Local certification of forbidden subgraphs

Nicolas Bousquet, Linda Cook, Laurent Feuilloley, Théo Pierron, Sébastien Zeitoun

February 26, 2025

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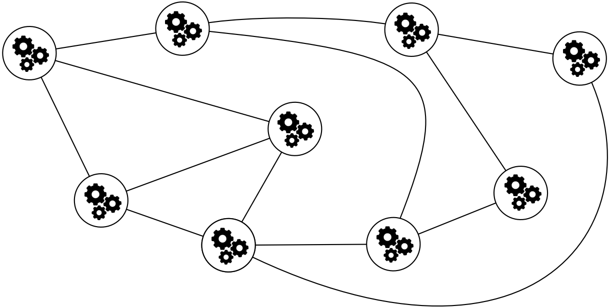
Context: distributed computing

Model: graph, $\begin{cases} \text{vertices} = \text{computation units} \\ \text{edges} = \text{communication channels} \end{cases}$



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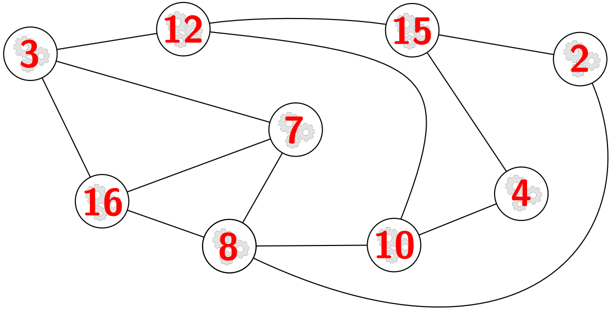

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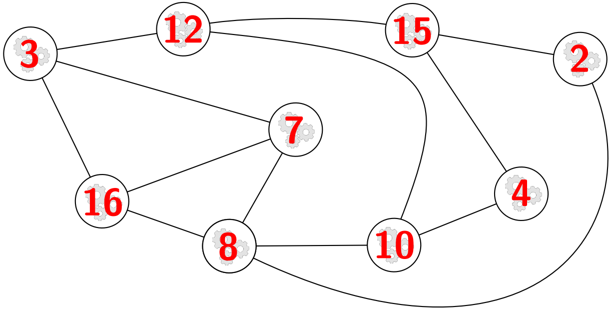

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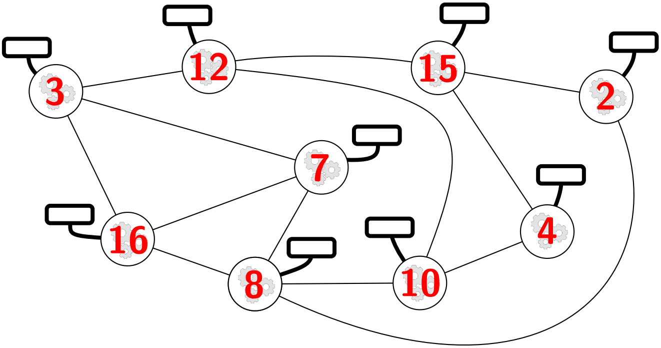

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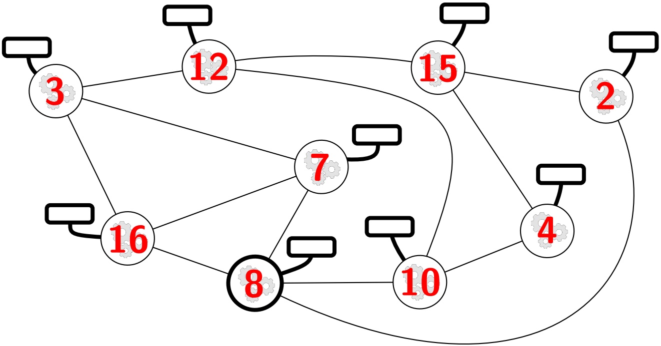

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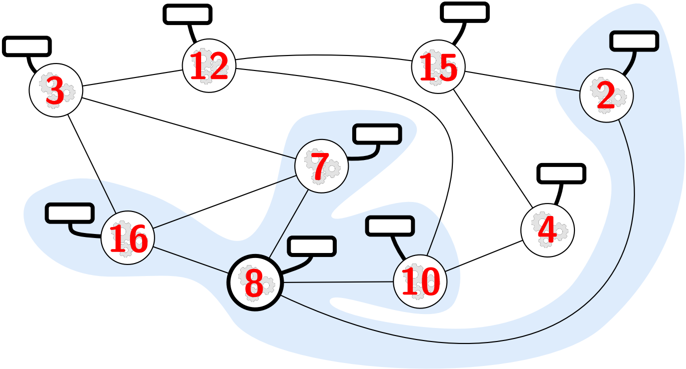

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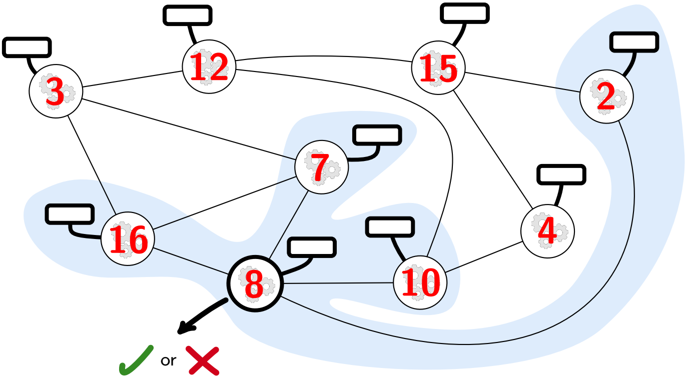

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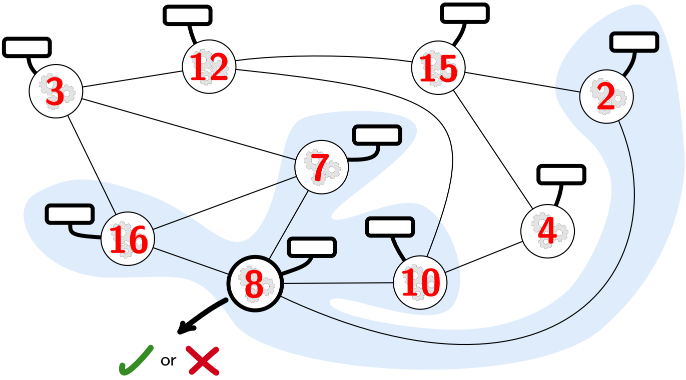

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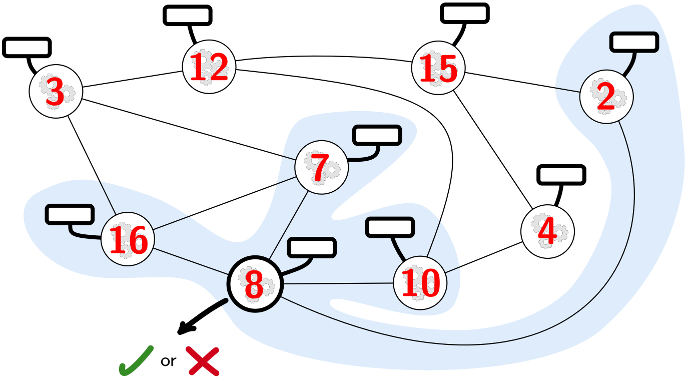

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G satisfies $\mathcal{P} \iff$ there exists an assignment of the certificates such that G is accepted

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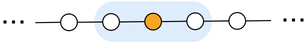

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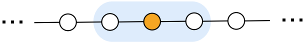

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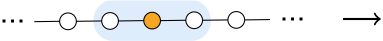

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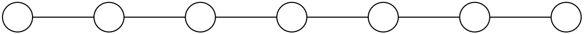

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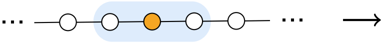

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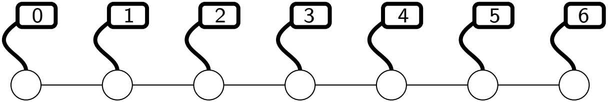

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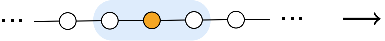

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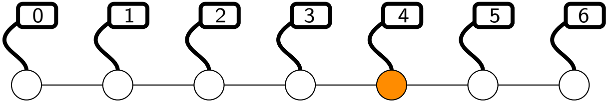

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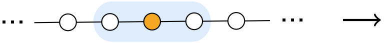

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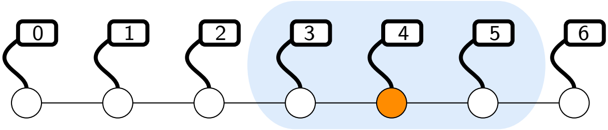

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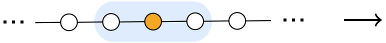

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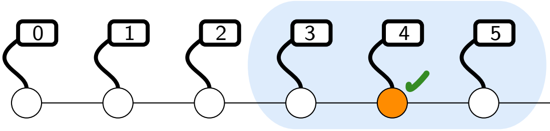

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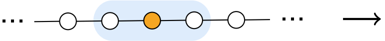

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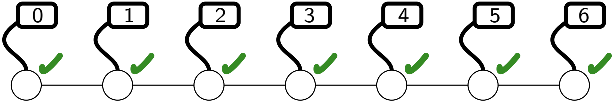

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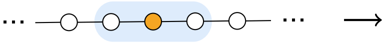

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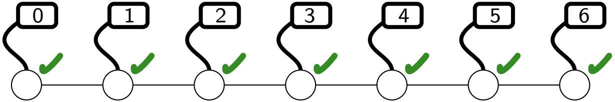

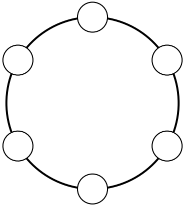

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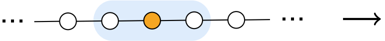

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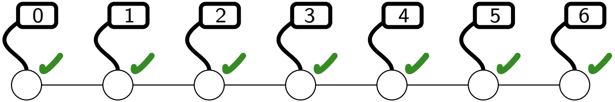

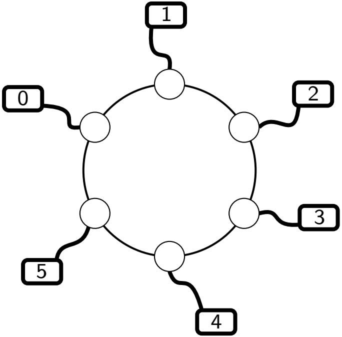

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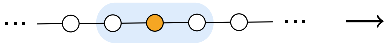

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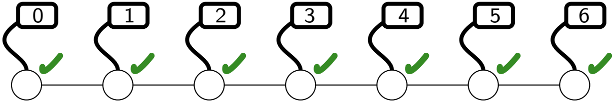

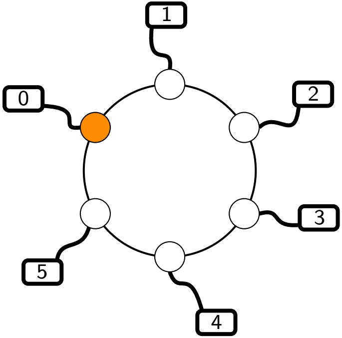

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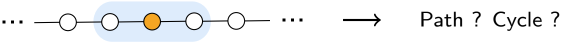

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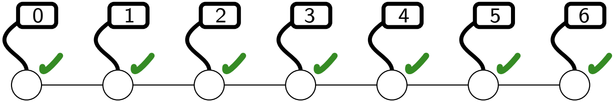

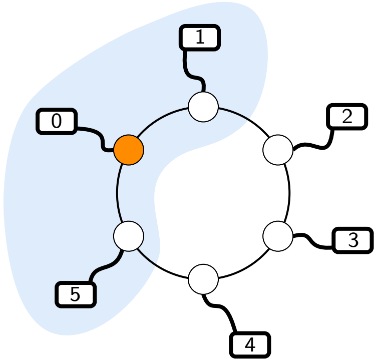

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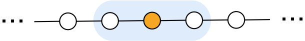

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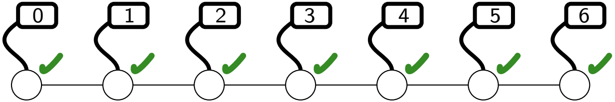

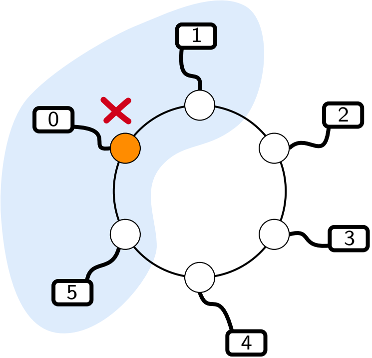

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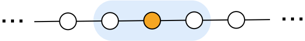

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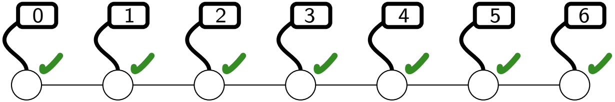

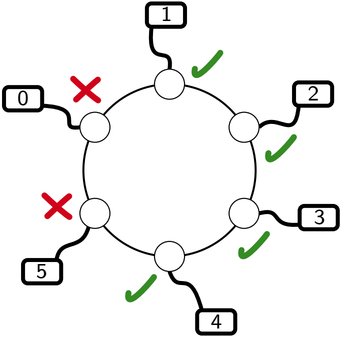

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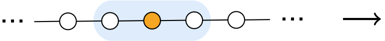

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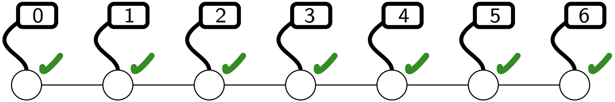

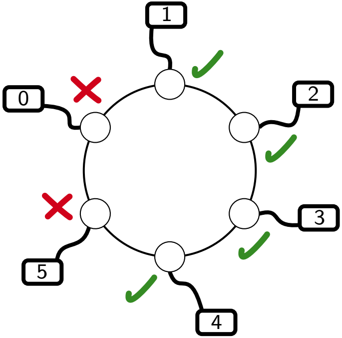

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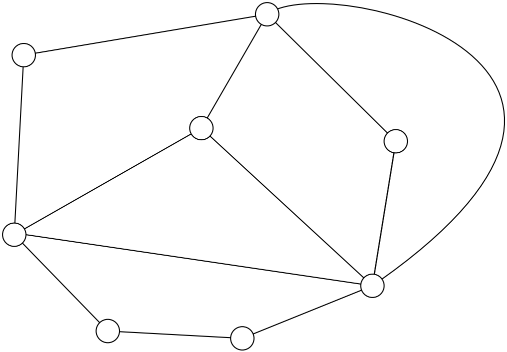

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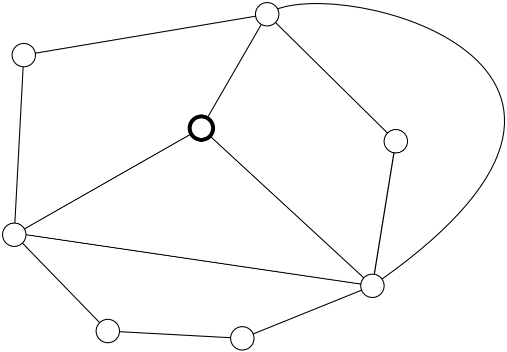

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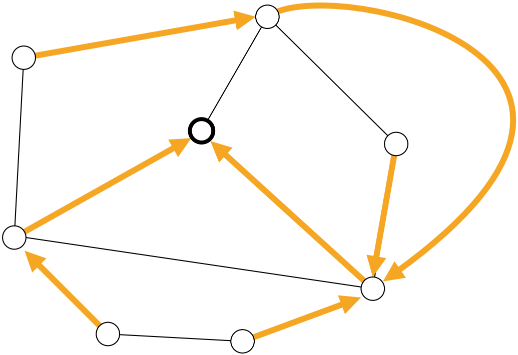

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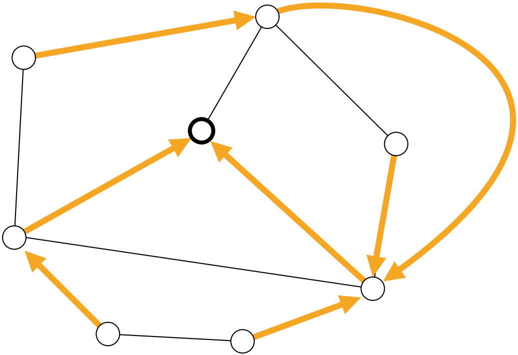

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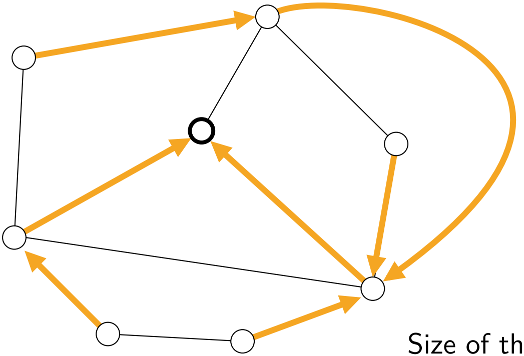

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Typical size of certificates :

$\tilde{\Theta}(n^2)$	$\Theta(\log n)$
▪ Non-3-colorability	▪ Paths
▪ Non-trivial automorphism	▪ Trees
	▪ Planar graphs

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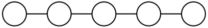

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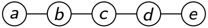

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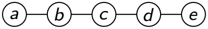

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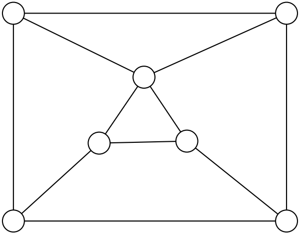

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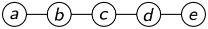

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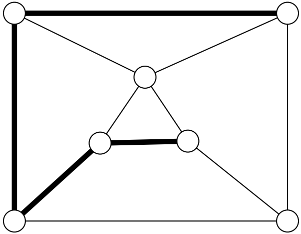

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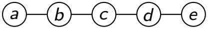

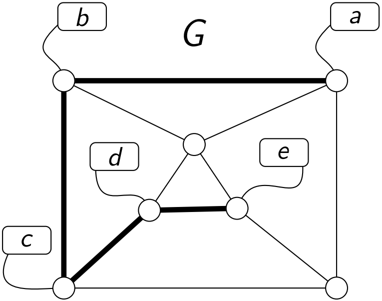

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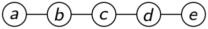

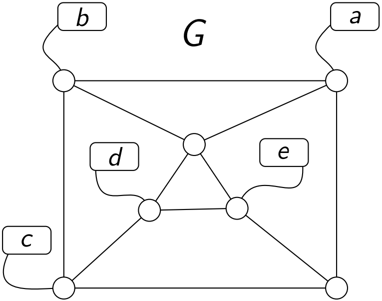

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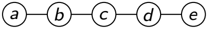

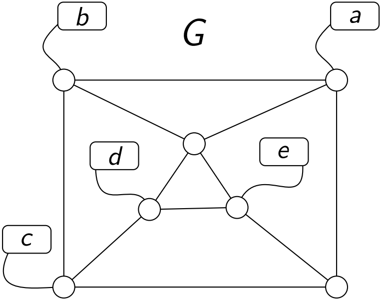

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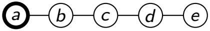

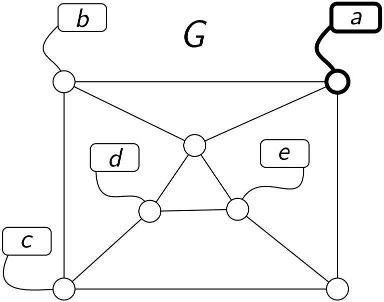

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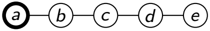

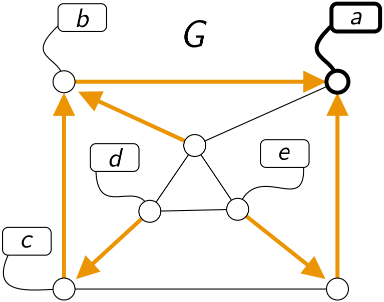

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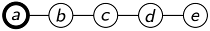

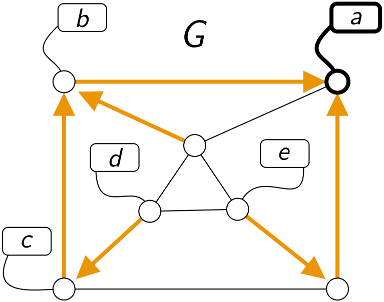

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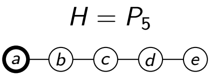

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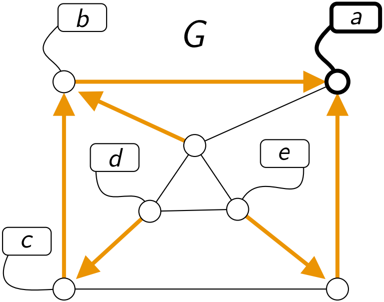

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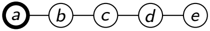

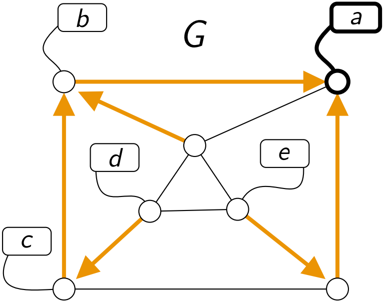

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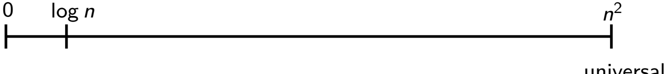

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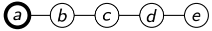

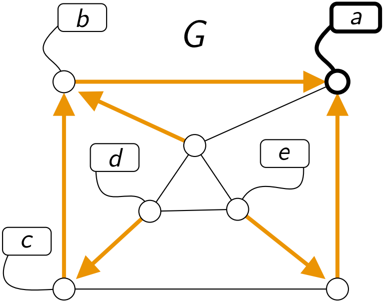

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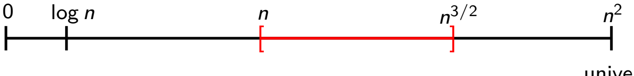

Lower bounds

Certification of P_k-freeness: lower bound

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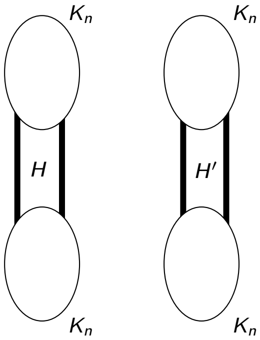

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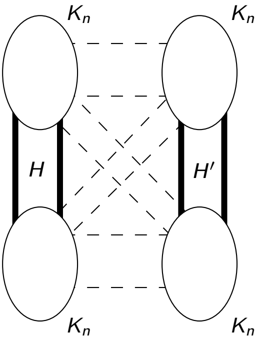

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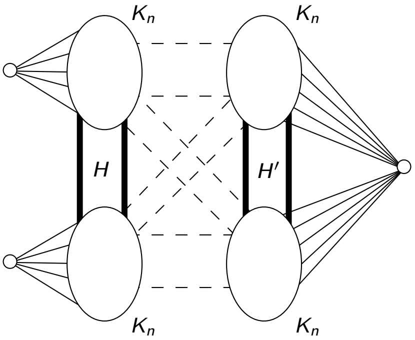

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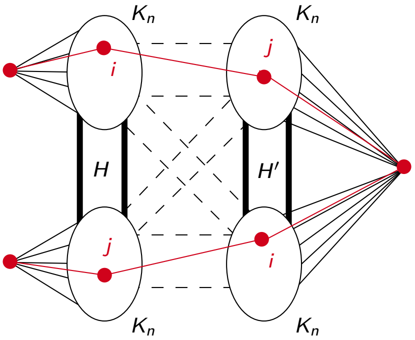

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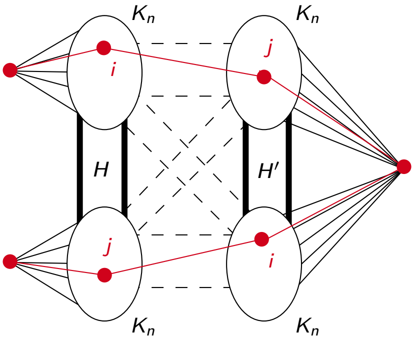

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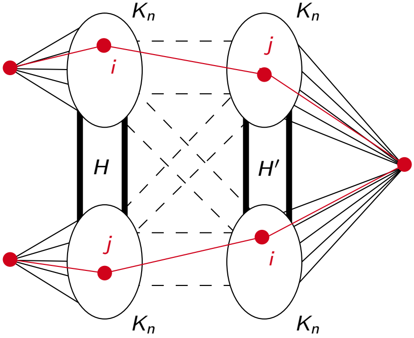

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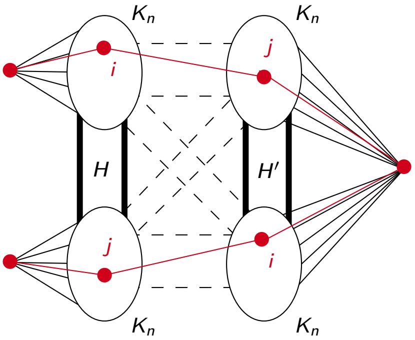

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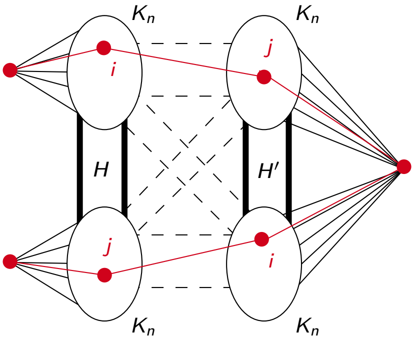

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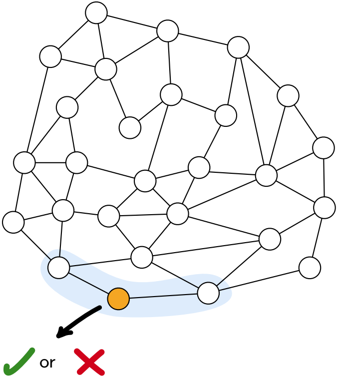

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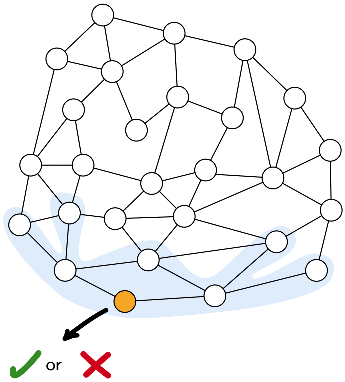

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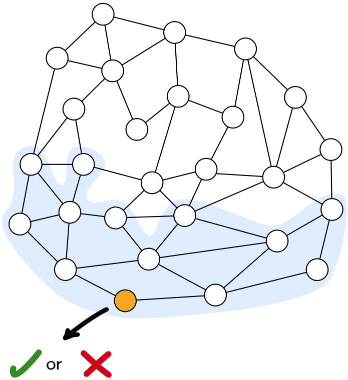

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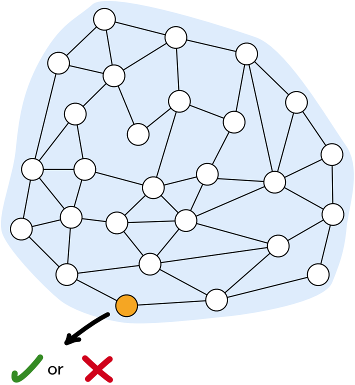

Upper bounds

Certification in graphs of minimum degree $O(n^{\delta})$

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Theorem (Bousquet, Cook, Feuilloley, Pierron, Z.)

Let $\delta < 1$. Any property can be certified with certificates of size $O(n^{2-\delta} \log n)$ in graphs of minimum degree n^δ, if vertices can see at distance 2.

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Idea of the proof:

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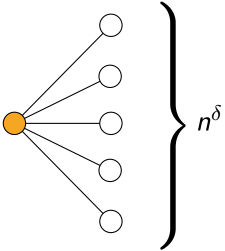

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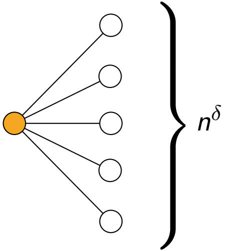

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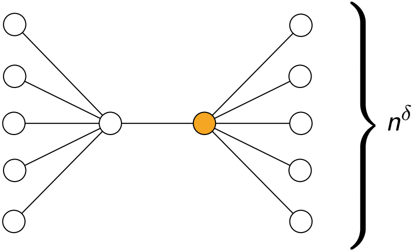

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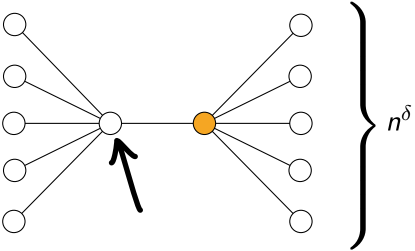

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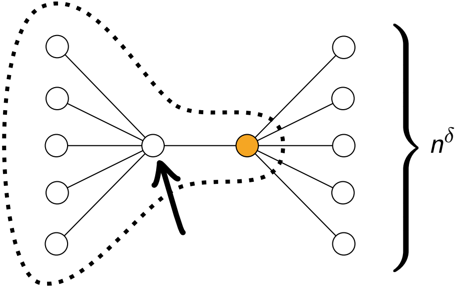

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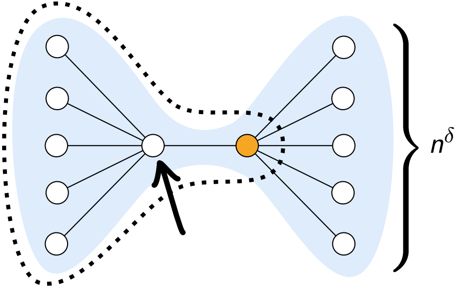

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Theorem (Bousquet, Cook, Feuilloley, Pierron, Z.)

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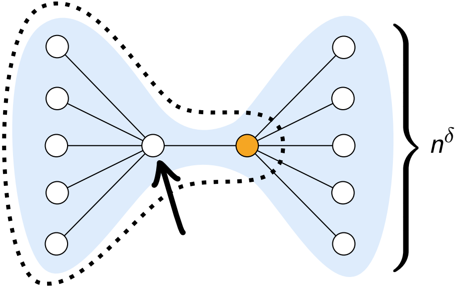

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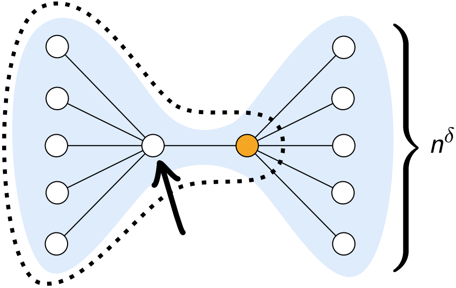

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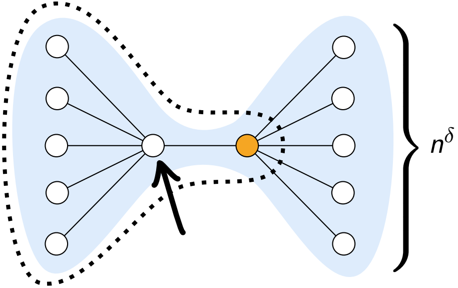
$\implies$ every vertex knows G



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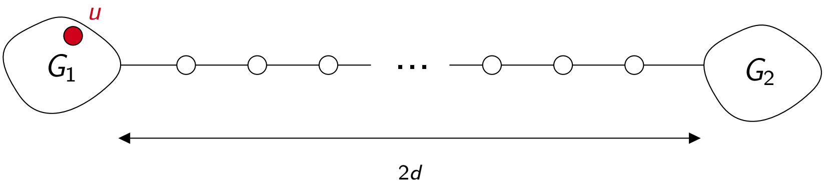

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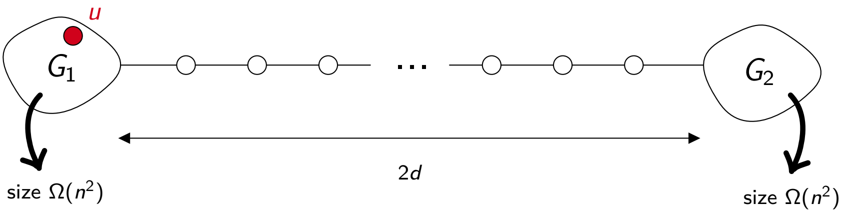

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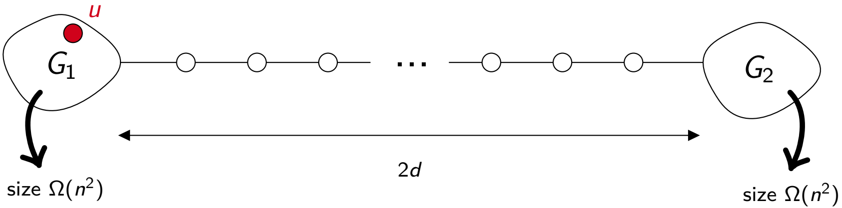

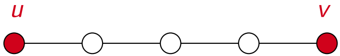

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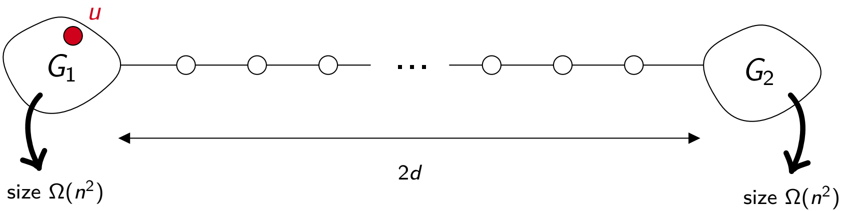

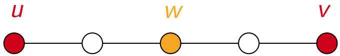

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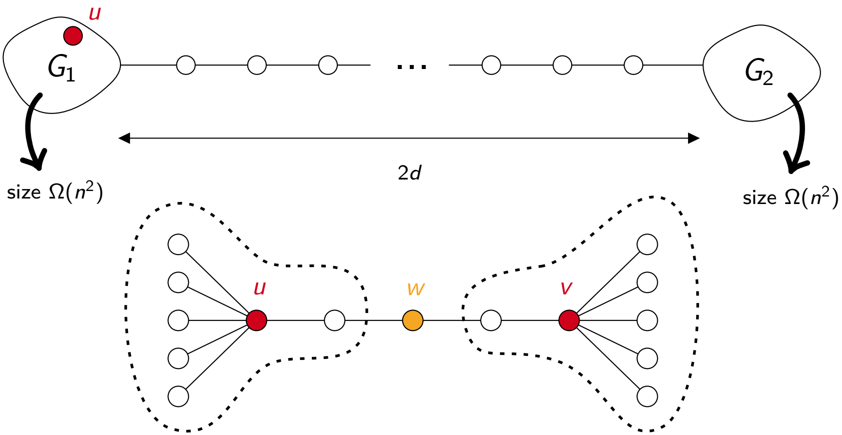

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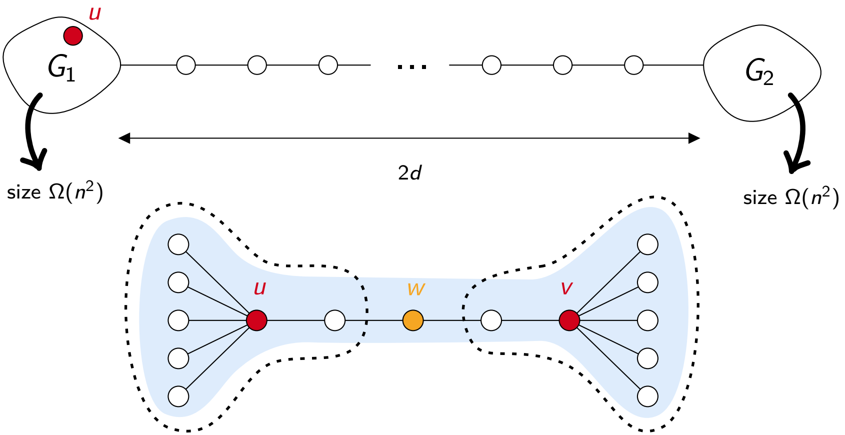

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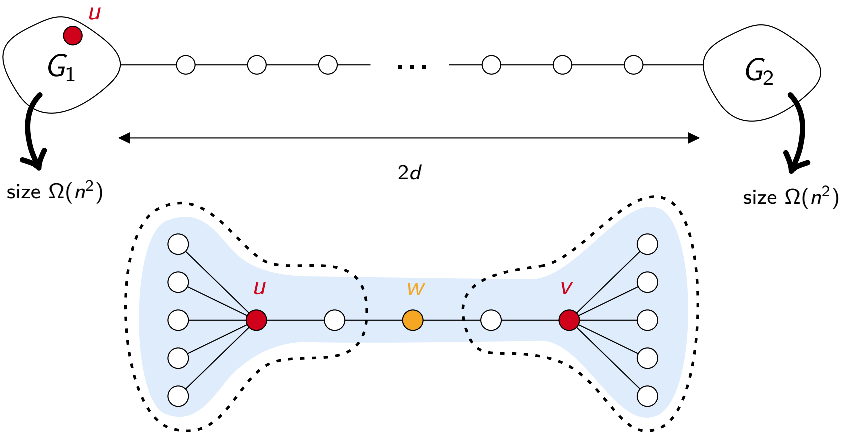

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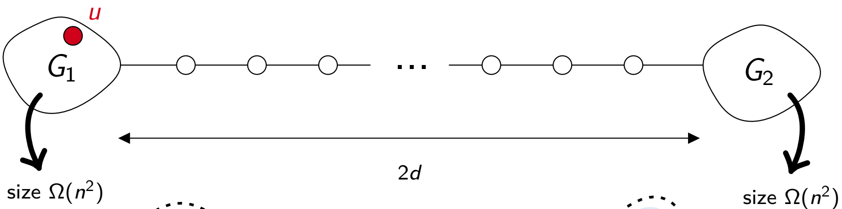

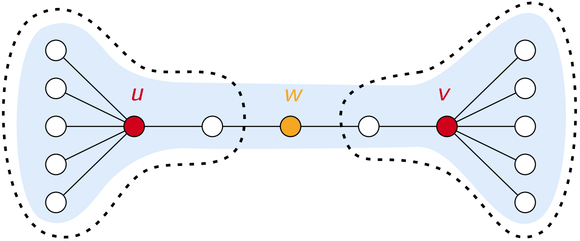

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Partition V^+ into components: set of vertices which reconstruct the same graph

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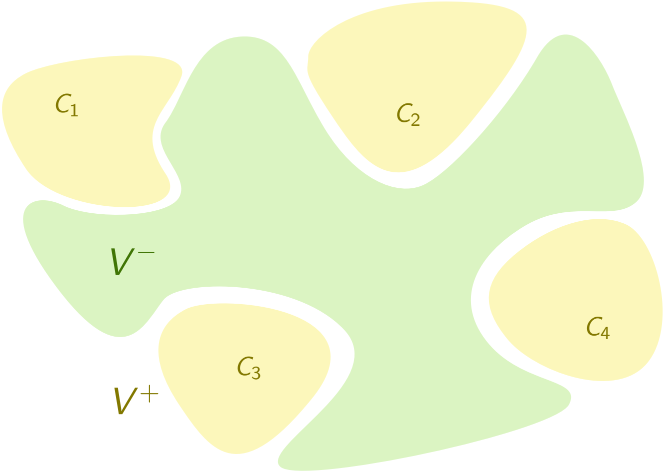

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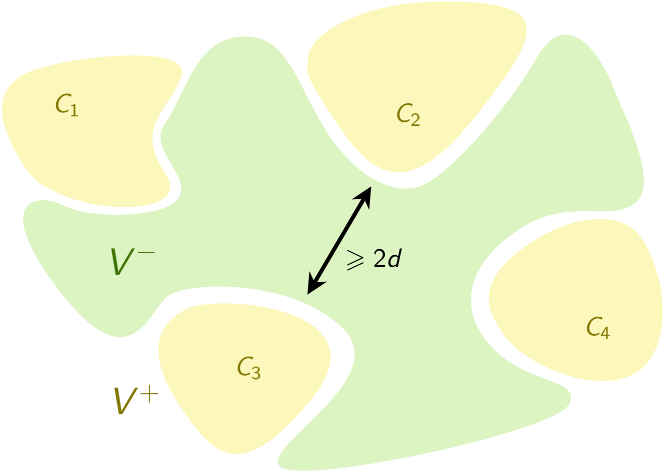

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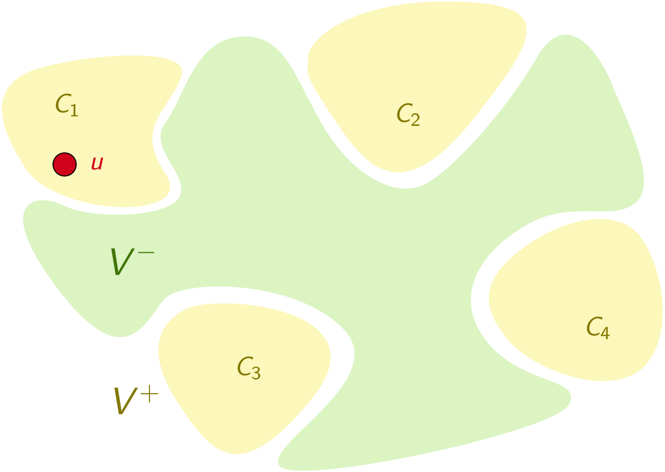

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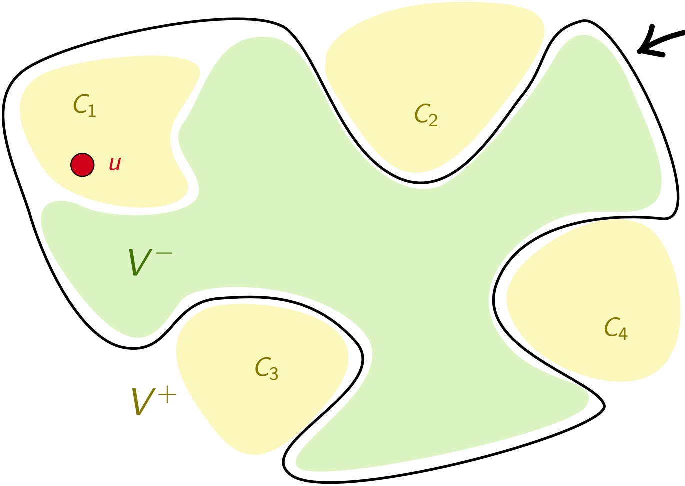

u knows all the edges and non-edges having an endpoint here

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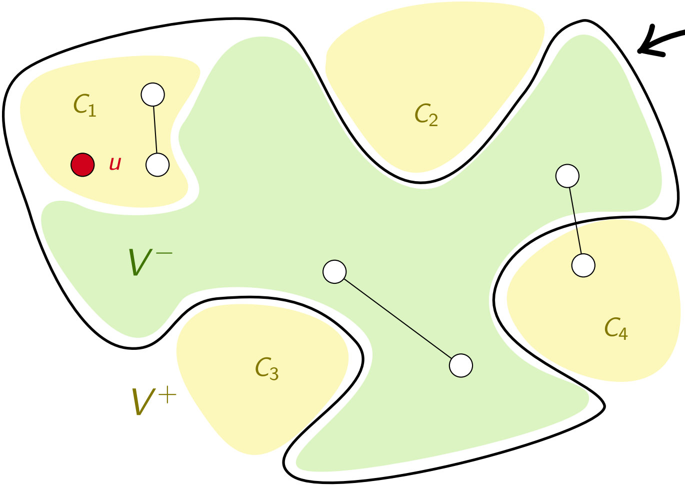

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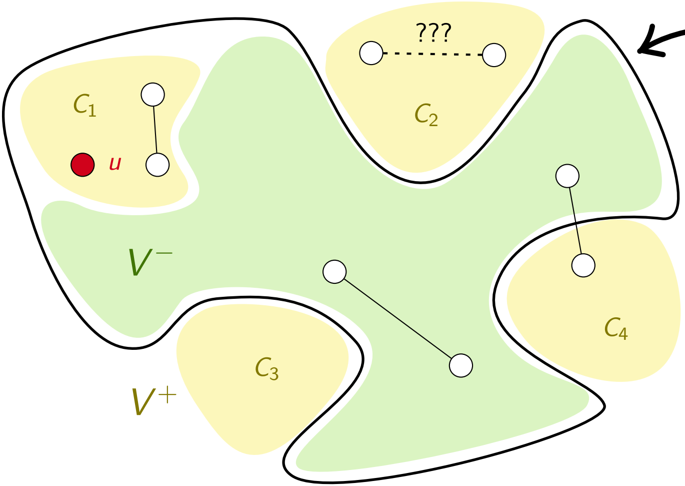

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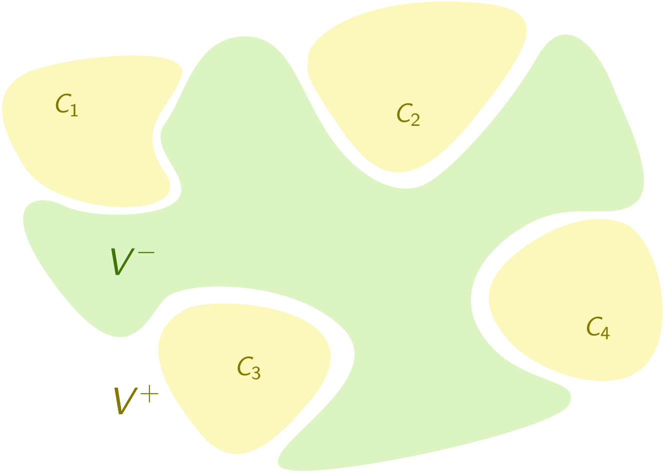
If there is a P_{4d-1}, which vertex detects it ?



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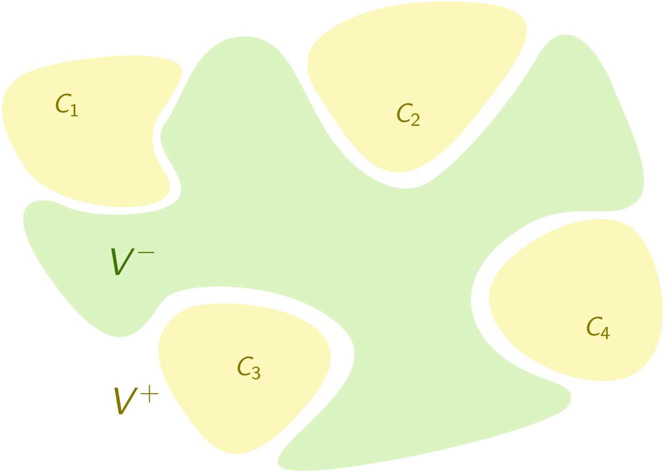

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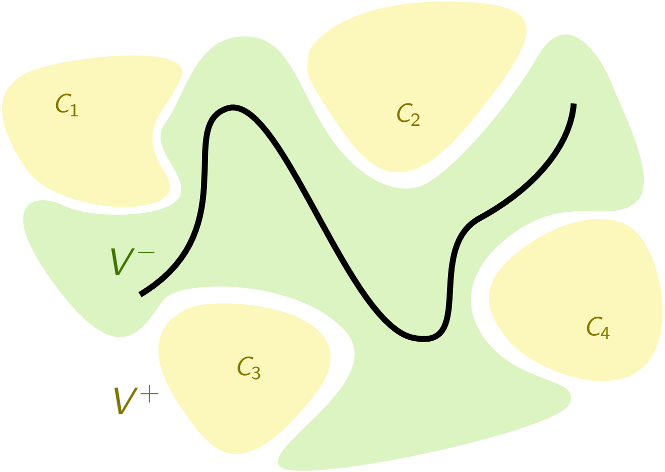

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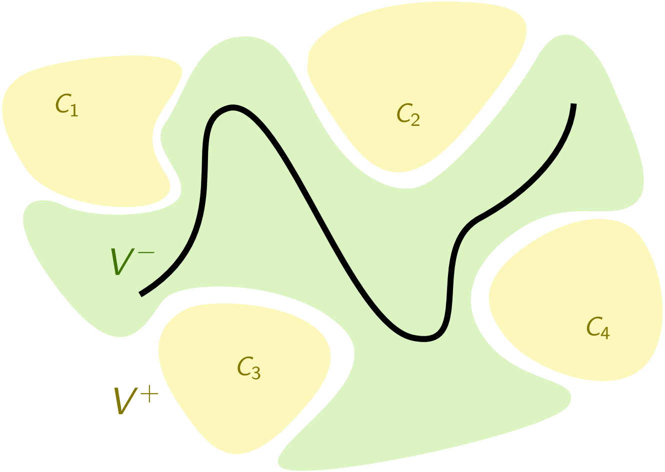

Every vertex detects it !

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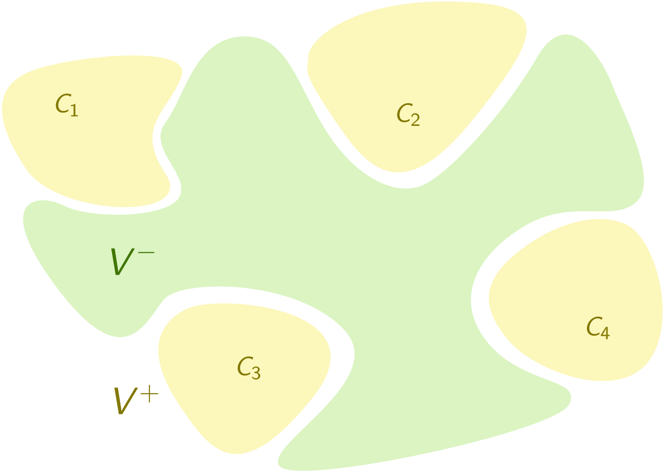

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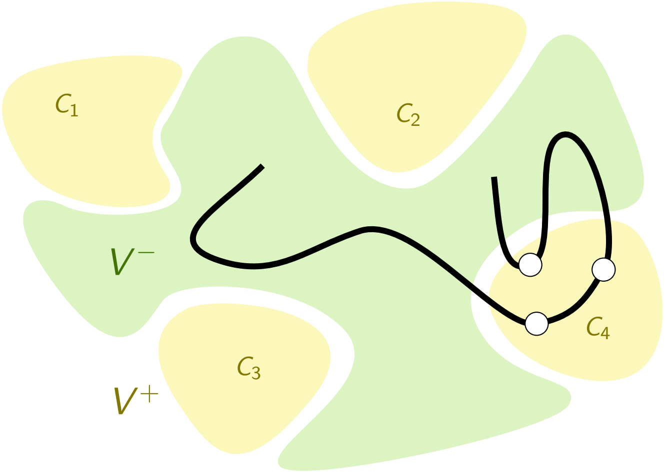

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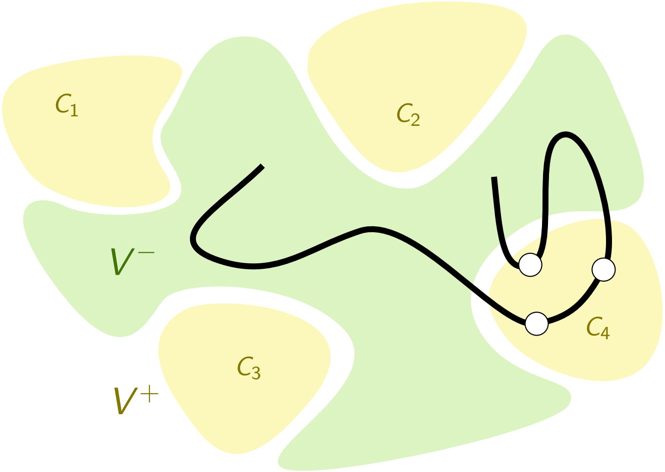

Every vertex in C_4 detects it !

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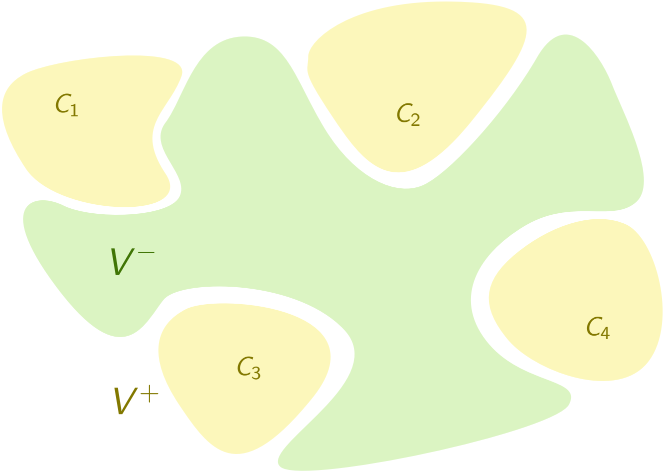

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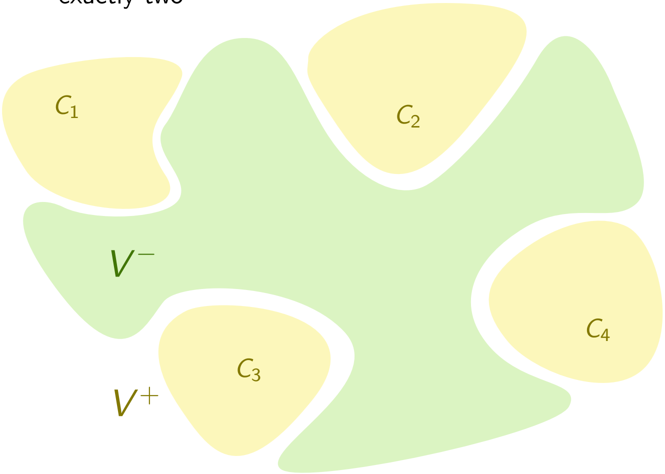

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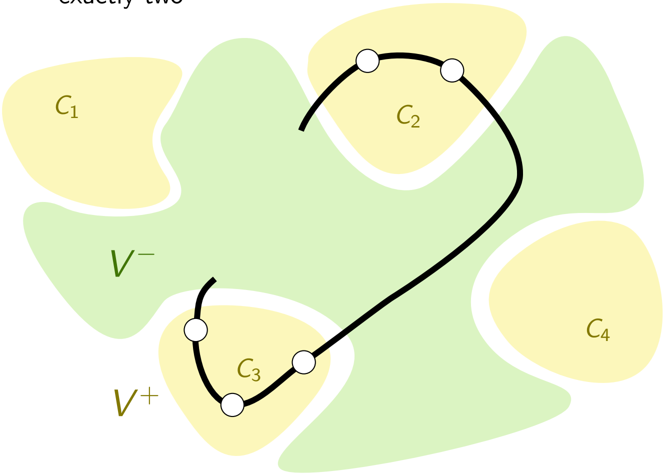

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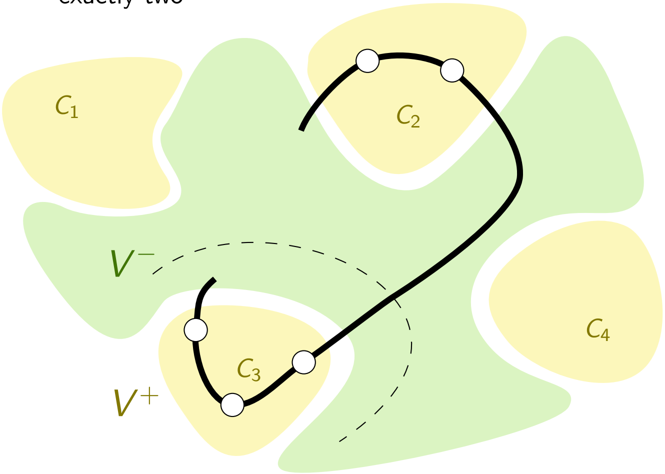

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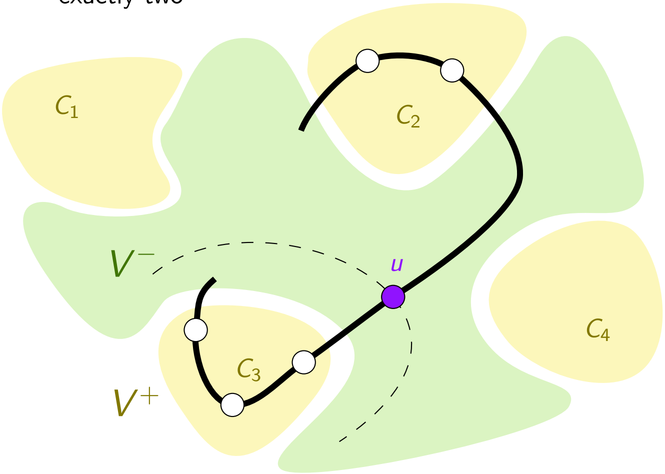

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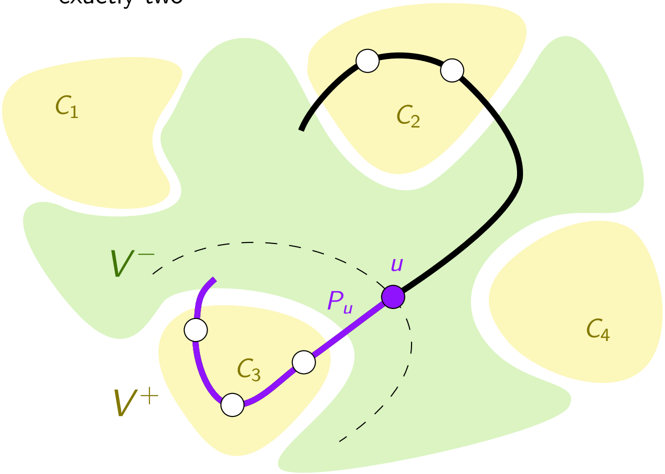

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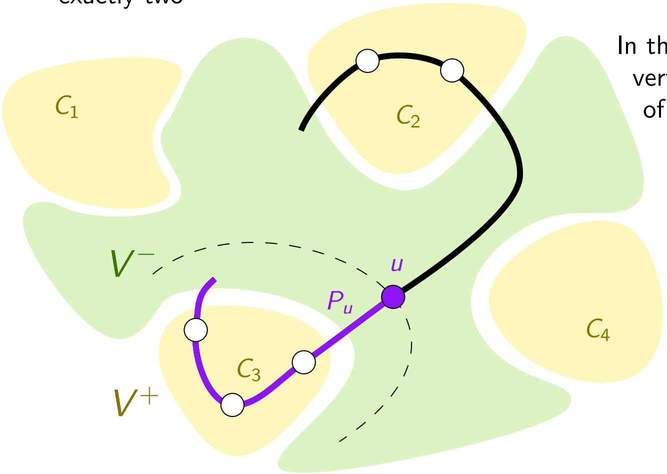

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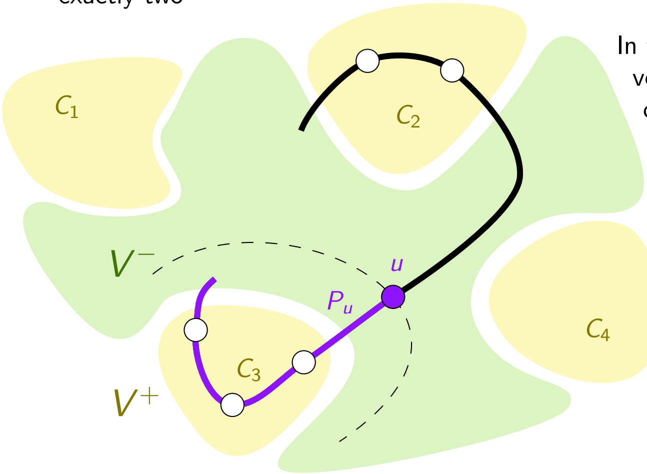

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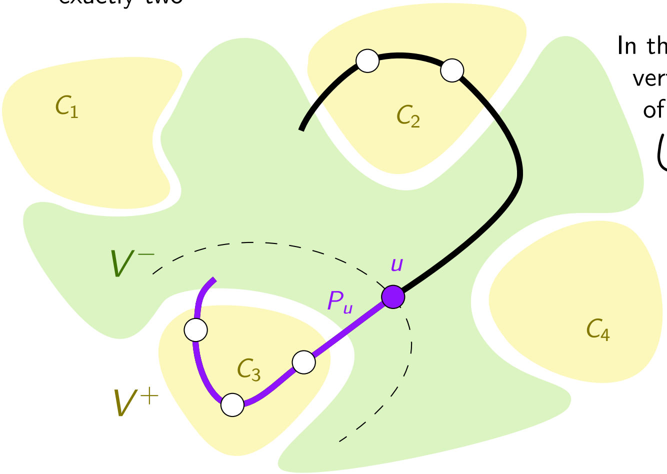

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Every vertex in C_2 detects it !

Conclusion: overview of our results for H-freeness and open questions

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Graph H	Bound

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$\|V(H)\| \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3 \rceil -1}$	$\tilde{O}(n^{3/2})$

Conclusion: overview of our results for H-freeness and open questions

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Open questions:

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Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5

Conclusion: overview of our results for H-freeness and open questions

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$\|V(H)\| \leqslant 4d-1$	$\tilde{O}(n^{3/2})$
$P_{\lceil 14d/3 \rceil -1}$	$\tilde{O}(n^{3/2})$
P_{3d-1}	$\tilde{O}(n)$

Open questions:

- what if $d = 1$? $\longrightarrow \tilde{O}(n^{3/2})$ for P_5
- Conjecture: for every $\alpha > 0$, there exists $\varepsilon > 0$ such that we can certify $P_{\alpha d}$-free graphs with certificates of size $O(n^{2-\varepsilon})$.

Thanks for your attention !