Transcendental Syntax

A toolbox for the interface logic-computation

LIPN – Université Sorbonne Paris Nord

Boris Eng

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Geometry of Interaction and Ludics (Girard). Reconstruction of logic from computation.

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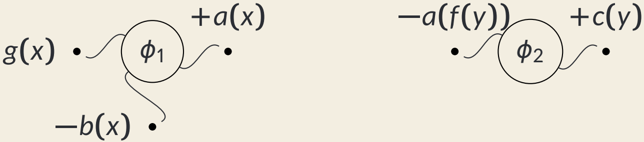
- types : formulas of linear logic and more.
- Speaks about the "logic" of a computational model.

Stellar Resolution

Girard's stars and constellations



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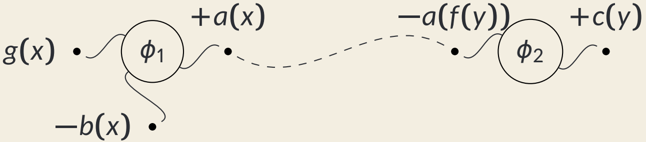

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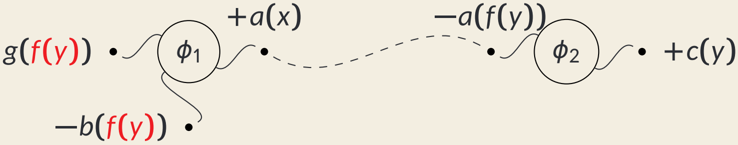

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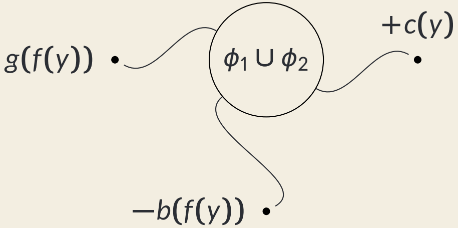

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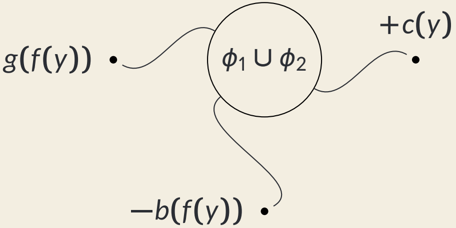

Constellation Φ (n stars)

= program

↓

Diagrams (maximal tilings)

↓

Constellation $\text{Ex}(\Phi)$

= normal form

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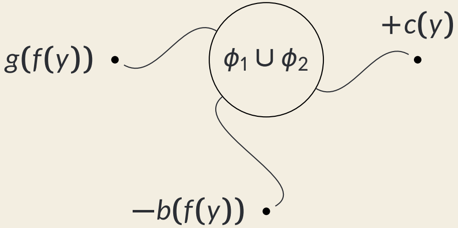

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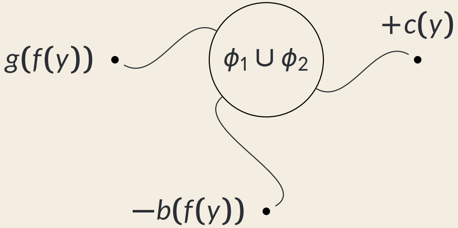

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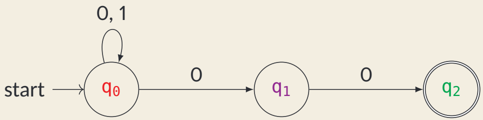
A reformulation of Robinson's first-order resolution / Query-free logic programming.

Stellar Resolution

Automata and circuits unified

Generalised automata.



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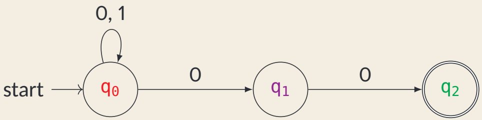

Generalised automata.

Transitions $\longleftrightarrow$ binary stars $[-a(c \cdot w, q), +a(w, q')]$.

Run on a word $\longleftrightarrow$ tiling/diagram.

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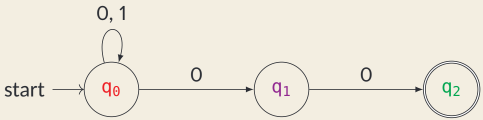

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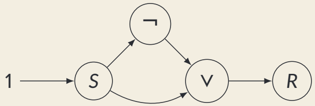

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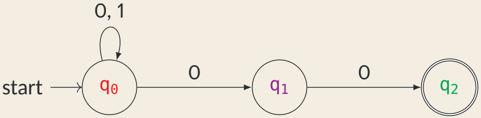

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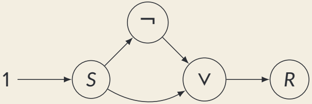

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Circuit evaluation $\longleftrightarrow$ execution of constellation.

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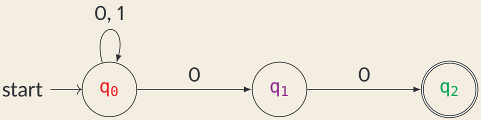

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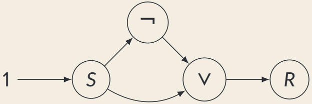

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Various models of linear logic + a logical description of a model of computation.

Vague ideas of applications

(Unit) testing in logic

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- Adequation : Φ is correct w.r.t. $A \Longrightarrow \Phi \in BH(A)$ with $BH(A) = BH(A)^{\perp\perp}$.

Atypic typing and complexity

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Descriptive complexity. Capture classes with formulas.

- **P** and **NP** as classes of formulas (Immerman, Fagin).

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Thank you for listening to my talk.