_j(c) = \mathbf{v}_j(c) - \mathbf{v}_j(c+2)$.

Conversely, let $\langle q_0, \mathbf{v}'_0\rangle, \ldots, \langle q_I, \mathbf{v}'_I\rangle$ be a finite run of M ending in q_F. We can build an infinite run of M' of the form

$$\langle q_0, \mathbf{0}\rangle, \langle (q_0)_{X_0}, \mathbf{v}_0\rangle, \langle (q_1)_{X_1}, \mathbf{v}_1\rangle, \ldots, \langle (q_I)_{X_I}, \mathbf{v}_I\rangle^\omega$$

(with $q_I = q_f$) such that for $j \in \{1, \ldots, I\}$ and for $c \in \{1,2\}$, $\mathbf{v}'_j(c) = \mathbf{v}_j(c) - \mathbf{v}_j(c+2)$ and satisfies the formula ϕ. □

□

The result of Theorem 10 can be refined by showing the undecidability of the strict fragment $\mathrm{MC}^\omega(\mathrm{LTL}^{\downarrow,\mathrm{s}}[\cdot, 4])$ restricted to reversal-bounded counter machines. Observe that we shall modify the above developments while we are dealing with a strict fragment for which each register is associated with a unique counter. First, we withdraw from M' the transitions of the form $((q_F)_X, \mathbf{true}, \mathbf{0}, (q_F)_X)$ and replace `zero` by `true` (zero-tests will be treated after). Then, we complete the counter machine M' (in order to obtain M'') with a mechanism that will be useful to test that counters have the same values. Figure 2 illustrates how M'' is built from M'; an edge labelled by **a** corresponds to a transition with action **a** and guard **true**. Similarly, an edge labelled by $-\mathbf{e}$ is a shortcut for four edges labelled respectively by $-\mathbf{e}_1$, $-\mathbf{e}_2$, $-\mathbf{e}_3$ and $-\mathbf{e}_4$. Moreover, an edge labelled by $x_c = 0?$ corresponds to a transition with action $\mathbf{0}$ and guard in which only the cth component has value `zero`. Since in M' the counters can only increase, the counter machine M'' is 4-reversal-0-bounded. Observe that any reachable configuration with control state $q_{1=3}$ [resp. $q_{2=4}$] satisfies that the first [resp. second] counter is equal to the third [resp. fourth] counter. So after reaching a final control state from M', the counter machine M'' will reach configurations with control states either $q_{1=3}$ or $q_{2=4}$ corresponding to counter values of previous configurations of M' in which zero-tests had to be performed. Now, let us build a formula $\phi' \in \mathrm{LTL}^{\downarrow,s}[Q'', 4]$ (assuming that Q'' is the set of control states of M'') such that $M'' \models^\omega \phi'$ if and only if the control state q_F is reachable in M. We use again the auxiliary formulae ϕ_1 and ϕ_2:

$$\phi' = \mathtt{F} q_{2=4} \wedge \bigwedge_{c \in \{1,2\}} \mathtt{G}\big(\downarrow^c_c \downarrow^{c+2}_{c+2} (\phi_c \Rightarrow \mathtt{F}(q_{c=c+2} \wedge \uparrow^c_c \wedge \uparrow^{c+2}_{c+2}))\big) \wedge \phi_{\mathrm{dec}} \wedge \phi_{\mathrm{fair}}$$

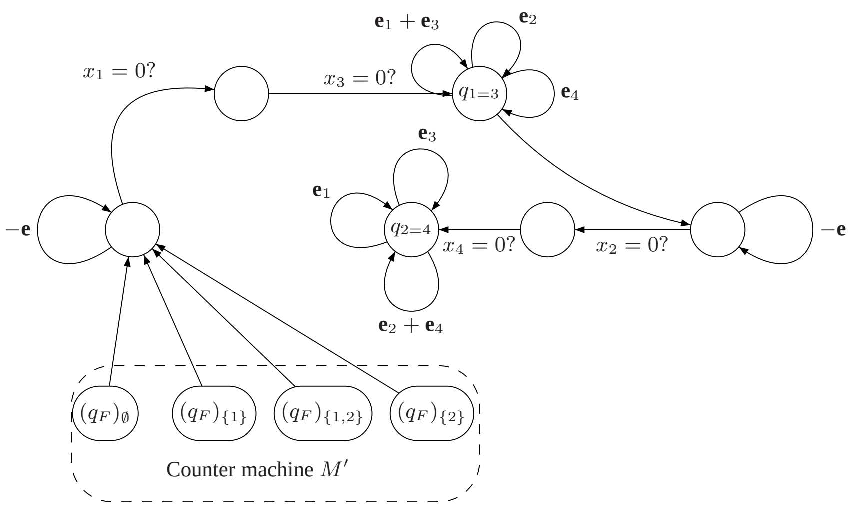

Fig. 2. Counter machine M'' built from M'

with ($\Downarrow \psi$ stands for $\downarrow_1^1\downarrow_2^2\downarrow_3^3\downarrow_4^4 \psi$)

$$\phi_{\rm fair} \stackrel{\rm def}{=} \mathtt{G} \bigwedge_{q \in Q'} (q \Rightarrow \Downarrow (\mathtt{F}(q_{1=3} \wedge \uparrow_2^2 \wedge \uparrow_4^4 \wedge \uparrow_1^1) \wedge \mathtt{F}(q_{2=4} \wedge \uparrow_1^1 \wedge \uparrow_3^3 \wedge \uparrow_2^2))))$$

$$\phi_{\rm dec} = \bigwedge_{c \in \{1,2\}} \mathtt{G}(\bigwedge_{(q, \mathbf{true}, -\mathbf{e}_c, q') \in \Delta} q_\emptyset \wedge \mathtt{X}\phi_{q'} \Rightarrow \Downarrow \neg \mathtt{F}(q_{c=c+2} \wedge \uparrow_1^1 \wedge \uparrow_2^2 \wedge \uparrow_3^3 \wedge \uparrow_4^4))$$

If a run ρ' of M'' satisfies ϕ', then for all the configurations of ρ' which are in a control state satisfying ϕ_1 [resp. ϕ_2], the values of the first [resp. second] and of the third [resp. fourth] counters are equal. This allows us to build a run ρ of M which is "correct". Hence, $M'' \models^\omega \phi'$ iff there exists a run ρ of M reaching the control state q_F. Observe that the correctness of the reduction heavily relies on the fact that all the counters in M' only increase, see the proof below.

Lemma 12. *$M'' \models^\omega \phi'$ iff the control state q_F can be reached in M.*

Proof. Let $\rho = \langle q_0, \mathbf{v}_0\rangle, \langle (q_0)_{X_0}, \mathbf{v}_0\rangle, \langle (q_1)_{X_1}, \mathbf{v}_1\rangle, \langle (q_2)_{X_2}, \mathbf{v}_2\rangle, \ldots$ be an infinite run of M'' satisfying ϕ'. We know that $\mathbf{v}_0 = \mathbf{0}$ and for some $I \geq 1$ and $X \subseteq \{1,2\}$, we have $q_I = (q_F)_X$. We now prove that the finite sequence $\langle q_0, \mathbf{v}'_0\rangle, \ldots, \langle q_I, \mathbf{v}'_I\rangle$ verifying for $j \in \{0, \ldots, I\}$ and for $c \in \{1,2\}$, $\mathbf{v}'_j(c) = \mathbf{v}_j(c) - \mathbf{v}_j(c+2)$ is a finite run of M. The proof is by induction on j. First, note that we have in M'' a transition of the form $((q_j)_{X_j}, \mathbf{true}, \mathbf{a}', (q_{j+1})_{X_{j+1}})$. We then consider the following cases.
Case 1: there is $c \in \{1,2\}$ such that $\mathbf{a}'(c+2) = 1$.
So, there exists a transition in M of the form $(q_j, \mathtt{true}, -\mathbf{e}_c, q_{j+1})$. This transition is firable from the configuration $\langle q_j, \mathbf{v}'_j\rangle$ if and only if $\mathbf{v}'_j(c) > 0$. *Ad absurdum*, suppose $\mathbf{v}'_j(c) = 0$. By the induction hypothesis, we get that $\mathbf{v}_j(c) = \mathbf{v}_j(c+2)$. Suppose $c = 1$ (the case $c = 2$ can be treated analogously). As observed before the proof, since $\rho \models \phi_{\rm fair}$, there is a configuration $\langle q_{1=3}, \mathbf{u}\rangle$ of ρ (occurring after the position j) such that $\mathbf{u}(1) = \mathbf{u}(3) = \max(\mathbf{v}_j(1), \mathbf{v}_j(3)) = \mathbf{v}_j(1)$, $\mathbf{u}(2) = \mathbf{v}_j(2)$ and $\mathbf{u}(4) = \mathbf{v}_j(4)$.

Since $\mathbf{v}_j(1) = \mathbf{v}_j(3)$, we conclude that for $c' \in \{1,2,3,4\}$, $\mathbf{v}_j(c') = \mathbf{u}(c')$, which leads to a contradiction with the satisfaction of $\rho \models \phi_{\rm dec}$. Consequently, $\mathbf{v}'_j(c) > 0$. By construction of M'', it is then obvious that the configuration $\langle q_{j+1}, \mathbf{v}'_{j+1}\rangle$ obtained after the firing of $(q_j, \mathtt{true}, -\mathbf{e}_c, q_{j+1})$ is such that for all $c \in \{1,2\}$, $\mathbf{v}'_{j+1}(c) = \mathbf{v}_{j+1}(c) - \mathbf{v}_{j+1}(c+2)$.
Case 2: $X_j = \{c\}$ for some $c \in \{1,2\}$
So, there exists a transition $(q_j, \mathbf{g}, \mathbf{0}, q_{j+1})$ in M such that $\mathbf{g}(c) = \mathtt{zero}$. Hence, we only need to check that $\mathbf{v}'_j(c) = 0$. By the induction hypothesis, we only need to check that $\mathbf{v}_j(c) = \mathbf{v}_j(c+2)$. This is ensured by the satisfaction of the first part of the formula ϕ' and by the fact that all the configurations in $\mathrm{Reach}(M'')$ of the form $\langle q_{c=c+2}, \mathbf{v}\rangle$ verify $\mathbf{v}(c) = \mathbf{v}(c+2)$.
Case 3: incrementation of counter c with guard $\mathbf{0}$.
This can be easily deduced from the way we build the counter machine M''.

Conversely, let $\langle q_0, \mathbf{v}'_0\rangle, \langle q_1, \mathbf{v}'_1\rangle \ldots, \langle q_I, \mathbf{v}'_I\rangle$ be a finite run of M ending in q_F. We can build an infinite run of M'' of the form

$$\begin{array}{l}\langle q_0, \mathbf{0}\rangle, \langle (q_0)_{X_0}, \mathbf{v}_0\rangle, \langle (q_1)_{X_1}, \mathbf{v}_1\rangle, \ldots, \langle (q_I)_{X_I}, \mathbf{v}_I\rangle, \ldots,\\ \langle q_{1=3}, \mathbf{0}\rangle, \langle q_{1=3}, \mathbf{u}_1\rangle, \ldots, \langle q_{1=3}, \mathbf{u}_{I_1}\rangle, \ldots,\\ \langle q_{2=4}, \mathbf{0}\rangle, \langle q_{2=4}, \mathbf{w}_1\rangle, \ldots, \langle q_{2=4}, \mathbf{w}_{I_2}\rangle, \ldots\end{array}$$

with $I_1 \leq I$ and $I_2 \leq I$ such that:

- $q_I = q_F$,
- for $j \in \{1, \ldots, I\}$, we have:
  1. for all $c \in \{1,2\}$, $\mathbf{v}'_j(c) = \mathbf{v}_j(c) - \mathbf{v}_j(c+2)$,
  2. there exists $j' \leq I_1$ such that $\mathbf{u}_{j'}(1) = \mathbf{u}_{j'}(3) = \max(\mathbf{v}_j(1), \mathbf{v}_j(3)) = \mathbf{v}_j(1)$, $\mathbf{u}_{j'}(2) = \mathbf{v}_j(2)$ and $\mathbf{u}_{j'}(4) = \mathbf{v}_j(4)$,
  3. there exists $j' \leq I_2$ such that $\mathbf{w}_{j'}(2) = \mathbf{w}_{j'}(4) = \max(\mathbf{v}_j(2), \mathbf{v}_j(2)) = \mathbf{v}_j(2)$, $\mathbf{w}_{j'}(1) = \mathbf{v}_j(1)$ and $\mathbf{w}_{j'}(3) = \mathbf{v}_j(3)$. .

Observe that the satisfaction of the above conditions (2)-(3) is mainly due to the fact that the four counters can only increase and for $j \leq I$, we have $\mathbf{v}_j(1) \geq \mathbf{v}_j(3)$ and $\mathbf{v}_j(2) \geq \mathbf{v}_j(4)$. It is then easy to check that such a run satisfies the formula ϕ'. □

This allows us to deduce the following results (using also Lemma 4).

Theorem 13. $\mathrm{MC}^\omega(\mathrm{LTL}^{\downarrow,\mathrm{s}}[\cdot, 4])$ *restricted to reversal-bounded counter machines are undecidable as well as* $\mathrm{MC}^\omega(\mathrm{LTL}^{\downarrow,\mathrm{s}}[\emptyset, 5])$ *(by using Lemma 4).*

6 Deterministic Counter Machines

In this section, we restrict ourselves to classes of deterministic counter machines. A class $\mathcal{C}$ of deterministic counter machines has the *PA-property* $\overset{\text{def}}{\Leftrightarrow}$ for each counter machine $M \in \mathcal{C}$, one can effectively build a formula $\phi_M(x_0, \ldots, x_{n+1})$ in Presburger arithmetic [44] such that for all $j_0, \ldots, j_{n+1} \in \mathbb{N}$, $\langle j_0, \langle j_1, \ldots, j_n\rangle\rangle$ is the j_{n+1}th configuration of the unique run of M iff $\langle j_0, \ldots, j_{n+1}\rangle \models \phi_M(x_0, \ldots, x_{n+1})$ (assuming that M has dimension n and its set of control states is viewed as a finite subset of $\mathbb{N}$).

We show below that model-checking restricted to counter machines can be sometimes reduced to the decidable satisfiability problem for Presburger arithmetic.

Lemma 14. *Let $\mathcal{C}$ be a class of deterministic counter machines. If $\mathcal{C}$ has the PA-property, then the model-checking problem* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ *over counter machines in $\mathcal{C}$ is decidable.*

The proof of Theorem 8(II) is partly based on the fact that the class of deterministic one-counter machines has the PA-property. Similarly, the class of deterministic flat counter macines has also the PA-property [23] (see Theorem 7).

Proof. Let $M \in \mathcal{C}$ and $\phi \in \mathrm{LTL}^{\downarrow}[Q,n]$. We write $\phi_M(t_0,\ldots,t_{n+1})$ to denote the Presburger formula encoding the unique run of M. For register r in ϕ, we consider the variable y_r in the built Presburger formula. Let us define the map $T(\cdot)$ that takes as arguments a formula ψ and $n+2$ variables $x_0,\ldots,x_{n+1}$ and returns a Presburger formula with free variables $x_0,\ldots,x_{n+1}$ internalizing the semantics of $\mathrm{LTL}^{\downarrow}[Q,n]$ formulae. The map $T(\cdot)$ is homomorphic for Boolean connectives and is inductively defined as follows:

- $T(q,x_0,\ldots,x_{n+1}) \stackrel{\text{def}}{=} x_0 = q$;
- $T(\uparrow_r^c, x_0,\ldots,x_{n+1}) \stackrel{\text{def}}{=} y_r = x_c$,
- $T(\mathtt{X}\psi, x_0,\ldots,x_{n+1}) \stackrel{\text{def}}{=} \exists x'_0,\ldots,x'_{n+1}(x'_{n+1} = x_{n+1}+1 \wedge \phi_M(x'_0,\ldots,x'_{n+1}) \wedge T(\psi,x'_0,\ldots,x'_{n+1}))$ ($x'_0,\ldots,x'_{n+1}$ are new variables),
- $T(\psi_1 \mathtt{U} \phi_2, x_0,\ldots,x_{n+1})$ is equal to the formula below:

$$\exists x'_0,\ldots,x'_{n+1}\ (x_{n+1} \leq x'_{n+1} \wedge \phi_M(x'_0,\ldots,x'_{n+1}) \wedge T(\psi_2,x'_0,\ldots,x'_{n+1}))\wedge$$
$$\forall x''_0,\ldots,x''_{n+1}((x_{n+1} \leq x''_{n+1} < x'_{n+1} \wedge \phi_M(x''_0,\ldots,x''_{n+1})) \Rightarrow$$
$$T(\psi_1,x''_0,\ldots,x''_{n+1}))$$

  ($x'_0,\ldots,x'_{n+1},x''_0,\ldots,x''_{n+1}$ are also new variables),
- the clause for the release operator $\mathtt{R}$ is similar,
- $T(\downarrow_r^c \psi, x_0,\ldots,x_{n+1}) = \exists y_r(y_r = x_c \wedge T(\psi,x_0,\ldots,x_{n+1}))$.

Let ϕ' be the formula

$$T(\phi,x_0,\ldots,x_{n+1}) \wedge x_0 = q_0 \wedge \bigwedge_{i\in\{1,\ldots,n+1\}} x_i = 0 \wedge \varphi_\infty,$$

with $\varphi_\infty \stackrel{\text{def}}{=} \forall\ x\ \exists t_1,\ldots,t_{n+1}\phi_M(x,t_1,\ldots,t_{n+1})$. Observe that recycling variables allow to obtain a formula equivalent to ϕ' with $\mathcal{O}(n)$ variables. One can then show that $M \models^{\omega} \phi$ iff $\langle\langle q_0,\mathbf{0}\rangle,0\rangle \models \phi'$. Since the satisfiability problem for Presburger formulae is decidable [44], we obtain that $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ over counter machines in $\mathcal{C}$ is decidable. □

Lemma 15.

(I) *The class of deterministic reversal-bounded counter machines has the PA-property.*
(II) *The class of deterministic VASS has the PA-property.*

Proof. (I) Let $M = (n, Q, \Delta, q_0)$ be a reversal-bounded deterministic counter machine. We transform it into a counter machine $M' = (n+1, Q, \Delta', q_0)$ in which we add an extra counter in order to count the number of steps. Since the additional counter only increases and since M is reversal-bounded, M' is also reversal-bounded. As the reachability set of reversal-bounded counter machines is a semi-linear set which can be effectively computed [24], there exists a Presburger formula $\phi_M(x_0, x_1, \ldots, x_{n+1})$ with free variables $x_0, x_1, \ldots, x_{n+1}$ such that for $j_0, \ldots, j_{n+1} \in \mathbb{N}$, we have $\langle j_0, \ldots, j_{n+1} \rangle \models \phi_M(x_0, \ldots, x_{n+1})$ iff $\langle j_0, \langle j_1, \ldots, j_n \rangle \rangle$ is a j_{n+1}th configuration of a run of M (assuming that its set of control states is a finite subset of $\mathbb{N}$). Furthermore, M being deterministic, by construction M' is also deterministic.
(II) Let M be a deterministic VASS. The Karp and Miller tree [38] for the deterministic counter machine M from the initial configuration $\langle q_0, \mathbf{0} \rangle$ is a finite path of the form $\langle q_0, \mathbf{u}_0 \rangle \xrightarrow{\mathbf{a}_0} \langle q_1, \mathbf{u}_1 \rangle \xrightarrow{\mathbf{a}_1} \ldots \xrightarrow{\mathbf{a}_{N-1}} \langle q_N, \mathbf{u}_N \rangle$, where $\langle q_0, \mathbf{u}_0 \rangle$ is the initial configuration; for $i \geq 1$, $\langle q_i, \mathbf{u}_i \rangle \in Q \times (\mathbb{N} \cup \{\omega\})^n$ and $\mathbf{a}_{i-1} \in \{-1, 0, 1\}^n$ is an action of M. Determinism of M entails that there is at most one i such that $\langle q_i, \mathbf{u}_i \rangle \in Q \times \mathbb{N}^n$ and $\langle q_{i+1}, \mathbf{u}_{i+1} \rangle \in Q \times ((\mathbb{N} \cup \{\omega\})^n \setminus \mathbb{N}^n)$. Moreover, by construction of such a path, either no transition can be fired from $\langle q_N, \mathbf{u}_N \rangle$ or there is $j < N$ such that $\langle q_j, \mathbf{u}_j \rangle = \langle q_N, \mathbf{u}_N \rangle$. In that latter case, the unique infinite run of M is made of a finite prefix followed by a loop of effects $\sum_{i=j}^{i=N-1} \mathbf{a}_i \geq \mathbf{0}$. Hence, assuming that M has the unique run $\langle q_0, \mathbf{v}_0 \rangle \langle q_1, \mathbf{v}_1 \rangle \ldots \langle q_i, \mathbf{v}_i \rangle, \ldots$ one can effectively build a Presburger formula $\phi(x_0, x_1, \ldots, x_{n+1})$ such that for all tuples $\langle j_0, \ldots, j_{n+1} \rangle$, we have $\langle j_0, \ldots, j_{n+1} \rangle \models \phi(x_0, x_1, \ldots, x_{n+1})$ iff there is i such that $\langle j_0, \ldots, j_{n+1} \rangle = \langle \langle q_i, \mathbf{v}_i \rangle, i \rangle$. $\square$

Corollary 16. $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ *is decidable when restricted to deterministic reversal-bounded counter machines and deterministic VASS.*

Checking whether a VASS is deterministic can be decided by using instances of the covering problem (the problem is actually PSPACE-complete [31]). Checking whether a reversal-bounded counter machine is deterministic is also decidable adding a counter which counts each step and using the fact that the reachability set can be expressed in Presburger arithmetic. By contrast, checking whether a counter machine is reversal-bounded is undecidable [24]. Observe that by combining the bounds in [31] and developments from the long version of [21], one can establish that $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ over deterministic VASS can be solved in EXPSPACE.

7 Flat Freeze LTL

In this section, we consider the restriction of the model-checking problem to flat formulae only. By Theorem 7, we already know that $\mathrm{MC}^\omega(\mathrm{flat}-\mathrm{LTL}^\downarrow[\cdot,\cdot])$ restricted to flat counter machines is decidable and that $\mathrm{MC}^\omega(\mathrm{flat}-\mathrm{LTL}^\downarrow[\cdot,\cdot])$ restricted to VASS is undecidable (the proof of Theorem 9 involves only flat formulae). It is worth observing that flat $\mathrm{LTL}^\downarrow[Q,n]$ strictly contains $\mathrm{LTL}[Q]$, and therefore we refine below decidability results from Section 4.2.

7.1 A detour to counter machines with parameterized tests

We introduce here parameterized counter machines in order to solve later model-checking problems restricted to flat formulae. First, let us fix some definitions. A *counter machine with parameterized tests* (shortly *parameterized* counter machine) is defined as a counter machine $M = (n, Q, \Delta, q_0, Z)$ extended with a finite set Z of integer variables such that the guards $\mathbf{g}$ are among $(\{\mathtt{zero}, \mathtt{true}\} \cup \{=(z), \neq(z), >(z), <(z) \mid z \in Z\})^n$. A *concretization* C of M is a map $\mathrm{C} : Z \to \mathbb{N}$. Given a parameterized counter machine M and a concretization C, we introduce the transition system $TS(M, \mathrm{C}) = (Q \times \mathbb{N}^n, \rightarrow)$ where $\rightarrow \subseteq (Q \times \mathbb{N}^n) \times (Q \times \mathbb{N}^n)$ is defined as follows: for $\langle q, \mathbf{v}\rangle, \langle q', \mathbf{v}'\rangle \in Q \times \mathbb{N}^n$, we have $\langle q, \mathbf{v}\rangle \rightarrow \langle q', \mathbf{v}'\rangle \overset{\text{def}}{\Leftrightarrow}$ there exists a transition $t = (q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ such that $\mathbf{v}' = \mathbf{v} + \mathbf{a}$, and for $1 \leq c \leq n$, $\mathbf{g}(c)$ equals $\mathtt{zero}$ implies $\mathbf{v}(c) = 0$, $\mathbf{g}(c)$ is equal to $=(z)$ implies $\mathbf{v}(c) = \mathrm{C}(z)$, $\mathbf{g}(c)$ is equal to $\neq(z)$ implies $\mathbf{v}(c) \neq \mathrm{C}(z)$, $\mathbf{g}(c)$ is equal to $>(z)$ implies $\mathbf{v}(c) > \mathrm{C}(z)$ and, $\mathbf{g}(c)$ is equal to $<(z)$ implies $\mathbf{v}(c) < \mathrm{C}(z)$. A finite [resp. infinite] *run* in $TS(M, \mathrm{C})$ is a finite [resp. infinite] sequence $\rho = \langle q_0, \mathbf{0}\rangle \rightarrow \langle q_1, \mathbf{v}_1\rangle \rightarrow \ldots$. The *parameterized reachability problem* for counter machines is defined as follows:

instance: a parameterized counter machine M and a configuration $\langle q, \mathbf{v}\rangle$.
question: is there a concretization C such that $\langle q_0, \mathbf{0}\rangle \xrightarrow{*} \langle q, \mathbf{v}\rangle$ in $TS(M, \mathrm{C})$?

Even if the parameterized reachability problem is obviously undecidable, we will see in this section that some restrictions lead to decidability. We will say that a parameterized counter machine is *Ibarra reversal-bounded* if the classical counter machine obtained by replacing each parameterized test by $\mathtt{true}$ is Ibarra reversal-bounded. We have then the following result.

Theorem 17. *[35] The parameterized reachability problem for Ibarra reversal-bounded parameterized counter machines is decidable.*

If a parameterized counter machine has no guard of the form either $\neq(z)$ or $<(z)$, we say it is *restricted*. In [29], parametric one-counter machines are defined as extensions of one-counter machines extended with actions consisting in incrementing or decrementing the unique counter with some parameterized integer constants. In [29], it is shown that the reachability problem for this class of one-counter machines is decidable. Here is a corollary.

Lemma 18. *The parameterized reachability problem for restricted parameterized one-counter machines is decidable.*

The proof of Lemma 18 consists in substituting each test of the form $=(z)$ by the following sequence of instructions: decrement by z, perform a zero-test and increment by z. In order to encode the test $>(z)$, we use the same technique except that we do not introduce a zero-test between the decrementation (in fact we also add a decrementation by 1 and an incrementation by 1) and the incrementation. Note that this method does not work if we allow guards of the form either $\neq(z)$ or $<(z)$, because the value of the counter cannot be negative, hence the decidability of the parameterized reachability problem for one-counter machines remains an open problem.

We introduce here a new problem which is needed to reduce the considered model-checking problem. The *parameterized generalized repeated reachability problem* for parameterized counter machines is defined as follows:

instance: a parameterized counter machine M, N sets $F_1, \ldots, F_N$ of control states
question: are there a concretization C and an infinite run of $TS(M, \mathrm{C})$ such that for $1 \leq i \leq N$, one control state in F_i is repeated infinitely often?

From the previous theorem and lemma, we deduce the following corollary.

Corollary 19. *The parameterized generalized repeated reachability problem is decidable when considering Ibarra reversal-bounded parameterized counter machines and restricted parameterized one-counter machines.*

Proof. Given a parameterized counter machine $M = (n, Q, \Delta, q_0, Z)$ and N sets of control states $F_1, \ldots, F_N$, we shall build a parameterized counter machine $M' = (n, Q', \Delta', q_0, Z \uplus \{z'_1, \ldots, z'_n\})$ such that there is a concretization C and an infinite run of $TS(M, \mathrm{C})$ such that for $1 \leq i \leq N$, one control state in F_i is repeated infinitely often if and only if the configuration $\langle q_{new}, \mathbf{0} \rangle$ can be reached in M'.

As done to reduce nonemptiness for generalized Büchi automata to nonemptiness for Büchi automata, we can build a parameterized counter machine $M_{\otimes N} = (n, Q \times \{1, \ldots, N\}, \Delta', \langle q_0, 1 \rangle, Z)$ such that $M_{\otimes N}$ is made of N copies of M and whenever the ith copy visits a control state in F_i, it jumps to the $(1 + (i \ mod \ N))$th copy. Consequently, there is a concretization C and an infinite run of $TS(M, \mathrm{C})$ such that for $1 \leq i \leq N$, one control state in F_i is repeated infinitely often iff there is a concretization C and an infinite run of $TS(M_{\otimes N}, \mathrm{C})$ such that one control state in $F_1 \times \{1\}$ is repeated infinitely often. Furthermore, $M_{\otimes N}$ is Ibarra reversal-bounded if and only if M is Ibarra reversal-bounded; the counters in $M_{\otimes N}$ evolves as in M. So, without any loss of generality, we can assume that $N = 1$.

Let $M = (1, Q, \Delta, q_0, Z)$ be a restricted parameterized one-counter machine and $F_1 \subseteq Q$. If $(\star)$ there is a concretization and an infinite run in which there is a control state in F_1 repeated infinitely often, then one of the conditions below is satisfied:

1. a test of the form either "`zero`" or "$=(z)$" is repeated infinitely often;
2. after some position, all the fired transitions have tests of the form either "`true`" or "$>(z)$".

Consequently, $(\star)$ iff one of the conditions below holds true:

1. there is a concretization and an infinite run in which there are two configurations $\langle q_i, v_i \rangle = \langle q_j, v_j \rangle$ with $i < j$ and $q_i \in F_1$,
2. (by Dickson's Lemma) there is a concretization and an infinite run in which there are two configurations $\langle q_i, v_i \rangle$, $\langle q_j, v_j \rangle$ with $i < j$, $v_i \leq v_j$, $q_i = q_j \in F_1$ and no test of the form either "`zero`" or "$=(z)$" is performed between $\langle q_i, v_i \rangle$ and $\langle q_j, v_j \rangle$.

Let us build the parameterized counter machine

$$M' = (1, Q', \Delta', q_0, Z \uplus \{z'_1\})$$

such that ($\star$) iff there is a concretization C′ such that $\langle q_0, 0\rangle \xrightarrow{*} \langle q_{new}, 0\rangle$ in $TS(M', \mathrm{C}')$, which can be decided thanks to [29]. Basically, M' is made of $5 \times \mathrm{card}(F_1)$ copies of M plus some extra control states such as q_{new} and q_0. Moreover, in the fourth and fifth copies, transitions with tests of the form either "`zero`" or "$=(z)$" are removed.

The control states of the $\langle q_f, i\rangle$th copy (with $\langle q_f, i\rangle \in F_1 \times \{1, \ldots, 5\}$) are among $Q \times \{q_f\} \times \{i\}$. For $\langle q_f, i\rangle \in F_1 \times \{1\}$, we consider the transitions $q_0 \xrightarrow{\mathtt{true},0} \langle q_0, q_f, i\rangle$. The $\langle q_f, 1\rangle$th copy of M in M' behaves as M except that nondeterministically we jump to the second or fourth copy when the control state q_f is visited (and we check that the counter value is equal to z'_1). So, we consider this additional sequence of transitions $\langle q_f, q_f, 1\rangle \xrightarrow{=(z'_1),0} \langle q_f, q_f, 2\rangle$ and $\langle q_f, q_f, 1\rangle \xrightarrow{=(z'_1),0} \langle q_f, q_f, 4\rangle$. As soon as a transition is performed in the $\langle q_f, 2\rangle$th copy (see the above strict inequality $i < j$), transitions jump to the $\langle q_f, 3\rangle$th copy. The $\langle q_f, 3\rangle$th copy of M in M' behaves as M except that we add the following transitions:

$$\langle q_f, q_f, 3\rangle \xrightarrow{=(z'_1),\mathbf{0}} q_{new} \text{ and } q_{new} \xrightarrow{\mathtt{true},-1} q_{new}$$

As soon as a transition is performed in the $\langle q_f, 4\rangle$th copy (see the above strict inequality $i < j$), transitions jump to the $\langle q_f, 5\rangle$th copy. The $\langle q_f, 5\rangle$th copy of M in M' behaves as M (remember some transitions have been also removed, those involving equality tests) except that we add the following transitions:

$$\langle q_f, q_f, 5\rangle \xrightarrow{=(z'_1),\mathbf{0}} q_{new} \text{ and } \langle q_f, q_f, 5\rangle \xrightarrow{>(z'_1),\mathbf{0}} q_{new}.$$

Let $M = (n, Q, \Delta, q_0, Z)$ be an Ibarra reversal-bounded parameterized counter machine and $F_1 \subseteq Q$. Without any loss of generality, we can assume that there is no guard of the form $\neq(z)$ since this can be replaced by transitions with guards $>(z)$ and $<(z)$. This may cause an exponential blow-up since there are n counters. If ($\star$) there is a concretization and an infinite run in which there is a control state in F_1 repeated infinitely often, then there is $X \subseteq \{1, \ldots, n\}$ such that the counters c in X are exactly those for which in this run, infinitely often there is a transition with guard on c of the form either `zero` or $<(z)$ or $=(z)$. Since M is Ibarra reversal-bounded, ($\star$) iff there is a concretization, $X \subseteq \{1, \ldots, n\}$ and an infinite run in which there are two configurations $\langle q_i, \mathbf{v}_i\rangle$, $\langle q_j, \mathbf{v}_j\rangle$ with $i < j$, $\mathbf{v}_i \leq \mathbf{v}_j$, for $c \in X$, $\mathbf{v}_i(c) = \mathbf{v}_j(c)$, and $q_i = q_j \in F_1$. Moreover, between $\langle q_i, \mathbf{v}_i\rangle$ and $\langle q_j, \mathbf{v}_j\rangle$, there are tests of the form either `zero` or $<(z)$ or $=(z)$ for exactly the counters from X. Indeed, any counter whose value is bounded during an infinite run takes a fixed value after some position by Ibarra reversal-boundedness. For the other counters, Dickson's Lemma allows us to obtain the condition $\mathbf{v}_i \leq \mathbf{v}_j$.

Let us build the parameterized counter machine $M' = (n, Q', \Delta', q_0, Z \uplus \{z'_1, \ldots, z'_n\})$ such that ($\star$) iff there is a concretization C′ such that $\langle q_0, \mathbf{0}\rangle \xrightarrow{*} \langle q_{new}, \mathbf{0}\rangle$ in $TS(M', \mathrm{C}')$, which is decidable by [35]. Basically, M' is made of $2^n \times \mathrm{card}(F_1) + 1$ copies of M plus some extra control states such as q_{new}. It includes an initial distinguished copy of M. For $X \subseteq \{1, \ldots, n\}$ and $q_f \in F_1$, the control states of the $\langle q_f, X\rangle$th copy are among $Q \times \{q_f\} \times \{X\} \times \mathcal{P}(X)$. For $X \subseteq \{1, \ldots, n\}$ and $q_f \in F_1$, we consider the transition $q_f \xrightarrow{\mathbf{g},\mathbf{0}} \langle q_f, X, \emptyset\rangle$: nondeterministically we jump to the $\langle q_f, X\rangle$th copy when the control state q_f is visited and for $1 \leq c \leq n$, $\mathbf{g}(c)$ is equal to $=(z'_c)$. In the $\langle q_f, X \rangle$th copy, $\langle q, q_f, X, Y \rangle \xrightarrow{\mathbf{g},\mathbf{a}} \langle q', q_f, X, Y' \rangle$ is a transition whenever there is a transition $q \xrightarrow{\mathbf{g},\mathbf{a}} q'$ in M such that for $c \in \{1, \ldots, n\} \setminus X$, $\mathbf{g}(c)$ is not of the form either $\mathtt{zero}$ or $<(z)$ or $=(z)$ and

$$Y' = Y \cup \{c : \mathbf{g}(c) \text{ is either of the form } \mathtt{zero} \text{ or } <(z) \text{ or } =(z)\}$$

As soon as in the $\langle q_f, X \rangle$th copy, all the counters in X have been property tested at least once, potentially we can jump to the final location q_{new}. Hence, in the $\langle q_f, X \rangle$th copy, we add the following transitions:

$$\langle q_f, q_f, X, X \rangle \xrightarrow{\mathbf{g},\mathbf{0}} q_{new} \text{ and } q_{new} \xrightarrow{\mathtt{true}, -\mathbf{e}_c} q_{new}$$

with $c \in \{1, \ldots, n\}$ and for $1 \leq c' \leq n$ either $\mathbf{g}(c')$ is equal to $=(z'_{c'})$ or ($c' \not\in X$ and $\mathbf{g}(c')$ is equal to $>(z'_{c'})$). Note that M' is also an Ibarra reversal-bounded parameterized counter machine. This is due to the fact that the counters in M' evolves as in M.

$\square$

7.2 Flat formulae and parameterized counter machines

For $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ restricted to flat formulae, we have the following result.

Theorem 20. *There is a reduction from* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ *restricted to flat formulae to the parameterized generalized repeated reachability problem for counter machines.*

Proof. Let $M = (n, Q, \Delta, q_0)$ be a counter machine and ϕ be a flat sentence belonging to $\mathrm{LTL}^{\downarrow}[Q, n]$. Without any loss of generality, we can assume that ϕ is in negation normal form (which means that all the occurrences of negation appear only in front of atomic formulae). Moreover, we can assume that if $\downarrow_r^c \psi$ and $\downarrow_{r'}^{c'} \psi$ are distinct occurrences of subformulae in ϕ, then $r \neq r'$ (this may just linearly increase the number of registers). Consequently, if $\psi_1 \mathtt{U} \psi_2$ [resp. $\psi_1 \mathtt{R} \psi_2$] is a subformula of ϕ, then the freeze operator $\downarrow$ cannot occur in ψ_1 [resp. ψ_2]. We shall effectively build a parameterized counter machine $M' = (n, Q', \Delta', q_0, Z')$ and sets $F_1, \ldots, F_N \subseteq Q'$ for which there is a concretization C and an infinite run of $TS(M', \mathrm{C})$ such that for $1 \leq i \leq N$, one control state in F_i is repeated infinitely often iff $M \models^{\omega} \phi$.
Let us fix some notations. As usual, the formula ϕ can be encoded as a finite tree whose leaves are labelled by atomic formulae and internal nodes are labelled by (Boolean, temporal or freeze) connectives. Each node of the formula tree corresponds naturally to a subformula and the set of nodes can be viewed as a finite prefix-closed subset $\mathrm{occ}(\phi) \subseteq (\mathbb{N} \setminus \{0\})^*$ (finite sequence of natural numbers). Each element in $\mathrm{occ}(\phi)$ corresponds to the occurrence of a subformula in ϕ; hence two occurrences may correspond to the same subformula since we do not bother herein with structure-sharing. For example, Figure 3 presents the formula tree of the flat formula $\downarrow_1^1 (\mathtt{X} \uparrow_1^1 \vee (q' \wedge (\downarrow_1^1 q)))$. The use of occurrences instead of subformulae is motivated by the need to provide formal and clear statements in which occurrences are crucial. For each occurrence $u \in \mathrm{occ}(\phi)$, we write $\phi(u)$ to denote the corresponding subformula in ϕ; for instance $\phi(\epsilon) = \phi$.

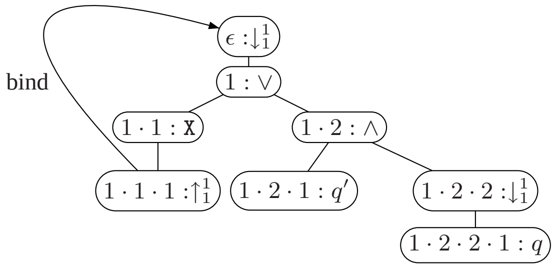

Fig. 3. Formula tree

Moreover, when u is a prefix of u', written $u \leq_{\rm pre} u'$, we know that $\phi(u')$ is a subformula of $\phi(u)$. We write ${\rm occ}^{\downarrow}(\phi)$ [resp. ${\rm occ}^{\uparrow}(\phi)$] to denote the set of occurrences corresponding to formulae whose outermost connective is of the form $\downarrow_r^c$ [resp. $\uparrow_r^c$]. For instance, ${\rm occ}^{\downarrow}(\downarrow_1^1 (\mathtt{X} \uparrow_1^1 \vee (q' \wedge (\downarrow_1^1 q)))) = \{\epsilon, 1 \cdot 2 \cdot 2\}$. Let $m = {\rm card}({\rm occ}^{\downarrow}(\phi))$. Observe that if $m = 0$, then we are in the case of ${\rm MC}^{\omega}({\rm LTL}[\cdot])$ which has been treated in Section 4.2. In the sequel, we assume that $m > 0$. Given $u \in {\rm occ}^{\uparrow}(\phi)$ with $\phi(u) =\uparrow_r^c$, we write ${\rm bind}(u)$ to denote the longest prefix of u (with respect to $\leq_{\rm pre}$) in ${\rm occ}^{\downarrow}(\phi)$ such that $\phi({\rm bind}(u))$ is of the form $\downarrow_r^{c'} \psi$ (i.e., with the same register). An *atom* X is a subset of ${\rm occ}(\phi)$ satisfying the conditions below (we abusively use subformulae to denote occurrences corresponding to formulae with the appropriate outermost connective):

1. if $\psi_1 \wedge \psi_2 \in X$, then $\psi_1, \psi_2 \in X$,
2. for all atomic formulae $\psi \in X$, $\{\psi, \neg\psi\} \not\subseteq X$,
3. if $\psi_1 \vee \psi_2 \in X$, then either $\psi_1 \in X$ or $\psi_2 \in X$,
4. if $\downarrow_r^c \psi \in X$, then $\psi \in X$.

The set of *atoms* of ϕ is denoted by ${\rm AT}(\phi)$. A pair of atoms $\langle X, X' \rangle$ is said to be *one-step consistent* iff the conditions below hold true:

(I) if $\psi_1 \mathtt{U} \psi_2 \in X$, then either $\psi_2 \in X$ or ($\psi_1 \in X$ and $\psi_1 \mathtt{U} \psi_2 \in X'$),
(II) if $\psi_1 \mathtt{R} \psi_2 \in X$, then $\psi_2 \in X$ and ($\psi_1 \in X$ or $\psi_1 \mathtt{R} \psi_2 \in X'$),
(III) if $\mathtt{X}\psi \in X$, then $\psi \in X'$,
(IV) No atom X'' strictly included in X' satisfies the conditions (I)–(III) (by replacing X' by X'').

We will now describe the construction of the parameterized counter machine M' which will use m integer variables $z_1, \ldots, z_m$. Intuitively, each integer variable will be used to store the value of a register. In order to make explicit this dependency, we shall use a one-to-one map $reg : {\rm occ}^{\downarrow}(\phi) \rightarrow \{1, \ldots, m\}$. We define also a function $counter : {\rm occ}^{\downarrow}(\phi) \cup {\rm occ}^{\uparrow}(\phi) \rightarrow \{1, \ldots, n\}$ that indicates the counter involved in the subformula. Given $u \in {\rm occ}^{\downarrow}(\phi)$ such that $\phi(u) =\downarrow_r^c \psi$, we have $counter(u) = c$ and given $u \in {\rm occ}^{\uparrow}(\phi)$ such that $\phi(u) =\uparrow_r^c$, we have $counter(u) = c$. The set Q' of control states is equal to $\{q_0\} \uplus Q \times {\rm AT}(\phi)$ plus some auxiliary control states that are introduced to perform tests. The relation Δ' is defined as follows. First, $(q_0, \mathbf{true}, \mathbf{0}, \langle q_0, Y \rangle) \in \Delta'$ whenever $\epsilon \in Y$ and no atom strictly included in Y contains ϵ (init). Then, for each transition $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ there is in Δ' the sequence of transitions

$$\langle q, Y \rangle \cdots q_1^{aux} \cdots q_T^{aux} \xrightarrow{\mathbf{g},\mathbf{a}} \langle q', Y' \rangle$$

assuming that:

1. $\mathrm{occ}^{\downarrow}(\phi) \cap Y$ contains T_1 elements, say $u_1, \ldots, u_{T_1}$; $\mathrm{occ}^{\uparrow}(\phi) \cap Y$ contains T_2 elements, say $u_{T_1+1}, \ldots, u_{T_1+T_2}$; $\{u \in Y \mid u{\cdot}1 \in \mathrm{occ}^{\uparrow}(\phi)$ and $\phi(u)$ is a negation$\}$ contains T_3 elements, say $u_{T_1+T_2+1}, \ldots, u_{T_1+T_2+T_3}$ with $T = T_1 + T_2 + T_3$,
2. $\langle Y, Y' \rangle$ is a one-step consistent pair,
3. $\{\phi(u) : u \in Y\} \cap Q \subseteq \{q\}$ and $\neg q \notin \{\phi(u) : u \in Y\}$
4. for $i \in \{1, \ldots, T_1\}$, before reaching q_i^{aux}, there is a transition testing equality between the counter $counter(u_i)$ and z_k with $k = reg(u_i)$,
5. for $i \in \{1, \ldots, T_2\}$, before reaching $q_{T_1+i}^{aux}$, there is a transition testing equality between the counter $counter(u_{T_1+i})$ and z_k with $k = reg(\mathrm{bind}(u_{T_1+i}))$,
6. for $i \in \{1, \ldots, T3\}$, before reaching $q_{T_1+T_2+i}^{aux}$, there is a transition testing inequality between the counter $counter(u_{T_1+T_2+i})$ and z_k with $k = reg(\mathrm{bind}(u_{T_1+T_2+i}))$.

Finally, let $u_1, \ldots, u_N$ be the occurrences in $\mathrm{occ}(\phi)$ such that the outermost temporal connective of $\phi(u_i)$ is the until operator `U`. Then, for $1 \leq i \leq N$, $F_i = \{\langle q, Y \rangle : u_i \notin Y$ or $(u_i \cdot 2) \in Y\}$. It remains to show the lemma below (whose proof follows).

Lemma 21. $M \models^{\omega} \phi$ *iff there exist a concretization* C *and an infinite run of* $TS(M', \mathrm{C})$ *s.t. for* $1 \leq i \leq N$*, one control state in* F_i *is repeated infinitely often.*

Proof. The formula ϕ is a flat sentence in negation normal form such that if $\downarrow_r^c \psi$ and $\downarrow_{r'}^{c'} \psi$ are distinct occurrences of subformulae in ϕ, then $r \neq r'$ (ϕ is then said to be *normalized*).

Let $\mathrm{C} : \{z_1, \ldots, z_m\} \to \mathbb{N}$ be a concretization and

$$\langle q_0, \mathbf{0} \rangle, \langle q_1', \mathbf{v}_1' \rangle, \ldots, \langle q_i', \mathbf{v}_i' \rangle, \ldots$$

be an infinite run of $TS(M', \mathrm{C})$ such that for $1 \leq i \leq N$, there is a control state in F_i that is repeated infinitely often. For the analysis below, we do not want to bother about the auxiliary control states. That is why, we introduce the map g below. Let $g : \mathbb{N} \to \mathbb{N}$ be the function such that $g(0) = 0$ and for all $i \in \mathbb{N} \setminus \{0\}$, $q'_{g(i)} \in Q \times \mathrm{AT}(\phi)$, $g(i) < g(i+1)$ and for all $g(i) < j < g(i+1)$, $q'_j \notin Q \times \mathrm{AT}(\phi)$. For all $i \in \mathbb{N} \setminus \{0\}$, we write $\langle \langle q_i, Y_i \rangle, \mathbf{v}_i \rangle$ to denote the $g(i)$th configuration $\langle q'_{g(i)}, \mathbf{v}'_{g(i)} \rangle$. By construction of the auxiliary control states, we also have that for $1 \leq i \leq N$, there is a control state in F_i that is repeated infinitely often in $\langle q_1, \mathbf{v}_1 \rangle, \ldots, \langle q_i, \mathbf{v}_i \rangle, \ldots$.

One can show that for $j \geq 1$, if $u \in Y_j \cap \mathrm{occ}^{\downarrow}(\phi)$, then for $k > j$, $u \notin Y_k$. In that way, for $i \in \{1, \ldots, m\}$, assuming $reg^{-1}(i) = u$ with $\phi(u) = \downarrow_r^c \psi$, the value $\mathrm{C}(z_i)$ is interpreted as the value of counter c at the unique position j (if it exists) for which u belongs to Y_j. This unicity property is a consequence of the facts that conditions (IV) and (init) imply that the disjunctions in conditions (3) –in the definitions for atoms–, (I) and (II) should be read exclusively in order to guarantee the minimality of the sets of formulae. The syntactic properties of ϕ then entail the desired property. *This is the crucial place where occurrences are easier to manipulate than subformulae.*

By construction of M' and g, the ω-sequence $\rho = \langle q_1, \mathbf{v}_1 \rangle, \ldots, \langle q_i, \mathbf{v}_i \rangle, \ldots$ is an infinite run of M. Remember that for each transition $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ there is in Δ' the sequence of transitions

$$\langle q, Y \rangle \cdots q_1^{aux} \cdots q_T^{aux} \xrightarrow{\mathbf{g},\mathbf{a}} \langle q', Y' \rangle$$

i.e., the same guards and actions are used (except for the intermediate and auxiliary transitions).

One can show that for $j > 0$ and $u \in Y_j$, we have $\rho, (j-1) \models_{f_{j-1}} \phi(u)$ for some register valuation f_{j-1} defined below. Consequently, $\rho, 0 \models_{f_0} \phi$ and therefore $M \models^{\omega} \phi$ where f_0 is the register valuation with empty domain. This can be shown by structural induction on $\phi(u)$ by using the properties of atoms, one-step consistent pairs and the sets of final control states (for satisfaction of until subformulae). This part is standard for plain LTL (we treat only the until subcase below). Let us first explain the cases with the freeze operators. We also need preliminary notations to explain how to define f_{j-1} for $j > 0$. Given a register r occurring in ϕ, we write $i(r, j)$ to denote the maximal position less than j for which there is $u \in Y_{i(r,j)} \cap \mathrm{occ}^{\downarrow}(\phi)$ such that $\phi(u)$ is of the form $\downarrow_r^c \psi$ if it exists (otherwise by convention $i(r, j) = 0$). Unicity is guaranteed since ϕ is normalized. So, for each r occurring in ϕ, the value $f_{j-1}(r)$ is undefined whenever $i(r, j) = 0$ otherwise $f_{j-1}(r) = \mathbf{v}_{i(r,j)}(c)$ for the unique counter c for which there is $u \in Y_{i(r,j)} \cap \mathrm{occ}^{\downarrow}(\phi)$ and $\phi(u)$ is of the form $\downarrow_r^c \psi$. Let us treat the cases in the induction that involve register valuations.

Case $\phi(u) = \downarrow_r^c \psi$:
By condition (4) (in the definition of atoms), $u \cdot 1 \in Y_j$ (since $\phi(u \cdot 1) = \psi$) and by the induction hypothesis, we have $\rho, (j-1) \models_{f_{j-1}} \phi(u \cdot 1)$. However, $i(r, j) = j$ since $u \in Y_j$ and therefore $f_{j-1}(r) = \mathbf{v}_j(c) = \mathbf{v}'_{g(j)}(c)$, whence by definition of the satisfaction relation $\rho, (j-1) \models_{f_{j-1}} \downarrow_r^c \phi(u \cdot 1)$ and $\phi(u) = \downarrow_r^c \phi(u \cdot 1)$.

Case $\phi(u) = \uparrow_r^c$:
By construction of atoms and one-step consistent pairs, we know that for $k \geq 0$, $Y_k \leq_{\mathrm{pre}} Y_{k+1}$ where $Y \leq_{\mathrm{pre}} Y' \stackrel{\text{def}}{\Leftrightarrow}$ for all $u' \in Y'$, there is $u \in Y$ such that $u \leq_{\mathrm{pre}} u'$ (u is a prefix of u'). Since ϕ is a normalized sentence, $i(r, j) \neq 0$ and $\mathrm{bind}(u) \in Y_{i(r,j)}$ with $\phi(\mathrm{bind}(u))$ of the form $\downarrow_r^{c'} \psi$ (i.e., with the same register). So, $f_{j-1}(r) = \mathbf{v}_{i(r,j)}(c') = \mathbf{v}'_{g(i(r,j))}(c')$. By condition (3) in the definition of Δ', we get $\mathbf{v}'_{g(i(r,j))}(c') = \mathrm{C}(z_{reg(\mathrm{bind}(u))})$ and $\mathbf{v}_j(c) = \mathbf{v}'_{g(j)}(c) = \mathrm{C}(z_{reg(\mathrm{bind}(u))})$. Hence, we deduce that $\mathbf{v}_j(c) = f_{j-1}(r)$, consequently $\rho, (j-1) \models_{f_{j-1}} \uparrow_r^c$. The case with $\phi(u)$ of the form $\neg \uparrow_r^c$ is analogous by observing again by condition (3), that $\mathbf{v}_j(c) \neq \mathrm{C}(z_{reg(\mathrm{bind}(u))})$.

Case $\phi(u) = \psi_1 \mathtt{U} \psi_2$ (standard):
There is a set F of final control states such that $F = \{\langle q, Y \rangle : u \notin Y \text{ or } (u \cdot 2) \in Y\}$ (remember $\phi(u \cdot 2) = \psi_2$). *Ad absurdum*, suppose that for $j' \geq j$, $u \cdot 2 \notin Y_{j'}$. By condition (I) in the definition for one-step consistent pairs, for $j' \geq j$, we have $u \cdot 1 \in Y_{j'}$ (remember $\phi(u \cdot 1) = \psi_1$) and $u \in Y_{j'}$. This is in contradiction with the fact that for some $\langle q, Y \rangle \in F$, the control state $\langle q, Y \rangle$ is repeated infinitely often in ρ. So, there is $j' \geq j$ such that $u \cdot 2 \in Y_{j'}$ and for $j \leq k < j'$, $u \cdot 2 \notin Y_k$. By induction, we can show that for $\alpha \in [0, j' - j - 1]$, $u \cdot 1 \in Y_{j+\alpha}$ and $u \in Y_{j+\alpha}$. When $j' = j$, this is trivial. Otherwise, take $\alpha = 0$. Since $u \cdot 2 \notin Y_j$, by (I) we get $u \cdot 1 \in Y_j$ and $u \in Y_{j+1}$. In

the induction step, suppose that $u \in Y_{j+\alpha}$ and $\alpha + 1 < j' - j$. Since $u \in Y_{j+\alpha}$ and $u \cdot 2 \notin Y_{j+\alpha}$, by (I), we get $u \cdot 1 \in Y_{j+\alpha}$ and $u \in Y_{j+\alpha+1}$. So, for $j \leq k < j'$, by induction hypothesis we obtain that $\rho, (k-1) \models \psi_1$. Moreover, we have $\rho, (j'-1) \models \psi_2$, whence $\rho, (j-1) \models \phi(u)$.

Conversely, let $\rho = \langle q_1, \mathbf{v}_1 \rangle, \ldots, \langle q_i, \mathbf{v}_i \rangle, \ldots$ be an infinite run of M such that $\rho, 0 \models \phi$. One can construct a concretization $\mathrm{C} : \{z_1, \ldots, z_m\} \rightarrow \mathbb{N}$, a sequence of register valuations $f_0, f_1, f_2, \ldots$ and a sequence of atoms $Y_1, Y_2, \ldots$ satisfying the conditions below:

(A) For $j \geq 0$, for $u \in Y_{j+1}$, we have $\rho, j \models_{f_j} \phi(u)$.
(B) For $j \geq 1$, the pair $\langle Y_j, Y_{j+1} \rangle$ is one-step consistent.
(C) $\epsilon \in Y_1$.
(D) For $j \geq 0$, f_{j+1} is an extension of f_j.
(E) For $j \geq 1$, for $u \in Y_j$ with $\phi(u) = \downarrow_r^c \psi$, we have $f_{j-1}(r) = \mathbf{v}_j(c)$.
(F) For $i \in \{1, \ldots, m\}$, if $reg^{-1}(i)$ belongs to some Y_j and $\phi(reg^{-1}(i)) = \downarrow_r^c \psi$, then $\mathrm{C}(z_i) = \mathbf{v}_j(c)$, i.e. $f_{j-1}(r) = \mathrm{C}(z_i)$.

The sequences can be built step by step on the model of the reduction rules introduced for e.g. in [19, Section 3.4]. This is a tedious and standard construction but its main idea is the following. We consider ω-sequences of the form $\langle Z_0, f_0' \rangle, \langle Z_1, f_1' \rangle, \ldots$ where $\bigcup Z_i \subseteq \mathrm{occ}(\phi)$ and each f_j' is a register valuation. We define a natural ordering on such sequences by checking component-wise set inclusion between sets of occurrences and extension relation between register valuations. We start by the bottom sequence $\langle \emptyset, f_\emptyset \rangle, \langle \emptyset, f_\emptyset \rangle, \ldots$ where $f_\emptyset$ is the register valuation with empty domain. Whenever one of the conditions among (B)-(F) is not satisfied or when some Y_j's is not an atom, we repair this *defect* either by adding an occurrence in some set (typically to satisfy condition (B)) or by extending some register valuation (typically to satisfy condition (E)). Repairing of defects is possible because the run ρ satisfies $\rho, 0 \models \phi$. By ordering the defects (for instance by ordering the occurrences and by choosing to repair defect at lowest position), it is possible to define a repair function h, that is monotonous with respect to the above-mentioned ordering. By Tarski-Knaster theorem on fixpoints, the map h has a least fixpoint from which can naturally define C and the sequences $f_0, f_1, f_2, \ldots$ and $Y_1, Y_2, \ldots$. Observe that sometimes, in order to repair a defect, $\rho, 0 \models \phi$ and (A) are used; for instance if a defect is present because condition (3) in the definition for atoms is not satisfied (disjunction), then the choice of the occurrence to be added in the adequate set of occurrences is made thanks to (A). When both disjuncts can be added, we use an arbitrary total ordering on occurrences to determine which occurrence to choose, so that h is indeed a function. By the way, (A) cannot cause any defect and is only useful to repair defects. Condition (D) is guaranteed because ϕ is flat and normalized whereas flatness guarantees that for $j \geq 1$, if $u \in Y_j \cap \mathrm{occ}^{\downarrow}(\phi)$, then for $k > j$, $u \notin Y_k$. So, when $reg^{-1}(i)$ does not occur in the sequence $Y_1, Y_2, \ldots, Y_k, \ldots$, the image $\mathrm{C}(z_i)$ can take an arbitrary value.

We can build an infinite run ρ' of $TS(M', \mathrm{C})$, say

$$\langle q_0, \mathbf{0} \rangle, \langle q_1', \mathbf{v}_1' \rangle, \ldots, \langle q_i', \mathbf{v}_i' \rangle, \ldots$$

such that there is a map $g : \mathbb{N} \to \mathbb{N}$ with $g(0) = 0$ and for all $i \in \mathbb{N} \setminus \{0\}$, $q'_{g(i)} \in Q \times \mathrm{AT}(\phi)$, $g(i) < g(i+1)$ and for all $g(i) < j < g(i+1)$, $q'_j \notin Q \times \mathrm{AT}(\phi)$. Observe that for $1 \leq i \leq N$, there is also a control state in F_i that is repeated infinitely often in ρ', which allows to conclude the other direction. In order to be more precise, a step

$$\langle q_j, \mathbf{v}_j \rangle \xrightarrow{\mathbf{g},\mathbf{a}} \langle q_{j+1}, \mathbf{v}_{j+1} \rangle$$

is replaced by a sequence of steps

$$\langle \langle q_j, Y_j \rangle, \mathbf{v}_j \rangle \cdots \langle q_1^{aux}, \ldots \rangle \cdots \langle q_T^{aux}, \ldots \rangle \xrightarrow{\mathbf{g},\mathbf{a}} \langle \langle q_{j+1}, Y_{i+1} \rangle, \mathbf{v}_{j+1} \rangle$$

Let us verify that this sequence of steps is valid. It is easy to check that $\langle Y_j, Y_{j+1} \rangle$ is a one-step consistent pair (by construction of the Y_k's), $\{\phi(u) : u \in Y\} \cap Q \subseteq \{q\}$ and $\neg q \notin \{\phi(u) : u \in Y\}$.

Let $u \in \mathrm{occ}^{\downarrow}(\phi) \cap Y_j$ with $\phi(u) = \downarrow_r^c \psi$ and $i = reg(u)$. By definition of C, we have $\mathrm{C}(z_i) = \mathbf{v}_i(c)$ and therefore the equality test between z_i and c is positive (before entering some adequate q_i^{aux}).

Similarly, let $u \in \mathrm{occ}^{\uparrow}(\phi) \cap Y_j$ with $\phi(u) = \uparrow_r^c$, $u' = \mathrm{bind}(u)$, $\phi(u') = \downarrow_r^{c'} \psi$ and $i = reg(u')$. Suppose $u' \in Y_{j'}$ for some $0 \leq j' \leq j-1$ (remember ϕ is a sentence). So $f_{j'}(r) = f_{j-1}(r) = \mathbf{v}_{j'}(c') = \mathrm{C}(z_i)$ and by construction of the Y_k's, we have $\rho, (j-1) \models_{f_{j-1}} \uparrow_r^c$, so the equality test between z_i and c is also positive (before entering some adequate q_i^{aux}).

The case $\phi(u)$ is a negation and $u \cdot 1 \in \mathrm{occ}^{\uparrow}(\phi) \cap Y_j$ is treated analogously. □

7.3 Decidability results

Remark that if the counter machine M is Ibarra reversal-bounded, then the parameterized counter machine M' built from M and the flat formula ϕ is Ibarra reversal-bounded. Using Corollary 19 and Theorem 20, we conclude that $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ restricted to Ibarra reversal-bounded counter machines and to flat formulae is decidable. Furthermore this can be extended to the class of reversal-bounded counter machines, using Lemma 22 below.

Lemma 22. *There is an exponential-time reduction from* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ *restricted to reversal-bounded counter machines into* $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ *restricted to Ibarra reversal-bounded counter machines. Furthermore this reduction preserves flatness of the formulae.*

Proof. Let $M = (n, Q, \Delta, q_0)$ be a reversal-bounded counter machine and ϕ be a formula. We assume that M is k-reversal-b-bounded (with $b > 0$). We build an Ibarra reversal-bounded counter machine M' and a formula ϕ' such that $M \models^{\omega} \phi$ iff $M' \models^{\omega} \phi'$. Before defining M', we would like to stress the following point: the result stated in Theorem 17 does also hold if we allow guards of the form $=(a)$ and $>(a)$ with $a \in \mathbb{N}$ understood as a constant (as it is proved in [35]). Hence, $\mathrm{MC}^{\omega}(\mathrm{LTL}^{\downarrow}[\cdot,\cdot])$ restricted to Ibarra reversal-bounded counter machines extended with tests to a constant and to flat formulae is also decidable. In order to improve the readability of this proof, we will consequently allow such tests in M'. In the same vein, we also allow M' to increment or decrement a counter by a units for some $a \in \mathbb{N}$.

We now give the construction of M' that is inspired by the one proposed in the proof of [24, Theorem 3] to establish that the reachability sets for reversal-bounded counter machines are semilinear. Let $M' = (n, Q \times B^n, \Delta', \langle q_0, \mathbf{0} \rangle)$ where $B = \{0, \ldots, b\} \uplus \{\omega_b\}$. Intuitively, the counter machine M' encodes the run of M and when a counter value in M is under the bound b, its value is stored into the control state of M'. The corresponding value of the counter in M' is 0, but when the value goes above b in M then it is restored in the counter in M'. The symbol ω_b is used to denote a value strictly greater than b. The transition relation Δ' is the smallest relation satisfying the following rules. We distinguish two types of transitions: either there is a counter whose value goes from $b + 1$ to b or not.

- for all $\mathbf{w} \in B^n$ and $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ such that for all $1 \leq c \leq n$, ($\mathbf{g}(c) = \mathtt{zero} \Leftrightarrow \mathbf{w}(c) = 0$) and ($\mathbf{w}(c) \neq \omega_b$ or $\mathbf{a}(c) \geq 0$), we include the transition $(\langle q, \mathbf{w} \rangle, \mathtt{true}, \mathbf{a}', \langle q', \mathbf{w}' \rangle)$ in Δ' where for all $1 \leq c \leq n$:
  - if $0 \leq \mathbf{w}(c) < b$, then $\mathbf{w}'(c) = \mathbf{w}(c) + \mathbf{a}(c)$ and $\mathbf{a}'(c) = 0$,
  - if $\mathbf{w}(c) = b$ and $\mathbf{a}(c) \leq 0$, then $\mathbf{w}'(c) = \mathbf{w}(c) + \mathbf{a}(c)$ and $\mathbf{a}'(c) = 0$,
  - if $\mathbf{w}(c) = b$ and $\mathbf{a}(c) = 1$, then $\mathbf{w}'(c) = \omega_b$ and $\mathbf{a}'(c) = b + 1$,
  - if $\mathbf{w}(c) = \omega_b$ and $\mathbf{a}(c) \geq 0$, then $\mathbf{w}'(c) = \omega_b$ and $\mathbf{a}'(c) = \mathbf{a}(c)$,
- for all $\mathbf{w} \in B^n$ and $(q, \mathbf{g}, \mathbf{a}, q') \in \Delta$ such that for all $1 \leq c \leq n$, ($\mathbf{g}(c) = \mathtt{zero} \Leftrightarrow \mathbf{w}(c) = 0$) and there exists $1 \leq c' \leq n$ such that $\mathbf{w}(c') = \omega_b$ and $\mathbf{a}(c') = -1$, then we include the two transitions $(\langle q, \mathbf{w} \rangle, \mathbf{g}', \mathbf{a}', \langle q', \mathbf{w}' \rangle)$ and $(\langle q, \mathbf{w} \rangle, \mathbf{g}'', \mathbf{a}'', \langle q', \mathbf{w}'' \rangle)$ in Δ' where for all $1 \leq c \leq n$:
  - if $0 \leq \mathbf{w}(c) < b$, then $\mathbf{g}'(c) = \mathbf{g}''(c) = \mathtt{true}$, $\mathbf{w}'(c) = \mathbf{w}''(c) = \mathbf{w}(c) + \mathbf{a}(c)$ and $\mathbf{a}'(c) = \mathbf{a}''(c) = 0$
  - if $\mathbf{w}(c) = b$ and $\mathbf{a}(c) \leq 0$, then $\mathbf{g}'(c) = \mathbf{g}''(c) = \mathtt{true}$, $\mathbf{w}'(c) = \mathbf{w}''(c) = \mathbf{w}(c) + \mathbf{a}(c)$ and $\mathbf{a}'(c) = \mathbf{a}''(c) = 0$,
  - if $\mathbf{w}(c) = b$ and $\mathbf{a}(c) = 1$, then $\mathbf{g}'(c) = \mathbf{g}''(c) = \mathtt{true}$, $\mathbf{w}'(c) = \mathbf{w}''(c) = \omega_b$ and $\mathbf{a}'(c) = \mathbf{a}''(c) = b + 1$,
  - if $\mathbf{w}(c) = \omega_b$ and $\mathbf{a}(c) \geq 0$, then $\mathbf{g}'(c) = \mathbf{g}''(c) = \mathtt{true}$, $\mathbf{w}'(c) = \mathbf{w}''(c) = \omega_b$ and $\mathbf{a}'(c) = \mathbf{a}(c)$,
  - if $\mathbf{w}(c) = \omega_b$ and $\mathbf{a}(c) = -1$, then $\mathbf{g}'(c)$ is equal to $>(b + 1)$, $\mathbf{w}'(c) = \omega_b$, $\mathbf{a}'(c) = \mathbf{a}(c)$, $\mathbf{g}''(c)$ is equal to $=(b + 1)$, $\mathbf{w}''(c) = b$ and $\mathbf{a}''(c) = -(b + 1)$.

The machine M' is then Ibarra reversal-bounded, because each counter performs the same number of alternations over b as in M and does not perform any alternation under this bound. We then define the relation $\sim \subseteq (Q \times \mathbb{N}^n) \times (Q \times B^n \times \mathbb{N}^n)$ as follows: $\langle q, \mathbf{v} \rangle \sim \langle \langle q', \mathbf{w} \rangle, \mathbf{v}' \rangle$ if and only if:

- $q = q'$
- for all $1 \leq c \leq n$:
  - if $\mathbf{w}(c) \in \{0, \ldots, b\}$ then $\mathbf{w}(c) = \mathbf{v}(c)$ and $\mathbf{v}'(c) = 0$,
  - if $\mathbf{w}(c) = \omega_b$ then $\mathbf{v}'(c) = \mathbf{v}(c)$ and $\mathbf{v}(c) > b$.

Let $TS(M) = (Q \times \mathbb{N}^n, \rightarrow)$ and $TS(M') = (Q \times B^n \times \mathbb{N}^n, \Rightarrow)$. By construction of M', one can easily prove that the relation $\sim$ enjoys the following property:

(⋆) Assume $\langle q, \mathbf{v}_1\rangle \sim \langle\langle q_1, \mathbf{w}_1\rangle, \mathbf{v}'_1\rangle$. For all $\langle q_2, \mathbf{v}_2\rangle \in Q \times \mathbb{N}^n$, we have $\langle q_1, \mathbf{v}_1\rangle \rightarrow \langle q_2, \mathbf{v}_2\rangle$ if and only if there exists $\langle\langle q_2, \mathbf{w}_2\rangle, \mathbf{v}'_2\rangle \in Q \times B^n \times \mathbb{N}^n$ such that $\langle q_2, \mathbf{v}_2\rangle \sim \langle\langle q_2, \mathbf{w}_2\rangle, \mathbf{v}'_2\rangle$ and $\langle\langle q_1, \mathbf{w}_1\rangle, \mathbf{v}'_1\rangle \Rightarrow \langle\langle q_2, \mathbf{w}_2\rangle, \mathbf{v}'_2\rangle$.

We shall now give the construction of the formula ϕ'. We define the map $T(\cdot,\cdot)$ that takes as arguments a subformula ψ of ϕ and a partial function g from the set of registers $\mathbb{N} \setminus \{0\}$ to $\{0, \ldots, b\}$. We set $\phi' = T(\phi, g_\emptyset)$ where $g_\emptyset$ has empty domain. The map $T(\cdot,\cdot)$ defined below recursively has a treatment for storing and testing registers that distinguishes the case when the counter value is below b:

- $T(q, g) \stackrel{\text{def}}{=} \bigvee_{q' \in \{q\} \times B^n} q'$,
- $T(\uparrow^c_r, g) \stackrel{\text{def}}{=} \uparrow^c_r \vee \bigvee_{q' \in \{\langle q, \mathbf{w}\rangle \in Q \times B^n \mid \mathbf{w}(c) = g(r)\}} q'$,
- $T(\neg\psi, g) \stackrel{\text{def}}{=} \neg T(\psi, g)$,
- $T(\psi \wedge \psi', g) \stackrel{\text{def}}{=} T(\psi, g) \wedge T(\psi', g)$,
- $T(\psi \vee \psi', g) \stackrel{\text{def}}{=} T(\psi, g) \vee T(\psi', g)$,
- $T(\psi \mathtt{U} \psi', g) \stackrel{\text{def}}{=} T(\psi, g) \mathtt{U} T(\psi', g)$,
- $T(\psi \mathtt{R} \psi', g) \stackrel{\text{def}}{=} T(\psi, g) \mathtt{R} T(\psi', g)$,
- $T(\mathtt{X}\psi, g) \stackrel{\text{def}}{=} \mathtt{X}\, T(\psi, g)$,
- $T(\downarrow^c_r \psi, g) \stackrel{\text{def}}{=} (\bigvee_{i \in \{0,\ldots,b\}} \bigvee_{q' \in \{\langle q, \mathbf{w}\rangle \in Q \times B^n \mid \mathbf{w}(c) = i\}} q' \wedge T(\psi, g[r \mapsto i]) \vee$ $(\bigvee_{q' \in \{\langle q, \mathbf{w}\rangle \in Q \times B^n \mid \mathbf{w}(c) = \omega_b\}} q' \wedge \downarrow^c_r T(\psi, g))$.

By taking advantage of (⋆), we can show by structural induction that $M \models^\omega \phi$ if and only $M' \models^\omega \phi'$. Observe that $T(\cdot,\cdot)$ requires exponential time in $|\phi| + |M|$ because of the clause about $T(\downarrow^c_r \psi, g)$. This exponential blow-up would persist even if we encode formulae as DAGs because of the presence of g in $T(\psi, g)$. □

Corollary 23. $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ *restricted to reversal-bounded counter machines and to flat formulae is decidable.*

Finally, assume the formula ϕ is a positively flat formula (see Section 3). For all atoms $Y \in \mathrm{AT}(\phi)$, the set $\{u \in Y \mid u \cdot 1 \in \mathrm{occ}^\uparrow(\phi)$ and $\phi(u)$ is a negation$\}$ is empty. So, in the construction of M' from M and ϕ, we only use parameterized tests of the form $=(z)$. Hence, if M is a one-counter machine and ϕ is a positively flat formula, we deduce that M' is a restricted parameterized one-counter machine. Using Corollary 19 and Theorem 20, we get the result below.

Theorem 24. $\mathrm{MC}^\omega(\mathrm{LTL}^\downarrow[\cdot,\cdot])$ *restricted to one-counter machines and to positively flat formulae is decidable.*

In order to extend Theorem 24 to the full flat fragment, one needs to perform inequality tests in parameterized one-counter machines, which is so far unclear how to perform while preserving decidability of the corresponding parameterized reachability problem. This generalization is left as an open problem.

8 Concluding Remarks

In this paper, we have studied the decidability status of model-checking freeze LTL over various subclasses of counter machines for which the reachability problem is known to be decidable. Our most remarkable technical contributions concern reversal-bounded counter machines and flat formulae. Besides, we have established an original link between reachability problems for parameterized counter machines and model-checking counter machines over the flat fragment of freeze LTL. Figure 4 contains a summary of the main results ($\mathbf{D}$ stands for decidability, $\mathbf{U}$ for undecidability) in which the columns referred to restriction either on the counter machines or on the formulae. Sometimes, an additional restriction between parentheses is indicated in order to emphasize that the result holds true for a stricter fragment. Bibliographical references in the table indicate that the related result is mainly due to the referred work. Here are a few rules of thumb:

	Det.	**NDet.**	**Flat formulae**	**No $\uparrow_r^c$**
RB	$\mathbf{D}$ **Cor. 16**	$\mathbf{U}$ **(strictness)** **Theo. 13**	$\mathbf{D}$ **Cor. 23**	$\mathbf{D}$ **[16]**
1CM	**PSPACE-C.** **[21]**	$\mathbf{U}$ **(1 reg.)** **[21]**	**open** $\mid$ $\mathbf{D}$ **for pos. flatness** **Theo. 24**	**PSPACE-C.** **[48,18]**
Flat CM	$\mathbf{D}$	$\mathbf{D}$ **Theo. 7**	$\mathbf{D}$	$\mathbf{D}$
VASS	**EXPSPACE** **Cor. 16**	$\mathbf{U}$ **(1 reg.)** **Theo. 9**	$\mathbf{U}$ **Theo. 9**	**EXPSPACE-C.** **[30]**

Fig. 4. Summary

determinism, flat counter machines and no freeze lead to decidability. However, flat formulae often guarantee decidability (except for VASS) whereas reversal-boundedness can lead to decidability (but the restriction with a single register leads to undecidability). Finally, throwing away the atomic formulae made of control states does not help for decidability. Even though we have established various decidability results in the paper, the complexity of the decision problems is far from being known, mainly because we use reductions to Presburger arithmetic. However, as a consequence of the effectiveness of our reductions, all the decidable decision problems we have considered, are known to have an elementary complexity. Similarly, the undecidability borders with respect to the number of registers in formulae and the number of counters in machines are not completely known, apart from mentioning the case with one-counter machines and flat formulae. Besides, we have not investigated the safety fragment as done in [40] (no until in the scope of an even number of negations).

Finally, other subclasses with decidable reachability problem are worth being studied; for instance $\mathrm{MC}^{\omega}(\mathrm{LTL}[\cdot])$ over lossy counter machines (with no freeze operators) is already known to be undecidable by [42] –see the reduction from repeated accessibility. Hence, this class behaves quite differently from the ones considered herein since we wished to study the effect of including the freeze operator in LTL. Last but not least, parameterized version of problems, in the lines of [29], would be worth being investigated.

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References

1. R. Alur and T. Henzinger. A really temporal logic. *Journal of the ACM*, 41(1):181–204, 1994.
2. C. Areces, P. Blackburn, and M. Marx. A road-map on complexity for hybrid logics. In *CSL'99*, volume 1683 of *Lecture Notes in Computer Science*, pages 307–321. Springer, 1999.
3. S. Bardin, A. Finkel, and J. Leroux. FASTer acceleration of counter automata in practice. In *TACAS'04*, volume 2988 of *Lecture Notes in Computer Science*, pages 576–590. Springer, 2004.
4. J. Berstel and L. Boasson. Context-free languages. In *Handbook of Theoretical Computer Science, Volume B, Formal models and semantics*, pages 59–102. Elsevier, 1990.
5. H. Björklund and M. Bojańczyk. Bounded depth data trees. In *ICALP'07*, volume 4596 of *Lecture Notes in Computer Science*, pages 862–874. Springer, 2007.
6. B. Boigelot. *Symbolic methods for exploring infinite state spaces*. PhD thesis, Université de Liège, 1998.
7. M. Bojańczyk, A. Muscholl, T. Schwentick, L. Segoufin, and C. David. Two-variable logic on words with data. In *LICS'06*, pages 7–16. IEEE, 2006.
8. A. Bouajjani, M. Bozga, P. Habermehl, R. Iosif, and P. M. nd T. Vojnar. Programs with lists are counter automata. In *CAV'06*, volume 4144 of *Lecture Notes in Computer Science*, pages 517–531. Springer, 2006.
9. P. Bouyer, N. Markey, J. Ouaknine, and J. Worrell. The cost of punctuality. In *LICS'07*, pages 109–118. IEEE, 2007.
10. M. Bozga, R. Iosif, and Y. Lakhnech. Flat parametric counter automata. *Fundamenta Informaticae*, 91(2):275–303, 2009.
11. A. Bradley and Z. Manna. *The Calculus of Computation*. Springer, 2007.
12. H. Comon and V. Cortier. Flatness is not a weakness. In *CSL'00*, volume 1862 of *Lecture Notes in Computer Science*, pages 262–276. Springer, 2000.
13. H. Comon and Y. Jurski. Multiple counters automata, safety analysis and Presburger analysis. In *CAV'98*, volume 1427 of *Lecture Notes in Computer Science*, pages 268–279. Springer, 1998.
14. U. Dal Lago, A. Montanari, and G. Puppis. On the equivalence of automaton-based representations of time granularities. In *TIME'07*, pages 82–93. IEEE, 2007.
15. D. Dams. Flat fragments of CTL and CTL*: separating the expressive and distinguishing powers. *Logic Journal of the IGPL*, 7(1):55–78, 1999.
16. Z. Dang, O. Ibarra, and P. San Pietro. Liveness verification of reversal-bounded multicounter machines with a free counter. In *FST&TCS'01*, volume 2245 of *Lecture Notes in Computer Science*, pages 132–143. Springer, 2001.
17. S. Demri, A. Finkel, V. Goranko, and G. van Drimmelen. Towards a model-checker for counter systems. In *ATVA'06*, volume 4218 of *Lecture Notes in Computer Science*, pages 493–507. Springer, 2006.
18. S. Demri and R. Gascon. The effects of bounding syntactic resources on Presburger LTL. *Journal of Logic and Computation*, 19(6):1541–1575, 2009.
19. S. Demri and P. Gastin. Specification and verification using temporal logics. In *Modern applications of automata theory*, volume 2 of *IIsc Research Monographs*. World Scientific, 2009.

20. S. Demri, R. Lazić, and D. Nowak. On the freeze quantifier in constraint LTL: decidability and complexity. *Information & Computation*, 205(1):2–24, 2007.
21. S. Demri, R. Lazić, and A. Sangnier. Model checking freeze LTL over one-counter automata. In *FOSSACS'08*, volume 4962 of *Lecture Notes in Computer Science*, pages 490–504. Springer, 2008.
22. J. Esparza, A. Finkel, and R. Mayr. On the verification of broadcast protocols. In *LICS'99*, pages 352–359. IEEE, 1999.
23. A. Finkel and J. Leroux. How to compose Presburger accelerations: Applications to broadcast protocols. In *FST&TCS'02*, volume 2256 of *Lecture Notes in Computer Science*, pages 145–156. Springer, 2002.
24. A. Finkel and A. Sangnier. Reversal-bounded counter machines revisited. In *MFCS'08*, volume 5162 of *Lecture Notes in Computer Science*, pages 323–334. Springer, 2008.
25. M. Fitting. Modal logic between propositional and first-order. *Journal of Logic and Computation*, 12(6):1017–1026, 2002.
26. S. Ginsburg and E. H. Spanier. Semigroups, presburger formulas, and languages. *Pacific Journal of Mathematics*, 16(2):285–296, 1966.
27. S. Göller, R. Mayr, and A. To. On the computational complexity of verifying one-counter processes. In *LICS'09*, pages 235–244. IEEE, 2009.
28. V. Goranko. Hierarchies of modal and temporal logics with references pointers. *Journal of Logic, Language, and Information*, 5:1–24, 1996.
29. C. Haase, S. Kreutzer, J. Ouaknine, and J. Worrell. Reachability in succinct and parametric one-counter automata. In *CONCUR'09*, volume 5710 of *Lecture Notes in Computer Science*, pages 369–383. Springer, 2009.
30. P. Habermehl. On the complexity of the linear-time mu -calculus for Petri nets. volume 1248 of *Lecture Notes in Computer Science*, pages 102–116. Springer, 1997.
31. R. Howell, P. Jančar, and L. Rosier. Completeness results for single-path Petri nets. *Information & Computation*, 106(2):253–265, 1993.
32. O. Ibarra. Reversal-bounded multicounter machines and their decision problems. *Journal of the ACM*, 25(1):116–133, 1978.
33. O. Ibarra. Restricted one-counter machines with undecidable universe problems. *Mathematical Systems Theory*, 13(181):181–186, 1979.
34. O. Ibarra and Z. Dang. On removing the stack from reachability constructions. In *ISAAC'01*, volume 2223 of *Lecture Notes in Computer Science*, pages 244–256. Springer, 2001.
35. O. Ibarra, J. Su, Z. Dang, T. Bultan, and R. Kemmerer. Counter machines and verification problems. *Theoretical Computer Science*, 289(1):165–189, 2002.
36. P. Jančar, A. Kučera, F. Moller, and Z. Sawa. Equivalence-checking with one-counter automata: a generic method for proving lower bounds. In *FOSSACS'02*, volume 2256 of *Lecture Notes in Computer Science*, pages 172–186. Springer, 2002.
37. P. Jančar. Decidability of a temporal logic problem for Petri nets. *Theoretical Computer Science*, 74(1):71–93, 1990.
38. R. M. Karp and R. E. Miller. Parallel program schemata. *JCSS*, 3(2):147–195, 1969.
39. P. Lafourcade, D. Lugiez, and R. Treinen. Intruder deduction for AC-like equational theories with homomorphisms. In *RTA'05*, volume 3467 of *Lecture Notes in Computer Science*, pages 308–322. Springer, 2005.
40. R. Lazić. Safely freezing LTL. In *FST&TCS'06*, volume 4337, pages 381–392. LNCS, 2006.
41. A. Lisitsa and I. Potapov. Temporal logic with predicate λ-abstraction. In *TIME'05*, pages 147–155. IEEE, 2005.
42. R. Mayr. Undecidable problems in unreliable computations. *Theoretical Computer Science*, 297(1–3):337–354, 2003.
43. M. Minsky. *Computation, Finite and Infinite Machines*. Prentice Hall, 1967.

44. M. Presburger. Über die Vollständigkeit eines gewissen Systems der Arithmetik ganzer Zahlen, in welchem die Addition als einzige Operation hervortritt. *Comptes Rendus du Premier Congrès de Mathématiciens des Pays Slaves, Warsaw*, pages 92–101, 1929.
45. C. Reutenauer. *The Mathematics of Petri nets*. Masson and Prentice, 1990.
46. A. Sangnier. *Vérification de systèmes avec compteurs et pointeurs*. Thèse de doctorat, LSV, ENS Cachan, France, 2008.
47. O. Serre. Parity games played on transition graphs of one-counter processes. In *FOSSACS'06*, volume 3921 of *Lecture Notes in Computer Science*, pages 337–351. Springer, 2006.
48. A. Sistla and E. Clarke. The complexity of propositional linear temporal logic. *Journal of the ACM*, 32(3):733–749, 1985.
49. A. To. Model checking FO(R) over one-counter processes and beyond. In *CSL'09*, volume 5771 of *Lecture Notes in Computer Science*, pages 485–499. Springer, 2009.
50. M. Vardi and P. Wolper. Reasoning about infinite computations. *Information & Computation*, 115:1–37, 1994.