Fault-tolerant coloring of the asynchronous cycle

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Setup

Takeaway

Contributions

- define the **asynchronous k-coloring problem** for asynchronous networks
- propose a **wait-free** algorithm for any ($n \geqslant 3$)-nodes cycle C_n
- using a **6-color** palette
- running in $O(\log^* n)$ (asynchronous) rounds

Unifies

- synchronous **graph k-coloring**
  - **LOCAL model**
  - $\rightsquigarrow$ $\Omega(\log^* n)$ temporal lower bound
- asynchronous **k-renaming**
  - **immediate snapshot** shared-memory model
  - $\rightsquigarrow$ coloring C_3 **requires ($k \geqslant 5$) colors**

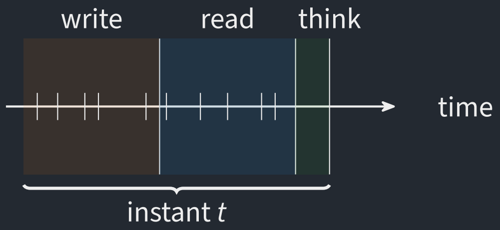
Model

async-LOCAL

- n asynchronous **processes** $p_1, \ldots, p_n$
- connected **graph** $G = (V := [n], E)$
- **schedule** $\sigma = \sigma(1), \sigma(2), \ldots \in \Sigma \subseteq 2^V$
- $i \in \sigma(t) \iff p_i$ **activated** at t:
  1. **writes** a value
  2. **reads** values of $p_j, j \sim_G i$
  3. privately **computes** a next state

Within one step t

1. **first** activated process **all write**
2. **then** activated process **all read**
   - if $i \sim j$ and $i, j \in \sigma(t)$, then i reads j's **step t value**



no schedule constraints (σ arbitrary)

Problem

async k-coloring

For a graph $G = (V, E)$ and scheduler $\Sigma \subseteq V^{\mathbb{N}}$.

- **uniform termination** $\exists B$: $|\sigma|_i \geqslant B \implies p_i$ outputs $c_i \neq \bot$
- **validity** if $p_i \sim p_j$ both output, then $c_i \neq c_j$
- **k-palette** $c_i = \bot \lor c_i \in \{1, \ldots, k\}$

Assuming initial unique identifiers $(X_u)_{u \in V}$.

Definition (wait-free)

An algorithm solves async-k-coloring **wait-free** over the graph G if it solves it for the complete scheduler $\Sigma = 2^{V(G) \times \mathbb{N}}$.

…also for a graph *class* $\mathcal{G}$:

- e.g., **cycles** $\mathcal{C} = \{C_n : n \geqslant 3\}$
- e.g., **cliques** $\mathcal{K} = \{K_n : n \geqslant 2\}$

$\rightsquigarrow$ **round complexity** (# of activations before a process returns)

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Related works

The LOCAL model

LOCAL

- n **nodes** $p_1, \ldots, p_n$ with **unique identifiers** $X_1, \ldots, X_n$
- connected graph $G = (V = [n], E)$
- in each round $t \geqslant 1$, every node p_i:
  1. **sends** a message to its neighbors
  2. **receives** each neighbor's **round t** message
  3. privately **computes** a next state
- all nodes run for T rounds, then output

- What can be computed **locally** ($\equiv T = o(n)$)

The k-coloring problem

graph k-coloring

For a graph $G = (V, E)$

- **termination** every node p_i outputs some color c_i
- **validity** if $i \sim j$, then $c_i \neq c_j$
- ***k*-palette** $c_i \in \{1, \ldots, k\}$

typically, $k = \Delta + 1$, $\Delta := \deg(G)$

Fundamental results on $\mathcal{C}$

- 2-coloring **non-local**
- 3-coloring C_n **requires** $1/2 \log^* n + O(1)$ rounds (Linial 92)
- 3-coloring C_n **can be solved** in $1/2 \log^* n + O(1)$ rounds (Cole+ 86)

Cole and Vishkin's algorithm

Algorithm 1: 3-coloring, code for p_i

Input : $X_i \in \mathrm{Poly}(n)$, unique identifier
for $T = \Theta(\log^* n)$ *rounds* **do**
  write(X_i) and **read** (X_{i-1}, X_{i+1})
  $X_i \leftarrow f(X_i, X_{i+1})$
▷ Here $X_i \leqslant 5$
for $k \in (5, 4, 3)$ **do**
  write(X_i) and **read** (X_{i-1}, X_{i+1})
  if $X_i = k$ **then**
   $X_i \leftarrow \min \mathbb{N} \setminus \{X_{i-1}, X_{i+1}\}$
return(X_i)

$$f(x,y) = 2\,\ell + x_\ell, \ \ell := \min\{i : x_i \neq y_i\}$$
$$x = \sum_{i \geqslant 0} 2^i x_i, \quad y = \sum_{i \geqslant 0} 2^i y_i$$

- each application of f logarithmically reduces $\max|X_i|$…
- …as long as $X_i \geqslant 6$
- final phase in $\mathrm{O}(1)$

The IS (*immediate snapshot*) model

immediate snapshot

- n asynchronous **processes** $p_1, \ldots, p_n$
- shared-memory **array** $\boldsymbol{M}$
- **schedule** $\sigma(1), \sigma(2), \ldots \subseteq [n]$
- $i \in \sigma(t) \iff p_i$ **activated** at t:
  1. **writes** in $\boldsymbol{M}[i]$
  2. **reads** entire array $\boldsymbol{M}$
  3. if $\boldsymbol{M} \models \mathcal{P}$ **terminates** with output $f(\boldsymbol{M})$
  4. else privately **computes** a next state

Within one step t

1. **first** activated process **all write**
2. **then** activated process **all read**
   - if $i, j \in \sigma(t)$, then i reads $\boldsymbol{M}[j](t)$

- **what can be computed** given (n, Σ)?
  e.g., no wait-free consensus

The k-renaming problem

async k-coloring

For initial unique names $X_1, \ldots, X_n < M$ and scheduler $\Sigma \subseteq V^{\mathbb{N}}$.

- **uniform termination** $\exists B :$ $|\sigma|_i \geqslant B \implies p_i$ outputs $c_i \neq \perp$
- **unicity** if $p_i \sim p_j$ both output, then $c_i \neq c_j$
- **k-palette** $c_i = \perp \vee c_i \in \{1, \ldots, k\}$

wait-free when $\Sigma = 2^{V \times \mathbf{N}}$

Fundamental results

- k-renaming **is impossible** when $k < 2n - 1$ and $n = p^m$, p prime (Herlihy+ 99, Castañeda+ 10)
- $(2n-1)$-renaming **can be solved** for all $n \geqslant 2$ (Attiya+ 90, Attiya+ 04)

Attiya and Welch's[†] algorithm

Algorithm 2: $2n - 1$ renaming, code for p_i

Input: $X_i \in \{0, 1, \ldots, M-1\}$
Initially: $c_i \leftarrow 0$
Forever:
  write(X_i, c_i)
  read $((X_1, c_1), \ldots, (X_n, c_n))$
  if $c_i \notin \{c_j : j \neq i\}$ **then** `return`(c_i)
  else
    $r_i \leftarrow \left|\{j : X_j < X_i\}\right|$
    $c_i \leftarrow r_i$-th min of $\mathbb{N} \setminus \{c_1, \ldots, c_n\}$

- $c_i \leqslant 2n - 1$
- active process p_i with smallest X_i cannot work forever
- all processes eventually terminate

A tale of two models

both problems inform our study:

LOCAL model

- coincides with our model when $\Sigma = (V, V, \ldots)$
- any async-k-coloring algorithm is a k-coloring algorithm
- $\Omega(\log^* n)$ **rounds** necessary to color the cycle C_n

IS model

- coincides with our model when $G = K_n$
- any async-k-coloring algorithm is a k-renaming algorithm for $G = C_3 = K_3$
- **5-color palette** necessary to color the cycle C_3

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Algorithmic contributions

async-6-coloring ("slow" worst-case)

Algorithm 3: async 6-coloring, code for p_i

Input: $X_i \in \mathbb{N}$ ▹ `proper coloring`
Initially: $c_i = (a_i, b_i) \leftarrow (0, 0)$
Forever:
- **write**(X_i, c_i) ▹ `immediate snapshot`
- **read**$((X, c), (X', c'))$
- **if** $c_i \notin \{c, c'\}$ **then** `return`(c_i)
- **else**
  - $a_i \leftarrow \min \mathbb{N} \setminus \{a_j : (X_j > X_i)\}$
  - $b_i \leftarrow \min \mathbb{N} \setminus \{b_j : (X_j < X_i)\}$

- $a_i + b_i \leqslant 2 \implies$ 6-colors palette
- local maxima/minima stubbornly keep $a_i = 0/b_i = 0$
- local extrema terminate in $\mathrm{O}(1)$
- process terminate in $\mathrm{O}(\ell)$, ℓ distance to a local extremum

Next?

- 5-coloration
- general graphs
- other problems

async-6-coloring ("fast" worst-case)

Algorithm 4: async 6-coloring, code for p_i

Input: $X_i \in \mathbb{N}$ ▹ `proper coloring`

Initially: $c_i = (a_i, b_i) \leftarrow (0,0), r_i \leftarrow 0$

Forever:

- **write**(X_i, c_i, r_i) ▹ `immediate snapshot`
- **read**$((X, c, r), (X', c', r'))$
- ▹ `update` c_i `as before`
- **if** $(r_i < \infty) \wedge (r_i \leqslant \min\{r, r'\})$ **then**
  - **if** $\min\{X, X'\} < X_p < \max\{X, X'\}$ **then**
    - $r_i \leftarrow r_i + 1$
    - $Y \leftarrow f(X_i, \min\{X, X'\})$ ▹ $f(x,y) = 2\,\ell + x_\ell, \quad \ell := \min\{|x|, |y|\} \cup \{i : x_i \neq y_i\}$
    - **if** $Y < \min\{X_q, X_{q'}\}$ **then** $X_p \leftarrow Y$
  - **else**
    - $r_i \leftarrow \infty$
    - **if** $X_i < \min\{X, X'\}$ **then**
      - $X_i \leftarrow \min\{X_i, \min(\mathbb{N} \setminus \{f(X, X_i), f(X', X_i)\})\}$