Continuity and rational functions

Michaël Cadilhac, Olivier Carton et Charles Paperman

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Continuity

Classic definition

Let X and Y be two topological spaces. A function $f : X \to Y$ is continuous if for any open set V of Y, $f^{-1}(V)$ is an open set of X.

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A function is continuous if it can't distinguish between similar stuff.

Regular continuous functions

Take $f : A^* \to B^*$ (partial), it is regular continuous if it preserves regular languages by inverse image.

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Examples:

1. $\mathrm{DeletePrefix}_{a^*} : a^* \cdot bu \mapsto bu$ with $b \neq a$
2. $\mathrm{DeleteEven} : u_0 u_1 u_2 u_3 \cdots \mapsto u_1 u_3 \cdots$
3. $\mathrm{Reverse} : u_0 u_1 \cdots u_n \mapsto u_n \cdots u_1 u_0$
4. $\mathrm{Square} : u \mapsto u \cdot u$

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4. $\mathrm{Square} : u \mapsto u \cdot u$

Co-examples:

5. $\mathrm{SquareRoot} : u \cdot u \mapsto u$

A sort of motivation

Descriptive and very concise description of regular languages:

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How to test conditions on those languages?

Conditions like starfree, AC^0 or other classes...

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(Sorry Jean-Éric.)

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- A more generic notion is the $\mathcal{V}, \mathcal{W}$-continuity:

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Our results

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We notably failed to understand the Burnside varieties

$(\llbracket x^2 = x^3 \rrbracket)$

The $\mathcal{V}$-profinite framework

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In the following, $\mathcal{V}$ is a variety of regular languages.

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Co-Example:

- The sequence $(w, w^2, w^3, \ldots, w^n, \ldots)$ is not fooling the language $(ww)^*$.

$\mathcal{V}$-profinite words

- Two sequences u and v that fool $\mathcal{V}$ are equivalent iff:
  $\forall L \in \mathcal{V}$ and n big enough $u[n] \in L \Leftrightarrow v[n] \in L$.

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Co-example:

- $(ab)^{\omega}$ and $(ba)^{\omega}$ are not equivalent since $(ab)^{\omega}$ ultimately is in aA^* and not $(ba)^{\omega}$
- $(ab)^{\omega}$ and $(ba)^{\omega}$ are Commutative-equivalent.

$\mathcal{V}$-profinite words (alternative)

A^*

$\mathcal{V} = \{L_1, \ldots, L_n, \ldots\}$

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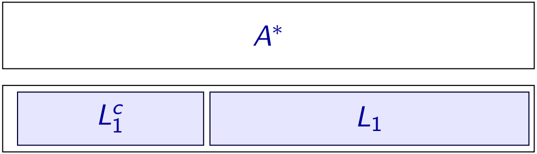

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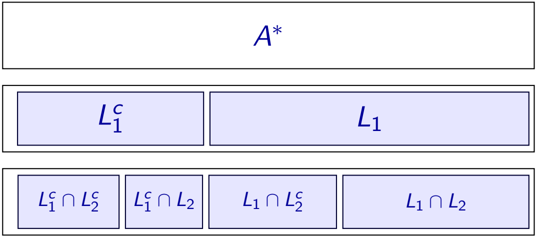

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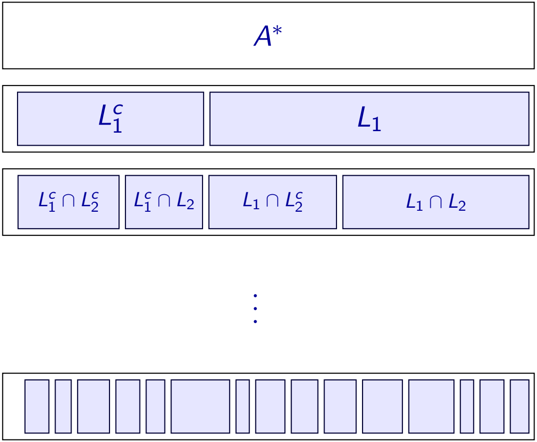

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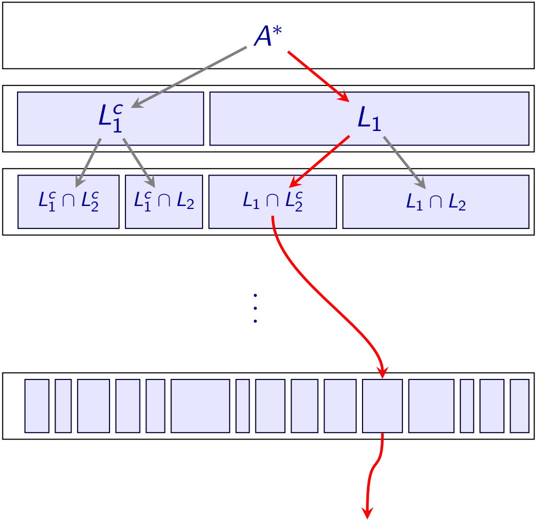

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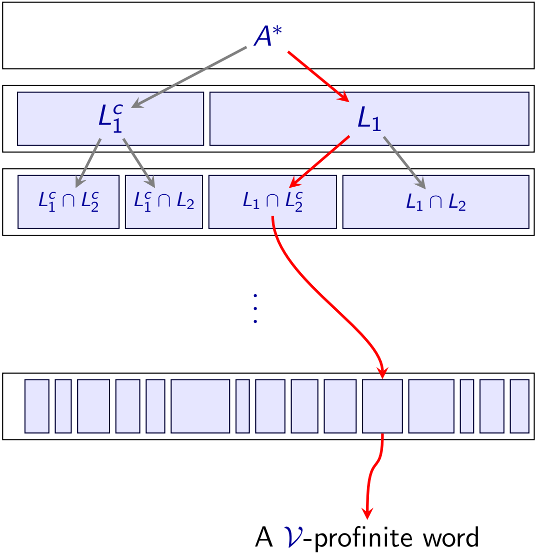

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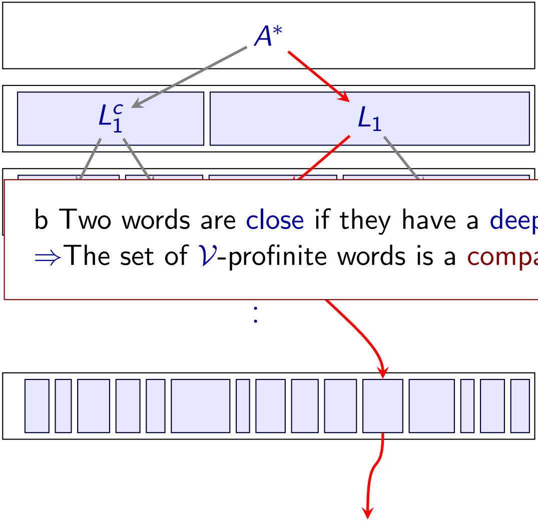

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b Two words are close if they have a deep common ancestor.
$\Rightarrow$The set of $\mathcal{V}$-profinite words is a compact topological monoid.

A $\mathcal{V}$-profinite word

Continuity is about continuity

Theorem (Pin, Silva, 11).

A function from A^* to B^* is $\mathcal{V}$-continuous if and only if it is continuous for the $\mathcal{V}$-profinite topology.

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Theorem (Pin, Silva, 11).

A function from A^* to B^* is $\mathcal{V}$-continuous if and only if it is uniformly-continuous for the $\mathcal{V}$-profinite topology.

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Co-Example:

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Continuity of rational functions

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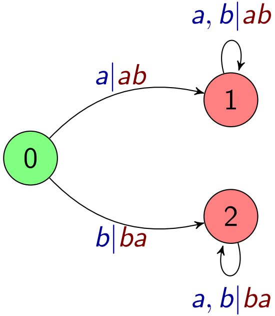

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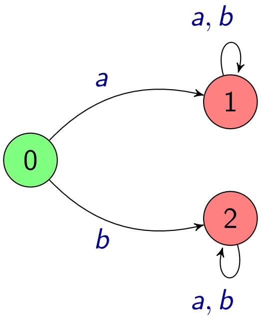

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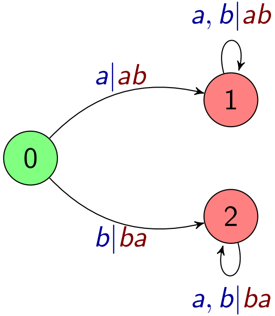

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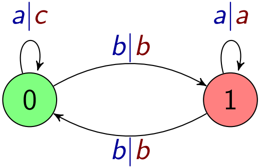

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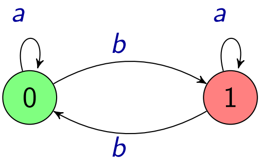

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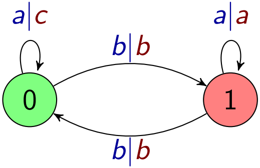

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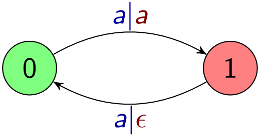

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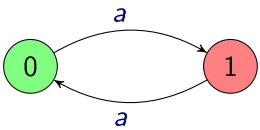

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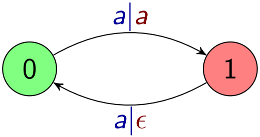

Even a are deleted

Transition monoid is not aperiodic but the function is aperiodic continuous.

Examples (Reutenauer-Schützenberger)

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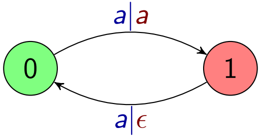

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Proposition (Folklore).

Functions computable by aperiodic transducers are aperiodic Continuous.

Conclusion

About decidability

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Further work

- Could (should?) be extended to $\mathcal{V}, \mathcal{W}$-Continuity.
- Could (should?) be extended to non-rational functions.
- Could (should?) be extended to non-regular classes.