When is containment decidable for probabilistic automata?

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Joint work with Marcin Jurdziński, Ranko Lazić, Filip Mazowiecki, Guillermo A.Pérez and James Worrell.

Delta, Paris, 27-03-2018

The containment problem

$$[\![\mathcal{A}]\!] \subseteq [\![\mathcal{B}]\!]$$

The containment problem

Boolean automata over Σ^*

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Languages over Σ^*

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Languages over Σ^*

Check whether:
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Functions: $\Sigma^* \to \mathbb{R}$

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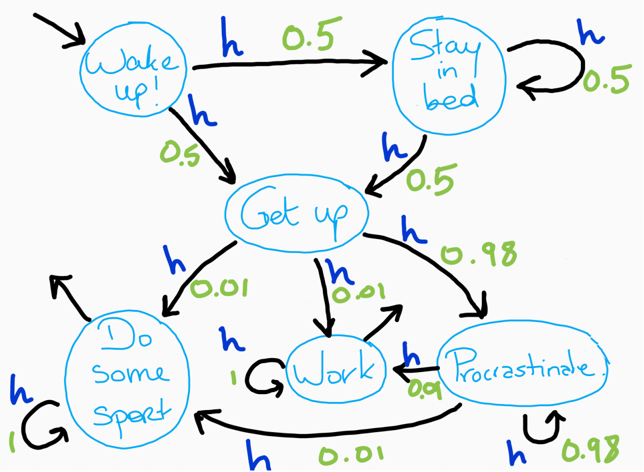
Functions: $\Sigma^* \to \mathbb{R}$

Check whether:
$[\![\mathcal{B}]\!] - [\![\mathcal{A}]\!] \geqslant 0$

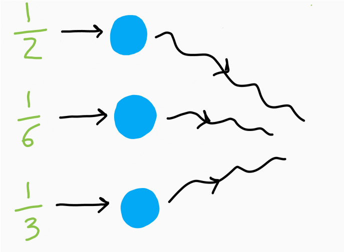
Probabilistic automata

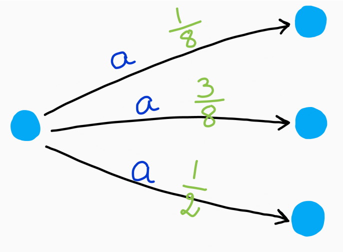


What is the probability that after 8 hours I have done some sport or work?

Probabilistic automata

Initial states and transitions are weighted with probability:





$[\![\mathcal{A}]\!]$: $w \mapsto$ probability to read w from an initial to a final state.

A few results

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Probabilistic automata

Max-plus automata

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Probabilistic automata

- Undecidable in general
  Post correspondence problem - Paz, Bertoni...

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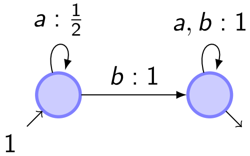

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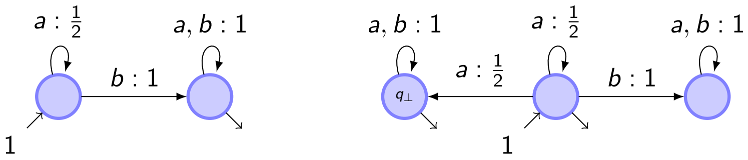

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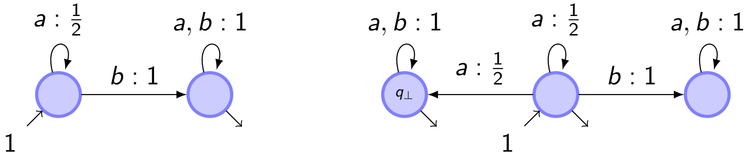

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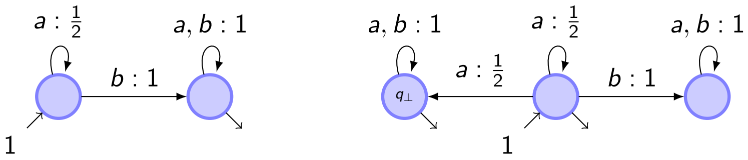

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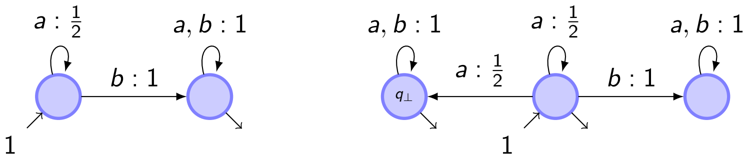

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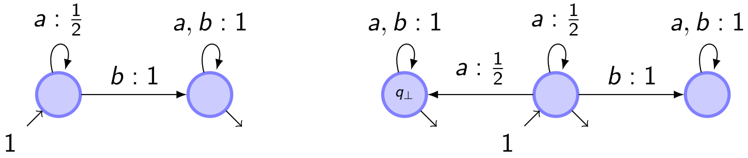

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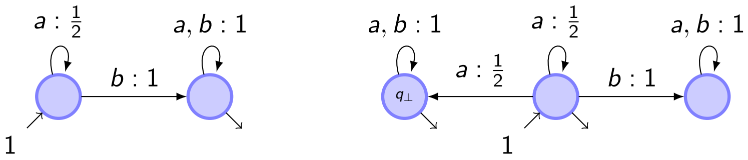

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- Polynomially ambiguous, exponentially ambiguous...

A few results

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  Post correspondence problem - Paz, Bertoni...

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- Decidable for finitely ambiguous
  Filiot-Gentilini-Raskin

When is containment decidable?

$$[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$$

Undecidable

When either $\mathcal{A}$ or $\mathcal{B}$ is at least linearly ambiguous.

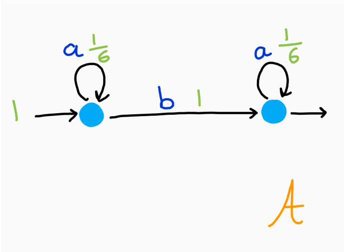
Decidable

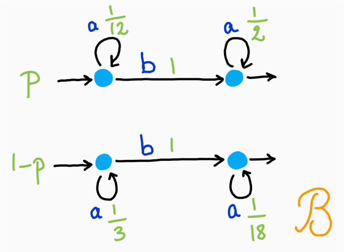
When $\mathcal{A}$ and $\mathcal{B}$ are finitely ambiguous and one is unambiguous.

Open

When $\mathcal{A}$ and $\mathcal{B}$ are finitely ambiguous.

Decidability: one example





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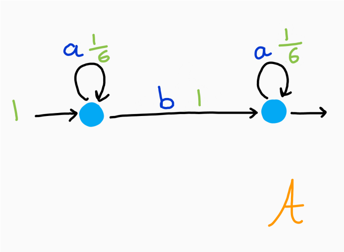

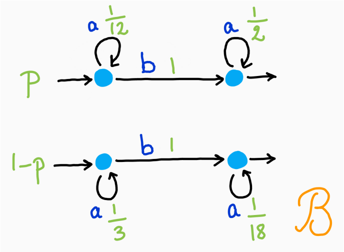

Are there positive integers x and y such that:

$$p \cdot \left(\frac{1}{12}\right)^x \cdot \left(\frac{1}{2}\right)^y + (1-p) \cdot \left(\frac{1}{3}\right)^x \cdot \left(\frac{1}{18}\right)^y < \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y$$

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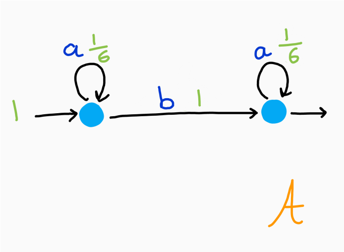

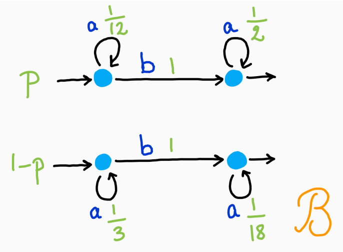

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Equivalently:

$$e^{\log(p) - x\log(2) + y\log(3)} + e^{\log(1-p) + x\log(2) - y\log(3)} < 1$$

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$$e^{\log(p)-x\log(2)+y\log(3)} + e^{\log(1-p)+x\log(2)-y\log(3)} < 1$$

Is there positive integers x, y s.t:

- $e^u + e^v < 1$ where:
- $u = \log(p) - x\log(2) + y\log(3)$
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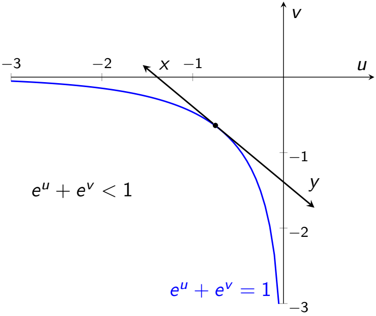

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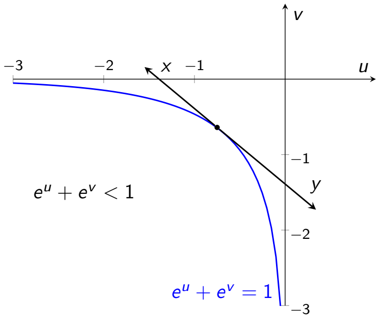

$\longrightarrow$ YES if and only if $p = \frac{1}{2}$.

Decidability: translating the problem

Is there a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$?

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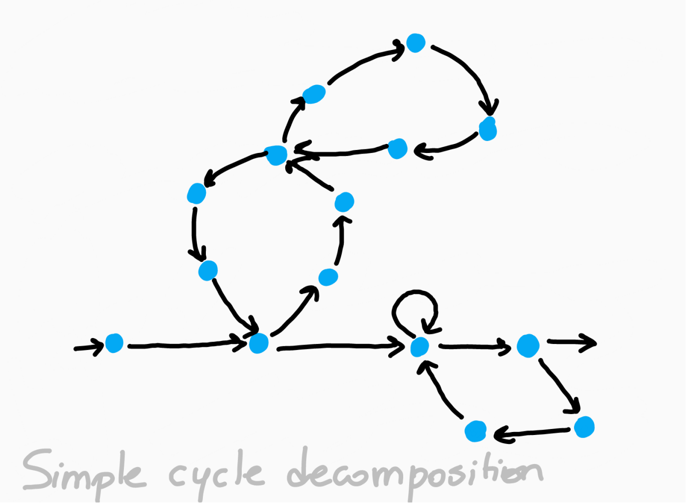

Decidability: translating the problem

Is there a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$?

Given $\mathcal{A}$ (k-ambiguous) and $\mathcal{B}$ (ℓ-ambiguous), one can compute:

- a positive integer n,
- a finite set of tuples $(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \boldsymbol{s})$ with
  - $\boldsymbol{p}$ in $\mathbb{Q}_{>0}^{k}$, $\boldsymbol{r}$ in $\mathbb{Q}_{>0}^{\ell}$, $\boldsymbol{q}$ in $\mathbb{Q}_{>0}^{k \times n}$, $\boldsymbol{s}$ in $\mathbb{Q}_{>0}^{\ell \times n}$,

such that for one of those tuples, there exist $\boldsymbol{x} \in \mathbb{N}^n$ such that:

$$\sum_{i=1}^{k} p_i q_{i,1}^{x_1} \dots q_{i,n}^{x_n} > \sum_{i=1}^{\ell} r_i s_{i,1}^{x_1} \dots s_{i,n}^{x_n}$$

if and only if there exist a word w such that $[\![\mathcal{A}]\!](w) > [\![\mathcal{B}]\!](w)$.

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- First case: there is i, j such that $q_{i,j} > s_j$
- Second case: for all i, j, $q_{i,j} \leqslant s_j$

Decidability: second case

Much more difficult!

Theorem

Determining whether $[\![\mathcal{A}]\!] \leqslant [\![\mathcal{B}]\!]$ is decidable when $\mathcal{A}$ is unambiguous and $\mathcal{B}$ is finitely ambiguous, assuming Schanuel's conjecture is true.

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- semi-decidable to find such $\boldsymbol{x}$
- if there is no such $\boldsymbol{x}$, there is a non-zero vector $\boldsymbol{d} \in \mathbb{Z}^n$ and $a, b \in \mathbb{Z}$ such that $\{\boldsymbol{d}^\top \boldsymbol{y} \mid \boldsymbol{y} \text{ is a real solution } \} \subseteq [a, b]$

$\rightarrow$ decrease the dimension by 1

Undecidability

Proposition

Given a two-counter machine, one can construct two linearly ambiguous probabilistic automata $\mathcal{A}$ and $\mathcal{B}$, such that the machine halts if and only if there exists a word w such that $[\![\mathcal{A}]\!](w) \leqslant [\![\mathcal{B}]\!](w)$.

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$\longrightarrow$ Simulate an execution with a word: $a^n b^m t_1 a^{n+1} b^m t_2 a^{n+1} b^{m'}$

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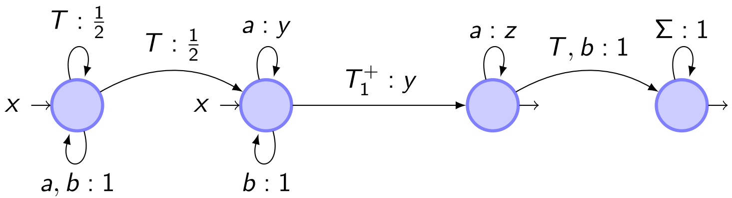

Conclusion

Containment problem
for finitely ambiguous probabilistic automata?