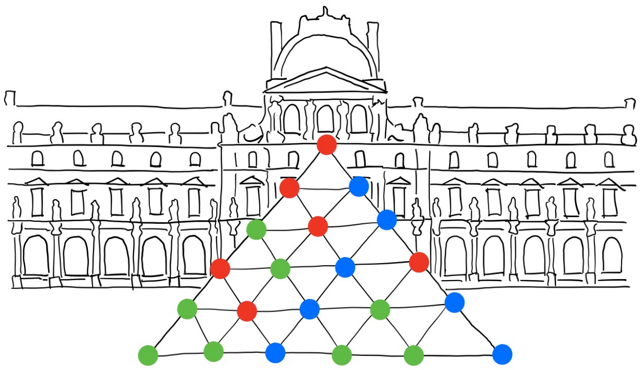

Workshop on Topological Combinatorics

Paris, 17–21 June 2024

Monday 17 June

Critical subgraphs of Schrijver graphs for the fractional chromatic number

Gábor Simonyi

Schrijver graphs are vertex-color-critical subgraphs of Kneser graphs having the same chromatic number. They also share the value of their fractional chromatic number but (just like Kneser graphs) Schrijver graphs are not critical for that. We present an induced subgraph of every Schrijver graph that is vertex-critical with respect to the fractional chromatic number. These subgraphs turn out to be isomorphic with certain rational (or circular) complete graphs.

Joint work with Anna Gujgiczer.

Shellability of balls is NP-complete

Pavel Paták

A d-dimensional simplicial complex is shellable, if it can be obtained as follows: one starts with a d-dimensional simplex and in each step, one attaches a new d-dimensional simplex to the complex by gluing it along its $(d-1)$-dimensional faces.

This procedure allows to control the topology of the resulting complex at each step; and underlies the proofs of Dehn–Sommerville relations or Upper and Lower bound theorems for simplicial polytopes.

In this talk, we show that it is NP-complete to decide whether a given triangulation of the 3D ball (or a higher dimensional ball) is shellable, and along the way also prove that collapsibility in 3D is NP-hard.

(Joint work with M. Tancer)

Holey Vietoris–Rips complex, Batman!

Chris Wells

Given a metric space and a positive number d, the Vietoris–Rips (VR) complex of scale d is the simplicial complex whose faces are all sets of diameter at most d. Recently, there's been a push to understand the VR complex of the Hamming cube. In this informal talk, we'll discuss (and give some partial and mostly unsatisfying answers to) the following questions:

- How many holes (non-trivial homologies) are there?
- How big is the largest facet?
- How small is the smallest facet?
- How many differently-sized facets are there?

Based on joint work with Joe Briggs and Ziqin Feng.

Simpler algorithmically unrecognizable 4-manifolds

Martin Tancer

Markov proved that there exists an unrecognizable 4-manifold, that is, a 4-manifold for which the homeomorphism problem is undecidable. Here we consider the question how close we can get to S^4 with an unrecognizable manifold. One of our achievements is that we show a way to remove so-called Markov's trick from the proof of existence of such a manifold. This trick contributes to the complexity of the resulting manifold. We also show how to decrease the deficiency (or the number of relations) in so-called Adian-Rabin set which is another ingredient that contributes to the complexity of the resulting manifold. Altogether, our approach allows to show that the connected sum $\#_9(S^2 \times S^2)$ is unrecognizable while the previous best result is the unrecognizability of $\#_1 2(S^2 \times S^2)$ due to Gordon.

During the talk, my aim will be to explain the main ideas how to obtain results of this type (skipping many technical details). In particular, I will emphasize combinatorial aspects of the proof.

MacPherson's conjecture on combinatorial Grassmannians, a new beginning

Pavle Blagojević

In 1991 Robert MacPherson, building on joint work with Gelfand, introduced combinatorial analogues of real vector bundles and identified the associated classifying spaces — nowadays called MacPhersonians. Additionally, he conjectured that the poset of all oriented matroids of fixed rank, the MacPhersonian, coincides with the corresponding real Grassmannian at least up to a homotopy — the MacPherson's conjecture.

The history of study of the MacPherson's conjecture is controversial, featuring an episode in 2003 with the Annals publication of Daniel Biss, where he claimed to settle the conjecture in full generality. Nikolai Mnëv, in his 2007 arXiv posting, pointed to an elementary but fatal error in Biss's proof, further mystifying the topology of MacPhersonians.

In this talk we will show how MacPherson's discretisation map can introduce a diagram of spaces over the fixed MacPhersonian whose colimit is exactly the corresponding real Grassmannian. Using the new construction we derive a spectral sequence converging to the (co)homology of the real Grassmannian, with the second page given as the (co)homology of the MacPhersonian with the coefficients in (co)homology of the diagram. If the diagram would have contractible entries our result would immediately settle MacPherson's conjecture. In particular, we recover all the known positive results on the MacPherson's conjecture in a uniform way.

(The lecture is based on the joint work with Bálint Zsigri)

Tuesday 18 June

The minimax property in infinite two-person win-lose games

Ron Holzman

We explore a version of the minimax theorem for two-person win-lose games with infinitely many pure strategies. In the countable case, we give a combinatorial condition on the game which implies the minimax property. In the general case, we prove that a game satisfies the minimax property along with all its subgames if and only if none of its subgames is isomorphic to the "larger number game." This generalizes a recent theorem of Hanneke, Livni and Moran. A key step in the proofs engages the topology of weak convergence of probability measures.

Flag complexes and Pfaffians

Ernest Chong

Let Δ be a flag complex, and let G be its underlying graph (i.e. Δ is the clique complex of G). In this talk, I will present how the f-vector of Δ can be computed in terms of the Pfaffians and generalized Pfaffian-characteristic polynomials of some skew-symmetric matrices constructed from G, thereby relating the f-vector of Δ to weighted perfect matchings of certain graphs associated to G.

From pre-cubical sets to chromatic simplicial complexes

Jérémy Ledent

This talk will explore the relationship between two combinatorial structures used to model the behaviour of concurrent programs. On the one hand, pre-cubical sets provide a model of directed spaces, which describes how a concurrent program evolves through time. In this approach, an execution of a program corresponds to a (directed) path in that space. I will describe two different combinatorial descriptions of paths in a pre-cubical set: ST-sequences and interval orders. Finally, I will explain how, by restricting to a special class of paths in an $(n+1)$-dimensional cube, we can recover another structure that arises in distributed computing: the chromatic subdivision of an n-dimensional simplex. The main focus of the talk will be the interplay between these abstract combinatorial structures; in particular I will not assume familiarity with distributed computing or concurrency theory.

Fáry's theorem in higher dimensions

Zuzana Patáková

We show that if a finite simplicial complex X can be piece-wise linearly embedded into a d-dimensional PL manifold M, then there is a triangulation of M containing X as a subcomplex. This can be seen as a higher-dimensional version of Fáry's theorem, which says that any simple planar graph can be drawn without crossings in the plane so that its edges are straight line segments.

Joint work with Karim Adiprasito.

Widely colorable graphs and their multichromatic numbers

Anna Gujgiczer

An s-wide coloring of a graph is a proper coloring where not just the color classes, but the first, second, ..., $(s-1)^{th}$ neighborhoods of any color class are also independent sets. Those graphs that admit an optimal s-wide coloring turned out to play a key role in constructing counterexamples to Hedetniemi's conjecture. In some of these counterexamples knowing the multichromatic number of these graphs became a relevant question.

This talk is mainly based on a joint work with Gábor Simonyi, where we answer this question.

Bisecting measures with hyperplane arrangements

Alfredo Hubard

I will discuss generalizations of the ham sandwich theorem in which a number of measures are bisected by an arrangement of hyperplanes.

Based on joint works with Roman Karasev and Pablo Soberón.

Wednesday 19 June

Degree criteria and stability for independent transversals

Penny Haxell

An *independent transversal* (IT) in a graph G with a given vertex partition $\mathcal{P}$ is an independent set of vertices of G (i.e. it induces no edges), that consists of one vertex from each part (block) of $\mathcal{P}$. Over the years, various criteria have been established that guarantee the existence of an IT, often given in terms of $\mathcal{P}$ being t-*thick*, meaning all blocks have size at least t. One such result, obtained recently by Wanless and Wood, is based on $b(G, \mathcal{P})$, the maximum of the degrees averaged over the blocks of $\mathcal{P}$. They proved that if $b(G, \mathcal{P})$ is at most $t/4$ then an IT exists. Resolving a problem posed by Groenland, Kaiser, Treffers and Wales (who showed that the ratio $1/4$ is best possible), we give a best possible criterion for the existence of an IT in terms of $b(G, \mathcal{P})$ and the maximum degree of G. This result interpolates between the criterion $b(G, \mathcal{P})$ being at most $t/4$ and the old and frequently applied theorem that if G has maximum degree at most $t/2$ then an IT exists. We also extend a theorem of Aharoni, Holzman, Howard and Sprüssel, by giving a stability version of their result. Our proofs make use of another previously known criterion for the existence of IT's that involves the topological connectedness of the independence complex of graphs.

This talk is based on joint work with R. Wdowinski.

Constructions of geometric hypergraphs with high chromatic number and transversal ratio

Seunghun Lee

Given a hypergraph $H = (V, E)$, we say that H is *(weakly) m-colorable* if there is a coloring $c : V \to [m]$ such that every hyperedge of H is not monochromatic. The *(weak) chromatic number* of H, denoted by $\chi(H)$, is the smallest m such that H is m-colorable. A vertex subset $T \subseteq V$ is called a *transversal* of H if for every hyperedge e of H we have $T \cap e \neq \emptyset$. The *transversal number* of H, denoted by $\tau(H)$, is the smallest size of a transversal in H. The *transversal ratio* of H is the quantity $\tau(H)/|V|$ which is between 0 and 1. Since a lower bound on the transversal ratio of H gives a lower bound on $\chi(H)$, these two quantities are closely related to each other.

We present constructions of geometric hypergraphs with high chromatic number and(or) transveral ratio. The ultimate conjecture on this line asks for a construction of d-polytopes for every $d \geq 4$ such that the supremum among the transversal ratios of the facet hypergraphs of those d-polytopes is eqaul to 1. As intermediate steps towards the conjecture, we will consider constructions regarding transversals and colorings coming from various types of simplicial complexes - neighborly spheres, simplicial complexes which are piecewise-linearly (or geometrically) embeddable in $\mathbb{R}^d$ and so on.

This presentation is based on two joint works; one with Joseph Briggs and Michael Gene Dobbins, and the other with Eran Nevo.

Colorful Borsuk–Ulam Theorems

Zoe Wellner

The classical Borsuk–Ulam theorem states that for any continuous map from the sphere to Euclidean space, $f\colon S^d \to R^d$, there is a pair of antipodal points that are identified, so $f(x) = f(-x)$. We prove a colorful generalization of the Borsuk–Ulam theorem. The classical result has many applications and consequences for combinatorics and discrete geometry and we in turn prove colorful generalizations of these consequences such as the colorful ham sandwich theorem, which allows us to prove a recent result of Bárány, Hubard, and Jerónimo on well-separated measures as a special case, and extend that case to the chessboard partition setting. The consequences on Brouwer's fixed point theorem additionally allow us to prove an alternative between KKM-covering results and Radon partition results.

This is joint work with Florian Frick.

Flag-no-square 4-manifolds

Eran Nevo

A simplicial complex is flag-no-square (fns) if it is the clique complex of a graph with no induced cycles of length four.

Which 4-manifolds admit a fns triangulation? We introduce the "star-connected-sum" operation on such triangulations, which preserves the fns property, from which we derive new constructions of fns 4-manifolds. In particular, we show the following: (i) there exist non-aspherical fns 4-manifolds, answering in the negative a question by Przytycki and Swiatkowski; (ii) for every large enough integer k there exists a fns 4-manifold M_{2k} of Euler characteristic $2k$, and further, (iii) M_{2k} admits a super-exponential number (in k) of fns triangulations.

Joint work with Daniel Kalmanovich and Gangotryi Sorcar.

Thursday 20 June

The Bárány-Kalai conjecture for certain families of polytopes

Shira Zerbib

Bárány and Kalai conjectured the following generalization of Tverberg's theorem: if f is a linear function from an m-dimensional polytope P to $\mathbb{R}^d$ and $m \geq (d+1)(r-1)$, then there are r pairwise disjoint faces of P whose images have a point in common. We show that the conjecture holds for cross polytopes, cyclic polytopes, and more generally for $(d+1)$-neighborly polytopes. Moreover, we show that for cross polytopes, the conjecture holds if the map f is assumed to be continuous (but not necessarily linear), and we give a lower bound on the number of sets of r pairwise disjoint faces whose images under f intersect. We also show that the conjecture holds for all polytopes when $d = 1$ and f is assumed to be continuous. Finally, when r is prime or large enough with respect to d, we prove that there exists a constant $c = c(d, r)$, depending only on d and r, such that the conjecture holds (with continuous functions) for the polytope obtained by taking c subdivisions of P. Joint with Pablo Soberón.

Transversal numbers of stacked spheres

Minho Cho

An innocent-looking but difficult problem is determining the transversal number of facets of a polytope. We investigate this problem for stacked polytopes which have the fewest possible facets with a given number of vertices.

The collection of facets of a $(d+1)$-dimensional stacked polytope is called a stacked d-sphere. A stacked sphere S is called linear if the polytope is constructed by taking cones over faces added in the previous step. The transversal ratio of S is the minimum proportion of vertices needed to cover all facets.

We construct linear stacked d-spheres with transversal ratio $\frac{6}{3d+8}$ and general stacked d-spheres with transversal ratio $\frac{2d+3}{(d+2)^2}$. Also we show that $\frac{6}{3d+8}$ is optimal for linear stacked 2-spheres, that is, the transversal ratio is at most $\frac{3}{7} + o(1)$ for linear stacked 2-spheres.

This is joint work with Jinha Kim.

About Gromov's ideas regarding the problem of convex bodies all whose sections or projection are linearly equivalent

Luis Montejano

Let $B \subset \mathbb{R}^N$ be a symmetric convex body, all of whose sections by n-dimensional linear subspaces, for some fixed integer n, $1 < n < N$, are linearly equivalent. Is it true that B is an ellipsoid?

The problem equivalent to the Banach Conjecture and is still open in general. In 1967, Gromov proved, using algebraic topology, that the answer is yes if $n > 1$ is even or if $N \geq n+2$. Montejano, et. al. proved in 2019 that the answer is yes if $n = 4k+1$. Gromov's ideas for the proof of the case $N \geq n+2$ are not yet well understood. He used the Stiefel Manifold of 2-frames in $\mathbb{R}^n + 2$. The purpose of this talk is to explain these ideas and adapt them to the solution of the following problem

Let $B \subset \mathbb{R}^N$ be a symmetric convex body, all of whose projection onto n-dimensional linear subspaces, for some fixed integer n, $1 < n < N$, are linearly equivalent. Is it true that B is an ellipsoid? In addition, we will also talk about the case $n = 4k+1$ of this last problem.

A canonical tree decomposition for order types

Xavier Goaoc

We introduce and study a notion of decomposition of planar point sets (or rather of their chirotopes) as trees decorated by smaller chirotopes. This decomposition is based on the concept of mutually avoiding sets and adapts in some sense the modular decomposition of graphs in the world of chirotopes. The associated tree always exists and is unique up to some appropriate constraints. We also show how to compute the number of triangulations of a chirotope efficiently, starting from its tree and the (weighted) numbers of triangulations of its parts

This is joint work with Mathilde Bouvel, Valentin Féray and Florent Koechlin (arXiv:2403.10311)

Stacks do not beat queues: an application of Topological Overlap Theorem

Michał Seweryn

Stack number and queue number are two graph invariants inspired by data structures. In 1992, Heath, Leighton and Rosenberg asked whether the stack number is bounded by a function of the queue number. This question has been answered negatively by Dujmović et al. in 2021. In my talk I will present another construction of graphs with bounded queue number and unbounded stack number. The key tool in the proof of the lower bound on the stack number is the Topological Overlap Theorem by Gromov. Joint work with David Eppstein, Robert Hickingbotham, Laura Merker, Sergey Norin, and David R. Wood.

Friday 21 June

Rainbow odd cycles

Zilin Jiang

The classical colorful Carathéodory theorem says that for every family of $d+1$ point sets in the d-dimensional Euclidean space, if each point set contains 0 in its convex hull, then there is a rainbow set with the same property. Here, if we drop the critical cardinality of the family to d, generically, such a rainbow set does not exist. However we start to see that rainbow problems in more combinatorial settings present stability. For example, Drisko's theorem on matchings in a bipartite graph still holds for $2n-2$ matchings of size n unless their union is a cycle of length $2n$. In this talk, I will illustrate a similar phenomenon for cycles with or without parity constraints on their lengths. Joint work with Ron Aharoni, Joseph Briggs and Ron Holzman.

Shannon capacity, Lovász theta number and Mycielski construction

Bence Csonka

We investigate the effect of the well-known Mycielski construction on the Shannon capacity of graphs and on one of its most prominent upper bounds, the (complementary) Lovász theta number. We prove that if the Shannon capacity of a graph, the distinguishability graph of a noisy channel, is attained by some finite power, then its Mycielskian has strictly larger Shannon capacity than the graph itself. For the complementary Lovász theta function we show that its value on the Mycielskian of a graph is completely determined by its value on the original graph, a phenomenon similar to the one discovered for the fractional chromatic number by Larsen, Propp and Ullman. We also consider the possibility of generalizing our results on the Sperner capacity of directed graphs and on the generalized Mycielski construction. Possible connections with what Zuiddam calls the asymptotic spectrum of graphs are discussed as well.

Criticality in Sperner's lemma

Tomáš Kaiser

Sperner's lemma states that if a labelling of the vertices of a triangulation K of the d-simplex Δ^d with labels $1, 2, \ldots, d+1$ has the property that (i) each vertex of Δ^d receives a distinct label, and (ii) any vertex lying in a face of Δ^d has the same label as one of the vertices of that face, then there exists a rainbow facet (a facet whose vertices have pairwise distinct labels).

Tibor Gallai asked in 1969 whether Sperner's Lemma is 'critical' in the sense that for every triangulation K as above and every facet σ of K, there is a labelling satisfying (i) and (ii) such that σ is the unique rainbow facet. (The question is included as Problem 9.14 in Jensen and Toft's collection *Graph Coloring Problems*.)

In this talk, we show that the answer is affirmative for $d \leq 2$ (as already proved by Gallai). For every $d \geq 3$, however, we answer Gallai's question in the negative by constructing an infinite family of examples where no labelling with the requested property exists. The construction is based on the properties of a convex 4-polytope which had been used earlier to disprove a claim of Theodore Motzkin on neighbourly polytopes.

Joint work with Matěj Stehlík and Riste Škrekovski.