Dynamic Networks through the Lens of Algebraic Topology

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DUCAT/ESTATE @ Cap Hornu – 17/03

LABORATOIRE
D'INFORMATIQUE
& SYSTÈMES

Outline

Modeling Dynamic Networks and Algebraic Topology :

- Pierre's talk follow up
- algebraic methods in distributed computing
  - (shared memory)
  - static networks
  - dynamic networks
  - message adversaries
- (after lunch) reminder
- recent techniques with algebraic topology

Motivations

- success of algebraic methods in distributed computing
- a model that can *model* (almost) all non-deterministic distributed executions
- **but** limits to effective generalization
- outline of some recent techniques
  - generalizing the standard chromatic subdivision [G.-Perdereau-2016]
  - full topology [Nowak-Schmid-Winkler-2019]
  - topology of the set of non-oblivious executions by geometrization [G.-Perdereau-2021]

some of the results were already presented at Descartes ...

The Dynamic Network or Message Adversary Model

- V is the set of processes
- *dynamic graph* = a sequence $G_0, G_1, G_2, \ldots, G_t, \ldots$
  G_t are **spanning digraphs** (ie over the same set of vertices V)
- *dynamic network* or *message adversary* = set of dynamic graphs

A Question of Terminology

- *Mobile faults* Santoro N., Wiedmayer P. Time is not a healer STACS 89
- *HeardOf Model* Charron-Bost B. and Schiper A. The Heard-Of model: computing in distributed systems with benign faults Dist. Computing 2009
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- *Time Varying Graphs and Dynamic Networks* Casteigts A., Flocchini P., Quattrociocchi W., Santoro N. Int. J. Parallel Emergent Distributed Syst. 2012

A Computation Step

The system evolves in synchronous rounds.

G_0 is the initialization.

In a given round t, a process v

- sends its message m_v to neighbours (`sendAll`(m_v)),
- it receives messages according to arcs in G_t :
  m_u is received if $uv \in A(G_t)$
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G_t is called the *instant digraph*

Message adversaries model (almost) everything

notation : infinite words (over an alphabet of graphs)

Examples with a network of two processes

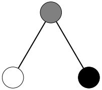
- at each round, everything is possible: $\{\circ\quad\bullet, \circ\!\leftrightarrow\!\bullet, \circ\!\leftarrow\!\bullet, \circ\!\rightarrow\!\bullet\}^\omega$.
- at each round, only one message can be lost:
  $\Gamma^\omega = \{\circ\!\leftrightarrow\!\bullet, \circ\!\leftarrow\!\bullet, \circ\!\rightarrow\!\bullet\}^\omega$.
- at most one of the processes can lose messages:
  $\{\circ\!\leftrightarrow\!\bullet, \circ\!\leftarrow\!\bullet\}^\omega \cup \{\circ\!\leftrightarrow\!\bullet, \circ\!\rightarrow\!\bullet\}^\omega$.
- at most one of the processes can crash:
  $C_1 = \{\circ\!\leftrightarrow\!\bullet\}^\omega \cup \{\circ\!\leftrightarrow\!\bullet\}^*(\{\circ\!\leftarrow\!\bullet\}^\omega \cup \{\circ\!\rightarrow\!\bullet\}^\omega)$.
- The communication system is *fair*:
  $Fair = \Gamma^\omega \setminus \Gamma^*(\{\circ\!\leftrightarrow\!\bullet, \circ\!\leftarrow\!\bullet\}^\omega \cup \{\circ\!\leftrightarrow\!\bullet, \circ\!\rightarrow\!\bullet\}^\omega)$.

no notion of faults but can *encode* faults

Other Examples

- Shared memory (IIS of this morning)
- $f-$omissions

P_3 :



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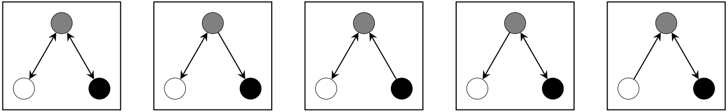

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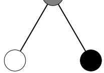

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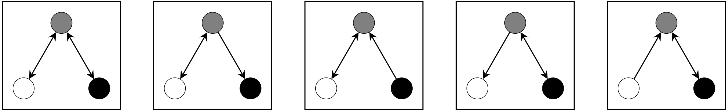

- an adversary is said *oblivious* (or *iterated*) if it is Σ^{ω} for some set Σ
- general dynamic networks
- async. static networks
- synchronous static network

IIS Model (as a dynamic network)

The set Γ_n of graphs with n vertices such that

- (loops : always omitted)
- containment: $u \longrightarrow v$ or $v \longrightarrow u$
- immediacy: $u \longrightarrow v$ and $v \longrightarrow w$ implies $u \longrightarrow w$.

They are the graphs of **pre-total orders**.

$$IIS(n) = \Gamma_n^\omega$$

Idea: $u \longrightarrow v$ means v has read the value written by u (in the immediate memory).

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$$IIS(n) = \Gamma_n^\omega$$

Idea: $u \longrightarrow v$ means v has read the value written by u (in the immediate memory). Initially proposed as a memory model, now can also be considered as a message model.

Topology : Simplicial Complex Modeling

identifier -> color

state -> label

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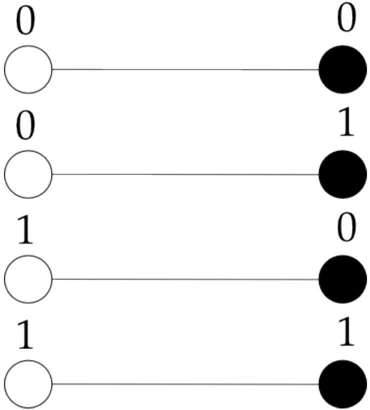

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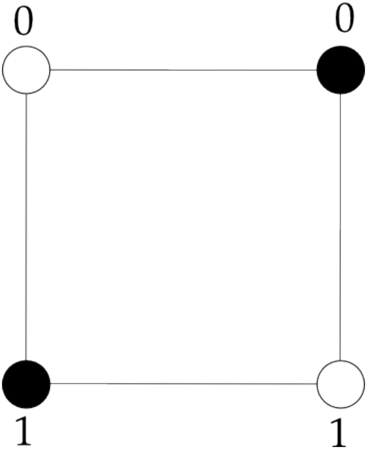
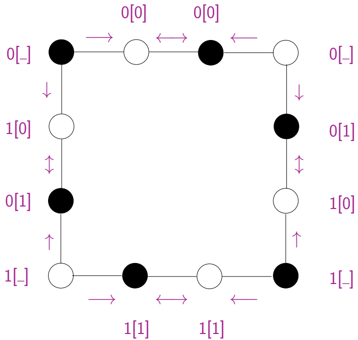
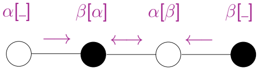


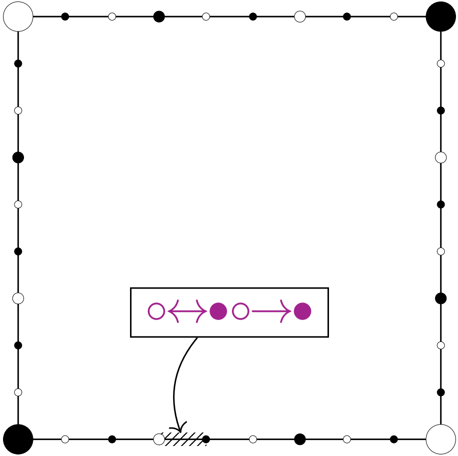
Topology : Operational View





Topology : Distributed Computations

Non-determinism => many following global states



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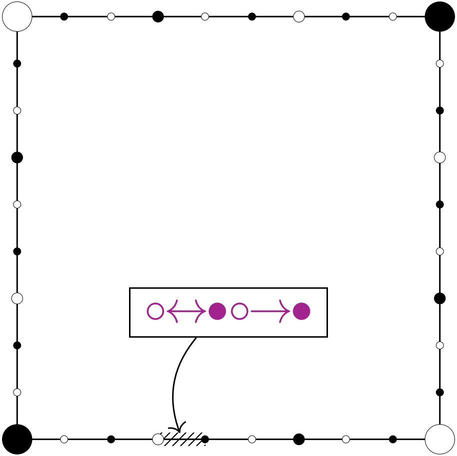
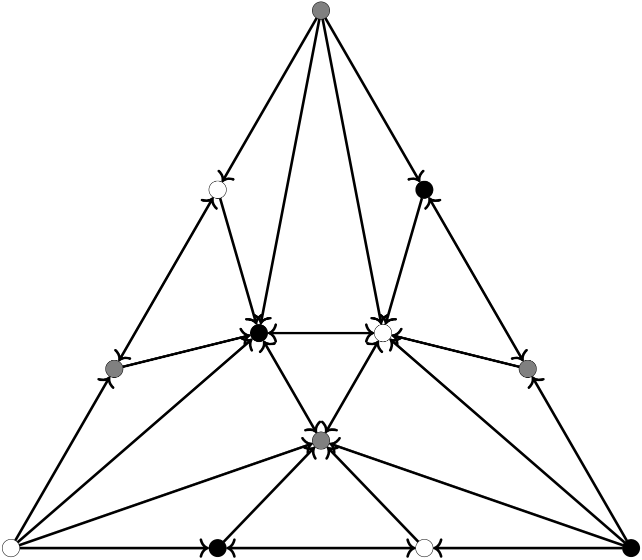


we drop labels in figures (only topology!)

Standard Chromatic Subdivision



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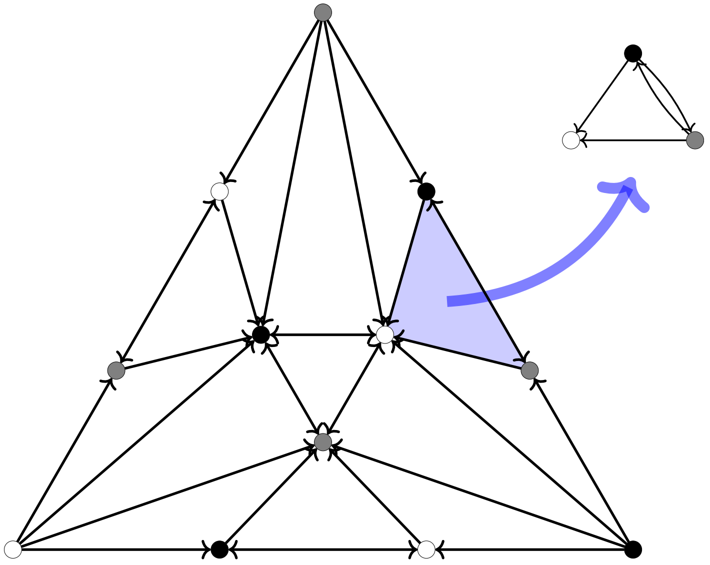
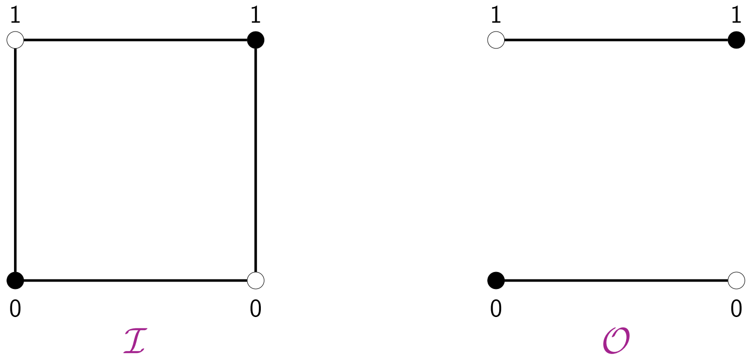


Topology : Executions

An execution gives a sequence of "simplicial complexes" that is algorithmically defined.
$=>$ *protocol complex* $\mathcal{P}$
the protocol complex when the algorithm is the *full information* protocol.

Topology : Tasks

- $\mathcal{I}$: set of inputs, $\mathcal{O}$: set of outputs
- Δ : relation between $\mathcal{I}$ and $\mathcal{O}$



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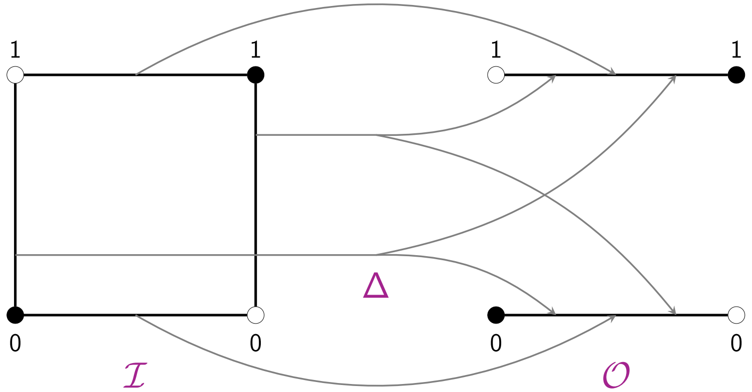

Binary consensus for 2 processes

Herlihy Shavit Distributed Computability Theorem

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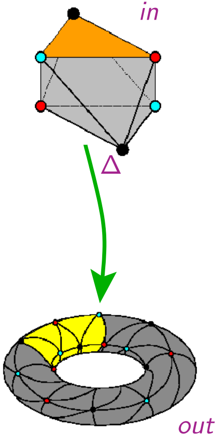

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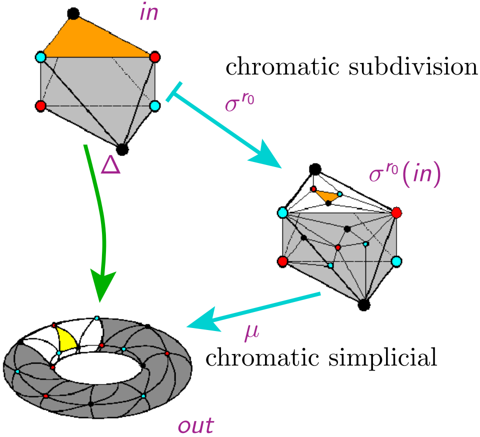

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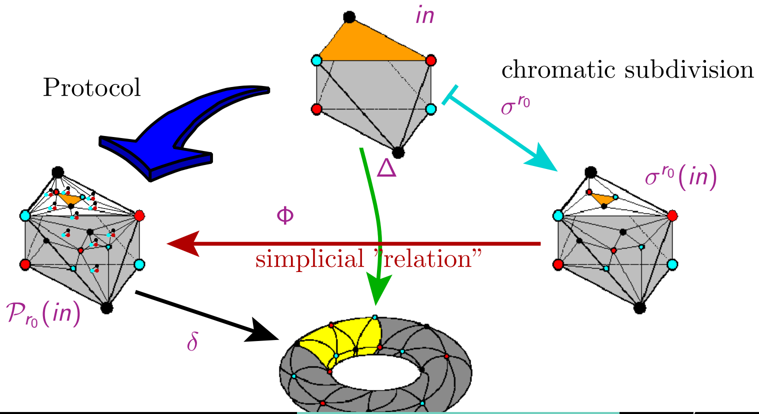

Survey of Old and Recent Results 1

See *Distributed Computing Through Combinatorial Topology* (Herlihy, Kozlov, Rajsbaum)

- distributed tasks are topologically constrained
- applications :
  - consensus: connexity
  - set-agreement : simple connexity
  - colored tasks : renaming bounds
  - "graph" tasks : DUCAT

Partial Survey of Old and Recent Results 2

Consensus
- oblivious adversaries F^ω [Coulouma-G.-Peters-2015]
- arbitrary adversaries :
  - $n = 2$ [Fevat-G.-2011], [G.-Perdereau-2020]
  - any n [Nowak-Schmid-Winckler-2019]
- know-all adversaries [CFPRRT-2021]

$k-$set agreement
- IIS : [Saks-Zaharoglou-2000]
- $f-$omissions : [G.-Perdereau-2016]
- know-all adversaries [CFPRRT-2019]

tasks
- IIS : [Herlihy-Shavit-1999], [Borowsky-Gafni-1993]
- affine models : [Kuznetsov-Rieutord-etal-2016,2018]

Partial Summary

Tasks \ Models	**sub-IIS**	**IIS**	**general graphs**
consensus	GP20(n=2)		NSW19 CFPRRT21
$k-$set agreement		SZ2000	GP16 CFPRRT19
general tasks	KR...16/18	BG93 HS99	???

Known Limits for the Topological Approach

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1. The protocol complex is too **complex**
2. focus on the IIS model = standard/chromatic subdivision
3. computability equivalences
   but *complexity ??*

IIS : When it Works Nicely

Iterated Immediate Snapshot (IIS)

IIS model is central see Pierre and ~~Hugues~~' talks, also Kozlov 2013

"regular" message adversaries oblivious / iterated

Direction for Generalization

- arbitrary underlying graphs
- arbitrary languages (not only oblivious)
- more direct characterizations

Generalizing : Graphical Subdivision $\zeta(G)$

Goal: define a subdivision for arbitrary (underlying) graphs
requirements :

- extension of the standard chromatic subdivision
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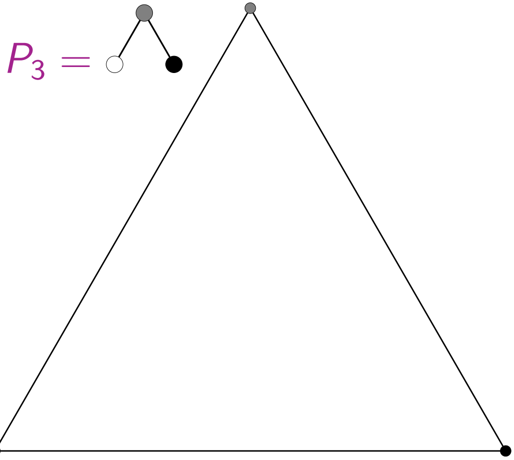
Definition by Recurrence for "G"

$$\zeta(G,\sigma) \stackrel{\text{def}}{=} \begin{cases} \emptyset & \text{if } V(G)=\emptyset \\ \sigma & \text{if } E(G)=\emptyset \\ \bigcup_{\tau \subsetneq \sigma} \text{join}(\zeta(G_{|\tau},\tau), \mu_\sigma(\sigma \setminus \tau)) & \text{otherwise} \end{cases}$$

Construction for Graphical Subdivision of $''P_3''$



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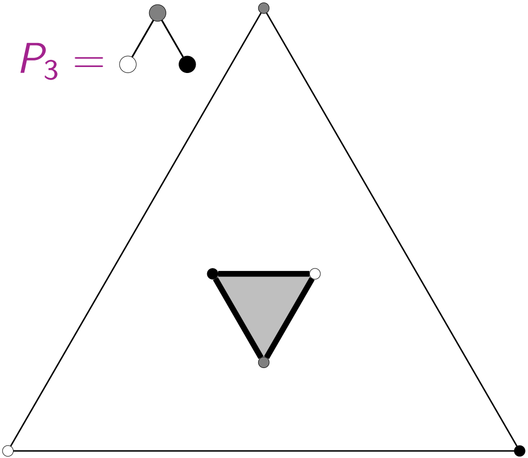

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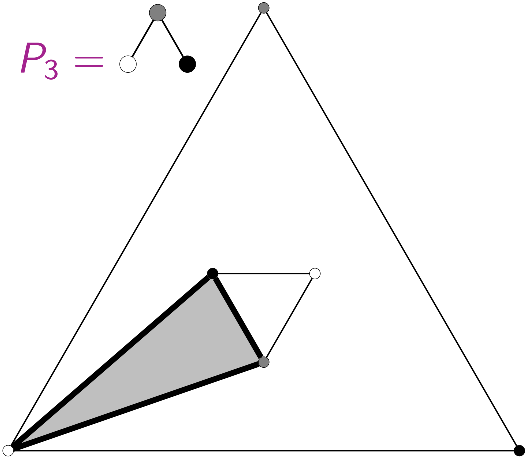

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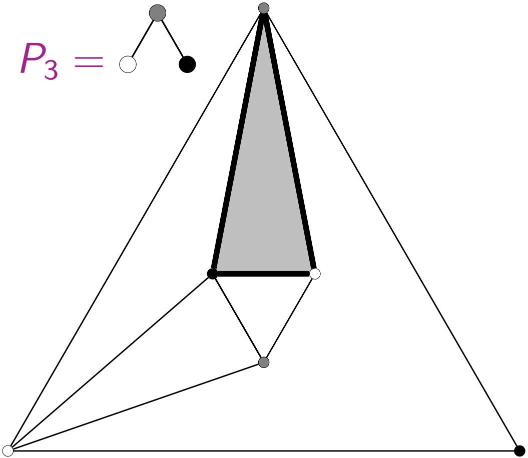

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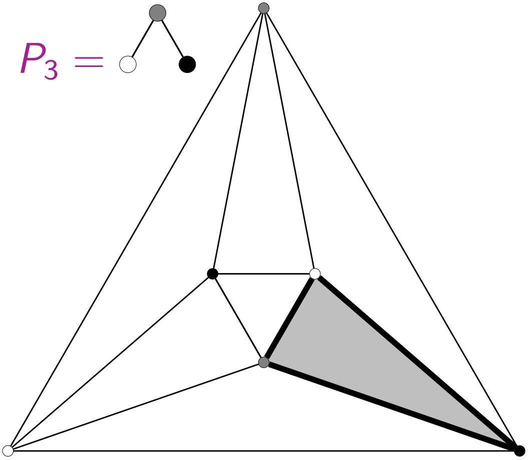

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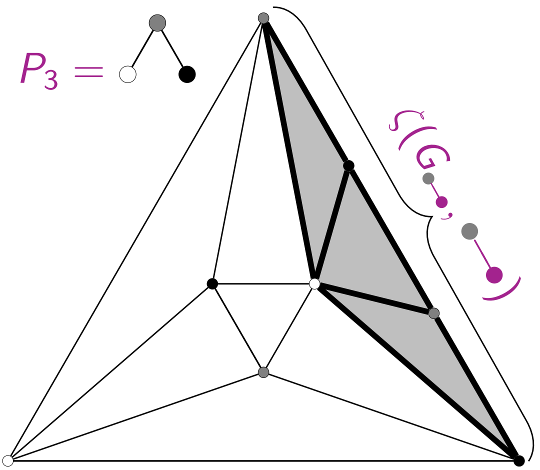

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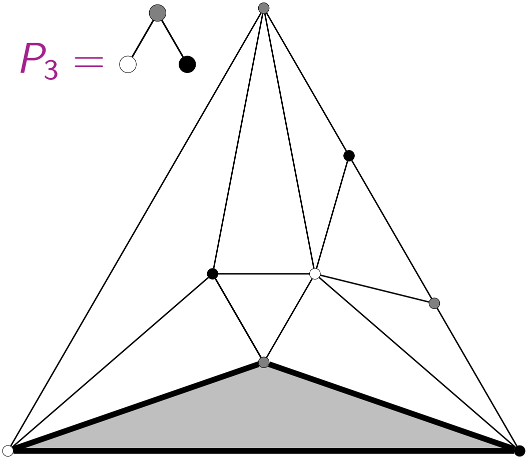

$\{\circ, \bullet\} \notin E$

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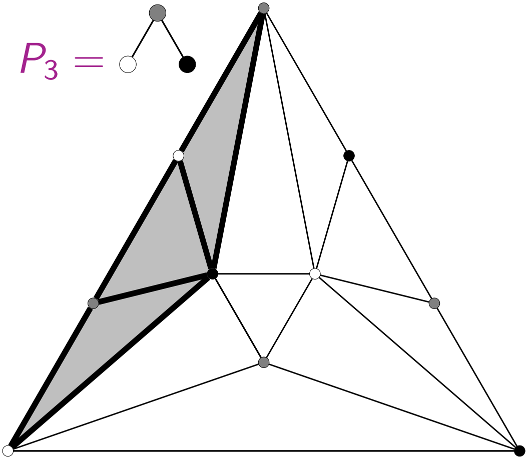
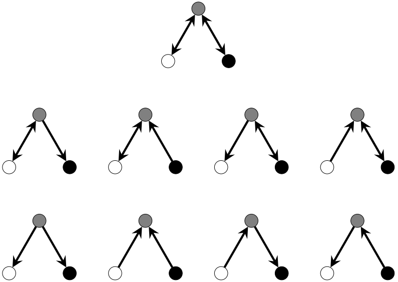


Corresponding Message Adversary

The standard correpsondance defines a special set of instant graphs: $\gamma(G)$

$n = |G|$.
D a subgraph of G, $\varphi_G(D)$ is defined by $uv \in A(\varphi_G(D))$ if $uv \in E(G)$,

$\gamma(G) = \varphi_G(IIS_n)$.



Application : Set-agreement

Proposition

It is impossible to solve set-agreement on $\gamma(P_3)^\omega$.

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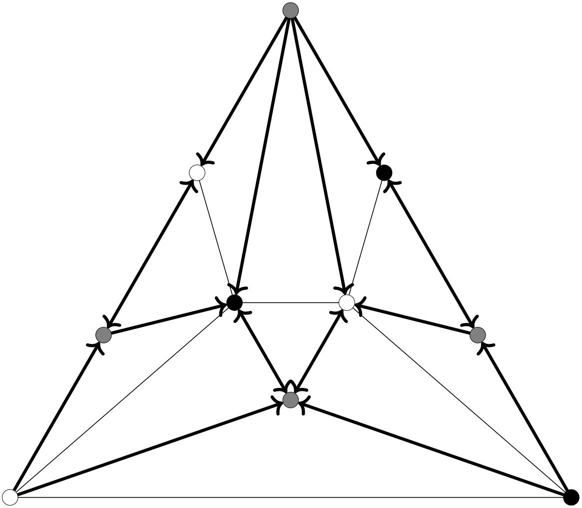
NB ne fonctionne que pour l'impossibilité

Actually Topological "Diagrams"



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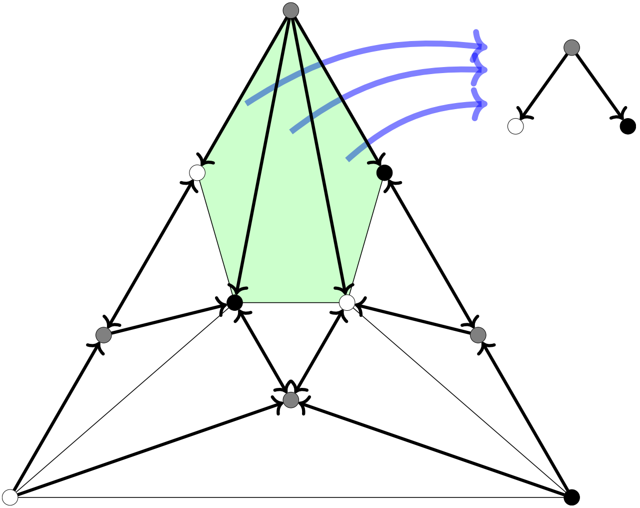

Power and Limits of Algebraic Tools

- elementary steps of modelization
  $D \longleftrightarrow S$
- $=>$ fully general
- how to handle in a simple way when generalizing
  - graphical subdivisions
  - geometrization

Perspectives

- diagrams => new applications of possibly more than one simplex per instant graphs
- non oblivious message adversaries and generalizing the induced "topology" => "removing"/"compacting" unnecessary executions

Questions ?

Thanks for your attention.