Two-Way Parikh Automata with a Visibly Pushdown Stack

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Delta - Mar 27 2018

Outline

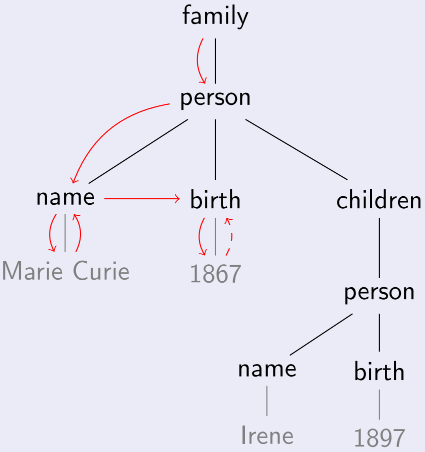
Outline

0 - Motivations

Well-nested words

Encoding

Well-nested words ≡ linearizations of (unranked) trees



⇒

< family >
< person >
< name > MarieCurie *< \name >*
< birth > 1867 *< \birth >*
< children >
< name > Irene *< \name >*
< birth > 1897 *< \birth >*
< \children >
< \person >
< \family >

Well-nested words

Definition (Structured Alphabet)

A *structured alphabet*, Σ, is a set $\Sigma = \Sigma_c \uplus \Sigma_r$ (*call* and *return* symbols respectively).

A word is *well-nested* if there is no pending call c_i nor return symbols r_i (Σ_{wn}^*)

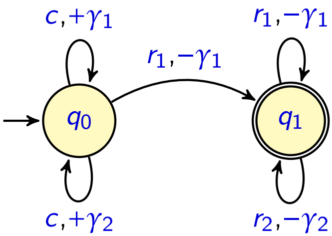
$c_1 \; c_2 \; r_2 \; r_1 c_1 r_1$ ✓ $\qquad$ $c_1 \; c_2 \; r_2 \; r_1 \underline{r_3} c_1 r_1 \underline{c_2}$ ✗

Well-nested words are generated by the grammar $u, v \in \Sigma_{\mathrm{wn}}^* \; : \; \epsilon \mid u.v \mid c u r$.

Visibly Pushdown Automata (VPAs) [Alur,Madhusudan,04]

VPAs = Pushdown Automata on *structured* alphabet $\Sigma = \Sigma_c \uplus \Sigma_r$:

- push **one** stack symbol on **call** symbols Σ_c
- pop **one** stack symbol on **return** symbols Σ_r
- accept on final state



$$L(A) = \{c^n \cdot r_1 \cdot (r_1 + r_2)^{n-1} \mid n > 0\}.$$

- The height of the stack at some position in the word does not depend on the considered computation, only on the position

Two-way VPA (2VPA) [Madhusudan,Viswanathan,09]

Main features :

- the head is placed between symbols
- directed states $Q \times \{\leftarrow, \rightarrow\}$ ($\rightarrow$: read the symbol on the right, ...)
- Behaviour :
  - Forward ($\rightarrow$): just as VPA
  - Backward ($\leftarrow$): dually, **pop** on call and **push** on return

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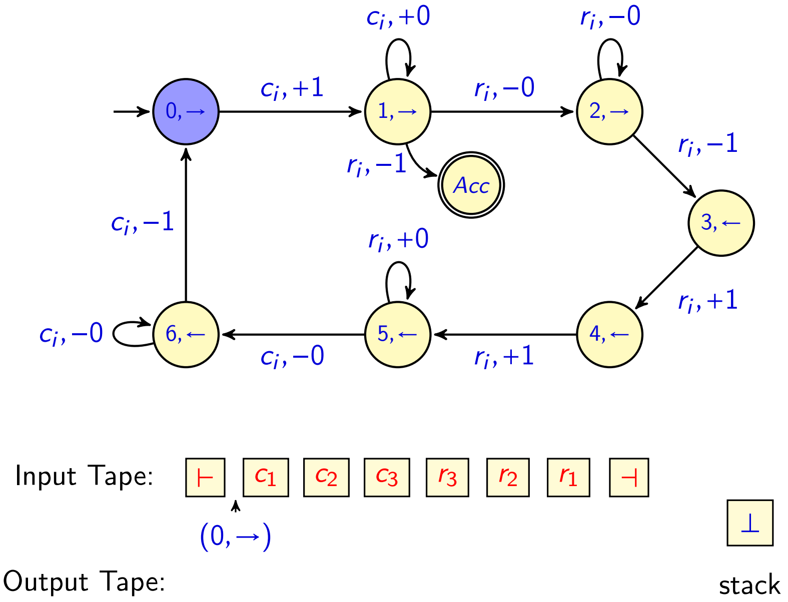
Theorem

2VPA are as expressive as VPA (reading from left to right).

2-way Visibly Pushdown Transducers (2VPT)



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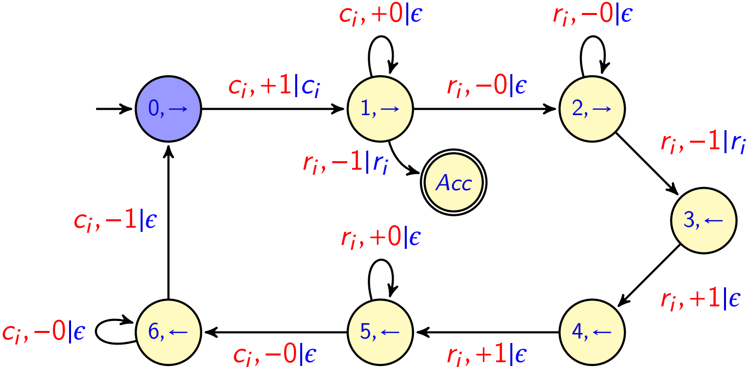

Input Tape: | ⊢ | c_1 | c_2 | c_3 | r_3 | r_2 | r_1 | ⊣ |

(0, →)

Output Tape:

⊥

stack

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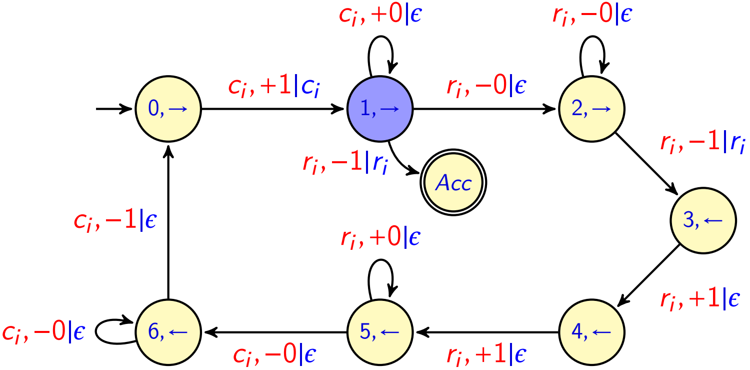

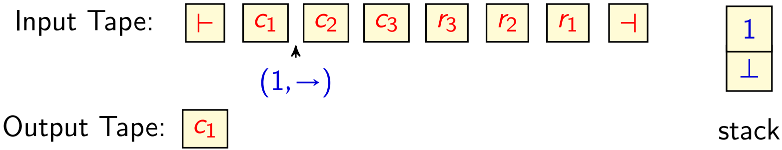

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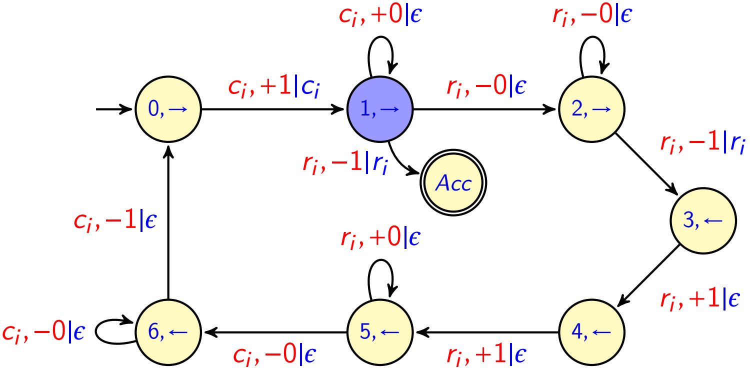

Input Tape: $\vdash$ c_1 c_2 c_3 r_3 r_2 r_1 $\dashv$

$(1, \rightarrow)$

Output Tape: c_1

stack: 0, 1, $\perp$

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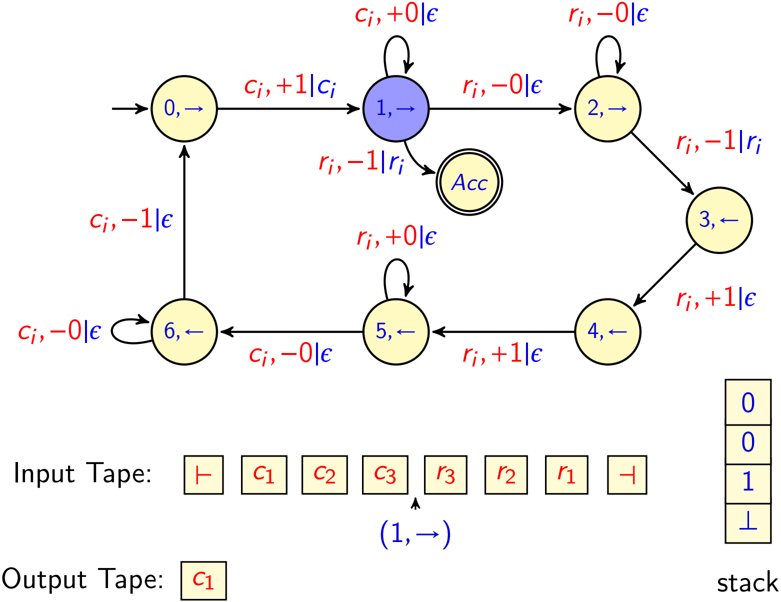

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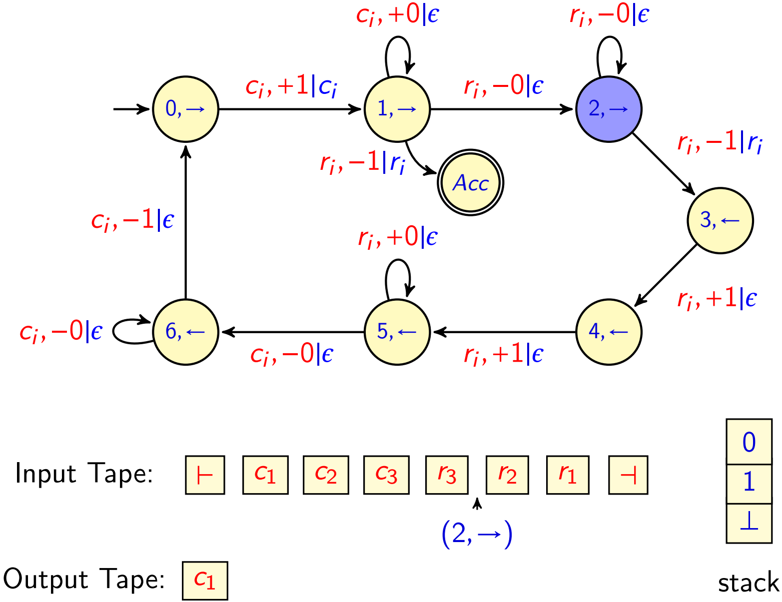

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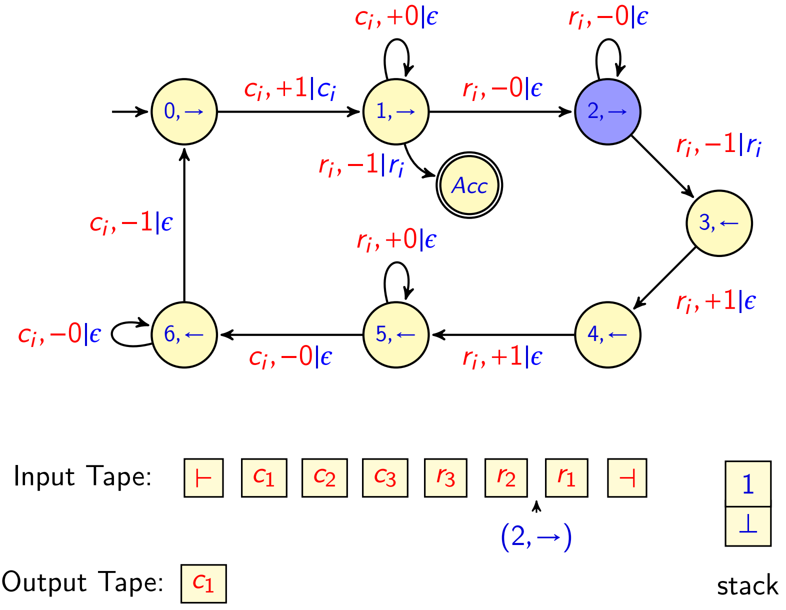

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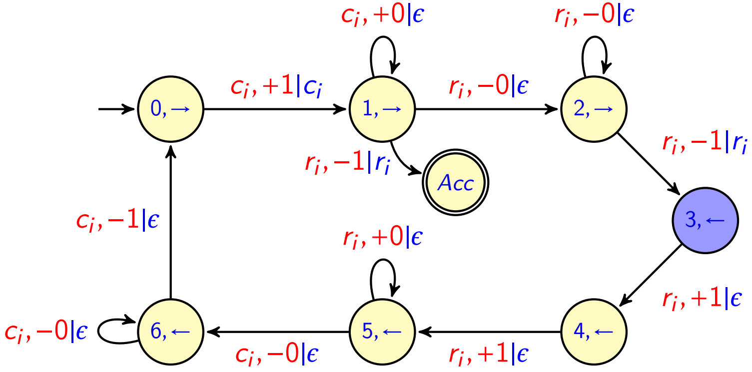

Input Tape: $\vdash$ c_1 c_2 c_3 r_3 r_2 r_1 $\dashv$

$(3, \leftarrow)$

$\perp$

Output Tape: c_1 r_1

stack

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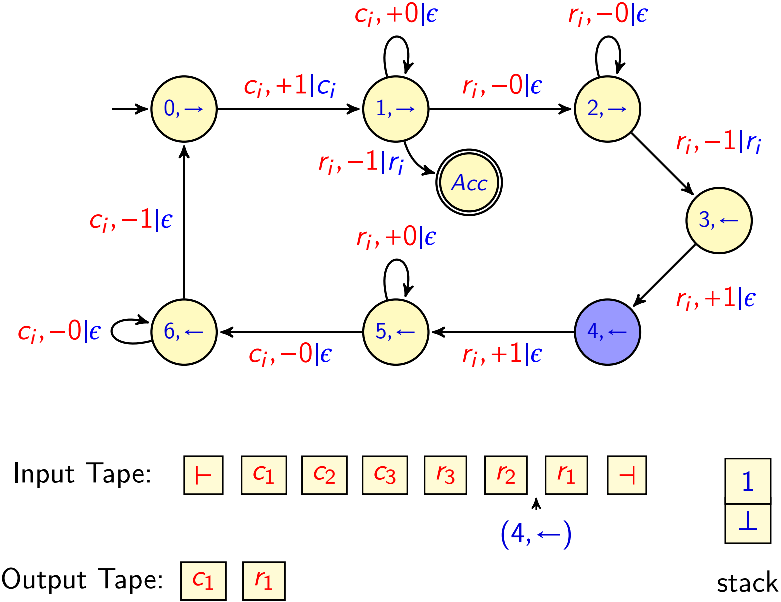

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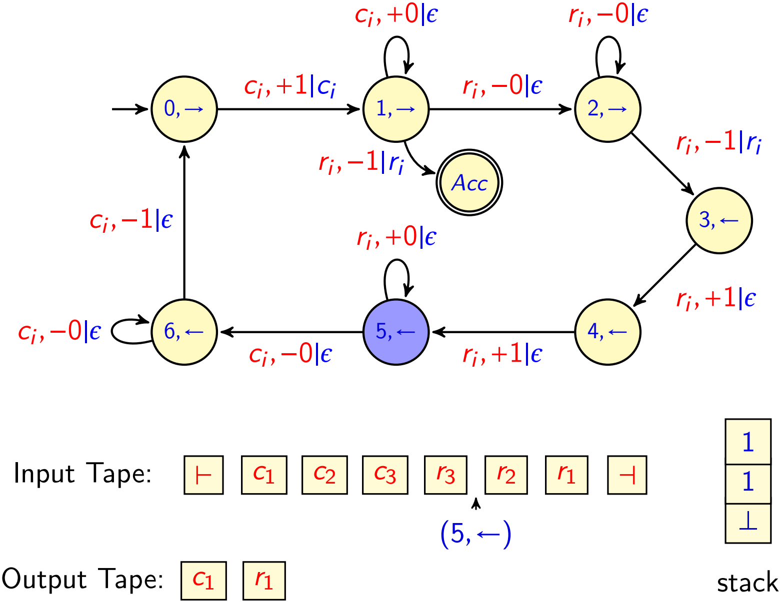

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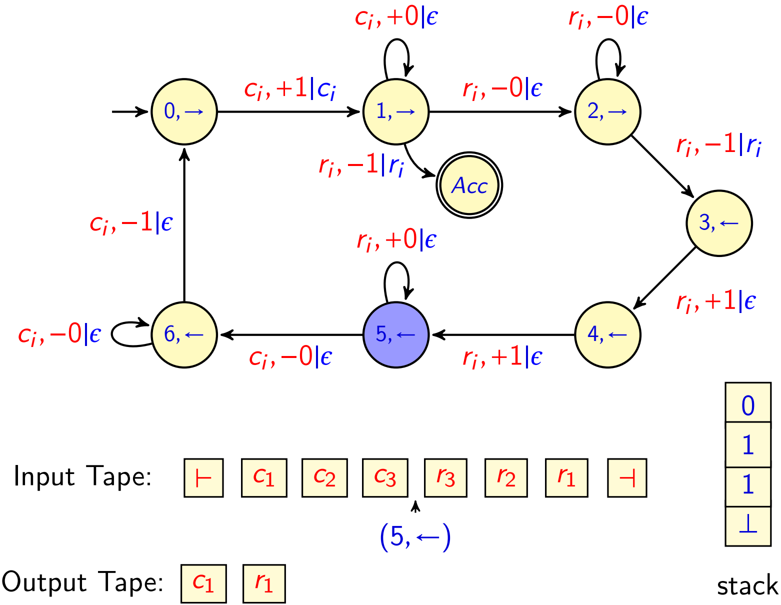

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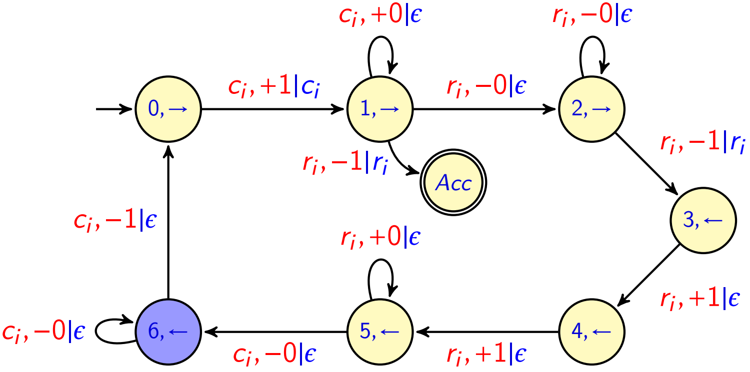

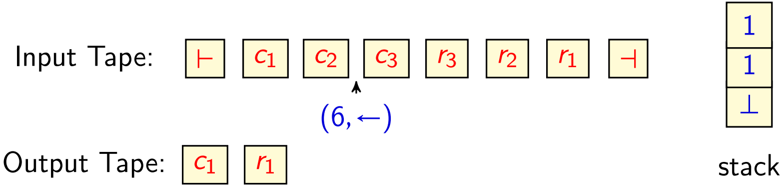

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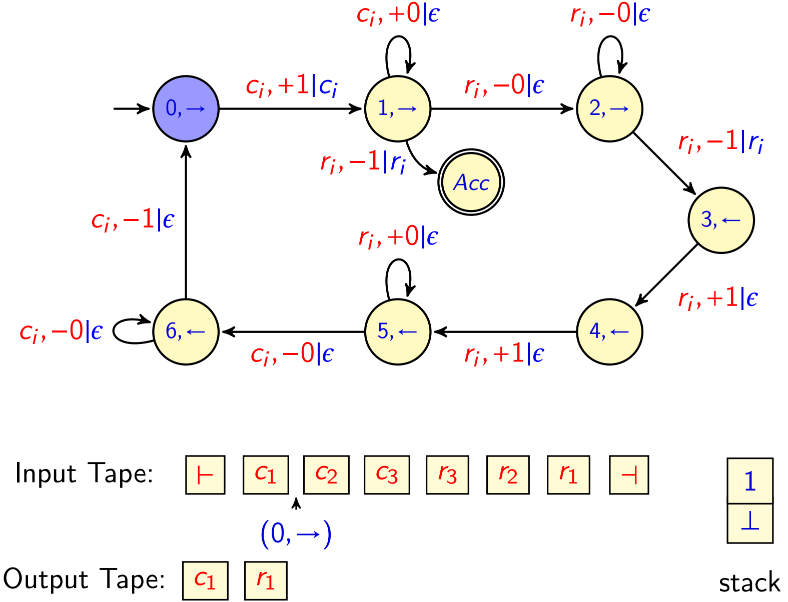

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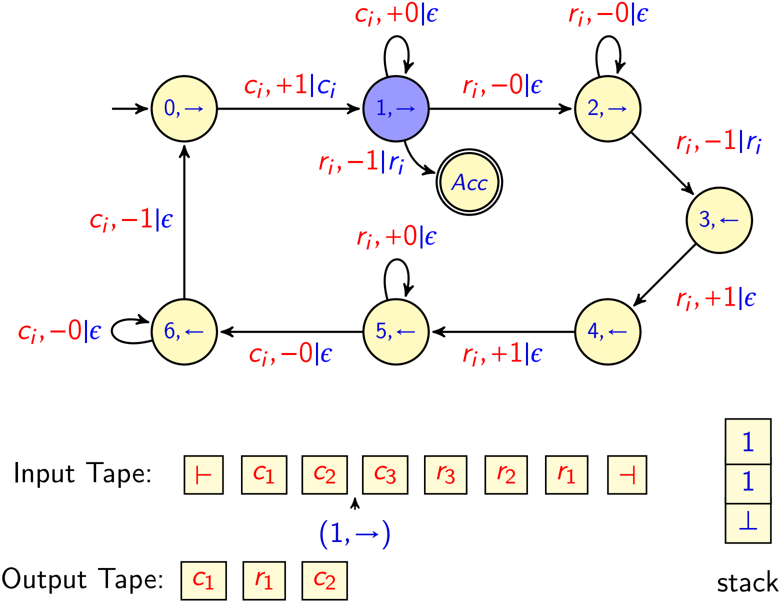

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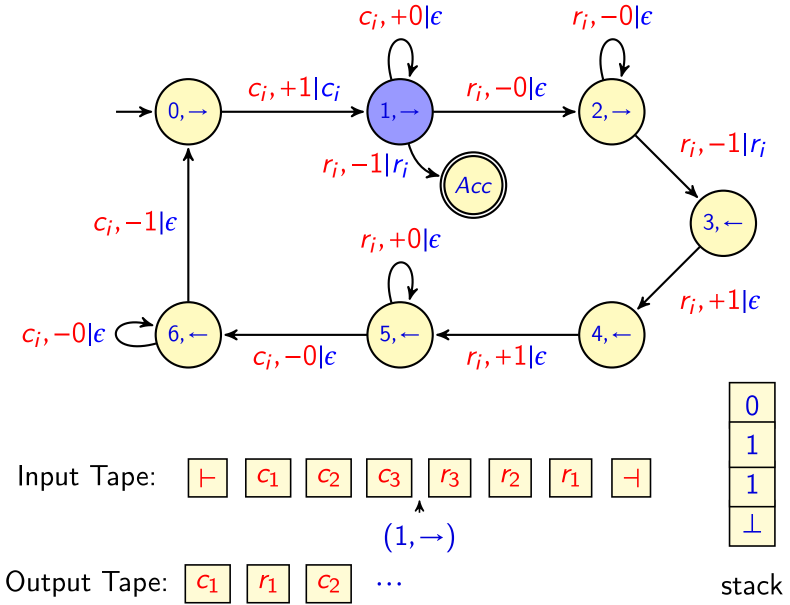

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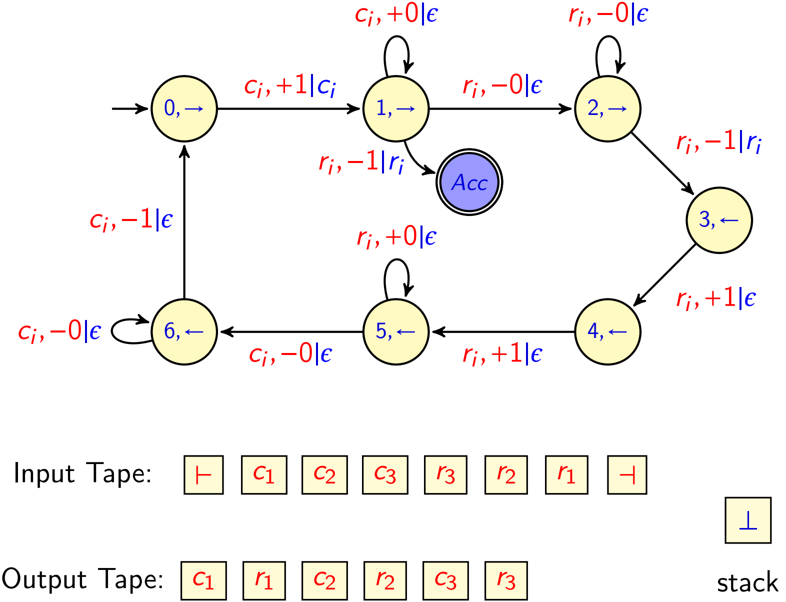
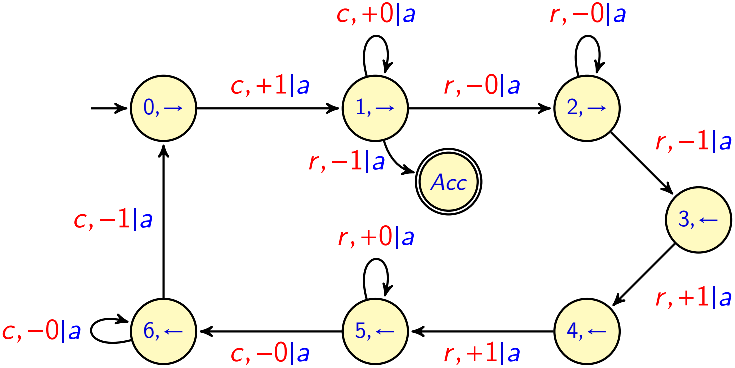


Output Growth

D2VPT are not linear size increase ($\nexists k \in \mathbb{N}, \forall w \in \Sigma_{wn}^*, |[\![A]\!](w)| \le k * |w|$).



$$|T(c^n r^n)| = O(n^2)$$

Single-use Property

Single-use restriction

A D2VPT is *single-use* (D2VPT$_{su}$) if in any accepting run, any producing transition occur at most once at a given position.

Decision

Given a D2VPT, deciding if it satisfies the single-use property is Exptime-complete.

Implication

D2VPT$_{su}$ are linear size increase.

Some Results [Dartois, Filiot, Reynier, T. 16]

Expressiveness

- D2VPT_{su} are as expressive as MSO[nw2w].
- Order-preserving MSO[nw2w] is equivalent to fVPT.

Decision results

- Equivalence problem for D2VPT is decidable (reduction to DTOP2S^{LA} equiv),
- Type-checking problem against a regular language is Exptime-complete.

Some Questions for $\mathrm{D2VPT_{su}}$

- functionality/k-valuedness problem
- well-nested output (when structured output alphabet) $[\![A]\!] \subseteq \Sigma_{wn}^*$

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Several questions but a single tool :

Two-way Visibly Pushdown Parikh Automata (2VPPA)

Outline

2 - (Two-way) Visibly Parikh Automata (2VPPA)

Parikh Automata

Definition

A Parikh automaton $P = (A, dim, \lambda, S)$ where $A = (\Sigma, Q, I, F, \Delta)$ is an NFA, dim is a natural number, $\lambda : \Delta \mapsto \mathbb{N}^{dim}$ and S is a semi-linear subset of $\mathbb{N}^{dim}$.

Here, S as a Presburger formula with dim free variables $x_1, \ldots, x_{dim}$

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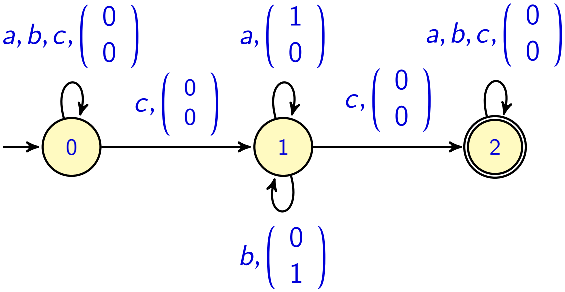

$$\varphi_S(x_1, x_2) = \ (x_1 = 2 * x_2 + 1)$$

Words containing a factor cwc with $w \in (a+b)^*$ and $|w|_a = 2|w|_b + 1$

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Here, S as a Presburger formula with dim free variables $x_1, \ldots, x_{dim}$

Equi-expressive to (non-deterministic) reversal-bounded counter machines [Ibarra] (weaker in the deterministic case).

In Parikh automata, counter values (and thus, updates) do not influence the control state evolution.

Two-way Visibly Pushdown ParikhParikh Automata

Definition

A two-way Visibly Pushdown Parikh automaton $P = (A, dim, \lambda, S)$ where $A = (\Sigma, Q, I, F, \Gamma, \Delta)$ is an 2VPA, dim is a natural number, $\lambda : \Delta \mapsto \mathbb{N}^{dim}$ and S is a semi-linear subset of $\mathbb{N}^{dim}$.

Two-way Visibly Pushdown ParikhParikh Automata

Definition

A two-way Visibly Pushdown Parikh automaton $P = (A, dim, \lambda, S)$ where $A = (\Sigma, Q, I, F, \Gamma, \Delta)$ is an 2VPA, dim is a natural number, $\lambda : \Delta \mapsto \mathbb{N}^{dim}$ and S is a semi-linear subset of $\mathbb{N}^{dim}$.

Reducing (D)2VPT problems to emptiness of 2VPPA

Well-nestedness of (D)2VPT

Globally as many calls as returns. Locally always more calls then returns.

$dim = 4$

$$q_1 \xrightarrow{\alpha,\pm\gamma|w} q_2 \ : \quad \begin{cases} q_1 \xrightarrow{\alpha,\pm\gamma,(|w|_{\Sigma_c},|w|_{\Sigma_r},|w|_{\Sigma_c},|w|_{\Sigma_r})} q_2 \\ q_1 \xrightarrow{\alpha,\pm\gamma,(|w|_{\Sigma_c},|w|_{\Sigma_r},|w|_{\Sigma_c},|w|_{\Sigma_r})} q_2' \\ q_1' \xrightarrow{\alpha,\pm\gamma,(0,0,|w|_{\Sigma_c},|w|_{\Sigma_r})} q_2' \end{cases}$$

$$\varphi = \ x_3 \neq x_4 \lor x_1 < x_2$$

Not well-nested $\Leftrightarrow$ Not empty

Functionality of 2VPT

On the same input word, there exists an output position where two different letters are output in two different computations (essentially).

$dim = 2$

- Compute a position in the output

$$q_1 \xrightarrow{\alpha,\pm\gamma|w} q_2 \quad \Rightarrow q_1 \xrightarrow{\alpha,\pm\gamma,(|w|,0)} q_2$$

- Guess the (first half of the) mismatched

$$q_1 \xrightarrow{\alpha,\pm\gamma|w} q_2 \quad \Rightarrow q_1 \xrightarrow{\alpha,\pm\gamma,(|w_1|,0)} q_2^a \qquad w = w_1 a w_2$$

- Rewind the input word and start reading again

$$q_1 \xrightarrow{\alpha,\pm\gamma|w} q_2 \quad \Rightarrow q_1^a \xrightarrow{\alpha,\pm\gamma,(0,|w|)} q_2^a$$

- Guess the (second half of the) mismatched

$$q_1 \xrightarrow{\alpha,\pm\gamma|w} q_2 \quad \Rightarrow q_1^a \xrightarrow{\alpha,\pm\gamma,(0,|w_1|)} q_F \qquad w = w_1 b w_2 \ , \ a \neq b$$

$$\varphi = \ x_1 = x_2$$

Not functional ⇔ Not empty

Outline

3 - Emptiness for 2VPPA

Some Existing Results

Emptiness for :

- deterministic two-way counter machines with 2 counters and k (fixed) counter-reversals is undecidable

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- deterministic two-way counter machines with 2 counters and k (fixed) counter-reversals is undecidable
- (non-deterministic) two-way reversal-bounded counter machines with finite crossing is decidable
- (non-deterministic) two-way pushdown automata with finite crossing is undecidable

Some Bad News

Theorem

Emptiness for deterministic two-way visibly pushdown Parikh automata is undecidable.

Proof Idea :

Reduction of solvability of Diophantine Eq. : $P = Q$ with $P, Q \in \mathbb{N}[\mathscr{X}]$, eg $2xy + zz = 4x + 2xz + 6$

The automaton part encodes only of monomials; sums and equality test encoded in Presburger accepting formula, a dimension for each monomial : $2x_1 + x_2 = 4x_3 + 2x_4 + 6$.

Input nested words represent valuations of variables

$$\begin{array}{ll} \text{Encode}(x * y * 1, [x \mapsto 3, y \mapsto 2]) = & x_c c^3 \, \text{Encode}(y * 1, [x \mapsto 3, y \mapsto 2]) \, r^3 x_r \\ \text{Encode}(1, [x \mapsto 3, y \mapsto 2]) = & \mathbb{1} \end{array}$$

The automaton checks well-formedness

- visibly pushdown "regular" sequence of monomials
- one dimension for each variable occurence in P and Q : test via the Presburger accepting formula that there are valuated the same way.

and evaluates monomials via counters :

The dimension updated as

$$[\![\text{Encode}(x_c c^3 \, \text{Encode}(y * 1, [x \mapsto 3, y \mapsto 2]) \, r^3 x_r)]\!] = 3 * [\![\text{Encode}(y * 1, [x \mapsto 3, y \mapsto 2])]\!]$$

Some (Modest) Good News

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Upperbound : let V be the values used for update in the PPA A

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- Step 1 : Define a NPA B accepting words of the form $\alpha_1 u^1_0 u^2_1 \alpha_2 u^1_3 u^2_2 \alpha_3 u^1_0 u^2_0 \alpha_4 u^1_0 u^2_1 \alpha_5 \ldots$ such that
  - $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \ldots$ has a run in A from an initial to a final configuration
  - each $u^1_{v_1} u^2_{v_2}$ corresponds to the update performed by the run at that position : $(v_1, v_2) \Rightarrow u^1_{v_1} u^2_{v_2}$
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- Step 3 : In NP [Scarpellini84], satisfiability test of

$$\psi((y_\sigma)_{\sigma \in \Sigma'}) \wedge \phi(x_1, \ldots, x_n) \wedge \bigwedge_{1 \leq i \leq dim} x_i = \sum_{v \in V} v.y_{u_v^i}$$

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Lower bound : (hardness does not rely on the acceptance set, eg Presburger satisfiability).

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Reduction of the 2-partition problem :

- I a finite set of natural numbers represented in binary.
- Does there exists $J \subseteq I$ such that

$$\sum_{e \in J} e = \sum_{e \notin J} e$$

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One dimension for J and one for $I \setminus J$.

Input words : $T_1 v_1 L_2 v_2 T_3 v_3 T_4 v_4 L_5 v_5 \ldots$

- v_k : representation of the unary encoding of the kth natural.
- T_i : take the ith natural (in J) - L_j : leave the jth natural

v_k is encoded as $u_n u_{n-1} \ldots u_1 u_0$ where

- $u_j = \epsilon$ if the jth bit of v_k is 0 and w_j otherwise
- recursively, $w_0 = 1$ and $w_l = c_l w_{l-1} r_l c_l w_{l-1} r_l$

and can be recognized by a small VPA.

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and can be recognized by a small VPA.

Thus, holds even if the automata are deterministic, with a fixed dimension 2, tuples of values in $\{0,1\}^2$ and with a fixed Presburger formula $(x_1, x_2) = \ x_1 = x_2$

Emptiness for single-use 2VPPA

Single-use 2VPPA : a position is visited only once per counter modifying transition.

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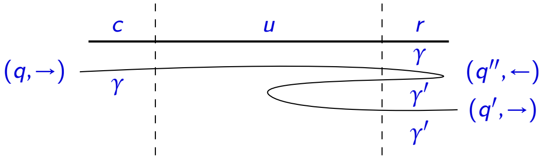
Proof Idea

- Extends [Dartois, Filiot, Reynier, T. 16] following Sherpherson's ideas (for FSA) on traversals T.

$((q,d)(q',d')) \in T(u)$ iff some run enters u by (q,d) and leaves it by (q',d')



Emptiness for single-use 2VPPA

Single-use 2VPPA : a position is visited only once per counter modifying transition.

Proposition

For any single-use 2VPPA, there exists an equivalent VPPA of at most exponential size.

Proof Idea

- Extends [Dartois, Filiot, Reynier, T. 16] following Sherpherson's ideas (for FSA) on traversals T.

  $((q,d)(q',d')) \in T(u)$ iff some run enters u by (q,d) and leaves it by (q',d')

- Productive and Non-productive traversals form a finite algebra.
- Consider possible decompositions of traversals $((q,\rightarrow)(q',\leftarrow))$ on $c_1 r w_2$ into productive traversals (finitely many as single-use) reachable from each other by non-productive traversals.
- Using commutativity of addition over $\mathbb{N}^{dim}$, extract a VPA and a mapping λ

Emptiness for single-use 2VPPA

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For any single-use 2VPPA, there exists an equivalent VPPA of at most exponential size.

Corollary

Emptiness for single-use 2VPPA is in NEXPtime.

Proposition

Emptiness for single-use 2VPPA is in NEXPtime-hard.

Implied Results

For well-nestedness, functionality and k-valuedness, single-use 2VPT yields single-use 2VPPA

Proposition

For single-use 2VPT, well-nestedness, functionality and k-valuedness are in NEXPtime.

Both known as EXPtime-hard.

Thank You !