Synchronous t-Resilient Consensus in Arbitrary Graphs

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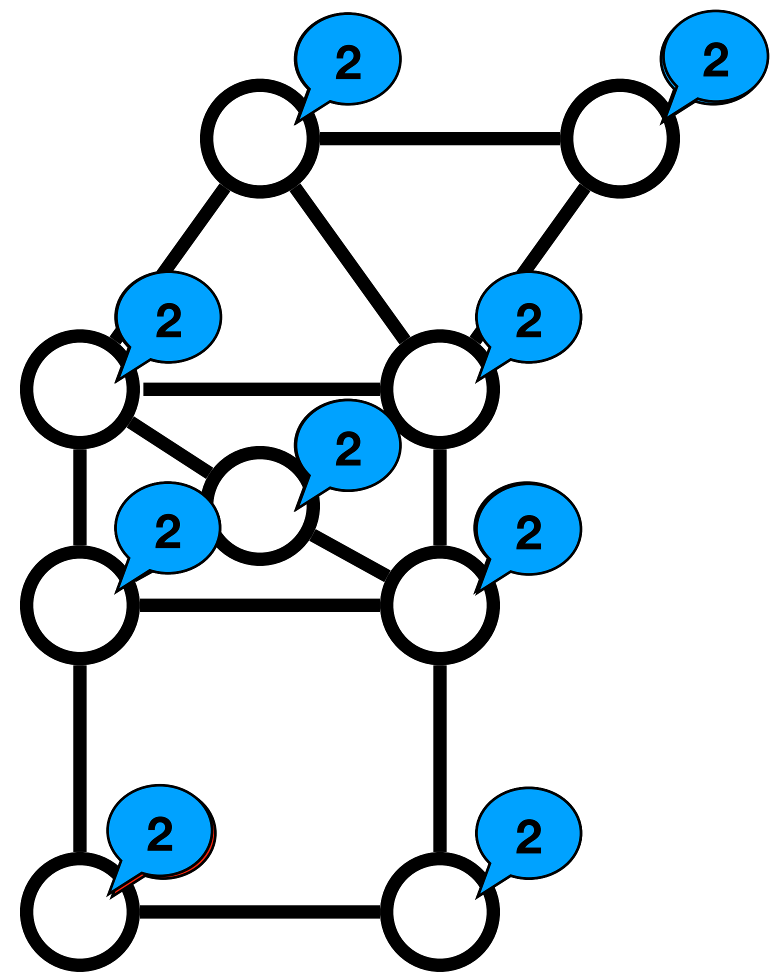
Consensus

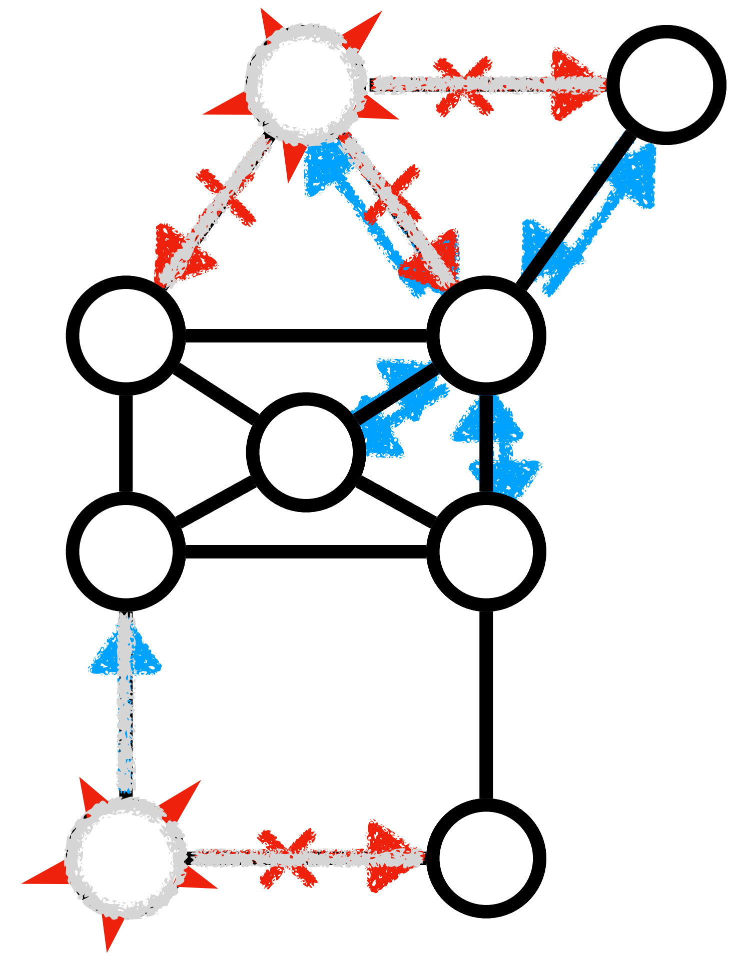
- **Agreement**:
  Decide the same value
- **Validity**:
  Decided values are input values
- **Termination**:
  Non-faulty processes decide



How fast consensus can be reached in arbitrary failure-prone networks?

Synchronous Failure-prone Networks

Synchronous rounds:
each node sends to/receive from neighbors

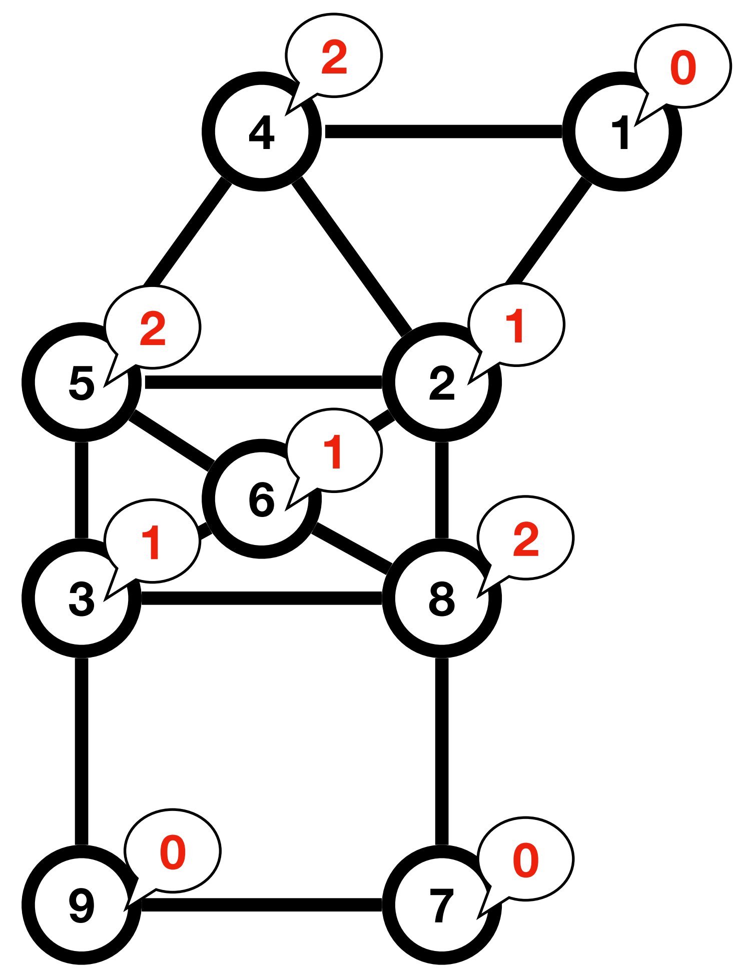


At most *t* nodes may crash

***clean*: no message sent**

***dirty*: messages sent to some neighbors**

Know-All model



- Each node has a unique id
- Graph G and ids assignment are known
- Only node i knows its input v_i
- At most t nodes fail

Given G and id assignment, design a consensus algorithm $\mathscr{A}_{G,id,t}$

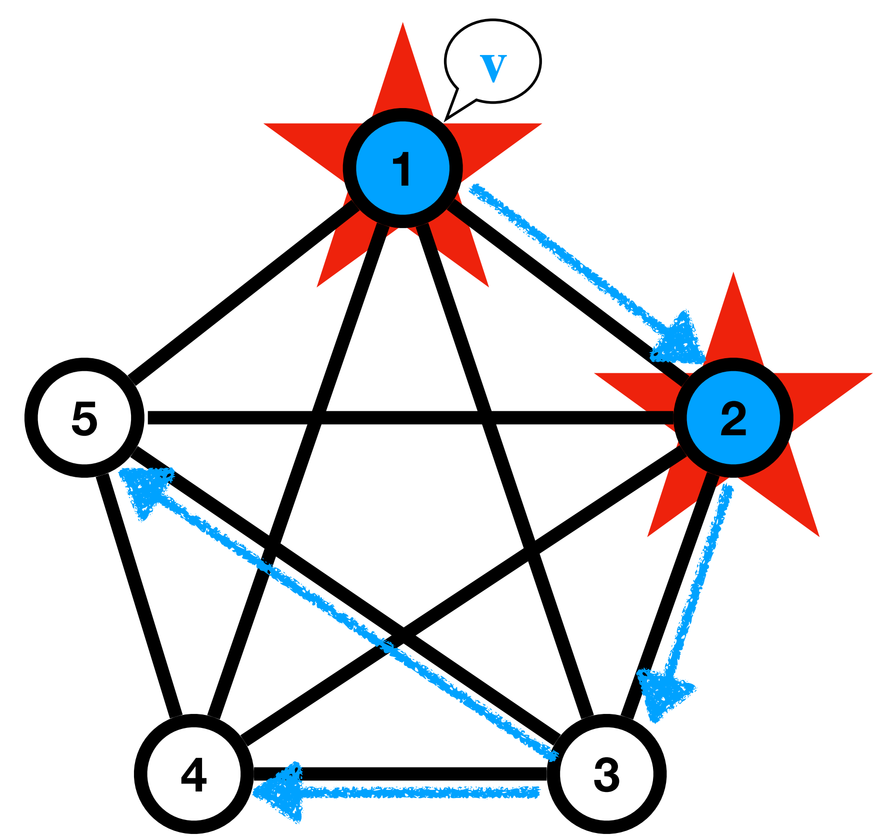
How many rounds are necessary to solve t resilient consensus ?

Synchronous Consensus in Complete Graphs

Theorem

t-resilient consensus in the clique:
$(t+1)$ rounds necessary and sufficient

Distributed Computing 101
[Lamport Fischer 82]
[Aguilera Toueg 99]
[Charron-Bost Schiper 00]
[Lamport 00]
[Moses Rajsbaum 02]
[Keidar Rajsbaum 03]
[Wang Teo Cao 05]
...
[Castaneda Gonczarowski Moses 14]



$(t+1)$ rounds for v to *flood* G
in the worst case

Synchronous Consensus in Arbitrary Graphs

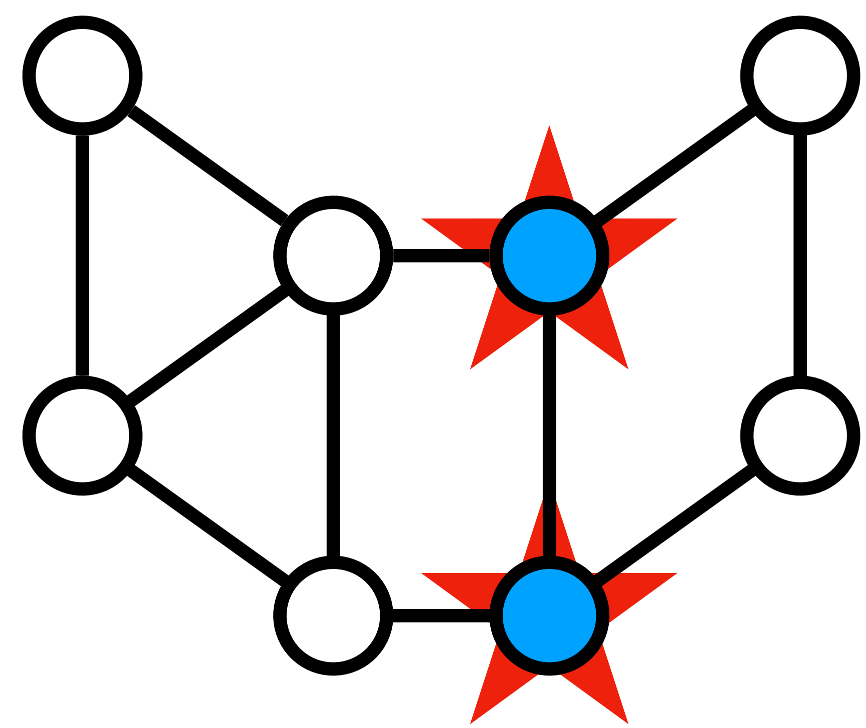
Solvability

t-resilient consensus solvable
iff
G is $(t+1)$-vertex connected

[Folklore]

Round complexity

??



Our Results

Definition

Dynamic notion of radius $\mathsf{Radius}(G, t)$ **taking into account failures**

Upper bound

Consensus is solvable in $\mathsf{Radius}(G, t)$ **rounds**

Lower bound

For symmetric graph, consensus cannot be solved in $\mathsf{Radius}(G, t) - 1$ **rounds**

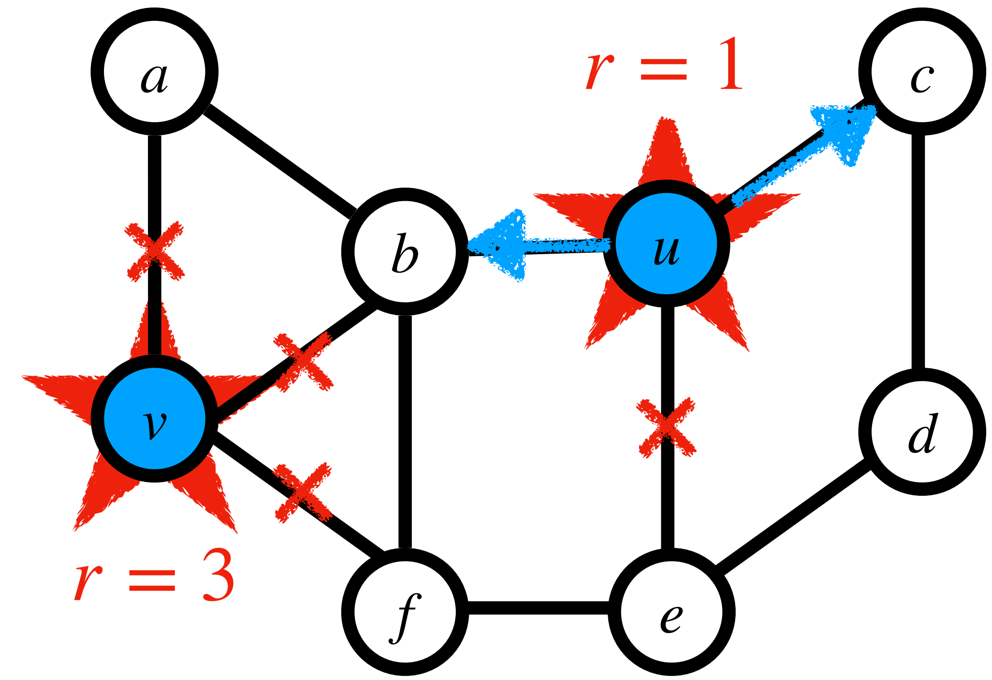
Roadmap

1. Failure-sensitive eccentricity and radius
2. A naive algorithm
3. An adaptive algorithm
4. Optimality for symmetric graphs

Failure Pattern

Failure pattern φ

- **Which node fails, and when?**
- **Which neighbors received messages in the failing round**



$$\varphi = \{(u,1,\{b,c\}),(v,3,\varnothing)\}$$

Faulty node

round of the failure

receiving neighbors

Failure Sensitive Eccentricity

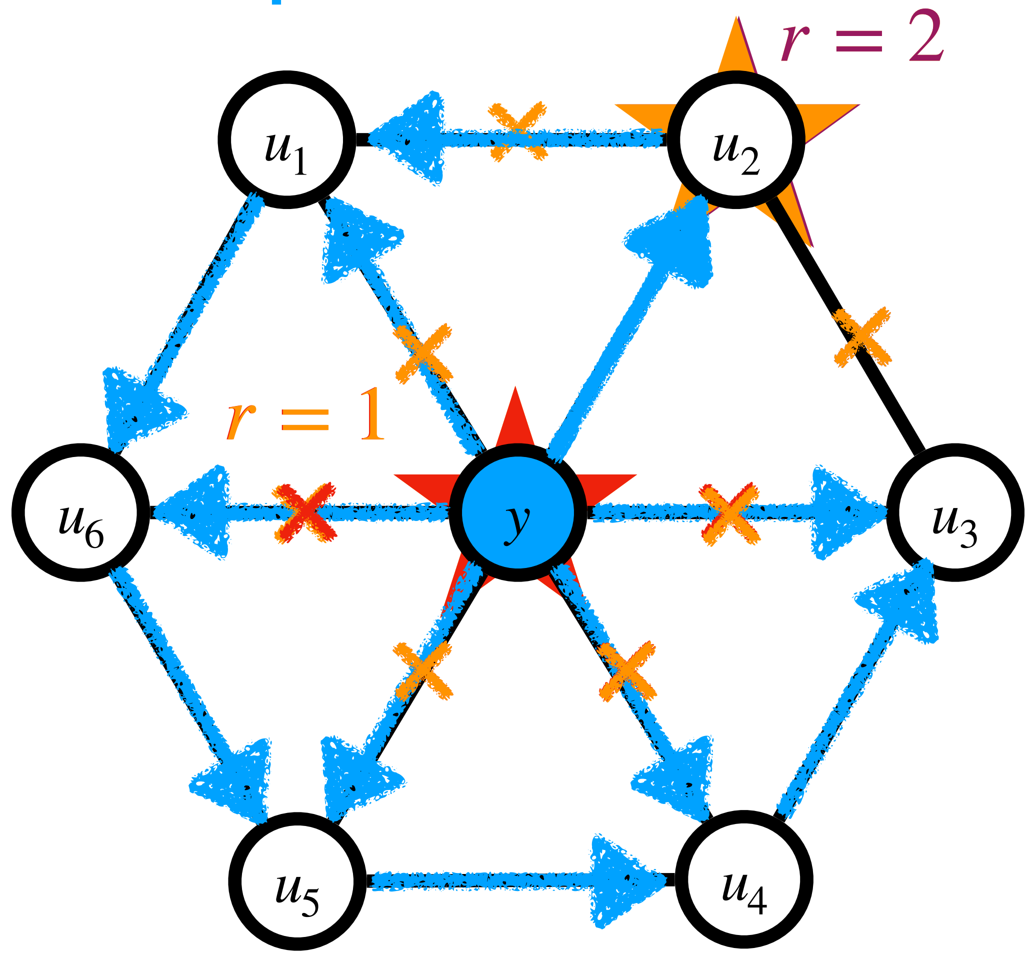
$\text{ecc}_G(v, \varphi) =$ **#round for v to flood G**

#rounds for every correct to receive input of v

$\text{ecc}(y, \varphi_\varnothing) = 1$

$\text{ecc}(y, \varphi_1) = +\infty$

$\text{ecc}(y, \varphi_2) = 6$



Radius

$$\text{Radius}(G, \Phi) = \min_{v \in V} \text{ecc}_G(v, \Phi)$$

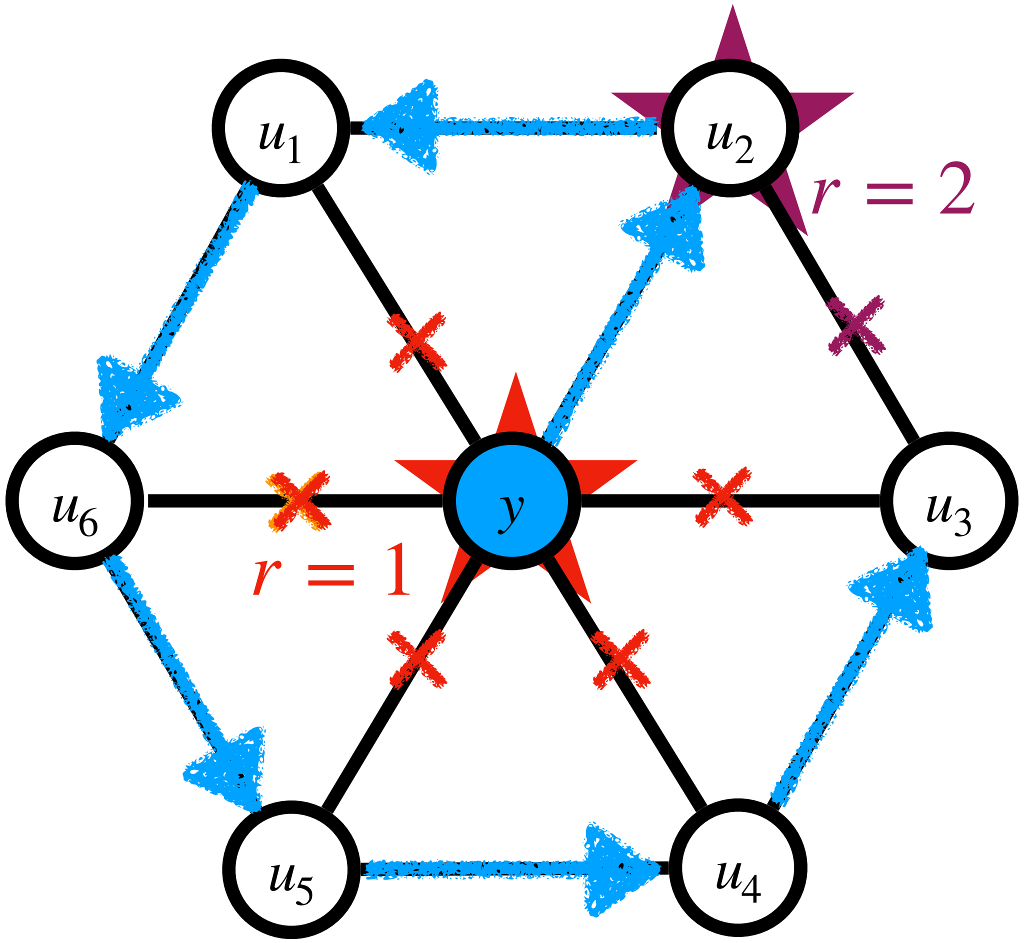
set of failure patterns

$$\max_{\varphi \in \Phi} \{\text{ecc}_G(v, \varphi) : \text{ecc}_G(v, \varphi) \textbf{ is finite}\}$$

$\text{ecc}(y, \Phi^2_{all}) = 6$

$\text{ecc}(u_i, \Phi^2_{all}) = 6$

$\text{Radius}(G, \Phi^2_{all}) = 6$



A Naive Algorithm

$\max_{\varphi \in \Phi} \{ \mathrm{ecc}_G(v_2, \varphi) : \mathrm{ecc}_G(v_2, \varphi) \text{ is finite} \}$

1. **Order node according to their eccentricity**

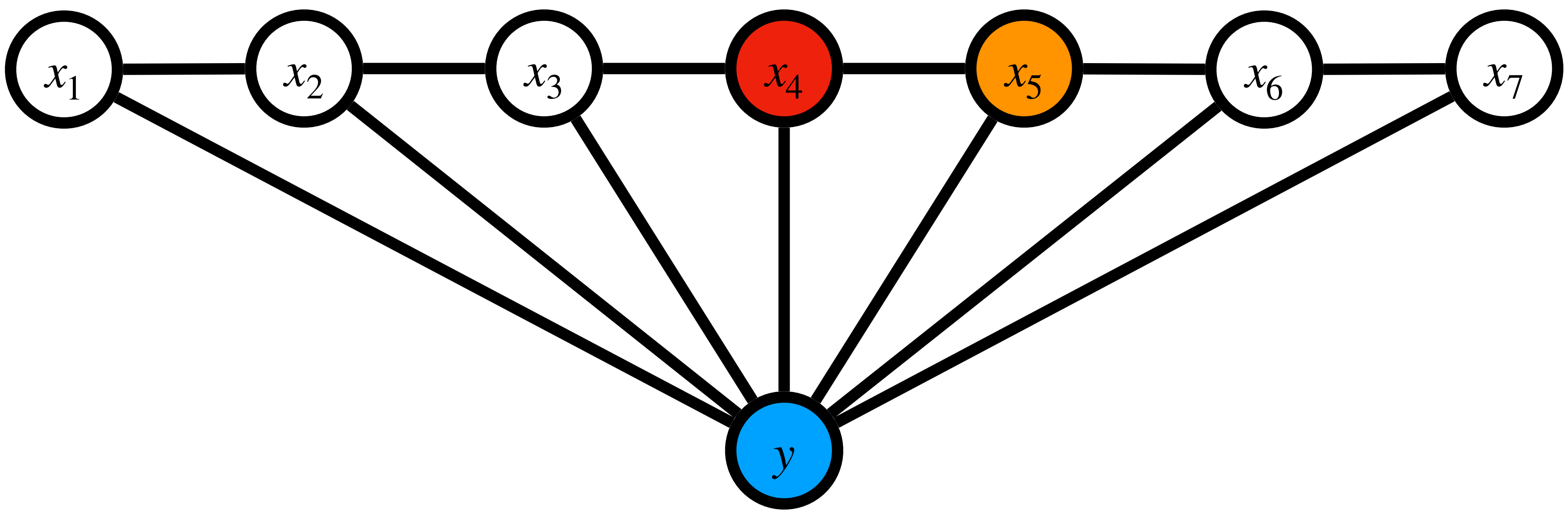
$$\mathrm{ecc}_G(v_1, \Phi^t_{all}) \leq \mathrm{ecc}_G(v_2, \Phi^t_{all}) \leq \cdots \leq \mathrm{ecc}_G(v_{t+1}, \Phi^t_{all})$$

2. **Perform flooding for $\mathrm{ecc}_G(v_{t+1}, \Phi^t_{all})$ rounds**

3. **Decide input of node with smallest index in $v_1, \ldots, v_{t+1}$**

Example

$t = 1$



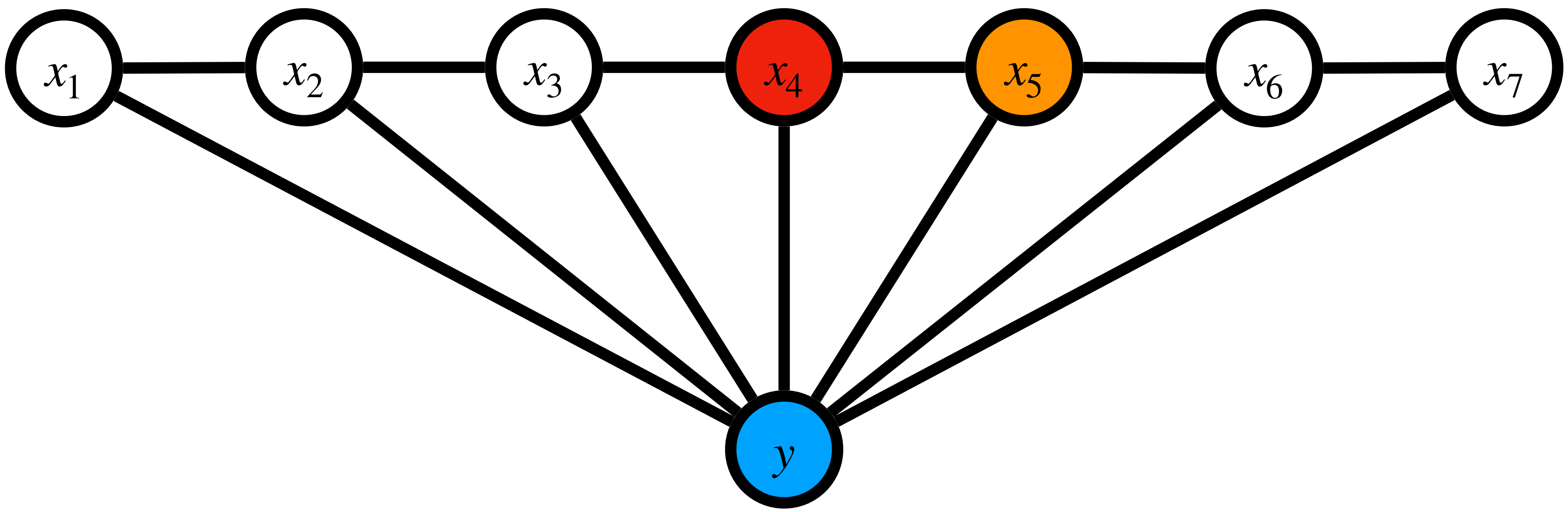
$\text{ecc}(x_4, \Phi^1_{all}) = 3 < \text{ecc}(x_5, \Phi^1_{all}) = 4$

Given $\varphi \in \Phi^1_{all}$, after 4 rounds:

- x_4 **input received by every correct, *or by none***
- x_5 **input received by every correct *or by none***
- **Every correct has received the input of x_4 or x_5, or both**

Non-optimality

$t = 1$



$\text{ecc}(x_4, \Phi^1_{all}) = 3 < \text{ecc}(x_5, \Phi^1_{all}) = 4 < \text{ecc}(y, \Phi^1_{all}) = 7$

let $\Phi_{x_4} = \{\varphi : x_4$ **fails** $\}$

$\text{ecc}(y, \Phi_{x_4}) = 1$

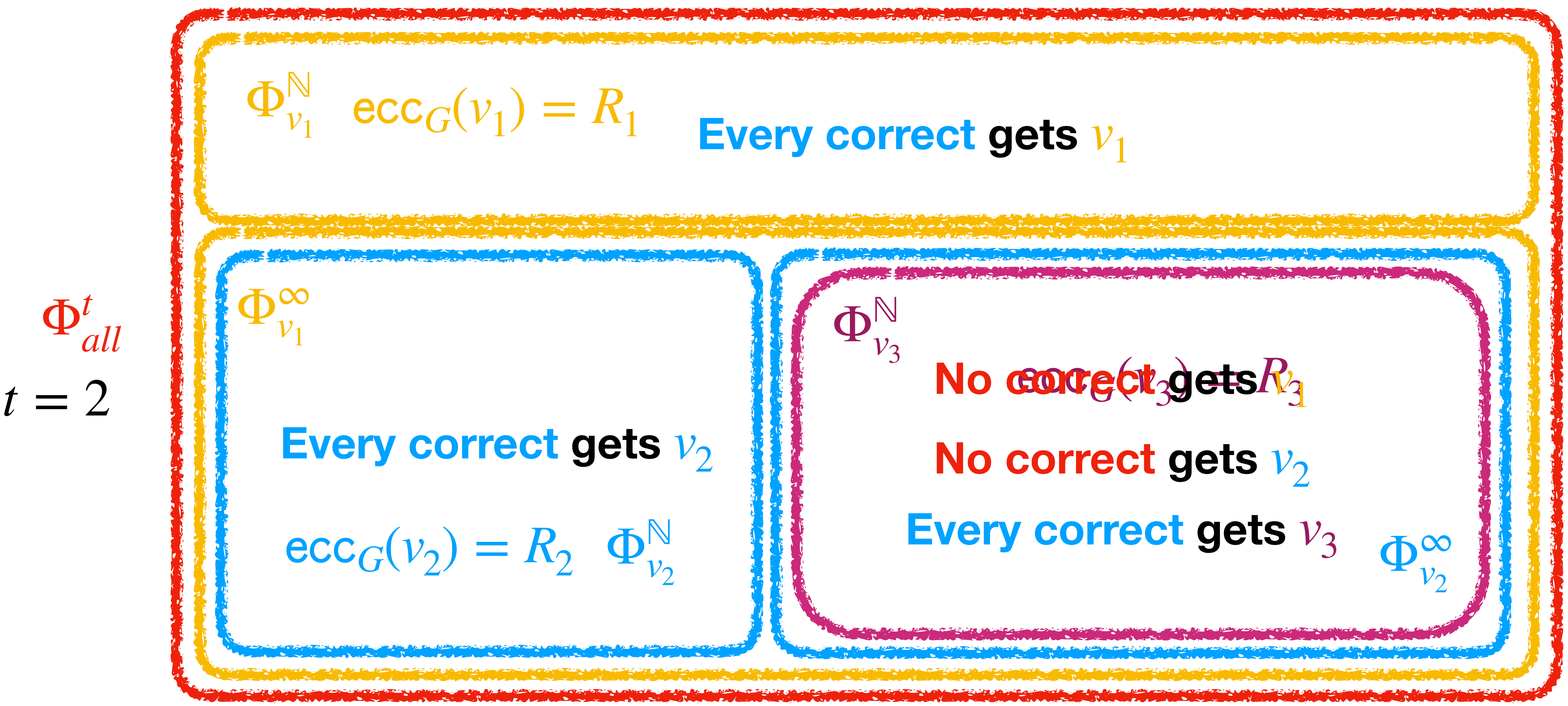
Given $\varphi \in \Phi^1_{all}$, **after 3 rounds**:

- x_4 **input received by every correct, *or by none***
- **if *no correct* has rcved** x_4 **input, every correct has received** y **input**

Consensus in Radius(G, Φ^t_{all}) Rounds

$\Phi^{\mathbb{N}}_v = \{\varphi \in \Phi^t_{all} : \text{ecc}_G(v, \varphi) < +\infty\}$ **Every correct gets v input**

$\Phi^{\infty}_v = \{\varphi \in \Phi^t_{all} : \text{ecc}_G(v, \varphi) = +\infty\}$ **No correct gets v input**



Consensus in $\max\{R_1, R_2, R_3\}$ rounds

Consensus in Radius(G, Φ^t_{all}) Rounds

Core sequence of $t+1$ nodes $v_1, v_2, \ldots, v_{t+1}$

v_1 : $\text{ecc}_G(v_1, \Phi^{\mathbb{N}}_{v_1}) = \text{Radius}(G, \Phi^t_{all})$

$\Phi_{i-1} = \Phi^{\infty}_{v_{i-1}} \cap \cdots \cap \Phi^{\infty}_{v_1}$

v_i : $\text{ecc}_G(v_i, \Phi^{\mathbb{N}}_{v_i} \cap \Phi_{i-1}) \leq \text{ecc}_G(v, \Phi^{\mathbb{N}}_{v} \cap \Phi_{i-1}) \forall v \neq v_1, \ldots, v_{i-1}$

Every correct gets v_i input

No correct gets $v_1, \ldots, v_{i-1}$ input

Key Lemma

$\text{ecc}_G(v_i, \Phi^{\mathbb{N}}_{v_i} \cap \Phi_{i-1}) > \text{ecc}_G(v_{i+1}, \Phi^{\mathbb{N}}_{v_{i+1}} \cap \Phi_i)$

Algorithm

Perform flooding for $\text{Radius}(G, \Phi^t_{all})$ **rounds**

Decide input of the core node with smallest index

Proof of Key Lemma

$v_1 : \mathrm{ecc}_G(v_1, \Phi^{\mathbb{N}}_{v_1}) = \mathrm{Radius}(G, \Phi^t_{all})$ $\qquad \Phi_1 = \Phi^{\infty}_{v_1}$

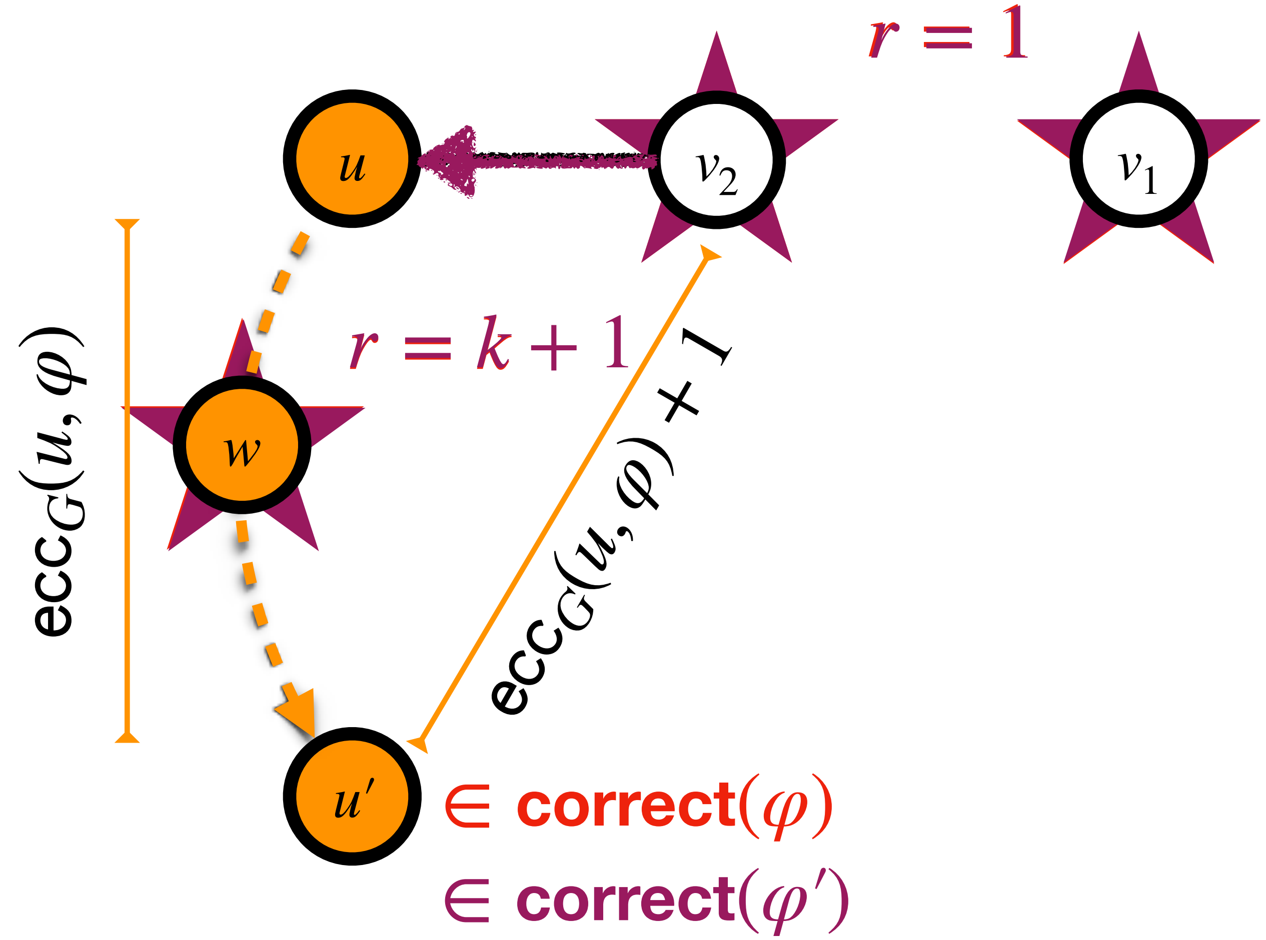
$v_2 : \mathrm{ecc}_G(v_2, \Phi^{\mathbb{N}}_{v_2} \cap \Phi_1) \leq \mathrm{ecc}_G(v, \Phi^{\mathbb{N}}_{v} \cap \Phi_1) \forall v \neq v_1$ $\qquad \Phi_2 = \Phi^{\infty}_{v_2} \cap \Phi^{\infty}_{v_1}$

$\exists u \neq v_1, v_2 : \mathrm{ecc}_G(u, \Phi^{\mathbb{N}}_{u} \cap \Phi_2) < \mathrm{ecc}_G(v_2, \Phi^{\mathbb{N}}_{v_2} \cap \Phi_1)$

$\varphi \in \Phi^{\mathbb{N}}_{u} \cap \Phi_2$

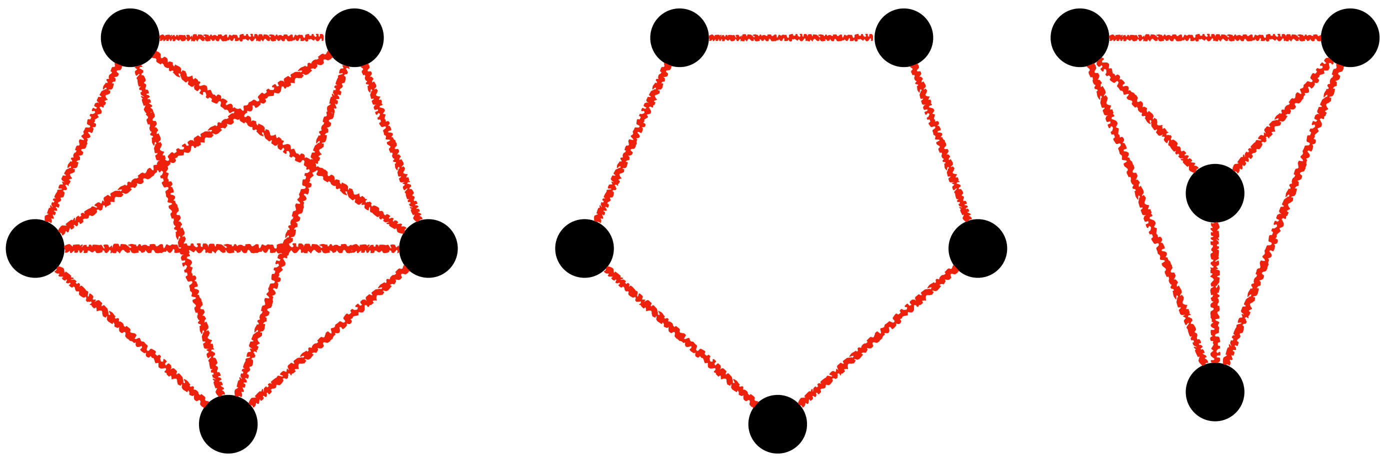
$\varphi' \in \Phi^{\mathbb{N}}_{v_2} \cap \Phi_1$

$\mathrm{ecc}_G(u, \varphi) + 1 \leq \mathrm{ecc}_G(v_2, \varphi')$



Lower Bound

- Symmetric graphs (vertex transitive)



- Oblivious algorithms

Perform R rounds of flooding

Decide: $\{(id_1, val_1), \ldots, (id_k, val_k)\} \rightarrow val$

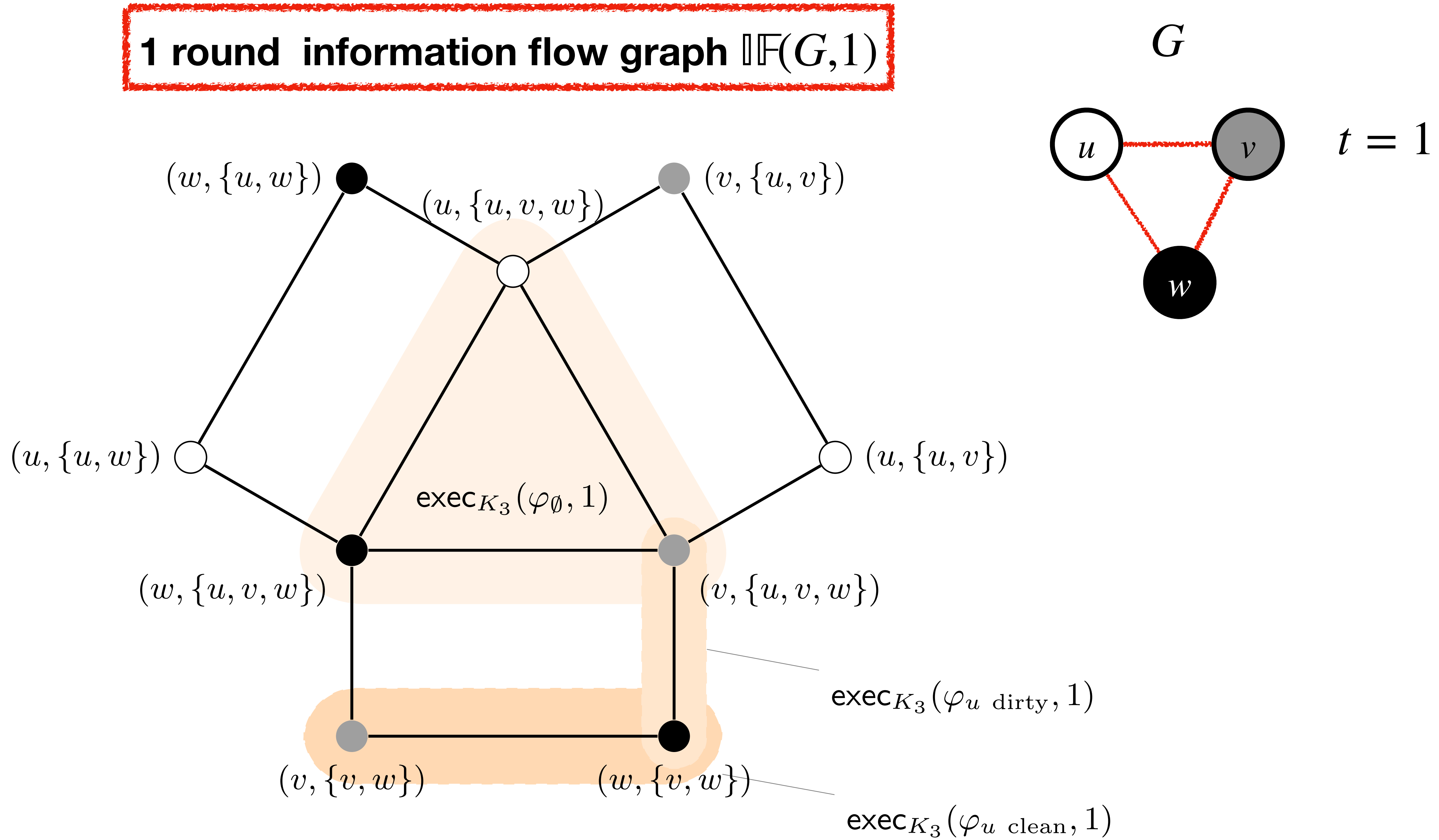
Lower Bound

Theorem

For any symmetric graph G, there is no oblivious algorithm that solves consensus in less than $\mathsf{Radius}(G, \Phi^t_{all})$ rounds

Information Flow Graph

1 round information flow graph $\mathbb{IF}(G,1)$



Consensus and Domination

Definition

Node $v \in V(G)$ *dominates* a connected component C of $\mathbb{IF}_G(\Phi, r)$

iff

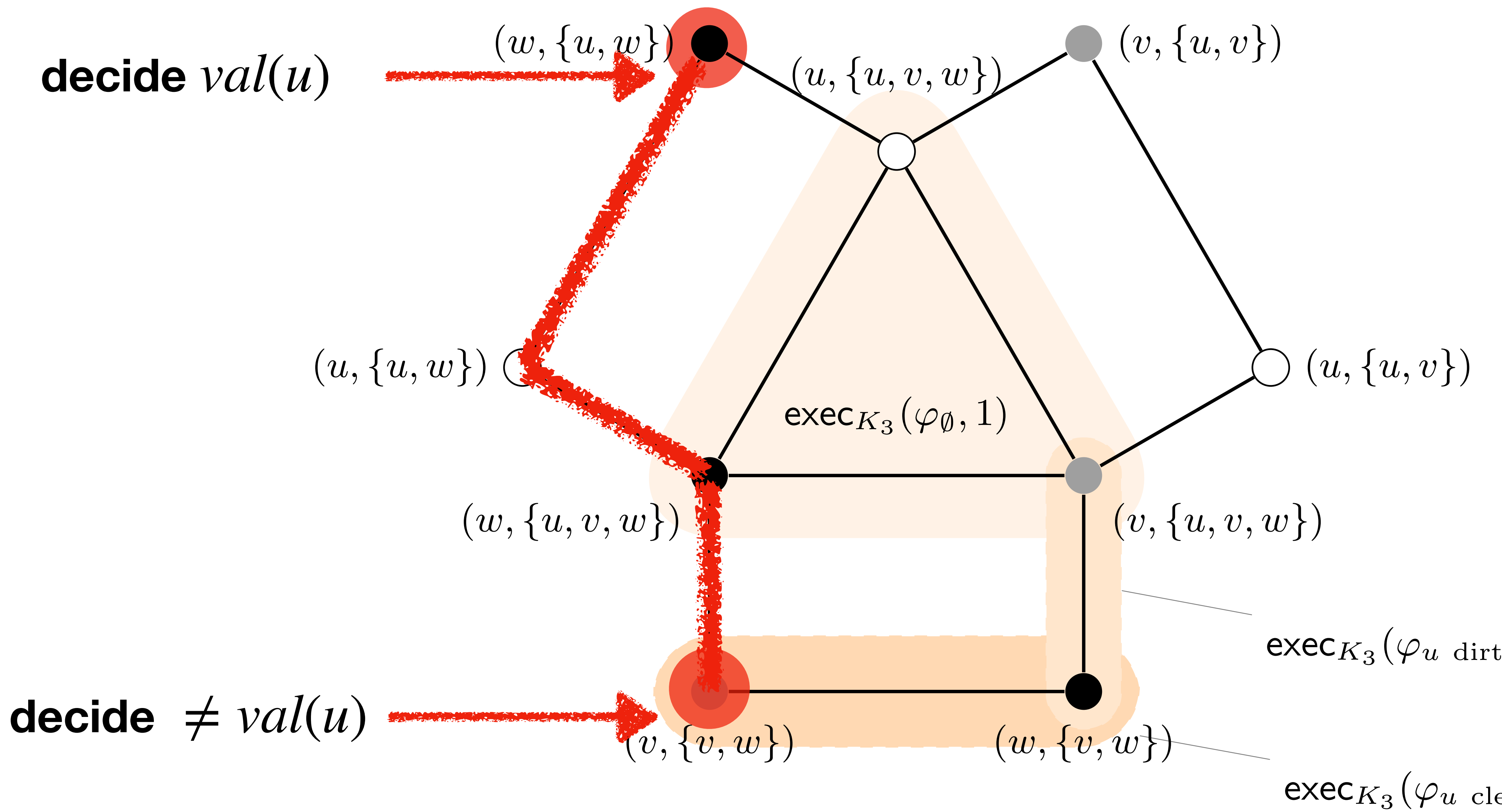
$\exists \varphi \in \Phi$ s.t. $(v, \text{view}_G(v, \varphi, r))$ dominates C

Theorem

There is an oblivious consensus algorithm in r rounds in G under failure patterns Φ iff each connected component of $\mathbb{IF}_G(\Phi, r)$ is dominated

Consensus and Domination

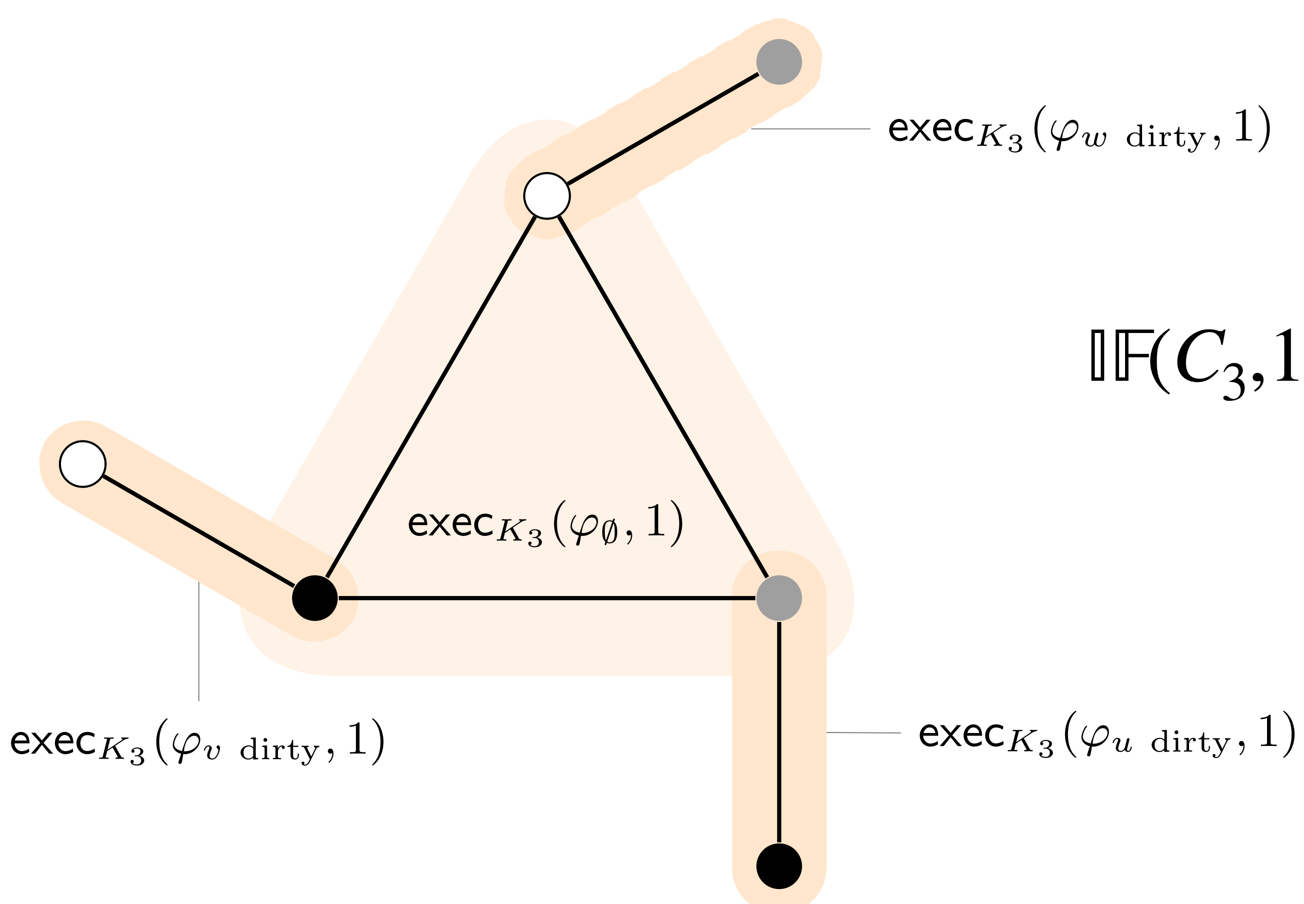
Suppose consensus solvable in r rounds
***and* there is a non-dominated CC in $\mathbb{IF}_G(\Phi, r)$**



Application: Symmetric Graphs

Theorem

For any symmetric graph G, there is no oblivious algorithm that solves consensus in less than $\text{Radius}(G, \Phi^t_{all})$ rounds



$\mathbb{IF}(C_3, 1, \Phi)$ **is not dominated**

Conclusion and Future Work

- **Tight complexity bound** for oblivious, crash-tolerant consensus in symmetric graph
- The information flow (a.k.a protocol complex) for study computability/complexity in **network**
- Are there faster **non-oblivious** algorithms ?
- What is the lower bound for **non-symmetric graphs** ?
- What are the round complexity of other **classical agreement tasks** in arbitrary graphs ?

Thanks!