# Lecture Series in IRIF: Games on Graphs and Linear Programming Abstractions

*Uri Zwick* *

Two-Player Turn-Based Stochastic Games (2TBSG) on graphs is an intriguing family of games. Mean-payoff Games (MPG), Energy Games (EG) and parity games (PG) are special classes of such games. Parity games are equivalent to some highly motivated problems in automata theory and automatic verification. The decisions problems corresponding to these games are in $NP \cap co\text{-}NP$. The question whether there is a polynomial time algorithm for solving these games is a major open problem.

The fastest known algorithms for 2TBSGs and MPGs are sub-exponential (Ludwig [12]) or pseudopolynomial ([16],[1]). For *discounted* 2TBSG with a fixed discount factor, a strongly polynomial time algorithm is known [5]. Very recently, in a breakthrough result, a quasi-polynomial time algorithm was obtained for PGs by Calude et al. [2]. Improving on these algorithms, or obtaining some hardness results is a major open problems.

An interesting feature of these games is that sub-exponential time algorithms known for solving them are local-improvement algorithms reminiscent of the celebrated *simplex algorithm* for solving Linear Programming (LP) problems. While it is not known how to cast these games as linear programs, which would lead, of course, to polynomial time algorithms, the problems do belong to a family of problems known as LP-type problems (Matoušek et al. [13]) and to other abstractions of linear programs such as Acyclic Unique Sink Orientations (AUSOs) of cubes (Szabó and Welzl [15]).

The study of these games is thus related to the study of *pivoting rules* for the simplex algorithm and to another major open problem: Is there a *strongly* polynomial time algorithm for linear programming. The fastest known randomized pivoting rules for the simplex algorithm are sub-exponential (Kalai [10, 11], Matoušek et al. [13], Hansen and Zwick [7]). A nice abstract setting that captures both linear programming and the above mentioned families of games, is the setting of Acyclic Unique Sink Orientations (AUSOs) of cubes (Szabó and Welzl [15]). There are many interesting open problems related to AUSOs: Can the known randomized sub-exponential algorithms for solving AUSOs, i.e., finding their sink, be improved? The ultimate question, of course, is whether there is such a polynomial time algorithm. Is there a *deterministic* sub-exponential time algorithm? Can a super-polynomial lower bound be obtained on the number of *queries* to an AUSO required in order to find its sink? The currently best lower bound is an essentially quadratic lower bound obtained by Schurr and Szabó [14].

The proposed lecture series will focus on these topics. The lectures would cover results in which I was involved, e.g., [3, 4, 5, 6, 7, 8, 9, 16], as well as results obtained by other researchers.

The following is a tentative list of lectures.

*Blavatnik School of Computer Science, Tel Aviv University, Israel. E-mail: `zwick@tau.ac.il`.

## Lecture 1: Two-player Turn-based Stochastic Games

In the first lecture we define the two-player Turn-Based Stochastic Games (TBSGs), the most general games considered in this lecture series. We define the objectives of the two players, the strategies that they can use, and define the values of the games. We then consider algorithms for finding the values and optimal strategies, first in one-player games and then in two-player games. While for one-player games polynomial time algorithms are known, for most two-player games no polynomial time algorithms are currently known.

## Lecture 2: Mean Payoff games and Energy Games

In this talk we consider non-stochastic versions of the games introduced in the first lecture, namely Mean Payoff Games (MPGs) and Energy Games (EGs). We show reductions between these two games. We then describe a pseudo-polynomial time algorithm of Brim et al. for EGs, and hence also MPGs.

## Lecture 3: Randomized sub-exponential time algorithm

In this talk we present an elegant randomized algorithm that solves all the games we consider in *sub-exponential* time. This is currently the fastest known algorithm for TBSGs, and also for MPGs and EGs, when there is no restriction on the edge costs. The algorithm is an adaptation of a randomized algorithms of Kalai and Matousek, Sharir and Welzl for solving linear programs. We also mention relations to abstractions of Linear Programming abstractions.

## Lecture 4: Acyclic Unique Sink Orientations (AUSOs)

In this lecture we consider Acyclic Unique Sink Orientations (AUSOs) an abstraction related to linear programming that also captures the games we are interested in. We present algorithms for finding the sink of an AUSOs, which directly translate to algorithms for solving games.

## Lecture 5: Parity Games

In this lecture we consider Parity Games (PGs), a very special case of MPGs, with many relations to automata on infinite words and to automatic verification. In a recent breakthrough Calude et al. recently obtained a quasi-polynomial time algorithm for solving PGs. We will describe one of the existing variants of their algorithm.

## Lecture 6: Lower bounds for Policy Iteration

Policy Iteration, introduced in the first lecture, is a very natural class of algorithms for solving TBSGs. Policy Iteration algorithms work very well in practice. However, we now know that their worst-case complexity is exponential. We will present such lower bounds.

## Lecture 7: Lower bounds for Random-Facet and Random-Edge

In this lecture we present sub-exponential lower bounds for Random-Facet (the randomized algorithm considered in Lecture 3) and Random-Edge.

## References


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