Using “Hilbert Methods” to decide Equivalence for Transducers

Adrien Boiret
joint work with Mikołaj Bojańczyk, Janusz Schmude, Radosław Piórkowski

March 27th 2018

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Polynomial Operations

Algebra $\mathcal{A}$, operations $\phi : \mathcal{A}^k \to \mathcal{A}$

Polynomial operations: $p : \mathcal{A}^n \to \mathcal{A}$ (or $p : \mathcal{A}^n \to \mathcal{A}^m$ by product)
Combination of operations ϕ

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Examples

$(\mathbb{Q}, +, \times)$, $p : (x, y) \mapsto (x^2 + xy, y)$

$(\mathbb{Q}[X], +, \times, -(-))$, $p : P \mapsto P(P)$

$(\Sigma^*, .)$, $p : (u, v) \mapsto u.v.u$

Register Automata

Bottom-up Tree Automata with Registers over $\mathcal{A}$ ($\mathcal{A}$-RA)
Finite ranked alphabet Σ, States Q, vector of n registers $\bar{r}$

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Examples

Tree to String transducers are register automata on finite words with concatenation.

Macro Tree Transducers are register automata on trees with leaf substitution.

Some Register Automata

On $(\Sigma^*, .)$

Example: Visibly Pushdown

One state q of dimension 1, output $q(r) \to r$
$a() \to q(< a >< /a >)$
$g(q(r_1), q(r_2)) \to q(< g > r_1 \cdot r_2 < /g >)$

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M register automata of dimension n over a ring $\mathcal{R}$

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The zeroness problem is decidable for $(\mathbb{Q}, +, \times)$ and $(\mathbb{Q}[X], +, \times)$

Polynomial Closure

For $S \subseteq \mathbb{Q}^n$

$\text{pol}(S) = \{p \mid \forall s \in S,\ p(s) = 0\}$ Ideal of $\overline{\mathbb{Q}}[X_1, \ldots, X_n]$

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- For p polynomial, $p(S_1) \subseteq S_2 \implies p(\overline{S_1}) \subseteq \overline{S_2}$
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Corollary

If a set of equations $\{S \supseteq p(S_1, \ldots, S_n) \ldots\}$ has a solution $S_1, \ldots, S_n$, then $\overline{S}_1, \ldots, \overline{S}_n$ is a solution.

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$\mathcal{A} \preceq_\pi \mathcal{B}$: $\mathcal{A}$ reduces to $\mathcal{B}$
$\preceq_\pi$ is a transitive relation

From Words to Integers

We reduce words on alphabet $\Sigma = \{0, 1\}$ into pairs of integers:

$(\Sigma^*, .)$	$(\mathbb{Z}, +, -, \times)$
u	$([u]_2, 2^{\|u\|})$

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Any Word-RA can be reduced to a $\mathbb{Z}$-RA

Reduction of Register Automata

$\mathcal{A}$ with operation $\phi_1, \ldots, \phi_l$, $\mathcal{B}$ such that $\mathcal{A} \preceq_\pi \mathcal{B}$,
Polynomial reduction: $f : \mathcal{A} \to \mathcal{B}^n$

$\mathcal{A}$-RA M into $\mathcal{B}$-RA States Q, m $\mathcal{A}$ registers: $m \times n$ $\mathcal{B}$ registers

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  $\exists p'$ polynomial in $\mathcal{B} \mid f(p) = p'(f, \ldots, f)$
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Decide Equivalence with "Hilbert Methods"

Theorem - Reduction

If $\mathcal{A} \preceq_\pi \mathcal{B}$ and functionality (equivalence) is decidable for register automata on $\mathcal{B}$, then functionality (equivalence) is decidable for register automata on $\mathcal{A}$.

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Theorem - "Hilbert Methods"

- Functionality is decidable for $(\mathbb{Q}[X], +, \times)$-RA
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Corollary (SMK 2015)

Equivalence is decidable for Tree to String transducers.

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Plan

Unranked Unordered Forests

Finite alphabet Σ, Algebra of forests $\mathcal{H}_\Sigma$.

Operations: Binary concatenation $\cdot$ (associative and commutative), for each $a \in \Sigma$, place a forest under a root of label a: unary root_a.

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Can be modeled with alphabet with one symbol.

$$\mathrm{root}_{a_i}(h) \to \mathrm{root}^i(\mathrm{root}\{\} \cdot \mathrm{root}(h))$$

Reduction from Forests to Polynomials

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Eisenberg Criterion

For $P(X) = a_0 + \cdots + a_k X^k \in \mathbb{Q}[X]$, if $\exists n$ prime such that

$$\forall 0 \leqslant i < k,\ n | a_i, \quad n \not| a_k, \quad n^2 \not| a_0$$

then $P(X)$ is irreducible in $\mathbb{Q}[X]$

0-1 Contexts

Unary alphabet, one variable y
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Can be extended to $0 - n$ with the same methods

Typed Algebra-RA

Typed algebra $\mathcal{A} = \mathcal{A}_1 \sqcup \cdots \sqcup \mathcal{A}_n$
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Polynomial reduction: $f : \mathcal{A} \to \mathcal{B}^n$
If a, a' same type in $\mathcal{A}$ **then** $f(a), f(a')$ same type in $\mathcal{B}$

Reduction from Contexts to Polynomials

Extend reduction to $\mathbb{Q}[X, Y]$ with $Y \leftarrow P$:

$(\text{Contexts}, \cdot, \text{root}, y \leftarrow h)$	$\mathbb{Q}[X, Y], +, \times, Y \leftarrow P$
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$h[y \leftarrow h']$	$f(h)[Y \leftarrow f(h')]$

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$$\begin{aligned} P(X, Y, Z) = P_0(X) + \quad & Y.P_1(X) + \quad & Y^2.P_2(X) \\ + \quad & Z.P_3(X) + \quad & Z^2.P_4(X) \\ & + \quad & Y.Z.P_5(X) \end{aligned}$$

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Theorem - Forests Register Automata

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Example: Vertical Exponential Blowup

One state q of dimension 1, output $q(r) \to r[y \leftarrow \alpha()]$
$a() \to (\beta(y)), \;\; g(q(r)) \to q(r[y \leftarrow r])$

MSO Unranked Unordered Forests Transformations

MSO on Unranked Unordered Forests: MSO + $\mathrm{Child}(x, y)$ $(+\mathrm{Sibling}(x, y))$

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Claim

Forests Register Automata can express all MSO Unranked Unordered Forests Transformations

MSO Unranked Unordered Forests Transformations

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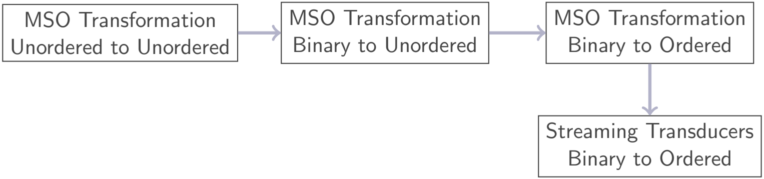
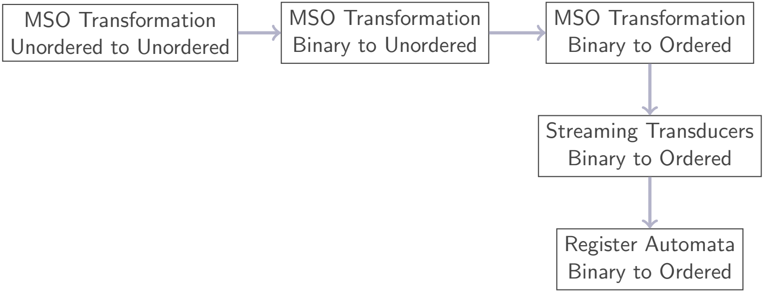
MSO Unranked Unordered Forests Transformations



MSO Unranked Unordered Forests Transformations



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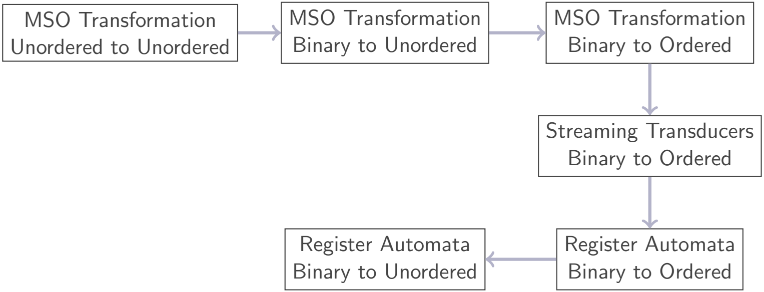

Plan

1. Polynomial Register Automata
   - Zeroness Problem
   - Polynomial Reduction

2. Positive and Negative Results
   - Unranked Unordered Forests are well-behaved
   - Polynomials with composition are not well-behaved

Polynomials with Composition

Algebra: $(\mathbb{Q}[X], +, \times, X \to P)$
Register automata equivalence on this algebra is undecidable.

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Reduction to accessibility in 2-counter machines

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Reduction to accessibility in 2-counter machines

2-counter machines (2CM)
$Q = \{q_0, \ldots, q_n\}$, configuration $(q, c_1, c2)$, $c_1, c_2 \in \mathbb{N}$
$\delta : (q, b_1, b_2) \to (q', -1/0/+1, -1/0/+1)$, $\;b_i : \; c_i = 0?$
Initial: $(q_0, 0, 0)$, Question: can we reach q_n?

This problem is undecidable.

Registers for 2-Counter Machine Simulation

Register word automata with alphabet δ

Reading $(q, b_1, b_2) \rightarrow (q', d_1, d_2)$: make the step in the 2CM

1-state Register Automaton:

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Registers to encode a 2CM configuration

- r_q: if the current state is q_i, then $r_q = i$
- r_1, r_2: if c_i is at value j, then $r_i = j$

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Update after reading $(i, b_1, b_2) \to (j, d_1, d_2)$

- $r_q \leftarrow j$
- $r_1 \leftarrow r_1 + d_1$, $r_2 \leftarrow r_2 + d_2$

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Were we **allowed** to read $(i, b_1, b_2) \to (j, d_1, d_2)$?

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r_w witness: $r_w = 0$ iff mistake.

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$r_w \leftarrow r_w . T_i(r_q) . T_1 . T_2$

Test for states: $T_i = \prod_{0 \leqslant j \leqslant n}^{i \neq j} X - j : \begin{cases} \neq 0 \text{ if } X = i \\ = 0 \text{ if } X \neq i \end{cases}$

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Test that $c_i \neq 0$: $T_i = r_i$

Test that $c_i = 0$: $T_i = \prod_{1 \leqslant j \leqslant k} X - j : \begin{cases} \neq 0 \text{ if } X = 0 \\ = 0 \text{ if } X \neq 0 \end{cases}$

Witness Register

r_w witness: $r_w = 0$ iff mistake.

Update after reading $(i, b_1, b_2) \rightarrow (j, d_1, d_2)$
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Test that $c_i = 0$ must be stored and updated into its own register

Polynomials with Composition are not well-behaved

Output: $P_n(r_q).r_w$

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This register automaton produces anything other than 0 $\iff$ q_n accessible.

Theorem

The equivalence problem for register automata over $(\mathbb{Q}[X], +, \times, X \to P)$ is undecidable.

Some Parting Words

What we've done:

- Abstraction of "Hilbert Methods" to decide some Transducer Equivalence
- Positive Result: Unranked Unordered Forests
- Negative Result: $\mathbb{Q}[X]$ with composition

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What's to come:

- New reductions? (DAG, Graphs with bounded tree width...)
- New targets for Zeroness results?
- "Stratified" registers with a dangerous operation
  Order on states, increased when using dangerous operation

That's it!

Thank you for your attention!