A concrete model of finitary/infinitary linear logic with fixed-points

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Overview

Non-uniform totality spaces

Finitary linear logic with fiexed points (μLL)

Infinitary linear logic with fixed-points (μLL_∞)

A denotational model of μLL_∞

Non-uniform totality spaces

Given a set E, and let us take $\mathcal{U} \subseteq \mathcal{P}(E)$.
We define:

$$\mathcal{U}^{\perp} = \{u' \subseteq E \mid \forall u \in \mathcal{U}(u \cap u' \neq \emptyset)\}$$

NUTS X: A pair $(|X|, \mathcal{T}X)$ such that

- $|X|$ is a set, (the web of X), and
- $(\mathcal{T}X)^{\perp\perp} = \mathcal{T}X$ (totality candidate) wehre $\mathcal{T}X \subseteq \mathcal{P}(|X|)$

Notation: $\mathrm{Tot}(X) = \{T \subseteq \mathcal{P}(X) \mid (T)^{\perp\perp} = T\}$

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An example: $N = (\mathbb{N}, U)$ where U is set of all infinite subsets of $\mathbb{N}$.

$$U^{\perp} = \{u \subseteq \mathbb{N} \mid \mathbb{N} \backslash u \text{ is finite}\}$$

Characterization of bi-orthogonality

Let $U \subseteq \mathcal{P}(E)$, then $(U)^{\perp\perp} = \uparrow U = \{v \subseteq E \mid \exists u \in U \ (u \subseteq v)\}$

Tensor product of two NUTS

Given two NUTS $A_i = (|A_i|, \mathcal{T}A_i)$
$A_1 \otimes A_2 := (|A_1 \otimes A_2|, \mathcal{T}(A_1 \otimes A_2))$ where

$$|A_1 \otimes A_2| = |A_1| \times |A_2|$$

$$\mathcal{T}(A_1 \otimes A_2) = \uparrow \{u_1 \otimes u_2 \mid u_i \in \mathcal{T}A_i\}$$

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Unit of tensor product:

$$1 = (\{*\}, \{\{*\}\})$$

Cartesian product of two NUTS

Given two NUTS $A_i = (|A_i|, \mathcal{T}A_i)$
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$$|A_1 \mathbin{\&} A_2| = \{1\} \times |A_1| \cup \{2\} \times |A_2|$$

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Unit of cartesian product:

$$T = (\emptyset, \{\emptyset\})$$

Exponentials

Given a NUTS $A = (|A|, \mathcal{T}A)$
$!A = (|!A|, \mathcal{T}(!A))$ where

$$|!A| = \mathcal{M}_{fin}(|A|)$$

$$\mathcal{T}(!A) = \uparrow \{\mathcal{M}_{fin}(u) \mid u \in \mathcal{T}A\}$$

Dual of a NUTS

Given a NUTS $A = (|A|, \mathcal{T}A)$
$A^{\perp} = (|A^{\perp}|, \mathcal{T}(A^{\perp}))$ where

$$|A^{\perp}| = |A|$$

$$\mathcal{T}(A^{\perp}) = (\mathcal{T}A)^{\perp}$$

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So, one can define:
dual of $\otimes$ "=" $\parr$
dual of & "=" $\oplus$

The category **NUTS**

Object: NUTS

Morphism:

$$\mathbf{NUTS}(A, B) = \{f \subseteq |A| \times |B| \mid \forall u \in \mathcal{T}A \ (f.u \in \mathcal{T}B)\}$$

where $f.u = \{y \in |B| \ \mid \ \exists x \in u \ (x, y) \in f\}$

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Example: $N = (\mathbb{N}, U)$ where

$$U = \{v \subseteq \mathbb{N} \mid u \neq \emptyset\}$$

Then

$$\begin{aligned}\mathbf{NUTS}(N,N) &= \{f \subseteq \mathbb{N} \times \mathbb{N} \mid \forall u \subseteq \mathbb{N}\ (u \neq \emptyset \Rightarrow f.u \neq \emptyset)\} \\ &= \{f \subseteq \mathbb{N} \times \mathbb{N} \mid \forall n\, \exists m \text{ s.t } (n,m) \in f\}\end{aligned}$$

Variable non-uniform totality spaces (VNUTS)

A VNUTS $\mathbb{E}$ is a pair $(|\mathbb{E}|, \mathcal{T}\mathbb{E})$ such that

- $|\mathbb{E}|$: **REL** $\to$ **REL** is a functor such that it is monotonic and continuous (both on objects and on morphisms)
  (**REL** is the category of sets and relations)
- $\mathcal{T}\mathbb{E}$ is an operation on **NUTS** such that

$$\mathcal{T}\mathbb{E}((|X|, \mathcal{T}X)) \in \mathsf{Tot}(|\mathbb{E}|(|X|))$$

- For any $f \in \mathbf{NUTS}(A, B)$,

$$|\mathbb{E}|(f) \in \mathbf{NUTS}((|\mathbb{E}|(|A|), \mathcal{T}\mathbb{E}(A)), (|\mathbb{E}|(|B|), \mathcal{T}\mathbb{E}(B)))$$

Fact: Any VNUTS induces a functor **NUTS** $\to$ **NUTS**.

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Fixed points of VNUTS

Let us say $\mathcal{E}$: **NUTS** $\to$ **NUTS** is the induced functor from $\mathrm{VNUTS}\ \mathbb{E} : \mathbb{E}$: **NUTS** $\to$ **NUTS**.
$\mu\mathbb{E} = (|\mu\mathbb{E}|, \mathcal{T}(\mu\mathbb{E}))$ where

$$\mu\mathbb{E} = \text{The initial algebra of the functor } \mathcal{E}$$

The existence of initial/final algebra is derived form a result in [1]

[1] M. Wand, Fixed-Point Constructions in Order-Enriched Categories.

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A denotational model of μLL_∞

Formulas and Inference rules of μLL based on [2]

$$A, B, \ldots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \parr B \mid ?A \mid !B \\ \mid X \mid \mu X.F \mid \nu X.F$$

$$(\mu X.F)^{\perp} = \nu X.(F^{\perp})$$

[2] D. Baelde, Least and Greatest Fixed Points in Linear Logic.

Inference rules of μLL are the one for LL plus

$$\frac{\vdash \Gamma, A[\mu X.A/X]}{\vdash \Gamma, \mu X.A}\ \mu \qquad \frac{\vdash ?\Gamma, B^{\perp}, A[B/X]}{\vdash ?\Gamma, B^{\perp}, \nu X.A}\ \nu - rec$$

An instance of $\nu - rec$ when $\nu X.A = \mathrm{nat}^{\perp}$ where $\mathrm{nat} = \mu X.(1 \oplus X)$:

$$\frac{\dfrac{\vdash ?\Gamma, B \qquad \vdash ?\Gamma, B, B^{\perp}}{\vdash ?\Gamma, B, \perp \,\&\, B^{\perp}}}{\vdash ?\Gamma, B, \mathrm{nat}^{\perp}}\ \nu - rec$$

The ? context of the $\nu - rec$ rule has not appeared in the original system [3].

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$\mu\mathrm{LL}_\infty$ syntax based on [4]

$$A, B, \ldots := 1 \mid 0 \mid \bot \mid \top \mid A \oplus B \mid A \otimes B \mid A \,\&\, B \mid A \parr B \mid ?A \mid !B$$
$$\mid X \mid \mu X.F \mid \nu X.F$$

A possibly infinite tree, generated by LL rules plus two following rules:

$$\frac{\vdash \Gamma, A[\mu X.A/X]}{\vdash \Gamma, \mu X.A}\,\mu \qquad \frac{\vdash \Gamma, A[\nu X.A/X]}{\vdash \Gamma, \nu X.A}\,\nu$$

Example: $\mathrm{nat} = \mu X.(1 \oplus X)$ $(\mathrm{nat}^\perp = \nu X.(\bot \,\&\, X))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash 1}{\vdash 1 \oplus \mathrm{nat}}\,\oplus}{\vdash \mathrm{nat}}\,\mu}{\vdash \bot, \mathrm{nat}}\,\bot \qquad \dfrac{\star}{\vdash \mathrm{nat}^\perp, \mathrm{nat}}}{\vdash \bot \,\&\, \mathrm{nat}^\perp, \mathrm{nat}}\,\&}{\star \;\; \vdash \mathrm{nat}^\perp, \mathrm{nat}}\,\nu$$

[4] David Baelde, Amina Doumane, Alexis Saurin: Infinitary Proof Theory: the Multiplicative Additive Case.

But...

$$\dfrac{\dfrac{\dfrac{\vdots}{\vdash \mu X.A}}{\vdash \mu X.A}\,\mu \qquad \dfrac{\dfrac{\vdots}{\vdash \nu X.A}}{\vdash \nu X.A}\,\nu}{\vdash}\,cut$$

Validity criteria based on [5]:

There is a validity criteria to distinguish valid proof from the ordinary ones.

[5] David Baelde, Amina Doumane, Alexis Saurin: Infinitary Proof Theory: the Multiplicative Additive Case.

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NUTS as a denotational model of μLL_∞

A formula $A(X) \mapsto$ A VNUTS $[\![A]\!]_X$: **NUTS** $\rightarrow$ **NUTS**.

Interpretation of proofs:
The interpretation of LL inference rules in **NUTS** is same as their interpretation in **REL**.

Let us take π as a possibly infinite proof in $\mu\mathrm{LL}_\infty$:
$[\![\pi]\!] =$ union of the interpretation of all finite approximation. (More elegant way in [6])

Theorem: If π and π' are μLL_∞ proofs of Γ and π reduces to π' by the cut-elimination rules of μLL_∞, then $[\![\pi]\!] = [\![\pi']\!]$.

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[6] Denis Kuperberg, Laureline Pinault, Damien Pous. Cyclic Proofs, System T, and the Power of Contraction

An example

A syntatic-free proof that any term of booleans has a defined boolean value true or false

Consider $1 \oplus 1$ (The type of booleans).
$[\![1 \oplus 1]\!] = (\{(1,\star),(2,\star)\}, \mathcal{T}[\![1 \oplus 1]\!])$ where

$$\mathcal{T}([\![1 \oplus 1]\!]) = \mathcal{P}(|[\![1 \oplus 1]\!]|) \backslash \emptyset$$

For any proof π of $1 \oplus 1$, we have $[\![\pi]\!] \in \mathcal{T}[\![1 \oplus 1]\!]$.
Hence $[\![\pi]\!] \neq \emptyset$.

Validity implies totality

Theorem: If π is a valid proof of the sequent $\vdash \Gamma$, then $[\![\pi]\!] \in \mathcal{T}[\![\Gamma]\!]$.

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The proof is similar to the proof of soundness of $LKID^\omega$ in [7].
However:
The system is classical logic with inductive definitions
and the proof is for a Tarskian semantic.
We need to adapt the proof in two aspects:
considering $\mu\mathrm{LL}_\infty$ instead of $LKID^\omega$,
and deal with the denotational semantic instead of Tarskian semantics.
Adapation for $\mu\mathrm{LL}_\infty$: somehow done in [8]
So, basically, the main point of this proof is adapting a Tarskian soundness theorem to a denotational semantic soundness.

[7] James Brotherston.Sequent Calculus Proof Systems for Inductive Def-initions. PhD thesis, University of Edinburgh, November 2006.

[8] Amina Doumane. On the infinitary proof theory of logics with fixedpoints. PhD thesis, Paris Diderot University, 2017.

Current and future work

- Working on a polarized calculus which corresponds μLL, and its categorical semantic.
- Categorical semantic of μLL_∞.
- Comparing the interpretation of proofs in different models such as coherence spaces, coherence spaces with totality, finiteness space, **NUTS**, **REL**,
- Connections between type theory with (co)inductive definitions and μLL.
  There are basically following styles for the fixpoints rules in the literature:
  - General fixpoint with the guarded conditions.
  - The elimination rule community.
  - The Park's rule (maybe the sequent calculus version of the elimination rule for the propositional part?).