A circular version of Gödel's T and its abstraction complexity

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Outline

1. Cyclic proofs: a Curry-Howard perspective
2. A circular version of Gödel's T
3. From models to interpretations
4. Conclusions

Motivating example: circular typing for Ackermann-Péter

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Consider the functions $I : (N \to N) \to N \to N$ and $A : N \to N \to N$ given by:

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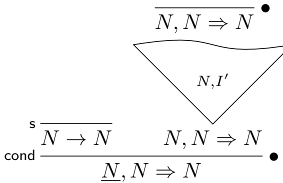

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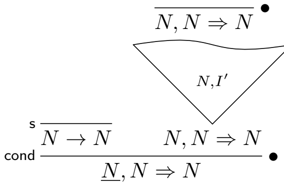

- Apparently non-wellfounded.
- *Why* is the function well-defined?

Landscape of cyclic proof theory

There are now several distinct communities studying non-wellfounded reasoning. *Some* of these include:

Algebra / Type systems	**Modal logic**	**Predicate logic**
Linear logic + μ, ν	μ-calculus	FOL + ind. dfns.
Kleene Algebra + $\cap, \backslash, /$	PDL & Game logic	Arithmetic

NB: formula expressivity *increases* left-to-right.

Some references:

- *Algebra and type systems*: [Santocanale '02], [Fortier & Santocanale '13], [Baelde, Doumane & Saurin '16], [D. & Pous '17, '18], [Kuperberg, Pinault & Pous '21].
- *Modal logics*: [Niwinski & Walukiewicz '96], [Afshari & Leigh '17], [Enqvist, Hansen, Kupke, Marti & Venema '19].
- *Predicate logic*: [Brotherston & Simpson '07], [Simpson '17], [Berardi & Tatsuta '17], [D. '20].

The Brotherston-Simpson conjecture

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The situation in arithmetic is now well-understood:

Theorem (Simpson '11)

Cyclic Arithmetic (CA) *is equivalent to Peano Arithmetic* (PA)*.*

Theorem (D. '20)

$I\Sigma_{n+1}$ *and* $C\Sigma_n$ *prove the same* Π_{n+1} *theorems.*

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What about type theories?

Outline

Church's simple type theory

Finite types:

$$\sigma, \tau \quad ::= \quad N \quad | \quad (\sigma \to \tau)$$

- **Language:** set of typed constants (always including equality $=_\sigma$ at all σ).
- **Terms:** formed by typed application.
- **Theory:** set of axioms and rules (always including intensional equality).

Example (Combinatory Algebra)

Language:

$$\begin{array}{lcl} K_{\sigma\tau} & : & \sigma \to \tau \to \sigma \\ S_{\rho\sigma\tau} & : & (\rho \to \sigma \to \tau) \to (\rho \to \sigma) \to \rho \to \tau \end{array}$$

Theory:

$$\begin{array}{rcl} K\,x\,y & = & x \\ S\,x\,y\,z & = & x\,z\,(y\,z) \end{array}$$

Standard model $\mathfrak{N}$:

$$\begin{array}{rcl} N^{\mathfrak{N}} & := & \mathbb{N} \\ (\sigma \to \tau)^{\mathfrak{N}} & := & \{f : \sigma^{\mathfrak{N}} \to \tau^{\mathfrak{N}}\} \end{array}$$

Interpretations: take equational axioms as definitions left-to-right.

System T

T extends combinatory algebra by **recursion cominators**:

$$\mathsf{rec}_\sigma : \sigma \to (N \to \sigma \to \sigma) \to N \to \sigma$$

and (quantifier-free) axioms and rules:

$$\begin{array}{rcl} \mathsf{rec}\, f\, g\, \mathsf{0} & = & g \\ \mathsf{rec}\, f\, g\, \mathsf{s}x & = & g\, x\, (\mathsf{rec}\, f\, g\, x) \end{array} \qquad \begin{array}{c} \neg \mathsf{s}x = \mathsf{0} \\ \mathsf{s}x = \mathsf{s}y \supset x = y \end{array} \qquad {}_{ind}\,\frac{\varphi(\mathsf{0}) \quad \varphi(x) \supset \varphi(\mathsf{s}x)}{\varphi(t)}$$

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T is equiconsistent with Peano Arithmetic.

$\leadsto$ we can *trade off* quantifier complexity for abstraction complexity.

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Question

Can we interpret cyclic arithmetic (directly) in a circular version of T?

T-terms typed in sequent style

$$\text{ex}\ \frac{\vec{\rho}, \sigma, \rho, \vec{\sigma} \Rightarrow \tau}{\vec{\rho}, \rho, \sigma, \vec{\sigma} \Rightarrow \tau} \qquad \text{wk}\ \frac{\vec{\sigma} \Rightarrow \tau}{\vec{\sigma}, \sigma \Rightarrow \tau} \qquad \text{cntr}\ \frac{\vec{\sigma}, \sigma, \sigma \Rightarrow \tau}{\vec{\sigma}, \sigma \Rightarrow \tau} \qquad \text{cut}\ \frac{\vec{\sigma} \Rightarrow \sigma \quad \vec{\sigma}, \sigma \Rightarrow \tau}{\vec{\sigma} \Rightarrow \tau}$$

$$\text{id}\ \frac{}{\sigma \Rightarrow \sigma} \qquad \text{L}\ \frac{\vec{\sigma} \Rightarrow \rho \quad \vec{\sigma}, \sigma \Rightarrow \tau}{\vec{\sigma}, \rho \to \sigma \Rightarrow \tau} \qquad \text{R}\ \frac{\vec{\sigma}, \sigma \Rightarrow \tau}{\vec{\sigma} \Rightarrow \sigma \to \tau}$$

$$0\ \frac{}{\Rightarrow N} \qquad \text{s}\ \frac{}{N \Rightarrow N} \qquad \text{cond}\ \frac{\vec{\sigma} \Rightarrow \tau \quad \vec{\sigma}, N \Rightarrow \tau}{\vec{\sigma}, N \Rightarrow \tau} \qquad \text{rec}_\tau\ \frac{\vec{\sigma} \Rightarrow \tau \quad \vec{\sigma}, N, \sigma \Rightarrow \tau}{\vec{\sigma}, N \Rightarrow \tau}$$

- Each instance of a rule is construed as a constant.
- ....the map (derivations $\to$ terms) is continuous.

Equational axiomatisation

$$\begin{aligned} \mathsf{id}\ x &= x \\ \mathsf{ex}\ t\ \vec{x}\ x\ y\ \vec{y} &= t\ \vec{x}\ y\ x\ \vec{y} \\ \mathsf{wk}\ t\ \vec{x}\ x &= t\ \vec{x} \\ \mathsf{cntr}\ t\ \vec{x}\ x &= t\ \vec{x}\ x\ x \end{aligned}$$

$$\begin{aligned} \mathsf{cut}\ s\ t\ \vec{x} &= t\ \vec{x}\ (s\ \vec{x}) \\ \mathsf{L}\ s\ t\ \vec{x}\ y &= t\ \vec{x}\ (y\ (s\ \vec{x})) \\ \mathsf{R}\ t\ \vec{x}\ x &= t\ \vec{x}\ x \end{aligned}$$

$$\begin{aligned} \mathsf{rec}\ s\ t\ \vec{x}\ 0 &= s\ \vec{x} \\ \mathsf{rec}\ s\ t\ \vec{x}\ \mathsf{s}z &= t\ \vec{x}\ z\ (\mathsf{rec}\ s\ t\ \vec{x}\ z) \end{aligned}$$

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NB: gives interpretations of constants in $\mathfrak{N}$, using meta-level induction.

We can generalise term trees and derivation trees to non-wellfounded counterparts:

Definition

- **coterms** are generated coinductively from constants and variables.
- **coderivations** are generated coinductively from the rules.

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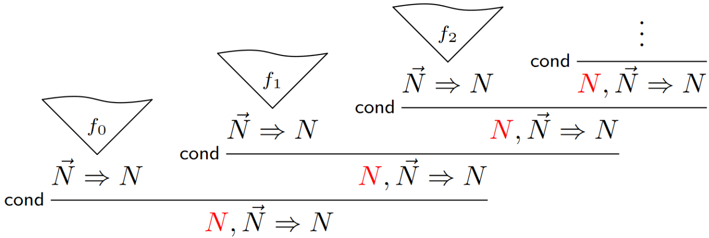
Semantics: Kleene-Herbrand-Gödel style partial functionals.

Example: type 1 completeness of coderivations

Let $f : \mathbb{N} \times \mathbb{N}^k \to \mathbb{N}$ and write $f_i(\vec{x}) := f(i, \vec{x})$.

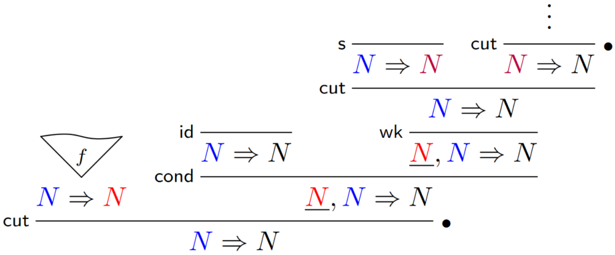


Example: Turing completeness of regular coderivations

Unbounded search $\mu x(f\,x = 0)$ is given by $H\,0$ with:

$$H\,x \;:=\; \mathsf{cond}\,(f\,x)\,x\,(H\,\mathsf{s}x)$$

H is computed by the following regular coderivation:



A totality criterion

σ^1 is an **immediate ancestor** of σ^2 if they are in the premiss and conclusion, resp., and have the 'same colour'.

Definition (Threads and progress)

- A **thread** is a maximal path in the graph of immediate ancestry.
- An N-thread is **progressing** if it is infinitely often principal for cond.
- A coderivation is **progressing** if each infinite branch has a progressing N-thread.

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Definition (Circular systems)

CT is the simple type theory that has a symbol for every progressing regular coderivation, and is axiomatised by all previous equations (over coterms).

- T_n is the restriction of T allowing only types of level n in typing derivations.
- CT_n is the restriction of CT allowing only types of level n in typing derivations.

Example: Ackermann-Péter

$$
\begin{aligned}
A(0, y) &:= y + 1 \\
A(x+1, 0) &:= A(x, 1) \\
A(x+1, y+1) &:= A(x, A(x+1, y))
\end{aligned}
$$

NB. Not representable in T_0!

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\begin{prooftree}
\AxiomC{}
\RightLabel{s}
\UnaryInfC{$N \Rightarrow N$}
\RightLabel{wk}
\UnaryInfC{$\underline{N}, N \Rightarrow N$}
\AxiomC{}
\LeftLabel{1}
\doubleLine
\UnaryInfC{$\Rightarrow N$}
\AxiomC{$\vdots(1)$}
\UnaryInfC{$N, N \Rightarrow N \bullet$}
\LeftLabel{cut}
\BinaryInfC{$N \Rightarrow N$}
\LeftLabel{wk}
\UnaryInfC{$N, \underline{N} \Rightarrow N$}
\AxiomC{$\vdots(2)$}
\UnaryInfC{$N, N \Rightarrow N \bullet$}
\AxiomC{$\vdots(3)$}
\UnaryInfC{$N, N \Rightarrow N \bullet$}
\LeftLabel{cut}
\BinaryInfC{$N, N, N \Rightarrow N$}
\LeftLabel{cond}
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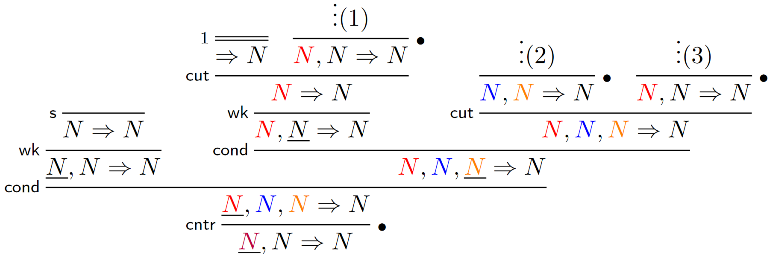

Question

What is the relative abstraction complexity of functionals in T and CT?

Proposition (Well-definedness)

A progressing coderivation computes a well-defined total functional.

Progress $\implies$ totality

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Proof sketch.

- Each rule preserves totality top-down, so preserves non-totality bottom-up.
- $\rightsquigarrow$ we may build a leftmost 'non-total' infinite branch.
- Assign to a progressing N-thread the least natural numbers witnessing non-totality of the corresponding coderivations.
- This sequence will be monotone decreasing but cannot converge. $\square$

Outline

Confluence of T

We may construe the equations of T and CT as a rewrite system:

$$
\begin{array}{rcl}
\mathsf{id}\ x & \rightsquigarrow & x \\
\mathsf{ex}\ t\ \vec{x}\ x\ y\ \vec{y} & \rightsquigarrow & t\ \vec{x}\ y\ x\ \vec{y} \\
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Write $\approx$ for reflexive symmetric transitive closure of $\rightsquigarrow$.

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Theorem (Confluence for CT, RCA_0)

If $s \rightsquigarrow^* t_0$ and $s \rightsquigarrow^* t_1$ then there is some t with $t_0 \rightsquigarrow^* t$ and $t_1 \rightsquigarrow^* t$.

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Theorem ($\mathsf{RCA}_0 + I\Sigma_{n+2}$)

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Corollary ($\mathsf{RCA}_0 + I\Sigma_{n+2}$)

t is weakly normalising wrt $\rightsquigarrow$.

Interpretation into T

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Theorem (Interpretation)

If $CT_n \vdash s = t$ *then* $T_{n+1} \vdash s \approx t$.

Corollary (Computation at type 1)

Any type 1 function representable in CT_n *is also representable in* T_{n+1}.

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T_{n+1} is interpreted into CT_n (over the level $n+1$ theory).

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T_{n+1} and CT_n are equiconsistent.

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By formalising termination of 'runs' along progressing coderivations in ACA_0, we recover recursion along progressing coderivations directly in T:

Theorem (Functional equivalence)

CT and T comptue the same functionals, at all types.

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