When is containment decidable for probabilistic automata?

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The containment problem

$[A] \subseteq [B]$
The containment problem

Boolean automata over $\Sigma^*$

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Languages over \( \Sigma^* \)

Boolean automata over \( \Sigma^* \)

\([A] \subseteq [B]\)
The containment problem

Boolean automata over $\Sigma^*$

Languages over $\Sigma^*$

Check whether:

$[B] \cap [A]^c = \emptyset$
The containment problem

Boolean automata over $\Sigma^*$

$\mathcal{A} \subseteq \mathcal{B}$

Functions: $\Sigma^* \rightarrow \mathbb{R}$

Check whether:
$[\mathcal{B}] \cap [\mathcal{A}]^c = \emptyset$
The containment problem

\[ [A] \subseteq [B] \]

\( \Sigma^* \rightarrow \mathbb{R} \)

Weighted automata over \( \Sigma^* \)

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The containment problem

Functions: $\Sigma^* \rightarrow \mathbb{R}$

Weighted automata over $\Sigma^*$

Check whether:

$[B] - [A] \geq 0$
What is the probability that after 8 hours I have done some sport or work?
Initial states and transitions are weighted with probability:

\[
\begin{bmatrix}
\frac{1}{2} & \rightarrow & \\
\frac{1}{6} & \rightarrow & \\
\frac{1}{3} & \rightarrow & 
\end{bmatrix}
\]

\[w \mapsto \text{probability to read } w \text{ from an initial to a final state.}\]
A few results
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Probabilistic automata

Max-plus automata
A few results

Probabilistic automata
- Undecidable in general
  Post correspondence problem - Paz, Bertoni...

Max-plus automata
- Undecidable in general
  Diophantine equations - Krob
Notion of ambiguity

How many accepting runs are labelled by a given word?
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Unambiguous: for all words, at most 1
Finitely ambiguous: for all words, at most $k$
Linearly ambiguous: for all words $w$, at most $k|w|
Quadratic: for all words $w$, at most $k|w|^2$
Polynomially ambiguous, exponentially ambiguous...
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- Undecidable for linearly ambiguous
  Halting problem of two-counter machines - Colcombet, Amalgor-Boker-Kupferman
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- Emptiness problem decidable for 2-ambiguous
  Fijalkow-Riveros-Worrell

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  Diophantine equations - Krob

- Undecidable for linearly ambiguous
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- Decidable for finitely ambiguous
  Filiot-Gentilini-Raskin
When is containment decidable?

\[
[A] \leq [B]
\]

Undecidable

When either \(A\) or \(B\) is at least linearly ambiguous.

Decidable

When \(A\) and \(B\) are finitely ambiguous and one is unambiguous.

Open

When \(A\) and \(B\) are finitely ambiguous.
Are there positive integers $x$ and $y$ such that:

$$p \cdot \left(\frac{1}{12}\right)^x \cdot \left(\frac{1}{18}\right)^y + \left(1 - p\right) \cdot \left(\frac{1}{3}\right)^x \cdot \left(\frac{1}{18}\right)^y < \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y$$

Equivalently:

$$e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1 - p) + x \log(2) - y \log(3)} < \frac{19}{15}$$

Decidability: one example
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Equivalently:

$$e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1-p) + x \log(2) - y \log(3)} < 1$$
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\[ e^{\log(p) - x \log(2) + y \log(3)} + e^{\log(1-p) + x \log(2) - y \log(3)} < 1 \]

Is there positive integers \( x, y \) s.t:

- \( e^u + e^v < 1 \) where:
  - \( u = \log(p) - x \log(2) + y \log(3) \)
  - \( v = \log(1-p) + x \log(2) - y \log(3) \)
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\[ \rightarrow \text{YES if and only if } p = \frac{1}{2}. \]
Decidability: translating the problem

Is there a word $w$ such that $[A](w) > [B](w)$?
Decidability: translating the problem

Is there a word \( w \) such that \([A](w) > [B](w)\)?
Decidability: translating the problem

Is there a word $w$ such that $\llbracket A \rrbracket(w) > \llbracket B \rrbracket(w)$?

Given $A$ ($k$-ambiguous) and $B$ ($\ell$-ambiguous), one can compute:
- a positive integer $n$,
- a finite set of tuples $(p, q, r, s)$ with
  - $p$ in $\mathbb{Q}^k_{>0}$, $r$ in $\mathbb{Q}^\ell_{>0}$, $q$ in $\mathbb{Q}^{k \times n}_{>0}$, $s$ in $\mathbb{Q}^{\ell \times n}_{>0}$,

such that for one of those tuples, there exist $x \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} > \sum_{i=1}^{\ell} r_i s_{i,1}^{x_1} \cdots s_{i,n}^{x_n}$$

if and only if there exist a word $w$ such that $\llbracket A \rrbracket(w) > \llbracket B \rrbracket(w)$. 
Decidability: first case

\([A] \leq [B]\) when \(B\) is unambiguous.
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\([A] \leq [B]\) when \(B\) is unambiguous.

Is there \(x \in \mathbb{N}^n\) such that:

\[
\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} > r s_{1}^{x_1} \cdots s_{n}^{x_n}
\]
Decidability: first case

\([A] \leq [B]\) when \(B\) is unambiguous.

Is there \(x \in \mathbb{N}^n\) such that:

\[
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\]

- First case: there is \(i, j\) such that \(q_{i,j} > s_j\)
- Second case: for all \(i, j\), \(q_{i,j} \leq s_j\)
Decidability: second case

Much more difficult!

**Theorem**

Determining whether $[A] \leq [B]$ is decidable when $A$ is un-ambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.
Decidability: second case

Much more difficult!

**Theorem**

Determining whether $\llbracket A \rrbracket \leq \llbracket B \rrbracket$ is decidable when $A$ is unambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $x \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} < 1$$
Decidability: second case

Much more difficult!

Theorem

Determining whether $[A] \leq [B]$ is decidable when $A$ is unambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $x \in \mathbb{N}^n$ such that:

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- semi-decidable to find such $x$
Decidability: second case

Much more difficult!

Theorem

Determining whether $[A] \leq [B]$ is decidable when $A$ is unambiguous and $B$ is finitely ambiguous, assuming Schanuel’s conjecture is true.

Is there $x \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \cdots q_{i,n}^{x_n} < 1$$

- semi-decidable to find such $x$
- if there is no such $x$, there is a non-zero vector $d \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$ such that $\{d^\top y \mid y \text{ is a real solution}\} \subseteq [a, b]$ 
  $\rightarrow$ decrease the dimension by 1
Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $A$ and $B$, such that the machine halts if and only if there exists a word $w$ such that $\|A\|(w) \leq \|B\|(w)$.
Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $A$ and $B$, such that the machine halts if and only if there exists a word $w$ such that $[A](w) \leq [B](w)$.

→ Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$
Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $\mathcal{A}$ and $\mathcal{B}$, such that the machine halts if and only if there exists a word $w$ such that $\mathcal{A}(w) \leq \mathcal{B}(w)$.

Proposition

Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$

Undecidability
Conclusion

Containment problem for finitely ambiguous probabilistic automata?