Coherence by Normalization for Linear Multicategorical Structures

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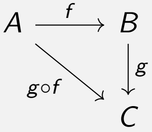
Outline

Lambek's Slogan

Categories are Deductive Systems.

Morphisms are proofs, composition is the *cut rule*:



$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ A \vdash B\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ B \vdash C\end{array}}{A \vdash B}$$

From Categories to Multicategories

Typed Calculi

Typed Terms

$$\begin{array}{c}\pi \\ \vdots \\ x_1 : A_1, \ldots, x_n : A_n \vdash M : B\end{array}$$

Substitution Operation

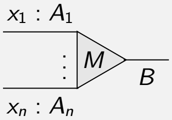
$$M \in \Lambda(A_1, \ldots, A_n; B), N_i \in \Lambda(\Gamma; A_i)$$

$$\mapsto$$

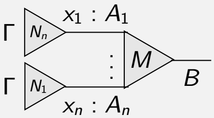
$$M\{N_1, \ldots, N_n / x_1, \ldots, x_n\} \in \Lambda(\Gamma, B)$$

Multicategories

Multimorphisms

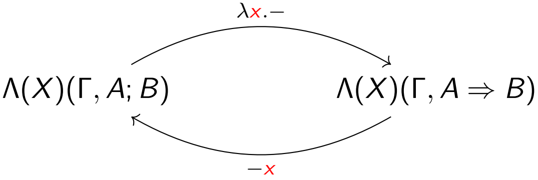


Composition



Calculi as Free Constructions

Free cartesian closed multicategories:



Inverse Maps

$$(\lambda x.M)x =_{\beta} M\{x/x\} = M \quad (\Gamma, x : A \vdash M : B)$$

$$M =_{\eta} \lambda x^{A}.(Mx) \quad (\Gamma \vdash M : A \Rightarrow B)$$

Triangular Equalities

$$\lambda y.(\lambda x.M)y =_{\beta\eta} \lambda x.M \quad (\Gamma, x : A \vdash M : B)$$

$$(\lambda x.(Mx))x =_{\beta\eta} Mx \quad (\Gamma, \vdash MA \Rightarrow B)$$

Calculi as Free Constructions

Free cartesian closed multicategories:

$$\Lambda(X)(\Gamma, A; B) \underset{-x}{\overset{\lambda x.-}{\rightleftarrows}} \Lambda(X)(\Gamma, A \Rightarrow B)$$

Free CCC

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & \Lambda(X) \\ & \searrow & \Big\downarrow {\scriptstyle [\![-]\!]} \\ & & \mathcal{C} \end{array}$$

Switching to Linear and Unbiased

We use the (rigid) *resource calculus* (ER08; MPV17).

$$\Lambda_r(X)(\gamma, a_1, \ldots, a_k; b) \xrightarrow{\lambda\langle x_1,\ldots,x_k\rangle.-} \Lambda_r(X)(\gamma, (a_1 \otimes \cdots \otimes a_k) \multimap b)$$

$$\Lambda_r(X)(\gamma, (a_1 \otimes \cdots \otimes a_k) \multimap b) \xrightarrow{-\langle x_1,\ldots,x_k\rangle} \Lambda_r(X)(\gamma, a_1, \ldots, a_k; b)$$

$$(\lambda x_1, \ldots, x_k.s)\langle t_1, \ldots, t_k\rangle \to_\beta s\{t_1, \ldots, t_k / x_1, \ldots, x_k\}$$

Substitution is Linear

$$(\lambda\langle x, y\rangle.x\langle y\rangle)\langle s, t\rangle \to s\langle t\rangle$$

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Coherence by Normalization

We shall consider (symmetric) *representable* and *closed* structures.

Adjoint POV

Structural adjunctions express an operational semantics.

Slogan

Structural morphisms are terms up to $\beta\eta$.

Theorem (General Coherence)

Two structural morphisms are equal iff their ***NF*** *is the same.*

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Representable Multicategories

Representable Maps

$$\mathsf{re}_{a_1,\ldots,a_k} : a_1, \ldots, a_k \to (a_1 \otimes \cdots \otimes a_k)$$

$$\mathrm{R}(\delta_1, (a_1 \otimes \cdots \otimes a_k), \delta_2; a) \xrightarrow{-\circ\langle \delta_1, \mathsf{re}, \delta_2 \rangle} \mathrm{R}(\delta_1, a_1, \ldots, a_k, \delta_2; a)$$

$$\mathrm{R}(\delta_1, a_1, \ldots, a_k, \delta_2; a) \xrightarrow{\mathsf{let}(-)} \mathrm{R}(\delta_1, (a_1 \otimes \cdots \otimes a_k), \delta_2; a)$$

Representable Multicategories

Representable Maps

$$x_1 : a_1, \dots, x_k : a_k \vdash \langle x_1, \dots, x_k \rangle : (a_1 \otimes \cdots \otimes a_k)$$

$$\mathrm{R}(\delta_1, (a_1 \otimes \cdots \otimes a_k), \delta_2; a) \xrightarrow{-\{\langle x_1,\dots,x_k\rangle / x\}} \mathrm{R}(\delta_1, a_1, \dots, a_k, \delta_2; a)$$

$$\mathrm{R}(\delta_1, (a_1 \otimes \cdots \otimes a_k), \delta_2; a) \xleftarrow{-[x_1^{a_1},\dots,x_k^{a_k} := x]} \mathrm{R}(\delta_1, a_1, \dots, a_k, \delta_2; a)$$

Syntax

Representable Terms

$$\frac{\gamma_1 \vdash s_1 : a_1 \ldots \gamma_k \vdash s_k : a_k}{\gamma_1, \ldots, \gamma_k \vdash \langle s_1, \ldots, s_k \rangle : (a_1 \otimes \cdots \otimes a_k)} \qquad \frac{}{x : a \vdash x : a}$$

$$\frac{\gamma \vdash s : (a_1 \otimes \cdots \otimes a_k) \qquad \delta, x_1 : a_1, \ldots, x_k : a_k, \delta' \vdash t : b}{\delta, \gamma, \delta' \vdash t[x_1^{a_1}, \ldots, x_k^{a_k} := s] : b}$$

Reduction: η and β

β and η

$$s[x_1, \ldots, x_k := \langle t_1, \ldots, t_k \rangle] \rightarrow_\beta s\{t_1, \ldots, t_k / x_1, \ldots, x_k\}$$

$$s \rightarrow_\eta \langle x_1, \ldots, x_k \rangle [x_1, \ldots, x_k := s] \quad (\gamma \vdash s : (a_1 \otimes \cdots \otimes a_k))$$

- η is *not* normalizing: restrict *base case* and *contexts*.

$$s \rightarrow s[\vec{x} := s] \rightarrow \vec{x}[\vec{x} := (\vec{y} := s)] \ldots$$

- Restricted η and β normalize *via* *combinatorial methods*.

Reduction: Let rules

$$\langle s_1, \ldots, s_i[\vec{x} := t], \ldots, s_n\rangle \to_{\mathrm{x}} \langle s_1, \ldots, s_n\rangle[\vec{x} := t]$$

$$s[\vec{x} := (t[\vec{y} := u])] \to_{\mathrm{x}} s[\vec{x} := t][\vec{y} := u]$$

- The reduction is sn but not *confluent*.
- we need *commutative equivalence*:

$$s[\vec{x} := t][\vec{y} := u] =_c s[\vec{y} := u][\vec{x} := t]$$

Coherence

Define a multicategory RM(X) :

- $\mathrm{ob}(\mathsf{RM}(X)) \ni a ::= o \in X \mid (a_1 \otimes \cdots \otimes a_k) \quad (k \in \mathbb{N})$.
- $\mathsf{RM}(X)(\gamma; a) = \{$ well-typed terms $\gamma \vdash s : a\} \, / \sim$.
- If $s \sim s'$ then $\mathsf{nf}(s) = \mathsf{nf}(s')$.
- If $\mathsf{nf}([s]) = \mathsf{nf}([s'])$ then $s \sim s'$.

Theorem (Mac Lane)

Let $s, s' \in \mathsf{nf}(\mathsf{RM}(X)(\gamma; a))$. *Then* $s = s$.

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Symmetric Case

We add symmetries to the calculus, staying *syntax directed.*

$$\frac{\gamma_1 \vdash s_1 : a_1 \ldots \gamma_k \vdash s_k : a_k \qquad \sigma \in \mathsf{shu}(\gamma_1, \ldots, \gamma_k)}{(\gamma_1, \ldots, \gamma_k) \cdot \sigma \vdash \langle s_1, \ldots, s_k \rangle : (a_1 \otimes \cdots \otimes a_k)}$$

$$\frac{\gamma \vdash s : \vec{a} \qquad \delta, \vec{x} : \vec{a}, \delta' \vdash t : b \qquad \sigma \in \mathsf{shu}(\delta, \gamma, \delta')}{(\delta, \gamma, \delta') \cdot \sigma \vdash t[x_1^{a_1}, \ldots, x_k^{a_k} := s] : b}$$

$$(a_1, \ldots, a_n) \cdot \sigma = a_{\sigma(1)}, \ldots, a_{\sigma(n)}$$

Theorem (Subject reduction)

If $\gamma \vdash s : a$ and $s \to s'$ then $\gamma \vdash s' : a$.

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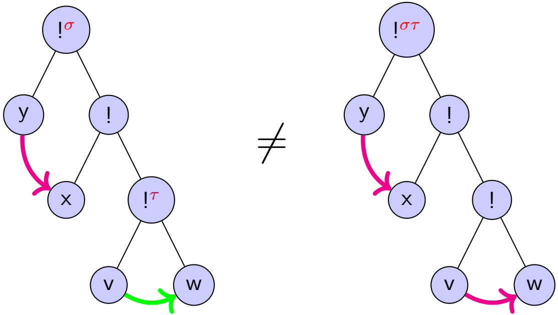
Canonicity *via* Shuffles



$$\dfrac{y \vdash y \qquad \dfrac{x \vdash x \qquad \dfrac{v \vdash v \qquad w \vdash w}{w, v \vdash \langle v, w\rangle}\,\tau}{x, w, v \vdash \langle x, \langle v, w\rangle\rangle}\,id}{x, y, w, v \vdash \langle x, \langle v, w\rangle\rangle}\,\sigma \qquad \dfrac{y \vdash y \qquad \dfrac{x \vdash x \qquad \dfrac{v \vdash v \qquad w \vdash w}{v, w \vdash \langle v, w\rangle}\,id}{x, v, w \vdash \langle x, \langle v, w\rangle\rangle}\,id}{x, y, w, v \vdash \langle x, \langle v, w\rangle\rangle}\,\sigma\tau$$

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- $\mathrm{ob}(\mathrm{SRM}(X)) \ni a ::= o \in X \mid (a_1 \otimes \cdots \otimes a_k) \quad (k \in \mathbb{N})$.
- $\mathrm{SRM}(X)(\gamma; a) = \{$ well-typed terms $\gamma \vdash s : a\} \,/\, \sim$.

Theorem

Let $s \in \mathrm{nf}(\mathrm{SRM}(X)(\gamma; a))$. There exists unique $\sigma \in \mathrm{Stab}(\gamma)$ and non-symmetric t s.t. $\sigma \cdot t = s$.

Theorem (Mac Lane)

Let $s, s' \in \mathrm{nf}(\mathrm{SRM}(X)(\gamma; a))$. If $\mathrm{sym}(s) = \mathrm{sym}(s')$ then $s = s'$.

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Outline

Autonomous Syntax

Closed Structure

$$\frac{\gamma, x_1 : a_1, \ldots, x_k : a_k \vdash s : b}{\gamma \vdash \lambda\langle x_1^{a_1}, \ldots, x_k^{a_k}\rangle.s : (a_1 \otimes \cdots \otimes a_k) \multimap b}$$

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Theorem

Let $[s], [s'] \in \mathrm{RM}(X)(\gamma; a)$. Then $s \sim s'$ iff $\mathrm{nf}(s) = \mathrm{nf}(s')$.

More general than (Kelly,Mac Lane)

No problems with types that contains $a \multimap ()$.

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Towards a Parametric Type Theory for Algebraic Theories

- We could generalize to arbitrary *structural rules*:

$$\mathsf{R}(\gamma, a_1, \ldots, a_k; a) \cong \mathsf{R}(\gamma, a_{\alpha(1)}, \ldots, a_{\alpha(k)}; a)$$

  where $\alpha : k \to k'$.
- Different choices of α gives different structures (affine, cartesian...).
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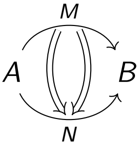
Normalization by Standardization

Two dimensional setting:



Theorem

Let $\alpha, \beta \in \mathsf{Free}(X)[M, N]$. $\alpha = \beta$ *iff* $\mathsf{nf}(\alpha) = \mathsf{nf}(\beta)$.

Thank You!